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We consider a model of intertemporal choice where time is a continuum, the set of instantaneous outcomes (e.g. consumption bundles) is a topological space, and where intertemporal plans (e.g. consumption streams) must be *continuous* functions of time. We assume the agent can form preferences over plans defined on open time intervals. We axiomatically characterize the intertemporal preferences that admit a representation via discounted utility integrals. In this representation, the utility function is continuous and unique up to positive affine transformations, and the discount structure is represented by a unique Riemann-Stieltjes integral plus a unique linear functional measuring the long-run asymptotic utility.

Key words: Intertemporal choice; intergenerational social choice; technological feasibility; continuous utility; Stone-Čech compactification.
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De natura Rationis est, res sub quadam æternitatis specie percipere.
—Spinoza

1. Introduction In most models of intertemporal choice, time is assumed to be discrete, and intertemporal preferences are represented by a discounted utility sum. But if time is a continuum, then such a discounted sum must be replaced with a discounted utility *integral*. Formally, let $\mathcal{X}$ be a space of instantaneous outcomes, and suppose we represent time by the interval $[0,T]$. Let us refer to a function $\alpha:[0,T]\longrightarrow\mathcal{X}$ as a *trajectory*; it represents a flow of outcomes over time. Let $u:\mathcal{X}\longrightarrow\mathbb{R}$ be a function representing instantaneous utility, and let $\delta:[0,T]\longrightarrow\mathbb{R}_+$ be a function representing pure time preferences. Then the *discounted utility integral* of the trajectory α is given by

$$\int_0^T \delta(t)\, u[\alpha(t)]\ \mathrm{d}t. \tag{1}$$

Recently, several papers have axiomatically characterized such discounted utility integral representations [13, 15, 18, 24, 30, 36, 45] (see Section 6 for a review). These papers assume a very large space of feasible trajectories, allowing all piecewise continuous functions or even all *measurable* functions from $[0,T]$ to $\mathcal{X}$. However, in many decision problems, discontinuous trajectories are not feasible. The purpose of the present paper is to axiomatically characterize discounted utility integral representations like (1) when only *continuous* trajectories are feasible.

In some cases, it is reasonable to truncate the planning interval, either by ignoring all outcomes *before* a certain moment in time, or by ignoring all outcomes *after* some other moment, or both, because the trajectory in these regions is fixed at some value and beyond the agent's control.

Formally, if $0<r<s<T$, then the agent may wish to restrict attention to continuous trajectories $\alpha:(r,s)\longrightarrow\mathcal{X}$. Her preferences over such trajectories would then be given by

$$\int_r^s \delta(t)\, u[\alpha(t)]\ \mathrm{d}t. \tag{2}$$

Some intertemporal decisions involve an infinite planning horizon —for example, decisions regarding economic or environmental policy that will have implications for many future generations. In this case, the representation (1) becomes

$$\int_0^\infty \delta(t)\, u[\alpha(t)]\ \mathrm{d}t. \tag{3}$$

But representations like (3) are poor at compromising between near future and far future generations. If δ decays too quickly, then the welfare of far future generations is essentially ignored, which seems ethically indefensible [35]. If δ decays too slowly, then the interests of near-future people are completely overwhelmed by their far more numerous descendants, which also seems unjust [28]. A sizable literature has evolved in response to this dilemma (see Section 6). In particular, Chichilnisky and Heal [5, 6, 7] have proposed intertemporal social welfare functions of the form

$$\int_0^\infty \delta(t)\, u[\alpha(t)]\ \mathrm{d}t \ +\ M\cdot \widetilde{\lim_{t\to\infty}}\ u[\alpha(t)]. \tag{4}$$

Here, the first summand is a standard discounted utility integral, biased towards the near future, while the second term is a correction factor that gives some weight to the long-term asymptotic trend in social welfare, reflecting the interests of far future generations. The tilde over the "lim" reflects the fact that the limit at infinity may not exist, in which case we must use a more sophisticated measure of the long-term asymptotic social welfare.

This paper considers intertemporal preferences over continuous trajectories, and provides an axiomatic characterization of discounted utility representations like (1) or (2) in the finite-horizon case, and (3) or (4) in the infinite-horizon case. The proofs of these characterization theorems use results from two earlier papers [32, 33]. The paper [32] develops a theory of integration for measures defined on the Boolean algebra of regular open subsets of a topological space. The paper [33] then applies this theory to develop subjective expected utility representations for preferences under uncertainty, when the state space and the outcome space are topological spaces. The present paper adapts the results of [32] and [33] to the special case when the underlying topological space is an interval in $\mathbb{R}$; using the topological properties of $\mathbb{R}$, we can reduce the SEU representations of [33] to discounted utility representations with a much simpler structure.

The remainder of this paper is organized as follows: Section 2 introduces notation and terminology. Section 3 introduces the axioms used in all our results. Section 4 gives two representation theorems for the finite-horizon case, while Section 5 gives three representation theorems for the infinite-horizon case. Section 6 reviews prior literature. All the proofs are in the Appendices.

2. Framework

Trajectories. Let $\mathcal{T}$ be a closed interval in $[0,\infty)$ with nonempty interior —either a bounded closed interval like $[0,T]$ (for some $T>0$) or an unbounded closed interval like $[0,\infty)$. Elements of $\mathcal{T}$ represent moments in time. Let $\mathcal{X}$ be any connected topological space (for example, a convex subset of $\mathbb{R}^N$); elements of $\mathcal{X}$ represent instantaneous outcomes. A function $\alpha:\mathcal{T}\longrightarrow\mathcal{X}$ is called a *trajectory*. Intertemporal preferences are preferences over such trajectories. However, we will suppose that only *bounded continuous* trajectories are feasible, as we now explain.

A subset $\mathcal{Y} \subseteq \mathcal{X}$ is *relatively compact* if its closure $\text{clos}(\mathcal{Y})$ is compact. In particular, if $\mathcal{X}$ is a metric space and $\mathcal{Y}$ is relatively compact, then $\mathcal{Y}$ is a bounded subset of $\mathcal{X}$. A function $\alpha : \mathcal{T} \longrightarrow \mathcal{X}$ is *bounded* if its image $\alpha(\mathcal{T})$ is relatively compact in $\mathcal{X}$. If $\mathcal{X}$ is a metric space, then this implies the usual definition of "bounded". But this definition makes sense even if $\mathcal{X}$ is nonmetrizable.

Let $\mathcal{C}(\mathcal{T}, \mathcal{X})$ be the set of all continuous functions from $\mathcal{T}$ into $\mathcal{X}$, and let $\mathcal{C}_\mathrm{b}(\mathcal{T}, \mathcal{X})$ be the set of all *bounded* continuous functions from $\mathcal{T}$ into $\mathcal{X}$. Let $\mathcal{A} \subseteq \mathcal{C}_\mathrm{b}(\mathcal{T}, \mathcal{X})$ denote the set of *feasible trajectories*. Trajectories may be subject to further feasibility constraints beyond continuity. For example, if $\mathcal{X} \subseteq \mathbb{R}^N$, then $\mathcal{A}$ may contain only differentiable functions from $\mathcal{T}$ to $\mathcal{X}$. In particular, $\mathcal{A}$ must satisfy a condition called (LV) that will be introduced below, which rules out excessively volatile trajectories. But $\mathcal{A}$ cannot be *too* small; it must be large enough to satisfy another structural condition called (R) (also introduced below). Also, $\mathcal{A}$ must contain all constant functions; these represent *static* trajectories.

Time intervals. A *time interval* is a subinterval $\mathcal{I} \subseteq \mathbb{R}$ (either open, closed, or half-open) that is *relatively open* as a subset of $\mathcal{T}$. We will indicate such a time interval by $\lfloor r, s \rceil$, where r and s denote the right-hand and left-hand end points. For example, if $\mathcal{T} = [0, \infty)$, then for any $r \in (0, \infty)$, we have $\lfloor 0, r \rceil := [0, r)$, while for any $s \in (r, \infty]$, we have $\lfloor r, s \rceil := (r, s)$; furthermore, any time interval in $(0, \infty)$ has one of these two forms. Meanwhile, if $\mathcal{T} = [a, b]$, then $\lfloor a, b \rceil := [a, b]$, and for any $r \in (a, b)$, we have $\lfloor a, r \rceil := [a, r)$ and $\lfloor r, b \rceil := (r, b]$, while for any $s \in (r, b)$, we have $\lfloor r, s \rceil := (r, s)$; furthermore, any time interval in $[a, b]$ has one of these four forms.[1] More generally, "$\lfloor r, s)$" represents $[a, s)$ if $r = a$, and (r, s) if $r > a$; likewise, "$(r, s\rceil$" represents $(r, b]$ if $s = b$, and (r, s) if $s < b$. Two time intervals $\mathcal{I}, \mathcal{J} \subset \mathcal{T}$ are *adjacent* if $\mathcal{I}$ and $\mathcal{J}$ are disjoint but share a common endpoint —that is, there exist $r < s < t$ such that either $\mathcal{I} = \lfloor r, s \rceil$ and $\mathcal{J} = \lfloor s, t \rceil$, or $\mathcal{J} = \lfloor r, s \rceil$ and $\mathcal{I} = \lfloor s, t \rceil$. In either case, we define $\mathcal{I} \vee \mathcal{J} := \lfloor r, t \rceil$. Let $\mathfrak{I}(\mathcal{T})$ denote the family of all time intervals of $\mathcal{T}$; when there is no ambiguity, we will simply call this set $\mathfrak{I}$.

Time spans. Let $\mathcal{S} \subseteq \mathcal{T}$ be an open subset, and let $\partial\mathcal{S}$ denote its boundary. We say that $\mathcal{S}$ is a *time span* if $\mathcal{S}$ is a disjoint union of a finite or countable collection of time intervals with distinct endpoints, and $\partial\mathcal{S}$ has no cluster points in $\mathbb{R}$. In other words, either

$$\mathcal{S} \quad = \quad \lfloor s_1, t_1) \sqcup (s_2, t_2) \sqcup \cdots \sqcup (s_N, t_N \rceil, \tag{5}$$

where $0 \leq s_1 < t_1 < s_2 < t_2 < \cdots < s_N < t_N \leq \infty$, or

$$\mathcal{S} \quad = \quad \lfloor s_1, t_1) \sqcup (s_2, t_2) \sqcup (s_3, t_3) \sqcup \cdots \tag{6}$$

where $0 \leq s_1 < t_1 < s_2 < t_2 < \cdots$ and $\lim_{n \to \infty} s_n = \infty$. (If $\mathcal{T}$ is bounded, then only option (5) is possible.) In either case, we refer to $\lfloor s_1, t_1)$, $(s_2, t_2) \ldots$ as the *component intervals* of $\mathcal{S}$. Let $\mathfrak{S}(\mathcal{T})$ denote the family of all time spans of $\mathcal{T}$; when there is no ambiguity, we will simply call this set $\mathfrak{S}$. Note that the intersection of two time spans is also a time span. However, the union of two time spans is generally not a time span. Likewise, the complement of a time span is generally not a time span. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are time spans, then we define $\mathcal{S}_1 \vee \mathcal{S}_2 := \text{int}[\text{clos}(\mathcal{S}_1 \cup \mathcal{S}_2)]$.[2] For example, if $r < s < t$, $\mathcal{S}_1 = \lfloor r, s)$ and $\mathcal{S}_2 = (s, t\rceil$, then $\mathcal{S}_1 \vee \mathcal{S}_2 = \lfloor r, t \rceil$; thus, this notation agrees with the notation for time intervals. Intuitively, the operation $\vee$ plays the role of "union", but with the extra proviso that a single isolated time instant is not perceptible, so we should treat the disjoint union $\lfloor r, s) \sqcup (s, t\rceil$ as indistinguishible from the interval $\lfloor r, t \rceil$.

[1] To understand this, note that $[a, r)$, $(s, b]$, and $[a, b]$ are relatively open subsets of $[a, b]$, even though they are not open subsets of $\mathbb{R}$ itself.

[2] Here, for any $\mathcal{S} \subseteq \mathcal{T}$, $\text{int}(\mathcal{S})$ and $\text{clos}(\mathcal{S})$ denotes the *relative interior* and *relative closure* of $\mathcal{S}$ in $\mathcal{T}$.

If $\mathcal{S}$ is a time span, then we define $\neg\mathcal{S} := \mathrm{int}[\mathcal{S}^{\complement}]$. For example, if $\mathcal{T} = [0,T]$ and $\mathcal{S}$ is as in (5), then $\neg\mathcal{S} = [0,s_1) \sqcup (t_1,s_2) \sqcup (t_2,s_3) \sqcup \cdots \sqcup (t_{N-1},s_N) \sqcup (t_N,T]$. Observe that $(\neg\mathcal{S}) \vee \mathcal{S} = \mathcal{T}$. Thus, $\neg\mathcal{S}$ plays the role of the "complement" of $\mathcal{S}$, once we regard $\vee$ as playing the role of "union".[3] For any $\mathcal{S} \in \mathfrak{S}$ and $\alpha \in \mathcal{A}$, let $\alpha_{\restriction\mathcal{S}}$ denote the restriction of α to $\mathcal{S}$. Let $\mathcal{A}(\mathcal{S}) := \{\alpha_{\restriction\mathcal{S}};\ \ \alpha \in \mathcal{A}\}$ be the set of $\mathcal{S}$-restricted trajectories. We will posit a preference order on $\mathcal{A}(\mathcal{S})$, as we now explain.

Intertemporal preferences. For any $\mathcal{I} \in \mathfrak{I}$, let $\succeq_{\mathcal{I}}$ be a preference order on $\mathcal{A}(\mathcal{I})$; we refer to these as *intertemporal preferences*.[4] Suppose $\mathcal{I} = \lfloor s,t \rceil$. As explained in Section 1, $\succeq_{\mathcal{I}}$ represents the intertemporal preferences of an agent over trajectories whose value outside of $\mathcal{I}$ is fixed and beyond her control.[5] For brevity, let $\succeq$ denote $\succeq_{\mathcal{T}}$ —that is, intertemporal preferences over $\mathcal{A}$ itself. We will refer to the collection $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ as an *intertemporal preference structure*; this will be the primitive data of the model. Our goal is to axiomatically characterize a discounted utility integral representation for $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$.

Separability and intermittent preferences. We will require the intertemporal preference structure $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ to satisfy the following separability axiom.

(LSep) (*Limited separability*) For any time span $\mathcal{S} \in \mathfrak{S}$, and all $\alpha, \alpha', \beta, \beta' \in \mathcal{A}$ such that $\alpha_{\restriction\mathcal{S}} = \alpha'_{\restriction\mathcal{S}}$ and $\beta_{\restriction\mathcal{S}} = \beta'_{\restriction\mathcal{S}}$, while $\alpha_{\restriction\mathcal{I}} \approx_{\mathcal{I}} \beta_{\restriction\mathcal{I}}$ and $\alpha'_{\restriction\mathcal{I}} \approx_{\mathcal{I}} \beta'_{\restriction\mathcal{I}}$ for every component interval $\mathcal{I}$ of $\neg\mathcal{S}$, we have $\alpha \succeq \beta$ if and only if $\alpha' \succeq \beta'$.

The normative justification for this axiom is straightforward: if the trajectories α and β are indifferent to one another when they are restricted to any component interval of $\neg\mathcal{S}$, then the agent's preferences between α and β should be entirely determined by comparing $\alpha_{\restriction\mathcal{S}}$ to $\beta_{\restriction\mathcal{S}}$. For the same reason, the agent's preferences between α' and β' should be entirely determined by comparing $\alpha'_{\restriction\mathcal{S}}$ to $\beta'_{\restriction\mathcal{S}}$. But if α and α' are identical when restricted to $\mathcal{S}$, and β and β' are identical when restricted to $\mathcal{S}$, then the comparison between $\alpha_{\restriction\mathcal{S}}$ and $\beta_{\restriction\mathcal{S}}$ is equivalent to a comparison between $\alpha'_{\restriction\mathcal{S}}$ and $\beta'_{\restriction\mathcal{S}}$. Thus, the agent's preferences between α and β should be identical to her preferences between α' and β'. Although its normative import is clear, the formal statement of (LSep) is somewhat obscure. Fortunately, it is equivalent to the following, more transparent axiom:

(LSep*) For any $\mathcal{S} \in \mathfrak{S}$, there is a preference order $\succeq_{\mathcal{S}}$ on $\mathcal{A}(\mathcal{S})$ such that, for any $\alpha, \beta \in \mathcal{A}$ with $\alpha_{\restriction\mathcal{I}} \approx_{\mathcal{I}} \beta_{\restriction\mathcal{I}}$ for every component interval $\mathcal{I}$ of $\neg\mathcal{S}$, we have $\alpha \succeq \beta$ if and only if $\alpha_{\restriction\mathcal{S}} \succeq_{\mathcal{S}} \beta_{\restriction\mathcal{S}}$.

Axiom (LSep*) has the same normative justification as (LSep): if $\alpha_{\restriction\mathcal{I}} \approx_{\mathcal{I}} \beta_{\restriction\mathcal{I}}$ for every component interval $\mathcal{I}$ of $\neg\mathcal{S}$, then the agent's preferences between α and β should be determined by comparing $\alpha_{\restriction\mathcal{S}}$ to $\beta_{\restriction\mathcal{S}}$. But axiom (LSep*) goes further, and asserts that this comparison between $\alpha_{\restriction\mathcal{S}}$ and $\beta_{\restriction\mathcal{S}}$ is itself governed by a preference order $\succeq_{\mathcal{S}}$ defined on $\mathcal{A}(\mathcal{S})$. This seems like a stronger requirement than (LSep). But under a mild richness condition (which will be part of the framework in Section 3), axioms (LSep) and (LSep*) are logically equivalent (see Lemma A.4 in Appendix A).

We will refer to the preference order $\succeq_{\mathcal{S}}$ defined by axiom (LSep*) as an *intermittent preference order*. If the intertemporal preference structure $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfies (LSep*), then we obtain a collection $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$, which we will refer to as an *intermittent preference structure*. Note that this

[3] In fact, $\mathfrak{S}$ is a Boolean algebra with these three operations. But this is not important for this discussion.

[4] Here by *preference order* we mean a binary relation that is complete, transitive, and reflexive. We do not require it to be antisymmetric.

[5] This involves an implicit separability assumption, since we suppose that $\succeq_{\mathcal{I}}$ is independent of the way in which trajectories are fixed outside of $\mathcal{I}$. One solution, previously used by [18], is to suppose that trajectories outside of $\mathcal{I}$ are always fixed at some "neutral" value, representing *status quo*, death, or unconsciousness. This may make trajectories discontinuous at the boundaries of $\mathcal{I}$, but it would be compatible with our feasibility constraints if we only required these constraints to apply *inside* $\mathcal{I}$.

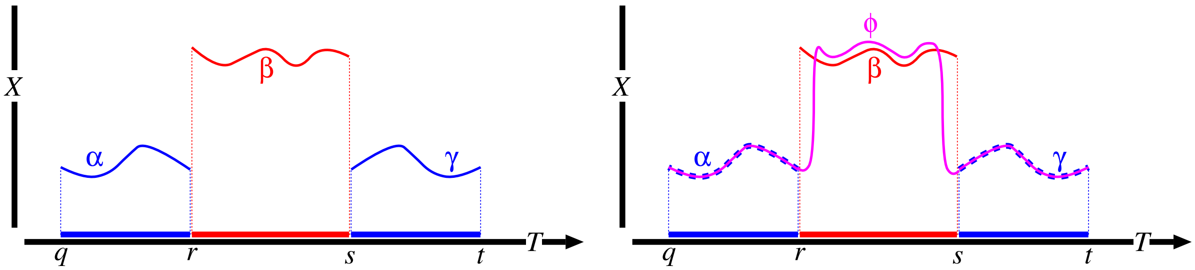

FIGURE 1. The richness condition (R).

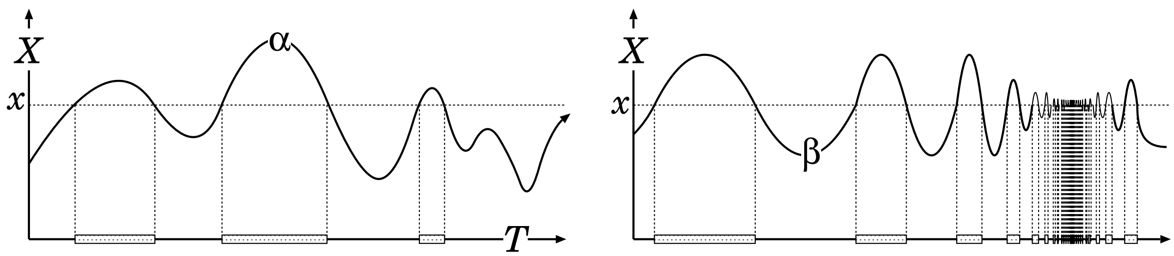

FIGURE 2. *Left.* The trajectory α satisfies axiom (LV). The horizontal axis represents $\mathcal{T}$, and the grey intervals on this axis represent $\mathcal{X}(\succ_{\text{sy}} x)$. (Here, the synchronic preference $\succeq_{\text{sy}}$ corresponds to height on the vertical axis representing $\mathcal{X}$.) *Right.* The trajectory β violates axiom (LV).

intermittent preference structure is entirely determined by $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$; we will say that $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ is *induced by* $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$. In the special case when $\mathcal{S}$ itself is a time interval, the intermittent preference order on $\mathcal{A}(\mathcal{S})$ induced by axiom (LSep*) is in fact identical with the intertemporal preference order (see Lemma A.5 in Appendix A). Thus, we will use the same symbol to refer to both, and regard the system $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ as an extension of the system $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$. The discounted utility integral representations we introduce below apply to all elements of $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$, not only those in $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$. Likewise, many of our axioms are stated directly in terms of $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ rather than $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$. Thus, we could just as easily take the intermittent preference structure $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ itself to be the primitive data of the model. We started with the intertemporal preference structure $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ only because it has a more transparent economic interpretation.

3. Axioms

We will assume that each order $\succeq_{\mathcal{I}}$ in the intertemporal preference structure $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ is *complete* (for any $\alpha,\beta\in\mathcal{A}(\mathcal{I})$, $\alpha\succeq_{\mathcal{I}}\beta$ or $\beta\succeq_{\mathcal{I}}\alpha$), *transitive* (for any $\alpha,\beta,\gamma\in\mathcal{A}(\mathcal{I})$, if $\alpha\succeq_{\mathcal{I}}\beta$ and $\beta\succeq_{\mathcal{I}}\gamma$, then $\alpha\succeq_{\mathcal{I}}\gamma$), and *nontrivial* (there exist $\alpha,\beta\in\mathcal{A}(\mathcal{I})$ such that $\alpha\succ_{\mathcal{I}}\beta$). We will need $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ to satisfy eight axioms. We will also impose two structural conditions that involve both $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ and $\mathcal{A}$.

Richness. The first structural condition says that $\mathcal{A}$ contains a rich enough variety of trajectories that certain vicissitudes of fortune are realizable by trajectories in $\mathcal{A}$.

(R) For all $q\le r\le s\le t$ in $\mathcal{T}$, and any $\alpha\in\mathcal{A}\lfloor q,r\rceil$, $\beta\in\mathcal{A}\lfloor r,s\rceil$, and $\gamma\in\mathcal{A}\lfloor s,t\rceil$, there exists some $\phi\in\mathcal{A}\lfloor q,t\rceil$ such that $\phi_{\restriction\lfloor q,r\rceil}=\alpha$, $\phi_{\restriction\lfloor r,s\rceil}\approx_{\lfloor r,s\rceil}\beta$, and $\phi_{\restriction\lfloor s,t\rceil}=\gamma$.

In other words, the values of a trajectory during $\lfloor q,r\rceil$ and $\lfloor s,t\rceil$ do not restrict the indifference class of that trajectory during $\lfloor r,s\rceil$, in spite of the continuity requirement on feasible trajectories. Figure 1 illustrates this idea.

Example 1. Let $\mathcal{X}=\mathbb{R}$, and let $\mathcal{A}$ be the set of all *piecewise linear* functions from $\mathcal{T}$ into $\mathbb{R}$. For any time interval $\mathcal{I}=\lfloor r,s\rceil$ in $\mathfrak{I}$, and any $\alpha,\beta\in\mathcal{A}$, suppose that $\alpha\succeq_{\mathcal{I}}\beta$ if and only if $\int_r^s\alpha(t)\ \mathrm{d}t\ge$

$\int_r^s \beta(t)\ \mathrm{d}t$. (Thus, $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ has a discounted utility integral representation, where the discount factor is constant, and the utility function is $u(x)=x$ for all $x\in\mathbb{R}$.) It is easily verified that $\mathcal{A}$ and $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfy (R). ◇

Synchronic preferences and limited variation. For any $x\in\mathcal{X}$, let κ^x be the constant x-valued trajectory on $\mathcal{T}$. Let $\mathcal{K}:=\{\kappa^x;\ x\in\mathcal{X}\}$. We have assumed $\mathcal{K}\subseteq\mathcal{A}$, so the preference order $\succeq$, restricted to $\mathcal{K}$, induces a preference order $\succeq_{\mathrm{sy}}$ on $\mathcal{X}$ as follows: for any $x,y\in\mathcal{X}$,

$$\Big(x\succeq_{\mathrm{sy}} y\Big)\quad\iff\quad\Big(\kappa^x\succeq\kappa^y\Big).\tag{7}$$

We interpret $\succeq_{\mathrm{sy}}$ as preferences over outcomes; we will call it the *synchronic preference order*. For any $x\in\mathcal{X}$, let $\mathcal{X}(\succeq_{\mathrm{sy}} x):=\{y\in\mathcal{X};\ y\succeq_{\mathrm{sy}} x\}$ and $\mathcal{X}(\preceq_{\mathrm{sy}} x):=\{y\in\mathcal{X};\ y\preceq_{\mathrm{sy}} x\}$ be the (weak) upper and lower contour sets of x with respect to these synchronic preferences. Our second structural condition says feasible trajectories have "limited variation" with respect to these contour sets.

(LV) For all $x\in\mathcal{X}$ and $\alpha\in\mathcal{A}$, $\mathrm{int}\left(\alpha^{-1}\left[\mathcal{X}(\succeq_{\mathrm{sy}} x)\right]\right)$ and $\mathrm{int}\left(\alpha^{-1}\left[\mathcal{X}(\preceq_{\mathrm{sy}} x)\right]\right)$ are elements of $\mathfrak{S}$.

This condition relates the objective variations of feasible trajectories to the agent's subjective experience of these variations. It says that α does not oscillate too wildly between good and bad outcomes *from the perspective of the agent*. In particular, α can only move back and forth between the interior of $\mathcal{X}(\succeq_{\mathrm{sy}} x)$ and the interior of $\mathcal{X}(\preceq_{\mathrm{sy}} x)$ *finitely* many times in any finite time interval. For example, let $\mathcal{T}=[0,T]$ and $\mathcal{X}=\mathbb{R}$; then (LV) rules out a situation where $u(x)=x$ for all $x\in\mathcal{X}$ while $\alpha(t)=t\sin(1/t)$ for all $t\in\mathcal{T}$. It also rules out a situation where $u(x)=x\sin(1/x)$ for all $x\in\mathcal{X}$ while $\alpha(t)=t$ for all $t\in\mathcal{T}$.[6] See Figure 2.

Example 2. Let $\mathcal{X}=\mathbb{R}$ and let $\mathcal{T}=[0,T]$. A function $\alpha:\mathcal{T}\longrightarrow\mathbb{R}$ is *piecewise polynomial* if there exists $t_0,\ldots,t_N\in\mathcal{T}$ with $0=t_0<t_1<\cdots<t_{N-1}<t_N=T$ such that $\alpha_{\upharpoonright[t_{n-1},t_n]}$ is a polynomial function for all $n\in[1\ldots N]$. (Piecewise linear functions are a special case.) Let $\mathcal{A}$ be the set of all piecewise polynomial functions on $\mathcal{T}$. Let $u:\mathbb{R}\longrightarrow\mathbb{R}$ be continuous and strictly increasing. For any time interval $\mathcal{I}=\lfloor r,s\rceil$ in $\mathfrak{I}$ and $\alpha,\beta\in\mathcal{A}$, suppose that

$$\Big(\alpha\succeq_{\mathcal{I}}\beta\Big)\quad\iff\quad\left(\int_r^s u[\alpha(t)]\ \mathrm{d}t\geq\int_r^s u[\beta(t)]\ \mathrm{d}t\right).$$

It is easily verified that $\mathcal{A}$ and $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfy both (LV) and (R). ◇

Note that (LV) need only be satisfied by the set of trajectories used in the axiomatic characterization of the discounted utilility integral representation. The intertemporal preferences themselves —and their discounted utilility integral representation —can apply to a much larger set of trajectories, as shown by Theorem 3 at the end of Section 4.

Continuity, Dominance and Separability. Having stated the structural conditions, we can now proceed with the axioms. We will require $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ to satisfy the axiom (LSep) stated in Section 2. Thus, $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ induces a unique intermittent preference structure $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ via the equivalent axiom (LSep*). Although $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ is the underlying primitive data of the model, it will be simpler to formulate most of the axioms directly in terms of $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$. The first four axioms are standard axioms in decision theory, so we present them without explanation.

(C) (*Continuity*) $\succeq_{\mathrm{sy}}$ is continuous in the topology on $\mathcal{X}$. That is: for all $x\in\mathcal{X}$, the contour sets $\mathcal{X}(\succeq_{\mathrm{sy}} x)$ and $\mathcal{X}(\preceq_{\mathrm{sy}} x)$ are closed subsets of $\mathcal{X}$.

[6] In both cases we adopt the convention that $0\cdot\sin(1/0):=0$.

(Dom) (*Dominance*) For any $\mathcal{S} \in \mathfrak{S}$ and any $\alpha, \beta \in \mathcal{A}(\mathcal{S})$, if $\alpha(s) \succeq_{\rm sy} \beta(s)$ for all $s \in \mathcal{S}$, then $\alpha \succeq_{\mathcal{S}} \beta$. Furthermore, if $\alpha(s) \succ_{\rm sy} \beta(s)$ for all $s \in \mathcal{S}$, then $\alpha \succ_{\mathcal{S}} \beta$.

(StEq) (*Static equivalents*) For any time span $\mathcal{S} \in \mathfrak{S}$ and any trajectory $\alpha \in \mathcal{A}(\mathcal{S})$, there exists some $x \in \mathcal{X}$ such that $\kappa^x_{\upharpoonright \mathcal{S}} \approx_{\mathcal{S}} \alpha$.

(ISep) (*Interval separability*) For any adjacent time intervals $\mathcal{I}, \mathcal{J} \subset \mathcal{T}$ with $\mathcal{H} := \mathcal{I} \vee \mathcal{J}$, and any $\alpha, \beta \in \mathcal{A}(\mathcal{H})$ with $\alpha_{\upharpoonright \mathcal{I}} \approx_{\mathcal{I}} \beta_{\upharpoonright \mathcal{I}}$, we have $\alpha \succeq_{\mathcal{H}} \beta$ if and only if $\alpha_{\upharpoonright \mathcal{J}} \succeq_{\mathcal{J}} \beta_{\upharpoonright \mathcal{J}}$.

Axiom (ISep) is relatively weak: it only requires separability between adjacent time intervals, rather than between arbitrary subsets of $\mathcal{T}$ (as would be the case with the standard separability axiom). But (ISep) and (LSep*) together imply the following, stronger separability property:

(Sep) For any time span $\mathcal{S} \in \mathfrak{S}$, any disjoint $\mathcal{Q}, \mathcal{R} \in \mathfrak{S}$ such that $\mathcal{Q} \vee \mathcal{R} = \mathcal{S}$, and any trajectories $\alpha, \beta \in \mathcal{A}(\mathcal{S})$ with $\alpha_{\upharpoonright \mathcal{Q}} \approx_{\mathcal{Q}} \beta_{\upharpoonright \mathcal{Q}}$, we have $\alpha \succeq_{\mathcal{S}} \beta$ if and only if $\alpha_{\upharpoonright \mathcal{R}} \succeq_{\mathcal{R}} \beta_{\upharpoonright \mathcal{R}}$.

We will not require (Sep) explicitly in our axiomatic framework, but it is worth noting that it is a consequence of (ISep) and (LSep*) (see Proposition A.1 in Appendix A). If $\mathcal{Q}, \mathcal{R} \in \mathfrak{S}$ are disjoint and $\mathcal{S} = \mathcal{Q} \vee \mathcal{R}$, then (Sep) says that the $\succeq_{\mathcal{S}}$-ranking of two trajectories $\alpha, \beta \in \mathcal{A}(\mathcal{S})$ is partly determined by the $\succeq_{\mathcal{Q}}$-ranking of $\alpha_{\upharpoonright \mathcal{Q}}$ versus $\beta_{\upharpoonright \mathcal{Q}}$ and the $\succeq_{\mathcal{R}}$-ranking of $\alpha_{\upharpoonright \mathcal{R}}$ versus $\beta_{\upharpoonright \mathcal{R}}$. The next axiom says that this dependency is continuous.

(CIP) (*Continuity of intertemporal preferences*) Let $\mathcal{S} = \mathcal{Q} \vee \mathcal{R}$ as in axiom (Sep). Let $\underline{\beta}, \alpha, \overline{\beta} \in \mathcal{A}(\mathcal{S})$ be three trajectories with $\underline{\beta} \prec_{\mathcal{S}} \alpha \prec_{\mathcal{S}} \overline{\beta}$. Then there exist $\underline{\delta}, \overline{\delta} \in \mathcal{A}(\mathcal{Q})$ and $\underline{\epsilon}, \overline{\epsilon} \in \mathcal{A}(\mathcal{R})$, with $\underline{\delta} \prec_{\mathcal{Q}} \alpha_{\upharpoonright \mathcal{Q}} \prec_{\mathcal{Q}} \overline{\delta}$ and $\underline{\epsilon} \prec_{\mathcal{R}} \alpha_{\upharpoonright \mathcal{R}} \prec_{\mathcal{R}} \overline{\epsilon}$ such that, for any $\alpha' \in \mathcal{A}(\mathcal{S})$, if $\underline{\delta} \prec_{\mathcal{Q}} \alpha'_{\upharpoonright \mathcal{Q}} \prec_{\mathcal{Q}} \overline{\delta}$ and $\underline{\epsilon} \prec_{\mathcal{R}} \alpha'_{\upharpoonright \mathcal{R}} \prec_{\mathcal{R}} \overline{\epsilon}$ then $\underline{\beta} \prec_{\mathcal{S}} \alpha' \prec_{\mathcal{S}} \overline{\beta}$.

The intuition here is that a small variation in $\alpha_{\upharpoonright \mathcal{Q}}$ and $\alpha_{\upharpoonright \mathcal{R}}$ (relative to the order topologies on $\mathcal{A}(\mathcal{Q})$ and $\mathcal{A}(\mathcal{R})$) should not affect the $\succeq_{\mathcal{S}}$- ranking of α versus $\underline{\beta}$ and $\overline{\beta}$.

Tradeoff consistency. The next axiom was introduced by Wakker, who used it to characterize additive representations in several settings [40, 41, 42, 43, 19].[7] First we need some notation. Let $\mathcal{S} \in \mathfrak{S}$, and let $\mathcal{Q} := \neg \mathcal{S}$. Consider an outcome $x \in \mathcal{X}$ and a trajectory $\alpha \in \mathcal{A}(\mathcal{Q})$. If $\mathcal{A}$ and $\{\succeq_{\mathcal{I}}\}_{\mathcal{I} \in \mathfrak{I}}$ satisfy (R), (ISep), and (LSep*), then there exists a trajectory $(x_{\mathcal{S}} \alpha) \in \mathcal{A}$ with two properties:

$$\textbf{(B1)} \quad (x_{\mathcal{S}}\alpha)_{\upharpoonright \mathcal{S}} \approx_{\mathcal{S}} \kappa^x_{\upharpoonright \mathcal{S}}, \qquad \text{and} \qquad \textbf{(B2)} \quad (x_{\mathcal{S}}\alpha)_{\upharpoonright \mathcal{Q}} \approx_{\mathcal{Q}} \alpha.$$

(See Lemma A.8 in Appendix A.) In general, $(x_{\mathcal{S}}\alpha)$ is not uniquely defined by (B1) and (B2). But if $(x_{\mathcal{S}}\alpha)$ and $(x_{\mathcal{S}}\alpha)'$ are two trajectories satisfying (B1) and (B2), then axiom (Sep) implies that $(x_{\mathcal{S}}\alpha) \approx (x_{\mathcal{S}}\alpha)'$. So we shall use "$(x_{\mathcal{S}}\alpha)$" to refer to any trajectory satisfying (B1) and (B2).

Now fix $x, y, v, w \in \mathcal{X}$, and $\mathcal{S} \in \mathfrak{S}$. Let $\mathcal{Q} := \neg \mathcal{S}$. We write $(x \overset{\mathcal{S}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{S}}{\rightsquigarrow} w)$ if there exist $\alpha, \beta \in \mathcal{A}(\mathcal{Q})$ such that $(x_{\mathcal{S}}\alpha) \preceq (y_{\mathcal{S}}\beta)$ while $(v_{\mathcal{S}}\alpha) \succeq (w_{\mathcal{S}}\beta)$. If $(x_{\mathcal{S}}\alpha) \preceq (y_{\mathcal{S}}\beta)$, then the "gain" obtained by changing x to y during $\mathcal{S}$ is at least enough to compensate for the "loss" incurred by changing α to β during $\mathcal{Q}$. In contrast, if $(v_{\mathcal{S}}\alpha) \succeq (w_{\mathcal{S}}\beta)$, then the gain obtained by changing v to w during $\mathcal{S}$ is at *most* enough to compensate for the loss incurred by changing α to β during $\mathcal{Q}$. Together, these two observations imply that the gain obtained from changing x to y during $\mathcal{S}$ is at least as large as the gain from changing v to w during $\mathcal{S}$; hence the notation $(x \overset{\mathcal{S}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{S}}{\rightsquigarrow} w)$. If $\succeq$ has a discounted utility integral representation with utility function u, then $(x \overset{\mathcal{S}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{S}}{\rightsquigarrow} w)$ means that $u(y) - u(x) \geq u(w) - u(v)$.

[7] In the first four publications, the axiom is called *triple cancellation*. In the last paper (jointly written with Köbberling), it is called *Tradeoff Consistency* and formulated slightly differently, but conceptually very similar.

Conversely, we write $(x \overset{\mathcal{S}}{\rightsquigarrow} y) \prec (v \overset{\mathcal{S}}{\rightsquigarrow} w)$ if there exist $\gamma, \delta \in \mathcal{A}(\mathcal{Q})$ such that $(x_{\mathcal{S}}\gamma) \succeq (y_{\mathcal{S}}\delta)$ while $(v_{\mathcal{S}}\gamma) \prec (w_{\mathcal{S}}\delta)$. If $\succeq$ had a discounted utility integral representation, then this means that $u(y) - u(x) < u(w) - u(v)$. Thus, if $\succeq$ had a discounted utility integral representation, then we could not have both $(x_{\mathcal{S}}\alpha) \preceq (y_{\mathcal{S}}\beta)$ and $(x \overset{\mathcal{S}}{\rightsquigarrow} y) \prec (v \overset{\mathcal{S}}{\rightsquigarrow} w)$. This observation motivates the next axiom.

(TC) (*Tradeoff Consistency*) For any two time spans $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{S}$, there are no outcomes $x, y, v, w \in \mathcal{X}$ such that $(x \overset{\mathcal{S}_1}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{S}_1}{\rightsquigarrow} w)$ while $(x \overset{\mathcal{S}_2}{\rightsquigarrow} y) \prec (v \overset{\mathcal{S}_2}{\rightsquigarrow} w)$.

Ephemera. If time is a continuum, then any single instant is ephemeral, and cannot have any importance for intertemporal preferences. The next axiom formalizes this observation.

(Eph) For any $t \in \mathcal{T}$, and any outcomes $w, x, y, z \in \mathcal{X}$ with $y \succ_{\rm sy} z$, there is some $\mathcal{S} \in \mathfrak{S}$ with $t \in \mathcal{S}$ such that $(w_{\mathcal{S}}y) \succ (x_{\mathcal{S}}z)$.

Thus, if $\mathcal{S}$ is small enough, then it is relatively unimportant for the agent's intertemporal assessments: her preferences between $(w_{\mathcal{S}}y)$ and $(x_{\mathcal{S}}z)$ are determined by the fact that $y \succ_{\rm sy} z$. In particular, this holds *even if* $w \prec_{\rm sy} x$. We can construct such a neighbourhood $\mathcal{S}$ for any $w, x, y, z \in \mathcal{X}$; thus the agent cannot place any special weight on t; it is ephemeral.

Stationarity. Our last axiom is not necessary to obtain a discounted utility representation. But by adding it to the other axioms, we ensure that the discounting is *exponential.* To state this axiom, we need some notation. For any time interval $\lfloor a, b\rceil$ in $\mathfrak{I}$, and any $t \in \mathbb{R}$, we define $\lfloor a, b\rceil^{+t} := \lfloor a+t, b+t\rceil$ whenever $\lfloor a+t, b+t\rceil$ itself is a time interval in $\mathfrak{I}$. (Thus, if $t > 0$, and $\mathcal{T} = [0, T]$ and $0 < a < b < T - t$, then $(a, b)^{+t} = (a+t, b+t)$, while $[0, b)^{+t} = (t, b+t)$. However, if $b = T - t$, then $(a, b)^{+t} = (a+t, T]$.) If $\mathcal{S} \in \mathfrak{S}$ has component intervals $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_N$, then we likewise define $\mathcal{S}^{+t} := \mathcal{I}_1^{+t} \sqcup \cdots \sqcup \mathcal{I}_N^{+t}$ —another element of $\mathfrak{S}$. In effect, this is the time span $\mathcal{S}$ "shifted forwards in time" by t time units. For any function $\alpha \in \mathcal{C}(\mathcal{S}, \mathcal{X})$, define $\alpha^{+t} \in \mathcal{C}(\mathcal{S}^{+t}, \mathcal{X})$ by setting $\alpha^{+t}(s) := \alpha(s - t)$ for all $s \in \mathcal{S}^{+t}$. Thus, if α is a trajectory on time span $\mathcal{S}$, then α^{+t} is the same trajectory "shifted forwards in time" by t time units, so as to become a trajectory on $\mathcal{S}^{+t}$.

The set of feasible trajectories may not be closed under such time-shifts; we must add an additional structural condition to guarantee this. Let $\mathcal{A}(\mathcal{S})^{+t} := \{\alpha^{+t};\ \ \alpha \in \mathcal{A}(\mathcal{S})\}$. If we required $\mathcal{A}(\mathcal{S})^{+t} \subseteq \mathcal{A}(\mathcal{S}^{+t})$, then we would be stipulating that trajectories feasible during time span $\mathcal{S}$ *remain* feasible at any later time span. On the other hand, if we required $\mathcal{A}(\mathcal{S})^{+t} \supseteq \mathcal{A}(\mathcal{S}^{+t})$, then we would be stipulating that any trajectory feasible during time span $\mathcal{S}^{+t}$ was *already* feasible at the earlier time span $\mathcal{S}$. Either assumption is too restrictive. Instead, we make the following assumption.

(Core) There is a subset $\mathcal{A}_0 \subseteq \mathcal{A}$ such that, for all $\mathcal{S} \in \mathfrak{S}$, and all $t \in [0, \infty)$ such that $\mathcal{S}^{+t} \subseteq \mathcal{T}$, we have $\mathcal{A}_0(\mathcal{S})^{+t} \subseteq \mathcal{A}(\mathcal{S}^{+t})$. Furthermore, $\mathcal{A}_0$ satisfies condition (R).

Intuitively, $\mathcal{A}_0$ is a stable "core" of trajectories that remain feasible at all times. Note that $\mathcal{A}_0$ is always nonempty, because all *constant* trajectories are in $\mathcal{A}_0$; the key part of (Core) is that $\mathcal{A}_0$ is large enough that it satisfies (R). Clearly, if $\mathcal{A} = \mathcal{C}_{\rm b}(\mathcal{T}, \mathcal{X})$, then (Core) is automatically true. Given this condition, we can formulate a stationarity axiom.

(Stat) (*Stationarity*) Let $\mathcal{A}_0$ be as in (Core). For all $\mathcal{S} \in \mathfrak{S}$, all $t \in [0, \infty)$ such that $\mathcal{S}^{+t} \subseteq \mathcal{T}$, and all $\alpha, \beta \in \mathcal{A}_0(\mathcal{S})$, we have $\left(\alpha \succeq_{\mathcal{S}} \beta\right) \iff \left(\alpha^{+t} \succeq_{\mathcal{S}^{+t}} \beta^{+t}\right)$.

4. Finite horizon intertemporal decisions Throughout this section, we suppose that $\mathcal{T} = [0, T]$ for some $T < \infty$. Let $\mathfrak{I} := \mathfrak{I}[0, T]$. Let $\mathcal{X}$ be a connected topological space, let $\mathcal{A} \subseteq \mathcal{C}_{\rm b}([0, T], \mathcal{X})$, and let $\{\succeq_{\mathcal{I}}\}_{\mathcal{I} \in \mathfrak{I}}$ be an intertemporal preference structure on $\mathcal{A}$, which induces an intermittent preference structure $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ through axiom (LSep*). A *discounted utility integral* (DUI) *representation* for $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ is given by a continuous, strictly increasing bijection $\rho:[0,T]\longrightarrow[0,1]$ and a continuous utility function $u:\mathcal{X}\longrightarrow\mathbb{R}$, such that, for all $\mathcal{S}\in\mathfrak{S}$ and all $\alpha,\beta\in\mathcal{A}(\mathcal{S})$,

$$\Big(\alpha\succeq_{\mathcal{S}}\beta\Big) \quad\iff\quad \left(\int_{\mathcal{S}} u\circ\alpha\ \mathrm{d}\rho\ \geq\ \int_{\mathcal{S}} u\circ\beta\ \mathrm{d}\rho\right), \tag{8}$$

where the expressions on the right-hand side are Riemann-Stieltjes integrals.[8] In particular, if $\mathcal{S}=[0,T]$, then (8) simplifies to

$$\Big(\alpha\succeq\beta\Big)\iff\left(\int_0^T u[\alpha(t)]\ \mathrm{d}\rho[t]\ \geq\ \int_0^T u[\beta(t)]\ \mathrm{d}\rho[t]\right).$$

Here, ρ is a "generalized discount factor"; it encodes the relative importance that the agent assigns to each moment in time. To be precise, the weight she assigns to the interval $\lfloor t_1,t_2\rceil$ is just $\rho(t_2)-\rho(t_1)$. Any time duration of nonzero length has nonzero weight (because ρ is strictly increasing). But any single instant in time always gets zero weight (because ρ is continuous). In the special case when $\succeq$ admits a representation like (1), ρ is an *antiderivative* of the discount function δ. But we do not assume in general that ρ is differentiable. We also do not require $\mathrm{d}\rho$ to be exponentially decaying, or even monotonically decreasing. Our first main result provides a characterization of DUI representations in terms of the axioms of Section 3.

Theorem 1. *Let $\mathcal{X}$ be a connected topological space, let $\mathcal{A}\subseteq\mathcal{C}([0,T],\mathcal{X})$, and let $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ be an intertemporal preference structure on $\mathcal{A}$ satisfying* (R) *and* (LV)*. Then $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfies* (ISep)*,* (LSep)*,* (StEq)*,* (C)*,* (Dom)*,* (CIP)*,* (TC) *and* (Eph) *if and only if it admits a discounted utility integral representation* (8)*. Furthermore, ρ is unique, and u is unique up to positive affine transformation.*

An *exponentially discounted utility integral representation* for $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ is given by a constant $\delta\in(0,1)$ and a continuous utility function $u:\mathcal{X}\longrightarrow\mathbb{R}$, such that, for all $\mathcal{S}\in\mathfrak{S}$ and all $\alpha,\beta\in\mathcal{A}(\mathcal{S})$,

$$\Big(\alpha\succeq_{\mathcal{S}}\beta\Big) \quad\iff\quad \left(\int_{\mathcal{S}} \delta^t\, u[\alpha(t)]\ \mathrm{d}t\ \geq\ \int_{\mathcal{S}} \delta^t\, u[\beta(t)]\ \mathrm{d}t\right). \tag{9}$$

We obtain such a representation if we replace (Eph) with (Stat) in Theorem 1.

Theorem 2. *Let $\mathcal{X}$ be a connected topological space, let $\mathcal{A}\subseteq\mathcal{C}([0,T],\mathcal{X})$, and let $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ be an intertemporal preference structure on $\mathcal{A}$ satisfying* (R)*,* (LV) *and* (Core)*. Then $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfies* (ISep)*,* (LSep)*,* (StEq)*,* (C)*,* (Dom)*,* (CIP)*,* (TC) *and* (Stat) *if and only if it admits an exponential DUI representation* (9)*. Furthermore, δ is unique, and u is unique up to positive affine transformation.*

Domain extension. Theorems 1 and 2 may appear limited due to the structural condition (LV), which restricts the set of feasible trajectories. For example, an intertemporal preference structure on $\mathcal{C}([0,T],\mathcal{X})$ will typically violate (LV), even if it has a DUI representation. But this problem is more apparent than real; (LV) is only required for the set of trajectories *used in the axiomatic characterization.* The intertemporal preference structure and its DUI representation can have a much larger scope, as we now explain.

[8] Note that these integrals are always well-defined, because α and β are bounded functions (by the definition of $\mathcal{A}$), so that $u\circ\alpha$ and $u\circ\beta$ are also bounded (because u is continuous). Since ρ ranges over $[0,1]$, the magnitudes of these integrals are bounded by $\|u\circ\alpha\|_\infty$ and $\|u\circ\beta\|_\infty$ respectively. For the same reason, all the other integrals that appear later in the paper are well-defined.

Let $(\mathcal{X},d)$ be a metric space, and let $\mathcal{A}\subseteq\mathcal{C}(\mathcal{T},\mathcal{X})$. For any $\mathcal{S}\in\mathfrak{S}$ and any $\alpha,\beta\in\mathcal{A}(\mathcal{S})$, let $d_{\mathcal{S}}(\alpha,\beta):=\sup_{s\in\mathcal{S}} d[\alpha(s),\beta(s)]$; this is the metric of uniform convergence on $\mathcal{A}(\mathcal{S})$. Let $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ be an intertemporal preference structure on $\mathcal{A}$. We will say that $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ is *uniformly continuous* if, for any $\mathcal{S}\in\mathfrak{S}$, and any $\alpha\in\mathcal{A}(\mathcal{S})$, the upper and lower contour sets of α with respect to $\succeq_{\mathcal{S}}$ are closed in the topology of uniform convergence. In other words, for any $\beta\in\mathcal{A}(\mathcal{S})$ and any sequence $\{\beta_n\}_{n=1}^{\infty}\in\mathcal{A}(\mathcal{S})$, if $\lim_{n\to\infty} d_{\mathcal{S}}(\beta_n,\beta)=0$ and $\beta_n\succeq_{\mathcal{S}}\alpha$ for all $n\in\mathbb{N}$, then $\beta\succeq_{\mathcal{S}}\alpha$; likewise if $\beta_n\preceq_{\mathcal{S}}\alpha$ for all $n\in\mathbb{N}$, then $\beta\preceq_{\mathcal{S}}\alpha$. For example, if $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ admits a DUI representation (8) in which u is uniformly continuous, then $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ is uniformly continuous.

Let $\succeq_*$ be a preference order on $\mathcal{X}$, and let $\mathcal{A}\subset\mathcal{A}'\subseteq\mathcal{C}([0,T],\mathcal{X})$. We will say that $\mathcal{A}$ is *uniformly $\succeq_*$-dense* in $\mathcal{A}'$ if, for any $\alpha'\in\mathcal{A}'$ there is a sequence $\{\alpha_n\}_{n=1}^{\infty}\in\mathcal{A}$ such that $\lim_{n\to\infty} d_{\mathcal{T}}(\alpha_n,\alpha')=0$ with $\alpha_n(t)\succeq_*\alpha(t)$ for all $n\in\mathbb{N}$ and $t\in\mathcal{T}$, and likewise a sequence $\{\alpha_n\}_{n=1}^{\infty}\in\mathcal{A}$ such that $\lim_{n\to\infty} d_{\mathcal{T}}(\alpha_n,\alpha')=0$ with $\alpha_n(t)\preceq_*\alpha(t)$ for all $n\in\mathbb{N}$ and $t\in\mathcal{T}$. In other words, there are sequences in $\mathcal{A}$ converging uniformly to any element of $\mathcal{A}'$ "from above" and "from below", relative to $\succeq_*$. (It follows that $\mathcal{A}(\mathcal{S})$ is uniformly $\succeq_*$-dense in $\mathcal{A}'(\mathcal{S})$ for every $\mathcal{S}\in\mathfrak{S}$.) For example, if $\mathcal{X}=\mathbb{R}$ and $\succeq_*$ is the standard ordering, then many familiar collections of continuous real-valued functions (e.g. the set of polynomials, the set of piecewise linear functions, etc.) are dense in $\mathcal{C}(\mathcal{T},\mathbb{R})$ in this sense.

THEOREM 3. *Let $\mathcal{X}$ be a metric space, let $\mathcal{A}'\subseteq\mathcal{C}([0,T],\mathcal{X})$ be a set containing all constant trajectories. Let $\{\succeq'_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ be a uniformly continuous intertemporal preference structure on $\mathcal{A}'$ that satisfies axiom* (Dom) *with respect to a preference order $\succeq_{\rm sy}$ on $\mathcal{X}$. Let $\mathcal{A}\subset\mathcal{A}'$, let $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ be the restriction of $\{\succeq'_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ to $\mathcal{A}$, and suppose $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ has a DUI representation* (8) *for some continuous functions ρ and u. If $\mathcal{A}$ is uniformly $\succeq_{\rm sy}$-dense in $\mathcal{A}'$, and u is uniformly continuous, then $\{\succeq'_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ also has a DUI representation* (8) *given by ρ and u.*

This result means that an intertemporal preference structure need only satisfy the conditions of Theorem 1 on the smaller domain $\mathcal{A}$, in order for us to obtain a DUI representation on a much larger domain $\mathcal{A}'$. For example, let $\mathcal{X}=\mathbb{R}$, and let $\{\succeq'_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ be a uniformly continuous intertemporal preference structure on $\mathcal{C}(\mathcal{T},\mathbb{R})$ that satisfies axiom (Dom) with respect to the standard ordering of $\mathbb{R}$. Thus, any restriction of $\{\succeq'_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfies (C). Let $\mathcal{A}$ be the set of piecewise polynomial functions from Example 2. Then $\mathcal{A}$ satisfies (R) and (LV), and it is uniformly $\geq$-dense in $\mathcal{C}(\mathcal{T},\mathbb{R})$ by the Stone-Weierstrass Theorem. Thus, if the restriction of $\{\succeq'_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ to $\mathcal{A}$ satisfies (ISep), (LSep), (StEq), (CIP), (TC) and (Eph), then Theorems 1 and 3 yield a discounted utility integral representation (8) that applies to all trajectories in $\mathcal{C}(\mathcal{T},\mathbb{R})$. If instead, the restriction of $\{\succeq'_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ to $\mathcal{A}$ satisfies (Core), (ISep), (LSep), (StEq), (CIP), (TC) and (Stat), then Theorems 2 and 3 yield an exponential DUI representation (9) that applies to all trajectories in $\mathcal{C}(\mathcal{T},\mathbb{R})$.

5. Infinite horizon intertemporal decisions Throughout this section, we assume $\mathcal{T}=[0,\infty)$. Let $\mathcal{A}\subseteq\mathcal{C}_{\rm b}([0,\infty),\mathcal{X})$ be a set of bounded, continuous trajectories, and let $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ be an intertemporal preference structure on $\mathcal{A}$. An *exponentially discounted utility integral representation* for $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ is given by a constant $\delta\in(0,1)$ and a continuous utility function $u:\mathcal{X}\longrightarrow\mathbb{R}$, such that statement (9) holds for all $\mathcal{S}\in\mathfrak{S}$ and all $\alpha,\beta\in\mathcal{A}(\mathcal{S})$. In particular

$$\Big(\alpha\succeq\beta\Big) \quad\Longleftrightarrow\quad \left(\int_0^{\infty}\delta^t\,u[\alpha(t)]\;\mathrm{d}t\;\geq\;\int_0^{\infty}\delta^t\,u[\beta(t)]\;\mathrm{d}t\right).$$

Our third result gives an axiomatic characterization for these preferences. But in an infinite-horizon environment, we must replace (ISep) with the following separability axiom.

(ISep*) Let $\mathcal{H}\in\mathfrak{I}$, and suppose $\mathcal{H}=\mathcal{Q}\vee\mathcal{R}$, where $\mathcal{Q},\mathcal{R}\in\mathfrak{S}$ are disjoint time spans. Let $\alpha,\beta\in\mathcal{A}(\mathcal{H})$. If $\alpha_{\upharpoonright\mathcal{I}}\succeq_{\mathcal{I}}\beta_{\upharpoonright\mathcal{I}}$ for every component interval $\mathcal{I}$ of $\mathcal{Q}$ and every component interval $\mathcal{I}$ of $\mathcal{R}$, then $\alpha\succeq_{\mathcal{H}}\beta$. If furthermore $\alpha_{\upharpoonright\mathcal{I}}\succ_{\mathcal{I}}\beta_{\upharpoonright\mathcal{I}}$ for some component interval $\mathcal{I}$ of $\mathcal{Q}$ or $\mathcal{R}$, then $\alpha\succ_{\mathcal{H}}\beta$.

If $\mathcal{T}$ is a finite interval, then (ISep*) and (ISep) are logically equivalent (Lemma A.3). But when $\mathcal{T}$ is infinite, (ISep*) is stronger. Likewise, when $\mathcal{T}$ is infinite, we must replace (R) with the following richness condition.

(R*) For any disjoint time spans $\mathcal{Q}, \mathcal{R} \in \mathfrak{S}$, with $\mathcal{S} = \mathcal{Q} \vee \mathcal{R}$, and any $\alpha, \beta \in \mathcal{A}(\mathcal{S})$, there exists $\phi \in \mathcal{A}(\mathcal{S})$ such that $\phi_{\mathcal{Q}} = \alpha_{\mathcal{Q}}$, while for every component interval $\mathcal{J}$ of $\mathcal{R}$, we have $\phi_{\upharpoonright \mathcal{J}} \approx_{\mathcal{J}} \beta_{\upharpoonright \mathcal{J}}$.

If $\mathcal{T}$ is a finite interval, then (R*) and (R) are logically equivalent (Lemma A.1). But when $\mathcal{T}$ is infinite, (R*) is stronger. This leads to a slightly modified version of (Core):

(Core*) There is a subset $\mathcal{A}_0 \subseteq \mathcal{A}$ such that, for all $\mathcal{S} \in \mathfrak{S}$, and all $t \in [0, \infty)$ such that $\mathcal{S}^{+t} \subseteq \mathcal{T}$, we have $\mathcal{A}_0(\mathcal{S})^{+t} \subseteq \mathcal{A}(\mathcal{S}^{+t})$. Furthermore, $\mathcal{A}_0$ satisfies axiom (R*).

Here is the extension of Theorem 2 to infinite-horizon decisions.

THEOREM 4. *Let $\mathcal{X}$ be a connected Hausdorff space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{L}}([0,\infty), \mathcal{X})$. Let $\{\succeq_{\mathcal{I}}\}_{\mathcal{I} \in \mathfrak{I}}$ be an intertemporal preference structure on $\mathcal{A}$ that satisfies* (R*), (LV) *and* (Core*). *Then $\{\succeq_{\mathcal{I}}\}_{\mathcal{I} \in \mathfrak{I}}$ satisfies* (ISep*), (LSep), (StEq), (C), (Dom), (CIP), (TC) *and* (Stat) *if and only if it has an exponential DUI representation. Here, δ is unique and u is unique up to positive affine transformation.*

The extension of Theorem 1 to infinite-horizon decisions is somewhat more complicated, because without the axiom (Stat), the intertemporal decision may be sensitive to the asymptotic behaviour of trajectories "at eternity". This sensitivity, in turn, depends on the structure of the time span under consideration. In a finite-horizon environment, every time span has only finitely many component intervals. But in an infinite horizon environment, we must distinguish between three types of time spans. Let $\mathfrak{S}_b$ be the set of *bounded* time spans in $[0, \infty)$ —that is, those of the form $\lfloor s_1, t_1 \rceil \sqcup \lfloor s_2, t_2 \rceil \sqcup \cdots \sqcup \lfloor s_N, t_N \rceil$ for some $0 \leq s_1 < t_1 < \cdots < s_N < t_N < \infty$. Meanwhile, let $\mathfrak{S}_u := \mathfrak{S}[0, \infty) \setminus \mathfrak{S}_b$ be the set of *unbounded* time spans in $[0, \infty)$. Let $\mathfrak{S}_a := \{\mathcal{S} \in \mathfrak{S};\ \ \neg\mathcal{S} \in \mathfrak{S}_b\}$; then $\mathfrak{S}_a \subseteq \mathfrak{S}_u$. We will refer to elements of $\mathfrak{S}_a$ as *abiding* time spans. A typical abiding time span has the form $\lfloor s_1, t_1) \sqcup (s_2, t_2) \sqcup \cdots \sqcup (s_N, \infty)$, where $0 \leq s_1 < t_1 < s_2 < t_2 < \cdots < s_N < \infty$. Finally, let $\mathfrak{S}_p := \mathfrak{S}_u \setminus \mathfrak{S}_c$; we will refer to elements of $\mathfrak{S}_p$ as *perennial* time spans. A typical perennial time span is an infinite disjoint union of open intervals, as shown in formula (6).

A *perennial partition* of $[0, \infty)$ is a collection $\mathcal{S}_1, \ldots, \mathcal{S}_N \in \mathfrak{S}_p$ of disjoint, perennial time spans such that $[0, \infty) = \mathcal{S}_1 \vee \cdots \vee \mathcal{S}_N$. For example, for each of the twelve calendar months of the year, let $\mathcal{S}_n$ be the set of all future moments in time occuring in month n (so $\mathcal{S}_1$ corresponds to all future Januaries, etc.). Then $\mathcal{S}_1, \ldots, \mathcal{S}_{12}$ is a perennial partition of $[0, \infty)$. A *coeternity structure* is a collection $\{c_{\mathcal{S}}\}_{\mathcal{S} \in \mathfrak{S}_p}$ of non-negative real numbers indexed by $\mathfrak{S}_p$, such that, for any perennial partition $\{\mathcal{S}_1, \ldots, \mathcal{S}_N\}$, we have $c_{\mathcal{S}_1} + \cdots + c_{\mathcal{S}_N} = 1$. Heuristically, a coeternity structure describes what "fraction of eternity" is covered by each of $\mathcal{S}_1, \ldots, \mathcal{S}_N$. For example, if λ is the Lebesgue measure on $[0, \infty)$, then we could *partly* define a coeternity structure by assigning to each $\mathcal{S} \in \mathfrak{S}_p$ its Cesàro density

$$c_{\mathcal{S}} \quad := \quad \lim_{T \to \infty} \frac{\lambda[\mathcal{S} \cap [0, T)]}{T}, \tag{10}$$

whenever this limit exists. But the limit (10) does not exist for all $\mathcal{S} \in \mathfrak{S}_p$, and for other perennial sets, we would need to define $c_{\mathcal{S}}$ in some other way.[9]

We will now present two DUI representations for infinite horizon intertemporal decisions. The first one is easier to state, and is a special case of the second one. For any Hausdorff space $\mathcal{X}$, let $\mathcal{C}_{\mathrm{L}}([0, \infty), \mathcal{X})$ be the set of all continuous functions $\alpha : [0, \infty) \longrightarrow \mathcal{X}$ that converge to a limit at infinity in the following sense: there is some $x \in \mathcal{X}$ such that, for any open neighbourhood $\mathcal{O} \subseteq \mathcal{X}$

[9] Also, nothing forces us to define c_s by the limit (10), even when this limit does exist.

around x, there exists $T>0$ with $\alpha(t)\in\mathcal{O}$ for all $t>T$. When it exists, this limit x is unique and denoted $\lim_{t\to\infty}\alpha(t)$.

Now, let $\mathcal{A}\subseteq\mathcal{C}_{\mathrm{L}}([0,\infty),\mathcal{X})$, and let $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ be an intertemporal preference structure on $\mathcal{A}$, which induces an intermittent preference structure $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ through axiom (LSep*). An *extended DUI representation* for $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ is given by a continuous, strictly increasing bijection $\rho:[0,\infty)\longrightarrow[0,1)$, a coeternity structure $\{c_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}_p}$, a constant $M\geq 0$, and a continuous utility function $u:\mathcal{X}\longrightarrow\mathbb{R}$, such that, for all $\alpha,\beta\in\mathcal{A}$, the following three statements hold:

$$\text{For all } \mathcal{S}\in\mathfrak{S}_b,\quad \Big(\alpha\succeq_{\mathcal{S}}\beta\Big) \iff \left(\int_{\mathcal{S}} u\circ\alpha\ \mathrm{d}\rho\ \geq\ \int_{\mathcal{S}} u\circ\beta\ \mathrm{d}\rho\right). \tag{11}$$

For all $\mathcal{S}\in\mathfrak{S}_a$,

$$\Big(\alpha\succeq_{\mathcal{S}}\beta\Big)\iff\left(\int_{\mathcal{S}} u\circ\alpha\ \mathrm{d}\rho+M\lim_{t\to\infty}u[\alpha(t)]\ \geq\ \int_{\mathcal{S}} u\circ\beta\ \mathrm{d}\rho+M\lim_{t\to\infty}u[\beta(t)]\right). \tag{12}$$

Finally, for all $\mathcal{S}\in\mathfrak{S}_p$,

$$\Big(\alpha\succeq_{\mathcal{S}}\beta\Big)\iff\left(\int_{\mathcal{S}} u\circ\alpha\ \mathrm{d}\rho+c_{\mathcal{S}}\,M\lim_{t\to\infty}u[\alpha(t)]\ \geq\ \int_{\mathcal{S}} u\circ\beta\ \mathrm{d}\rho+c_{\mathcal{S}}\,M\lim_{t\to\infty}u[\beta(t)]\right). \tag{13}$$

As in Section 4, the integrals on the right hand sides of these statements are Riemann-Stieltjes integrals. In particular, for any $\alpha,\beta\in\mathcal{A}$, formula (12) yields:

$$\Big(\alpha\succeq\beta\Big)\iff\left(\int_0^\infty u\circ\alpha\ \mathrm{d}\rho+M\lim_{t\to\infty}u[\alpha(t)]\geq\int_0^\infty u\circ\beta\ \mathrm{d}\rho+M\lim_{t\to\infty}u[\beta(t)]\right).$$

Thus, the discount structure consist of three components. As in Section 4, ρ is a "generalized discount factor" encoding the relative importance of each moment in $[0,\infty)$. Meanwhile, M is an additional coefficient weighting the asymptotic utility of the trajectories "at eternity". If $M>0$, then the agent assigns a nonzero weight to the asymptotic utility of the trajectories at eternity, *even if* $\lim_{t\to\infty}\ \mathrm{d}\rho(t)=0$. An abiding time span assigns a weight of M to eternity, but a perennial time span may assign less. The coeternity structure encodes how much weight each perennial time span assigns to eternity. This weighting could be decided by a natural formula like (10), but it is essentially arbitrary. For example, if $\{\mathcal{S}_1,\ldots,\mathcal{S}_{12}\}$ is the perennial partition with respect to the twelve calendar months, then it may be that $c_{\mathcal{S}_1}=1$ while $c_{\mathcal{S}_n}=0$ for all $n\in[2\ldots12]$. Note that M could be zero (meaning that the agent totally ignores the asymptotic utilities of trajectories); in this case, the formulae (12) and (13) reduce to formula (11), and $\{c_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}_p}$ plays no role. Here is the first extension of Theorem 1 to infinite-horizon decisions.

Theorem 5. *Let $\mathcal{X}$ be a connected Hausdorff space, and let $\mathcal{A}\subseteq\mathcal{C}_{\mathrm{L}}([0,\infty),\mathcal{X})$. Let $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ be an intertemporal preference structure on $\mathcal{A}$ that satisfies* (R*) *and* (LV). *Then $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfies Axioms* (ISep*), (LSep), (StEq), (C), (Dom), (CIP), (TC) *and* (Eph) *if and only if it has an extended DUI representation* (11)-(13). *Here, ρ and M are unique, $\{c_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}_p}$ is unique* (if $M>0$), *and u is unique up to positive affine transformation.*

The problem with Theorem 5 is that the intertemporal preference structure can only compare trajectories that converge to a limit at infinity. This is because an intertemporal preference structure defined over any larger domain of trajectories may be sensitive to the asymptotic behaviour of these trajectories in a way that eludes an extended DUI representation. In particular, it could respond in different ways to the asymptotic behaviour of trajectories along different sequences in $[0,\infty)$ —for example, it may give different weight to $\lim_{n\to\infty}u\circ\alpha(2n)$ and $\lim_{n\to\infty}u\circ\alpha(2n+1)$ (assuming these limits exist). Intuitively, to capture such sensitivity with a DUI representation, we

would need to introduce distinct "limit points" for these two sequences, and then assign different discount weights to these endpoints. But *no such distinct limit points exist* in $[0,\infty]$. To solve this problem, we must add a plethora of new endpoints to $[0,\infty)$, each acting like a distinct "moment at the end of time". To be precise, we must extend $[0,\infty)$ to its *Stone-Čech compactification.* This is a unique compact Hausdorff space $\widehat{\mathcal{T}}$ with the following properties.
(SČ1) $[0,\infty)$ is an open, dense subset of $\widehat{\mathcal{T}}$, and the native topology of $[0,\infty)$ is the same as the subspace topology it inherits from $\widehat{\mathcal{T}}$.
(SČ2) For any compact Hausdorff space $\mathcal{X}$, and any continuous function $\alpha:[0,\infty)\longrightarrow\mathcal{X}$, there is a unique continuous function $\widehat{\alpha}:\widehat{\mathcal{T}}\longrightarrow\mathcal{X}$ such that $\widehat{\alpha}_{\upharpoonright[0,\infty)}=\alpha$.
Let $\Omega:=\widehat{\mathcal{T}}\setminus[0,\infty)$; heuristically, this is the set of limit points "at eternity".[10] It is an uncountably infinite, compact topological space. For any $\mathcal{S}\in\mathfrak{S}_p$, let $\mathrm{clos}_{\widehat{\mathcal{T}}}(\mathcal{S})$ be its closure as a subset of $\widehat{\mathcal{T}}$, and then let $\widehat{\mathcal{S}}:=\mathrm{int}_{\widehat{\mathcal{T}}}[\mathrm{clos}_{\widehat{\mathcal{T}}}(\mathcal{S})]$ be the interior of this closure in $\widehat{\mathcal{T}}$. Then $\widehat{\mathcal{S}}$ is an open subset of $\widehat{\mathcal{T}}$, and $\widehat{\mathcal{S}}\cap[0,\infty)=\mathcal{S}$ [32, Lemma 7.4(a)]. Let $\mathcal{S}_\infty:=\Omega\cap\widehat{\mathcal{S}}$; this is a relatively open subset of Ω. Let $\partial\widehat{\mathcal{S}}$ be the boundary of $\widehat{\mathcal{S}}$ in $\widehat{\mathcal{T}}$, and then let $\partial_\infty\mathcal{S}:=\Omega\cap\partial\widehat{\mathcal{S}}$; this is a relatively closed subset of Ω. If Ω is the set of "limit points at eternity", then $\partial_\infty\mathcal{S}$ is the set of such limit points that can be reached both by a sequence of times in $\mathcal{S}$ and a sequence of times in $\neg\mathcal{S}$. In contrast, $\mathcal{S}_\infty$ is the set of the set of such limit points that can be reached by a sequence in $\mathcal{S}$, but *not* a sequence in $\neg\mathcal{S}$. If $\mathcal{S}$ is abiding, then $\mathcal{S}_\infty=\Omega$. Meanwhile, $\partial_\infty\mathcal{S}\neq\emptyset$ if and only if $\mathcal{S}$ is perennial.

Let η be a Borel measure on Ω, and for all $\mathcal{S}\in\mathfrak{S}_p$, let $\phi_{\mathcal{S}}:\partial_\infty\mathcal{S}\longrightarrow\mathbb{R}_+$ be a Borel measurable function. We will call the collection $\{\phi_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}_p}$ an *amaranthine structure* if, for any perennial partition $\{\mathcal{S}_1,\ldots,\mathcal{S}_N\}$ of $[0,\infty)$, we have $\phi_{\mathcal{S}_1}(\omega)+\cdots+\phi_{\mathcal{S}_N}(\omega)=1$ for η-almost all $\omega\in(\partial_\infty\mathcal{S}_1)\cup\cdots\cup(\partial_\infty\mathcal{S}_N)$. This is like the coeternity structure that appeared in the extended DUI representation (13), but now we must contend with the fact that Ω is not a single point, but an uncountably infinite topological space.

Let $\mathcal{X}$ be another Hausdorff space. For any $\alpha\in\mathcal{C}_{\mathrm{b}}([0,\infty),\mathcal{X})$, assertion (SČ2) yields a unique function $\widehat{\alpha}\in\mathcal{C}(\widehat{\mathcal{T}},\mathcal{X})$ such that $\widehat{\alpha}_{\upharpoonright[0,\infty)}=\alpha$. Let $\mathcal{A}\subseteq\mathcal{C}_{\mathrm{b}}([0,\infty),\mathcal{X})$, and let $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ be an intertemporal preference structure on $\mathcal{A}$, which induces an intermittent preference structure $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ through axiom (LSep*). A *Stone-Čech DUI representation* for $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ is given by a continuous utility function $u:\mathcal{X}\longrightarrow\mathbb{R}$, a continuous, strictly increasing bijection $\rho:[0,\infty)\longrightarrow[0,1)$, a normal Borel measure η on Ω, and an amaranthine structure $\{\phi_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}_p}$ such that, for all $\alpha,\beta\in\mathcal{A}(\mathcal{S})$, the following three statements hold:

$$\text{For all } \mathcal{S}\in\mathfrak{S}_b,\quad \Big(\alpha\succeq_{\mathcal{S}}\beta\Big)\iff\left(\int_{\mathcal{S}}u\circ\alpha\ \mathrm{d}\rho\ \geq\ \int_{\mathcal{S}}u\circ\beta\ \mathrm{d}\rho\right). \tag{14}$$

$$\text{For all } \mathcal{S}\in\mathfrak{S}_a,\quad \Big(\alpha\succeq_{\mathcal{S}}\beta\Big)\iff\left(\int_{\mathcal{S}}u\circ\alpha\ \mathrm{d}\rho+\int_{\Omega}u\circ\widehat{\alpha}\ \mathrm{d}\eta\ \geq\ \int_{\mathcal{S}}u\circ\beta\ \mathrm{d}\rho+\int_{\Omega}u\circ\widehat{\beta}\ \mathrm{d}\eta\right). \tag{15}$$

Finally, for all $\mathcal{S}\in\mathfrak{S}_p$,

$$\begin{aligned}\Big(\alpha\succeq_{\mathcal{S}}\beta\Big)\iff\Bigg(&\int_{\mathcal{S}}u\circ\alpha\ \mathrm{d}\rho+\int_{\mathcal{S}_\infty}u\circ\widehat{\alpha}\ \mathrm{d}\eta+\int_{\partial_\infty\mathcal{S}}(u\circ\widehat{\alpha})\cdot\phi_{\mathcal{S}}\ \mathrm{d}\eta\\ &\geq\ \int_{\mathcal{S}}u\circ\beta\ \mathrm{d}\rho+\int_{\mathcal{S}_\infty}u\circ\widehat{\beta}\ \mathrm{d}\eta+\int_{\partial_\infty\mathcal{S}}(u\circ\widehat{\beta})\cdot\phi_{\mathcal{S}}\ \mathrm{d}\eta\Bigg).\end{aligned} \tag{16}$$

Here, the ρ-integrals over subsets of $[0,\infty)$ are Riemann-Stieltjes integrals, while in (15) and (16), the η-integrals over subsets of Ω are Lebesgue integrals. In particular, for any $\alpha,\beta\in\mathcal{A}$, formula (15) becomes

$$\Big(\alpha\succeq\beta\Big)\iff\left(\int_0^\infty u\circ\alpha\ \mathrm{d}\rho+\int_{\Omega}u\circ\widehat{\alpha}\ \mathrm{d}\eta\ \geq\ \int_0^\infty u\circ\beta\ \mathrm{d}\rho+\int_{\Omega}u\circ\widehat{\beta}\ \mathrm{d}\eta\right).$$

[10] Formally, Ω is called the *corona* of the space $[0,\infty)$.

There is a clear analogy between formulae (11), (12) and (13) and formulae (14), (15) and (16). But the latter are more complex, due to the greater complexity of the Stone-Čech compactification. The measure η plays the same role in representations (15) and (16) that the coefficient M played in the representation (12) and (13): it assigns some weight to the asymptotic utility of trajectories at eternity. But η is much more complicated than a single coefficient, because now we can assign a different weight to each point in Ω; heuristically, this means that the intertemporal preference structure can exhibit different sensitivity to different aspects of the asymptotic utility of trajectories. If $\mathcal{S}$ is an abiding time span, then η completely describes the agent's attitudes towards these asymptotic utilities, as shown in formula (15). But if $\mathcal{S}$ is a perennial time span, then we also need the amaranthine structure $\{\phi_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}_p}$, as shown in formula (16). To understand this, note that $\partial_\infty\mathcal{S}=\partial_\infty(\neg\mathcal{S})$. Heuristically, the amaranthine structure describes the way that $\mathcal{S}$ and $\neg\mathcal{S}$ "share" the η-mass of their common boundary at eternity. For example, suppose that $\mathcal{S}=[0,1)\sqcup(2,3)\sqcup(4,5)\sqcup\cdots$, so that $\neg\mathcal{S}=(1,2)\sqcup(3,4)\sqcup(5,6)\sqcup\cdots$. Suppose that $\phi_{\mathcal{S}}=1_\Omega$ while $\phi_{\neg\mathcal{S}}=0$. Then $\succeq_{\mathcal{S}}$ is quite sensitive to the asymptotic utilities of trajectories, while $\succeq_{\neg\mathcal{S}}$ is totally insensitive to this information. Note that η could be trivial (if the agent ignores the asymptotic utilities of trajectories); in this case, the formulae (15) and (16) reduce to (14), and $\{c_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}_p}$ is irrelevant. Here is our last main result.

THEOREM 6. *Let $\mathcal{X}$ be a connected Hausdorff space, and let $\mathcal{A}\subseteq\mathcal{C}_{\mathrm{b}}([0,\infty),\mathcal{X})$. Let $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ be an intertemporal preference structure on $\mathcal{A}$ satisfying* (R*) *and* (LV)*. Then $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ satisfies* (ISep*)*,* (LSep)*,* (StEq)*,* (C)*,* (Dom)*,* (CIP)*,* (TC) *and* (Eph) *if and only if it admits a Stone-Čech DUI representation* (14)-(16)*. Furthermore, η and ρ are unique, each element of $\{\phi_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}_p}$ is unique η-almost everywhere, and u is unique up to positive affine transformation.*

6. Prior literature The first axiomatic characterization of exponentially discounted utility sums was by Koopmans [21, 22]. Since then, there have been other axiomatizations, such as [23], [27] or [41, Theorem IV.4.4]; see Bleichrodt et al. [4] for a good summary of this literature. Meanwhile, empirically observed deviations from exponential discounting led to the investigation of *non*exponentially discounted utility sums, such as quasihyperbolic discounting [31], including axiomatic characterizations [3, 16, 29]. Likewise, dissatisfaction with exponential discounting as a normative criterion led to axiomatic characterizations of discounted sum representations with more slowly decaying discount terms [9, 14].

However, this literature is entirely concerned with *discrete* time. Only recently has there been an investigation of preference axiomatizations of continuous-time intertemporal preferences. As observed by Wakker [44], a subjective expected utility (SEU) representation can be interpreted as a discounted utility integral representation, if the underlying state space $\mathcal{T}$ is isomorphic to a real interval. In the particular case where the outcome space $\mathcal{X}$ is a connected topological space, Wakker [39] axiomatically characterized such an SEU representation with a continuous utility function; see also [44, Corollary 2.14]. But Wakker did not impose a topology on $\mathcal{T}$, so he did not require trajectories to be continuous. Likewise, Kopylov [24] obtained a representation of intertemporal preferences on $\mathbb{R}$ as an exponentially discounted utility integral, by adapting his axiomatization of SEU representations with countably additive probability measures. But Kopylov assumed that $\mathcal{A}$ contains all *simple* functions (i.e. measurable step functions with finite range). Also, in his model, $\mathcal{X}$ has no topology, so it is meaningless to ask whether the utility function u is continuous.

Kahneman et al. [18, Theorem A.1] axiomatically characterized utility integral representations with a continuous utility function and *no* discounting, in the special case when $\mathcal{X}$ is an interval of real numbers. Like the present paper, they consider not only trajectories defined on time intervals (which they call *episodes*), but also trajectories defined on disjoint unions of time intervals (which they call *temporally extended outcomes*). Harvey and Østerdal [15] and Sagara [36] assumed that $\mathcal{X}=\mathbb{R}^N$, and obtained discounted utility integral representations with continuous utility functions.

However, all three axiomatizations require the possibility to "splice" two trajectories together at some moment in time, so they cannot restrict to continuous functions, and must instead allow the space $\mathcal{A}$ of feasible trajectories to include *piecewise* continuous functions. (In fact, Sagara allows $\mathcal{A}$ to be any subset of $\mathcal{L}^p([0,\infty),\mathbb{R}^N)$ that is closed under splicing.) Harvey and Østerdal emphasize the importance of such domain restrictions, writing: "....it may be impossible for the analyst to visualize an outcome stream that is discontinuous at an uncountable number of times and thus impossible for him or her to judge whether the conditions on preferences are satisfied by such functions." They insist: "...in a prescriptive policy study the analyst should choose a set of alternatives that encompasses the possible alternatives and excludes alternatives that differ greatly from the possible alternatives and thus are difficult to compare with them." Harvey and Østerdal obtain a discounted utility representation in terms of a Riemann integral. Because Sagara and Kahneman et al. allow much larger sets of trajectories, their representations use Lebesgue integrals.

Hara [13] allows $\mathcal{X}$ to be any separable metric space, and restricts $\mathcal{A}$ to *càdlàg*[11] functions from $[0,\infty)$ into $\mathcal{X}$. He characterizes exponentially discounted utility integral representations, along with a generalization of the recursive discounting model of Uzawa [38]. Finally, Webb [45] characterizes quasihyperbolic discounted utility integrals on $[0,\infty)$. But in his model, $\mathcal{A}$ contains only step functions, and like Kopylov [24], the outcome space $\mathcal{X}$ has no topology, so the utility function cannot be continuous. Pan et al. [30] also characterize quasihyperbolic discounting in continuous time, but in their model, alternatives are *dated outcomes* (as in [10]) rather than consumption streams. So their results are not strictly comparable to the present paper.

As Spinoza observed, "It is in the nature of Reason to perceive things from the aspect of eternity." Suppose we interpret an infinite-horizon intertemporal choice problem as an *intergenerational* choice problem. Ethical considerations of impartiality suggest that we should not discount future utility relative to present utility. But starting with Koopmans [23] and Diamond [8], a series of papers showed that such "intertemporal impartiality" is inconsistent with other compelling axioms, such as weak Pareto and continuity. This has led to a large literature on intergenerational social choice based on weaker impartiality criteria; see Asheim [2] for a good review. To respect the concerns of the very far future, Chichilnisky and Heal [5, 6, 7] introduced an intergenerational social welfare function like formula (4), which combined a discounted utility sum with a limit term obtained from the Stone-Čech compactification of $\mathbb{N}$. More recently, Sakai [37] has characterized a similar construction. This is philosophically similar to Stone-Čech DUI representation that we characterize in Theorem 6, but there are important differences. First, Chichilnisky and Sakai work with discrete time, while we assume time is a continuum. Second, they assume $\mathcal{X}=\mathbb{R}$ or $\mathbb{R}^N$, whereas we allow $\mathcal{X}$ to be any connected Hausdorff space. Third, their representations do not use an amaranthine structure —this is because they only consider preferences over trajectories defined on all of $\mathbb{N}$, and not on perennial time spans.

Unfortunately, intergenerational preferences defined using the Stone-Čech compactification are not constructible. Indeed, to construct the Stone-Čech compactification itself, we need the Axiom of Choice. This does not mean that such preferences are useless. We can prove that they exist and have certain properties, and in many cases we can decide whether one trajectory is prefered to another. But there will always exist pairs of trajectories for which the comparison cannot be decided. Indeed, this is inevitable in any infinite-horizon intertemporal preference order satisfying desirable properties [25, 26, 46].

Appendix A: Proofs of minor claims from Sections 2, 3 and 5 This appendix collects the proofs of some claims that were made informally in the text, and that are important for the proofs of our main results. The first of these claims appeared in Section 5. It says we can use the

[11] That is: functions that are continuous from the right at have a left-hand limit at each point in $[0,\infty)$.

hypothesis (R*) in place of (R) without any loss of generality when proving results from Section 4. As a consequence, conditions (Core) and (Core*) are equivalent when proving these results. Thus, it is useful to state it first.

LEMMA A.1. *If $\mathcal{T}$ is a finite interval, then* (R*) *and* (R) *are equivalent.*

Proof. Clearly, (R*) implies (R), because (R) is just the special case when $\mathcal{R} = \lfloor r, s\rceil$ and $\mathcal{Q} = \lfloor q, r\rceil \vee \lfloor s, t\rceil$. We must show the reverse implication. Since $\mathcal{T}$ itself is an interval of finite length, $\mathcal{R}$ and $\neg\mathcal{R}$ must each have a finite number of intervals. Find $\alpha^*, \beta^* \in \mathcal{A}$ such that $\alpha = \alpha^*_{\upharpoonright\mathcal{S}}$ and $\beta = \beta^*_{\upharpoonright\mathcal{S}}$. By inductive application of (R), we obtain $\phi^* \in \mathcal{A}$ such that $\phi^*_{\upharpoonright\neg\mathcal{R}} = \alpha^*_{\upharpoonright\neg\mathcal{R}}$, while for every component interval $\mathcal{J}$ of $\mathcal{R}$, we have $\phi^*_{\upharpoonright\mathcal{J}} \approx_{\mathcal{J}} \beta^*_{\upharpoonright\mathcal{J}}$. Now let $\phi := \phi^*_{\upharpoonright\mathcal{S}}$. Then $\phi_{\mathcal{Q}} = \alpha_{\mathcal{Q}}$ (because $\mathcal{Q} \subseteq \neg\mathcal{R}$), and for every component interval $\mathcal{J}$ of $\mathcal{R}$, we have $\phi_{\upharpoonright\mathcal{J}} = \phi^*_{\upharpoonright\mathcal{J}} \approx_{\mathcal{J}} \beta^*_{\upharpoonright\mathcal{J}} = \beta_{\upharpoonright\mathcal{J}}$, as desired. □

The next lemma yields a slightly enhanced version of the axiom (ISep).

LEMMA A.2. *Suppose $\mathcal{A}$ and $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfy* (R) *and* (ISep)*. Let $\mathcal{I}, \mathcal{J} \in \mathfrak{I}$ be adjacent time intervals and let $\mathcal{H} = \mathcal{I} \vee \mathcal{J}$. Let $\alpha, \beta \in \mathcal{A}(\mathcal{H})$.*

(a) *If $\alpha_{\upharpoonright\mathcal{I}} \succeq_{\mathcal{I}} \beta_{\upharpoonright\mathcal{I}}$ and $\alpha_{\upharpoonright\mathcal{J}} \succeq_{\mathcal{J}} \beta_{\upharpoonright\mathcal{J}}$, then $\alpha \succeq_{\mathcal{H}} \beta$.*

(b) *If $\alpha_{\upharpoonright\mathcal{I}} \succ_{\mathcal{I}} \beta_{\upharpoonright\mathcal{I}}$ and $\alpha_{\upharpoonright\mathcal{J}} \succeq_{\mathcal{J}} \beta_{\upharpoonright\mathcal{J}}$, then $\alpha \succ_{\mathcal{H}} \beta$.*

Proof. (b) Use (R) to find $\gamma \in \mathcal{A}(\mathcal{H})$ such that $\gamma_{\upharpoonright\mathcal{J}} = \beta_{\upharpoonright\mathcal{J}}$ and $\gamma_{\upharpoonright\mathcal{I}} \approx_{\mathcal{I}} \alpha_{\upharpoonright\mathcal{I}}$. Since $\gamma_{\upharpoonright\mathcal{I}} \approx_{\mathcal{I}} \alpha_{\upharpoonright\mathcal{I}} \succ_{\mathcal{I}} \beta_{\upharpoonright\mathcal{I}}$, thus $\gamma_{\upharpoonright\mathcal{I}} \succ_{\mathcal{I}} \beta_{\upharpoonright\mathcal{I}}$ by transitivity. Since $\gamma_{\upharpoonright\mathcal{J}} = \beta_{\upharpoonright\mathcal{J}}$, (ISep) yields $\gamma \succ_{\mathcal{H}} \beta$. Meanwhile, as $\alpha_{\upharpoonright\mathcal{J}} \succeq_{\mathcal{J}} \beta_{\upharpoonright\mathcal{J}} = \gamma_{\upharpoonright\mathcal{J}}$, thus $\alpha_{\upharpoonright\mathcal{J}} \succeq_{\mathcal{J}} \gamma_{\upharpoonright\mathcal{J}}$. Since $\alpha_{\upharpoonright\mathcal{I}} \approx_{\mathcal{I}} \gamma_{\upharpoonright\mathcal{I}}$, (ISep) yields $\alpha \succeq_{\mathcal{H}} \gamma$. Now we have $\alpha \succeq_{\mathcal{H}} \gamma \succ_{\mathcal{H}} \beta$, and thus $\alpha \succ_{\mathcal{H}} \beta$ by transitivity.

(a) Suppose $\alpha_{\upharpoonright\mathcal{J}} \succeq_{\mathcal{J}} \beta_{\upharpoonright\mathcal{J}}$. If $\alpha_{\upharpoonright\mathcal{I}} \approx_{\mathcal{I}} \beta_{\upharpoonright\mathcal{I}}$, then (ISep) yields $\alpha \succeq_{\mathcal{H}} \beta$. On the other hand, if $\alpha_{\upharpoonright\mathcal{I}} \succ_{\mathcal{I}} \beta_{\upharpoonright\mathcal{I}}$, then part (b) yields $\alpha \succ_{\mathcal{H}} \beta$. □

The next lemma says we can use (ISep*) in place of (ISep) when proving results from Section 4.

LEMMA A.3. (ISep*) *implies* (ISep)*. Furthermore, suppose $\mathcal{T}$ is a finite interval, and $\mathcal{A}$ and $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfy* (R)*. Then they satisfy* (ISep) *if and only if they satisfy* (ISep*).

Proof. "(ISep*)$\Longrightarrow$(ISep)" Let $\mathcal{I}, \mathcal{J} \in \mathfrak{I}$ be adjacent time intervals, let $\mathcal{H} := \mathcal{I} \vee \mathcal{J}$, and let $\alpha, \beta \in \mathcal{A}(\mathcal{H})$ with $\alpha_{\upharpoonright\mathcal{I}} \approx_{\mathcal{I}} \beta_{\upharpoonright\mathcal{I}}$. We must show that $(\alpha \succeq_{\mathcal{H}} \beta) \iff (\alpha_{\upharpoonright\mathcal{J}} \succeq_{\mathcal{J}} \beta_{\upharpoonright\mathcal{J}})$.

"$\Longleftarrow$" As $\alpha_{\upharpoonright\mathcal{I}} \approx_{\mathcal{I}} \beta_{\upharpoonright\mathcal{I}}$, thus $\alpha_{\upharpoonright\mathcal{I}} \succeq_{\mathcal{I}} \beta_{\upharpoonright\mathcal{I}}$. So if $\alpha_{\upharpoonright\mathcal{J}} \succeq_{\mathcal{J}} \beta_{\upharpoonright\mathcal{J}}$, then (ISep*) yields $\alpha \succeq_{\mathcal{H}} \beta$.

"$\Longrightarrow$" Suppose $\alpha \succeq_{\mathcal{H}} \beta$. If $\alpha_{\upharpoonright\mathcal{J}} \prec_{\mathcal{J}} \beta_{\upharpoonright\mathcal{J}}$, then invoking $\alpha_{\upharpoonright\mathcal{I}} \approx_{\mathcal{I}} \beta_{\upharpoonright\mathcal{I}}$ and the second part of (ISep*) yields $\alpha \prec_{\mathcal{H}} \beta$, a contradiction. Thus, we must have $\alpha_{\upharpoonright\mathcal{J}} \succeq_{\mathcal{J}} \beta_{\upharpoonright\mathcal{J}}$.

"(ISep)$\Longrightarrow$(ISep*)" If $\mathcal{T}$ is finite, and $\mathcal{Q}, \mathcal{R} \in \mathfrak{S}$, then $\mathcal{Q}$ and $\mathcal{R}$ must each have a finite number of component intervals. Furthermore, if $\mathcal{Q}$ and $\mathcal{R}$ are disjoint, and $\mathcal{Q} \vee \mathcal{R}$ is the interval $\mathcal{H}$, then the component intervals of $\mathcal{Q}$ and $\mathcal{R}$ must alternate and touch at their endpoints.

First suppose that $\alpha_{\upharpoonright\mathcal{I}} \succeq_{\mathcal{I}} \beta_{\upharpoonright\mathcal{I}}$ for every component interval $\mathcal{I}$ of $\mathcal{Q}$ and $\mathcal{R}$. By inductively applying Lemma A.2(a), we conclude that $\alpha \succeq_{\mathcal{H}} \beta$.

Now, further suppose that $\alpha_{\upharpoonright\mathcal{I}} \succ_{\mathcal{I}} \beta_{\upharpoonright\mathcal{I}}$ for some component interval $\mathcal{I}$ of $\mathcal{Q}$ or $\mathcal{R}$. Then at some step in the previous inductive argument, use Lemma A.2(b) to get a strict preference. Then invoke Lemma A.2(b) for all remaining steps in the argument, to eventually conclude that $\alpha \succ_{\mathcal{H}} \beta$. □

LEMMA A.4. *If $\mathcal{A}$ and $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfy* (R*)*, then* (LSep*) *and* (LSep) *are equivalent.*

Proof. It is obvious that (LSep*) implies (LSep). We must show the reverse implication. So, suppose (LSep) holds. For any $\mathcal{S} \in \mathfrak{S}$, define $\succeq_{\mathcal{S}}$ on $\mathcal{A}(\mathcal{S})$ as follows. Let $\alpha, \beta \in \mathcal{A}(\mathcal{S})$. By definition of $\mathcal{A}(\mathcal{S})$, there is some $\alpha' \in \mathcal{A}$ such that $\alpha'_{\upharpoonright\mathcal{S}} = \alpha$. Condition (R*) yields $\beta' \in \mathcal{A}$ such that $\beta'_{\upharpoonright\mathcal{S}} = \beta$, while $\beta'_{\upharpoonright\mathcal{I}} \approx_{\mathcal{I}} \alpha'_{\upharpoonright\mathcal{I}}$ for every component interval $\mathcal{I}$ of $\neg\mathcal{S}$. Then define $\alpha \succeq_{\mathcal{S}} \beta$ if and only if $\alpha' \succeq \beta'$. By axiom (LSep), the relation $\succeq_{\mathcal{S}}$ defined in this way is independent of the choice of extensions α' and β'. Thus, it determines a well-defined binary relation on $\mathcal{A}(\mathcal{S})$.

The binary relation $\succeq_{\mathcal{S}}$ defined in this way is reflexive and complete because $\succeq$ itself is reflexive and complete. To see that $\succeq_{\mathcal{S}}$ is transitive, let $\alpha,\beta,\gamma\in\mathcal{A}(\mathcal{S})$, and suppose that $\alpha\succeq_{\mathcal{S}}\beta$ and $\beta\succeq_{\mathcal{S}}\gamma$. By the definition of $\succeq_{\mathcal{S}}$ in the previous paragraph, there exist $\alpha',\beta'\in\mathcal{A}$ such that $\alpha=\alpha'_{\upharpoonright\mathcal{S}}$, $\beta=\beta'_{\upharpoonright\mathcal{S}}$, and $\alpha'_{\upharpoonright\mathcal{I}}\approx_{\mathcal{I}}\beta'_{\upharpoonright\mathcal{I}}$ for every component interval $\mathcal{I}$ of $\neg\mathcal{S}$, and $\alpha'\succeq\beta'$. Condition (R*) yields $\gamma'\in\mathcal{A}$ such that $\gamma=\gamma'_{\upharpoonright\mathcal{S}}$, while $\gamma'_{\upharpoonright\mathcal{I}}\approx_{\mathcal{I}}\beta'_{\upharpoonright\mathcal{I}}$ for every component interval $\mathcal{I}$ of $\neg\mathcal{S}$. Then $\beta'\succeq\gamma'$, because $\beta\succeq_{\mathcal{S}}\gamma$ and this is how $\succeq_{\mathcal{S}}$ was defined in the previous paragraph. Thus, by transitivity, $\alpha'\succeq\gamma'$. But for every component interval $\mathcal{I}$ of $\neg\mathcal{S}$, we have $\gamma'_{\upharpoonright\mathcal{I}}\approx_{\mathcal{I}}\beta'_{\upharpoonright\mathcal{I}}\approx_{\mathcal{I}}\alpha'_{\upharpoonright\mathcal{I}}$ by construction, and hence $\gamma'_{\upharpoonright\mathcal{I}}\approx_{\mathcal{I}}\alpha'_{\upharpoonright\mathcal{I}}$ by transitivity. Thus, we must have $\alpha\succeq_{\mathcal{S}}\gamma$, by the definition of $\succeq_{\mathcal{S}}$ given in the previous paragraph. □

The next lemma shows that $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ is an extension of $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$.

Lemma A.5. *Suppose $\mathcal{A}$ and $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfy* (R*), (ISep), *and* (LSep*)*. Let $\mathcal{J}\in\mathfrak{I}$. The intertemporal preference order associated with $\mathcal{J}$ is equal to the intermittent preference order defined on $\mathcal{A}(\mathcal{J})$ by Axiom* (LSep*).

Proof. Let $\succeq_{\mathcal{J}}$ be the intertemporal preference order associated with $\mathcal{J}$. Let $\succeq^*_{\mathcal{J}}$ be the intermittent preference order defined on $\mathcal{A}(\mathcal{J})$ by Axiom (LSep*). We must show they are the same.

Either $\mathcal{T}=[q,t]$ for some $q<t<\infty$, or $\mathcal{T}=[q,\infty)$, in which case we define $t:=\infty$. Suppose $\mathcal{J}=\lfloor r,s\rceil$, where $q\le r<s\le t$. Let $\mathcal{I}:=\lfloor q,r\rceil$ and $\mathcal{K}:=\lfloor s,t\rceil$ (one of these may be empty). Then $\neg\mathcal{J}=\mathcal{I}\vee\mathcal{K}$, and these are the component intervals of $\neg\mathcal{J}$. Let $\alpha,\beta\in\mathcal{A}$, and suppose that $\alpha_{\upharpoonright\mathcal{I}}\approx_{\mathcal{I}}\beta_{\upharpoonright\mathcal{I}}$ and $\alpha_{\upharpoonright\mathcal{K}}\approx_{\mathcal{K}}\beta_{\upharpoonright\mathcal{K}}$. Then according the definition of $\succeq^*_{\mathcal{J}}$ in Axiom (LSep*), we have

$$\Big(\alpha_{\upharpoonright\mathcal{J}}\succeq^*_{\mathcal{J}}\beta_{\upharpoonright\mathcal{J}}\Big)\iff\Big(\alpha\succeq\beta\Big).\tag{A1}$$

Now, let $\mathcal{H}:=\mathcal{I}\vee\mathcal{J}=\lfloor q,s\rceil$. By axiom (ISep), we have

$$\Big(\alpha_{\upharpoonright\mathcal{H}}\succeq_{\mathcal{H}}\beta_{\upharpoonright\mathcal{H}}\Big)\iff\Big(\alpha_{\upharpoonright\mathcal{J}}\succeq_{\mathcal{J}}\beta_{\upharpoonright\mathcal{J}}\Big).\tag{A2}$$

Next, note that $\mathcal{T}=\mathcal{H}\vee\mathcal{K}$. Thus, by applying (ISep) again (this time to $\mathcal{H}$), we have

$$\Big(\alpha\succeq\beta\Big)\iff\Big(\alpha_{\upharpoonright\mathcal{H}}\succeq_{\mathcal{H}}\beta_{\upharpoonright\mathcal{H}}\Big).\tag{A3}$$

Combining (A1), (A2) and (A3), we see that $\alpha_{\upharpoonright\mathcal{J}}\succeq^*_{\mathcal{J}}\beta_{\upharpoonright\mathcal{J}}$ if and only if $\alpha_{\upharpoonright\mathcal{J}}\succeq_{\mathcal{J}}\beta_{\upharpoonright\mathcal{J}}$. □

Proposition A.1. *If $\mathcal{A}$ and $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfy* (R*), (ISep*) *and* (LSep*)*, then they satisfy* (Sep)*.*

The proof of Proposition A.1 requires two lemmas.

Lemma A.6. *Suppose $\mathcal{A}$ and $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfy* (R*), (ISep), *and* (LSep*)*. Let $\mathcal{S}\in\mathfrak{S}$, and let $\mathcal{Q},\mathcal{R}\subseteq\mathcal{S}$ be disjoint time spans such that $\mathcal{S}=\mathcal{Q}\vee\mathcal{R}$. Let $\alpha,\beta\in\mathcal{A}(\mathcal{S})$ be such that $\alpha_{\upharpoonright\mathcal{I}}\approx_{\mathcal{I}}\beta_{\upharpoonright\mathcal{I}}$ for every component interval $\mathcal{I}$ of $\mathcal{R}$. Then $\alpha\succeq_{\mathcal{S}}\beta$ if and only if $\alpha_{\upharpoonright\mathcal{Q}}\succeq_{\mathcal{Q}}\beta_{\upharpoonright\mathcal{Q}}$.*

Proof. Find $\alpha^*\in\mathcal{A}$ with $\alpha=\alpha^*_{\upharpoonright\mathcal{S}}$. Condition (R*) yields $\beta^*\in\mathcal{A}$ such that $\beta=\beta^*_{\upharpoonright\mathcal{S}}$, while

$$\alpha^*_{\upharpoonright\mathcal{J}}\quad\approx_{\mathcal{J}}\quad\beta^*_{\upharpoonright\mathcal{J}}\qquad\text{for every component interval }\mathcal{J}\text{ of }\neg\mathcal{S}.\tag{A4}$$

From the definition of $\succeq_{\mathcal{S}}$ in Axiom (LSep*), it follows that

$$\Big(\alpha\succeq_{\mathcal{S}}\beta\Big)\iff\Big(\alpha^*\succeq\beta^*\Big).\tag{A5}$$

From the hypothesis of the lemma, we also have

$$\alpha^*_{\upharpoonright\mathcal{I}}\quad\approx_{\mathcal{I}}\quad\beta^*_{\upharpoonright\mathcal{I}},\quad\text{for every component interval }\mathcal{I}\text{ of }\mathcal{R},\tag{A6}$$

because $\alpha^*_{\upharpoonright\mathcal{I}}=\alpha_{\upharpoonright\mathcal{I}}$ and $\beta_{\upharpoonright\mathcal{I}}=\beta^*_{\upharpoonright\mathcal{I}}$ for any such component interval, because $\mathcal{R}\subseteq\mathcal{S}$.

Note that $\neg\mathcal{Q}=(\neg\mathcal{S})\vee\mathcal{R}$. Thus, if $\mathcal{K}$ is any component interval of $\neg\mathcal{Q}$, then either

(i) $\mathcal{K}$ is a component interval of $\neg\mathcal{S}$; or
(ii) $\mathcal{K}$ is a component interval of $\mathcal{R}$; or
(iii) $\mathcal{K} = \mathcal{I} \vee \mathcal{J}$, where $\mathcal{I}$ is a component interval of $\mathcal{R}$ and $\mathcal{J}$ is a component interval of $\neg\mathcal{S}$, and these two intervals are adjacent.
In Case (i), statement (A4) says $\alpha^*_{\upharpoonleft\mathcal{K}} \approx_{\mathcal{K}} \beta^*_{\upharpoonleft\mathcal{K}}$. In Case (ii), statement (A6) says $\alpha^*_{\upharpoonleft\mathcal{K}} \approx_{\mathcal{K}} \beta^*_{\upharpoonleft\mathcal{K}}$. In Case (iii), combining statements (A4), (A6) and Axiom (ISep) yields $\alpha^*_{\upharpoonleft\mathcal{K}} \approx_{\mathcal{K}} \beta^*_{\upharpoonleft\mathcal{K}}$. So we conclude that

$$\alpha^*_{\upharpoonleft\mathcal{K}} \quad \approx_{\mathcal{K}} \quad \beta^*_{\upharpoonleft\mathcal{K}}, \quad \text{for every component interval } \mathcal{K} \text{ of } \neg\mathcal{Q}. \tag{A7}$$

From statement (A7) and the definition of $\succeq_{\mathcal{Q}}$ in Axiom (LSep*), it follows that

$$\left(\alpha_{\upharpoonleft\mathcal{Q}} \succeq_{\mathcal{Q}} \beta_{\upharpoonleft\mathcal{Q}}\right) \iff \left(\alpha^* \succeq \beta^*\right). \tag{A8}$$

Combining (A5) and (A8), we see that $\alpha \succeq_{\mathcal{S}} \beta$ if and only if $\alpha_{\upharpoonleft\mathcal{Q}} \succeq_{\mathcal{Q}} \beta_{\upharpoonleft\mathcal{Q}}$. □

Lemma A.7. *Suppose $\mathcal{A}$ and $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfy* (R*), (ISep*) *and* (LSep*). *Let $\mathcal{Q} \in \mathfrak{S}$ and let $\alpha, \beta \in \mathcal{A}(\mathcal{Q})$. If $\alpha_{\upharpoonleft\mathcal{I}} \approx_{\mathcal{I}} \beta_{\upharpoonleft\mathcal{I}}$ for every component interval $\mathcal{I}$ of $\mathcal{Q}$, then $\alpha \approx_{\mathcal{Q}} \beta$.*

Proof. Find $\alpha' \in \mathcal{A}$ with $\alpha'_{\upharpoonleft\mathcal{Q}} = \alpha$. Let $\mathcal{R} := \neg\mathcal{Q}$. Condition (R*) yields $\beta' \in \mathcal{A}$ such that

$$\beta'_{\upharpoonleft\mathcal{Q}} \;=\; \beta, \quad \text{and} \quad \beta'_{\upharpoonleft\mathcal{I}} \;\approx_{\mathcal{I}}\; \alpha'_{\upharpoonleft\mathcal{I}}, \quad \text{for every component interval } \mathcal{I} \text{ of } \mathcal{R}. \tag{A9}$$

Combining (A9) and Lemma A.6, we have $\alpha'_{\upharpoonleft\mathcal{Q}} \approx_{\mathcal{Q}} \beta'_{\upharpoonleft\mathcal{Q}}$ if and only if $\alpha' \approx \beta'$. Since $\alpha'_{\upharpoonleft\mathcal{Q}} = \alpha$ and $\beta'_{\upharpoonleft\mathcal{Q}} = \beta$, we deduce that $\alpha \approx_{\mathcal{Q}} \beta$ if and only if $\alpha' \approx \beta'$. Thus, it remains to show that $\alpha' \approx \beta'$. Now, for every component interval $\mathcal{I}$ of $\mathcal{Q}$, we have

$$\alpha'_{\upharpoonleft\mathcal{I}} \;=\; \alpha_{\upharpoonleft\mathcal{I}} \;\approx_{\mathcal{I}}\; \beta_{\upharpoonleft\mathcal{I}} \;=\; \beta'_{\upharpoonleft\mathcal{I}}, \tag{A10}$$

where "$\approx_{\mathcal{I}}$" is by hypothesis, and the equalities are because $\alpha'_{\upharpoonleft\mathcal{Q}} = \alpha$ and $\beta'_{\upharpoonleft\mathcal{Q}} = \beta$. Meanwhile, for every component interval $\mathcal{I}$ of $\mathcal{R}$, we have $\alpha'_{\upharpoonleft\mathcal{I}} \approx_{\mathcal{I}} \beta'_{\upharpoonleft\mathcal{I}}$ by statement (A9). At this point, axiom (ISep*) yields $\alpha' \approx \beta'$. □

Proof of Proposition A.1. Let $\mathcal{S} = \mathcal{Q} \vee \mathcal{R}$ and $\alpha, \beta \in \mathcal{A}(\mathcal{S})$ be as in the statement of (Sep). Condition (R*) yields $\alpha' \in \mathcal{A}(\mathcal{S})$ such that

$$\alpha'_{\upharpoonleft\mathcal{R}} \;=\; \alpha_{\upharpoonleft\mathcal{R}}, \tag{A11}$$

$$\text{while} \quad \alpha'_{\upharpoonleft\mathcal{I}} \approx_{\mathcal{I}} \beta_{\upharpoonleft\mathcal{I}}, \quad \text{for every component interval } \mathcal{I} \text{ of } \mathcal{Q}. \tag{A12}$$

Lemma A.7 and statement (A12) imply that $\alpha'_{\upharpoonleft\mathcal{Q}} \approx_{\mathcal{Q}} \beta_{\upharpoonleft\mathcal{Q}}$. Meanwhile, $\alpha_{\upharpoonleft\mathcal{Q}} \approx_{\mathcal{Q}} \beta_{\upharpoonleft\mathcal{Q}}$ by hypothesis. Thus, by transitivity, we get

$$\alpha'_{\upharpoonleft\mathcal{Q}} \quad \approx_{\mathcal{Q}} \quad \alpha_{\upharpoonleft\mathcal{Q}}. \tag{A13}$$

Meanwhile, equation (A11) implies that

$$\alpha'_{\upharpoonleft\mathcal{I}} = \alpha_{\upharpoonleft\mathcal{I}} \quad \text{and hence} \quad \alpha'_{\upharpoonleft\mathcal{I}} \;\approx_{\mathcal{I}}\; \alpha_{\upharpoonleft\mathcal{I}}, \quad \text{for every component interval } \mathcal{I} \text{ of } \mathcal{R}. \tag{A14}$$

Combining statements (A13) and (A14) with Lemma A.6, we deduce that

$$\alpha' \quad \approx_{\mathcal{S}} \quad \alpha. \tag{A15}$$

Meanwhile, statement (A12) and Lemma A.6 (with the roles of $\mathcal{R}$ and $\mathcal{Q}$ reversed) yields

$$\left(\alpha' \succeq_{\mathcal{S}} \beta\right) \iff \left(\alpha'_{\upharpoonleft\mathcal{R}} \succeq_{\mathcal{R}} \beta_{\upharpoonleft\mathcal{R}}\right). \tag{A16}$$

By (A15), the left hand side of (A16) is equivalent to "$\alpha \succeq_{\mathcal{S}} \beta$". By equation (A11), the right hand side of (A16) is equivalent to "$\alpha_{\upharpoonleft\mathcal{R}} \succeq_{\mathcal{R}} \beta_{\upharpoonleft\mathcal{R}}$". With these transformations, statement in (A16) becomes "$\alpha \succeq_{\mathcal{S}} \beta$ if and only if $\alpha_{\upharpoonleft\mathcal{R}} \succeq_{\mathcal{R}} \beta_{\upharpoonleft\mathcal{R}}$," as desired. □

In the formulation of the Tradeoff Consistency axiom (TC) at the end of Section 3, we claimed that it was possible to construct a trajectory $(x_{\mathcal{S}}\alpha)$ satisfying properties (B1) and (B2). In fact, this claim is a special case of a richnesss property that will be important in the proofs in Appendix B, and which is closely related to condition (R*).

LEMMA A.8. *If $\mathcal{A}$ and $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfy* (R*), (LSep*) *and* (ISep*), *then they satisfy the following axiom:*

(Rch) *Let $\mathcal{Q},\mathcal{R}\in\mathfrak{S}$ be disjoint time spans, and let $\mathcal{S}:=\mathcal{Q}\vee\mathcal{R}$. For any $\alpha\in\mathcal{A}(\mathcal{Q})$ and $\beta\in\mathcal{A}(\mathcal{R})$, there exists some $\gamma\in\mathcal{A}(\mathcal{S})$ such that $\gamma_{\upharpoonright\mathcal{Q}}=\alpha$ and $\gamma_{\upharpoonright\mathcal{R}}\approx_{\mathcal{R}}\beta$.*

Proof. Condition (R*), yields some $\gamma\in\mathcal{A}$ such that $\gamma_{\upharpoonright\mathcal{Q}}=\alpha_{\upharpoonright\mathcal{Q}}$, while $\gamma_{\upharpoonright\mathcal{I}}\approx_{\mathcal{I}}\beta_{\mathcal{I}}$ for every component interval $\mathcal{I}$ of $\mathcal{R}$. Then Lemma A.7 implies that $\gamma_{\upharpoonright\mathcal{R}}\approx_{\mathcal{R}}\beta$. □

Appendix B: Technical background on contents This appendix introduces some machinery needed for the proofs of the main results. Let $\mathcal{X}$ be a topological space. A *regular open subset* of $\mathcal{X}$ is a subset $\mathcal{R}\subseteq\mathcal{X}$ such that $\mathcal{X}=\mathrm{int}[\mathrm{clos}(\mathcal{X})]$. Every regular open set is open. But not every open set is regular. For example, if $\mathcal{Z}\subset\mathbb{R}$ is closed and nowhere dense (in particular, if $\mathcal{Z}$ is finite or countable), then $\mathbb{R}\setminus\mathcal{Z}$ is open, but not regular. Let $\mathfrak{RO}(\mathcal{X})$ be the set of all regular open subsets of $\mathcal{X}$. For any $\mathcal{Q},\mathcal{R}\in\mathfrak{RO}(\mathcal{X})$, let $\mathcal{Q}\vee\mathcal{R}:=\mathrm{int}[\mathrm{clos}(\mathcal{Q}\cup\mathcal{R})]$ and $\neg\mathcal{R}:=\mathrm{int}[\mathcal{X}\setminus\mathcal{R}]$. Then $\mathfrak{RO}(\mathcal{X})$ is a Boolean algebra with operations $\vee$, $\cap$ and $\neg$ [12, Thm. 314P].

Now let $\mathcal{T}\subseteq\mathbb{R}$, and let $\mathfrak{S}$ be the family of all time spans on $\mathcal{T}$. As noted in footnote 3, $\mathfrak{S}$ is a Boolean algebra under the operations $\vee$, $\cap$ and $\neg$ —indeed, it is a Boolean *subalgebra* of $\mathfrak{RO}(\mathcal{T})$. If $\mathfrak{B}$ is any Boolean subalgebra of $\mathfrak{RO}(\mathcal{T})$, then a *content* on $\mathfrak{B}$ is a function $\mu:\mathfrak{B}\longrightarrow[0,1]$ with $\mu(\mathcal{T})=1$, which is "finitely additive" in the following sense:

$$\mu[\mathcal{R}\vee\mathcal{S}]=\mu[\mathcal{R}]+\mu[\mathcal{S}],\qquad\text{for all } \mathcal{R},\mathcal{S}\in\mathfrak{B} \text{ with } \mathcal{R}\cap\mathcal{S}=\emptyset. \tag{B1}$$

For any $\mathcal{S}\in\mathfrak{B}$, a $\mathfrak{B}$-*partition* of $\mathcal{S}$ is a collection $\mathcal{S}_1,\ldots,\mathcal{S}_N\in\mathfrak{B}$ of disjoint sets such that $\mathcal{S}=\mathcal{S}_1\vee\cdots\vee\mathcal{S}_N$. It follows from (B1) that $\mu[\mathcal{S}]=\mu[\mathcal{S}_1]+\cdots+\mu[\mathcal{S}_N]$.

The proofs in this appendix draw heavily on results from a companion paper [32], which studies contents and their representations by classical probability measures, as well as "integration" with respect to contents. We will refer to results in the companion paper with the prefix "PV". Thus, "Theorem PV-4.4" should be read as, "Theorem 4.4 from [32]."

A content μ has *full support* if $\mu[\mathcal{S}]>0$ for any nonempty $\mathcal{S}\in\mathfrak{S}$. We say that μ is *nonatomic* if, for any $t\in\mathcal{T}$ and $\epsilon>0$, there is some element $\mathcal{S}\in\mathfrak{S}$ with $t\in\mathcal{S}$ and $\mu[\mathcal{S}]<\epsilon$. Finally, we say that μ is *exponential* if, for any $\mathcal{R},\mathcal{S}\in\mathfrak{S}$ with $\mathcal{R}\subseteq\mathcal{S}$ and any $t\in[0,\infty)$ such that $\mathcal{S}^{+t}\in\mathfrak{S}$, we have

$$\frac{\mu[\mathcal{R}]}{\mu[\mathcal{S}]}\quad=\quad\frac{\mu[\mathcal{R}^{+t}]}{\mu[\mathcal{S}^{+t}]}, \tag{B2}$$

where $\mathcal{R}^{+t}$ and $\mathcal{S}^{+t}$ are defined as in the formulation of the axiom (Stat) in Section 3.

A function $f:\mathcal{T}\longrightarrow\mathbb{R}$ is *comeasurable* with respect to $\mathfrak{S}$ if $\mathrm{int}\,(f^{-1}(-\infty,r])\in\mathfrak{S}$ and $\mathrm{int}\,(f^{-1}[r,\infty))\in\mathfrak{S}$ for all $r\in\mathbb{R}$. Let $\mathcal{C}_{\mathrm{b}}(\mathcal{T},\mathbb{R})$ be the Banach space of bounded, continuous, real-valued functions, with the uniform norm $\|\cdot\|_\infty$. Let $\mathcal{G}_{\mathfrak{S}}(\mathcal{T})$ be the linear subspace of $\mathcal{C}_{\mathrm{b}}(\mathcal{T},\mathbb{R})$ spanned by all $\mathfrak{S}$-comeasurable functions in $\mathcal{C}_{\mathrm{b}}(\mathcal{T},\mathbb{R})$. For any subset $\mathcal{S}\subseteq\mathcal{T}$, let $\mathcal{G}_{\mathfrak{S}}(\mathcal{S}):=\{g_{\upharpoonright\mathcal{S}};\ g\in\mathcal{G}_{\mathfrak{S}}(\mathcal{T})\}$. An *integrator* on $\mathfrak{S}$ is a collection $\mathbf{I}:=\{\mathbb{I}_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$, where for all $\mathcal{S}\in\mathfrak{S}$,

- $\mathbb{I}_{\mathcal{S}}:\mathcal{G}_{\mathfrak{S}}(\mathcal{S})\longrightarrow\mathbb{R}$ is a bounded linear functional that is *weakly monotonic* —that is, for any $f,g\in\mathcal{G}_{\mathfrak{S}}(\mathcal{S})$, if $f(s)\leq g(s)$ for all $s\in\mathcal{S}$, then $\mathbb{I}_{\mathcal{B}}[f]\leq\mathbb{I}_{\mathcal{B}}[g]$;
- For any $\mathfrak{S}$-partition $\{\mathcal{S}_n\}_{n=1}^N$ of $\mathcal{S}$, and for any $g\in\mathcal{G}_{\mathfrak{S}}(\mathcal{S})$, we have

$$\mathbb{I}_{\mathcal{S}}[g]\quad=\quad\sum_{n=1}^N\mathbb{I}_{\mathcal{S}_n}\left[g_{\upharpoonright\mathcal{S}_n}\right]. \tag{B3}$$

If $g \in \mathcal{G}_{\mathfrak{S}}(\mathcal{T})$ and $\mathcal{S} \in \mathfrak{S}$, we will abuse notation and write "$\mathbb{I}_{\mathcal{S}}[g]$" to mean $\mathbb{I}_{\mathcal{S}}[g_{\upharpoonleft\mathcal{S}}]$. Meanwhile, we we will write "$\mathbb{I}[g]$" to mean $\mathbb{I}_{\mathcal{T}}[g]$. Given a content μ on $\mathfrak{S}$, we can define an integrator $\mathbf{I}^\mu = \{\mathbb{I}^\mu_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ as follows. Let $\mathcal{S} \in \mathfrak{S}$. A function $f : \mathcal{S} \longrightarrow \mathbb{R}$ is ***simple*** if there is some $\mathfrak{S}$-partition $\{\mathcal{S}_1, \ldots, \mathcal{S}_N\}$ of $\mathcal{S}$ such that f is constant on each of the sets $\mathcal{S}_1, \ldots, \mathcal{S}_N$. (We place no constraints on the behaviour of f on the boundaries of these sets; it is not important.) For any such simple function, we define

$$\int_{\mathcal{S}}^{\diamond} f \ \mathrm{d}\mu \quad := \quad \sum_{n=1}^{N} r_n \, \mu[\mathcal{S}_n], \tag{B4}$$

where r_n is the value that f takes on $\mathcal{S}_n$. Let $\mathcal{F}(\mathcal{S})$ be the set of all simple functions on $\mathcal{S}$. Given any $g \in \mathcal{G}_{\mathfrak{S}}(\mathcal{S})$, define

$$\mathbb{I}^\mu_{\mathcal{S}}[g] \ := \sup_{f\in\mathcal{F}_g(\mathcal{S})} \int_{\mathcal{S}}^{\diamond} f \ \mathrm{d}\mu, \quad \text{where} \quad \mathcal{F}_g(\mathcal{S}) \ := \ \{f \in \mathcal{F}(\mathcal{S})\, ; \, f(s) \le g(s), \text{ for all } s \in \mathcal{S}\}. \tag{B5}$$

Theorem PV-4.4 states that this is the unique integrator that is "compatible" with μ in the sense that $\mathbb{I}_{\mathcal{S}}[\mathbf{1}] = \mu[\mathcal{S}]$ for all $\mathcal{S} \in \mathfrak{S}$ (where $\mathbf{1}$ is the constant unit function).

Let $u : \mathcal{X} \longrightarrow \mathbb{R}$ be a continuous function representing the synchronic preference relation $\succeq_{\rm sy}$, and let μ be a content on $\mathfrak{S}$. The pair (u, μ) is a ***content-utility*** representation for the intermittent preference structure $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ if, for any $\mathcal{S} \in \mathfrak{S}$ and any $\alpha, \beta \in \mathcal{A}(\mathcal{S})$, we have

$$\Big(\alpha \succeq_{\mathcal{S}} \beta\Big) \iff \Big(\mathbb{I}^\mu_{\mathcal{S}}[u\circ\alpha] \ge \mathbb{I}^\mu_{\mathcal{S}}[u\circ\beta]\Big). \tag{B6}$$

One minor issue here is that it is not clear, *a priori*, that the expressions on the right-hand side are well-defined, because it is not clear that the functions $u\circ\alpha$ and $u\circ\beta$ are in $\mathcal{G}_{\mathfrak{S}}(\mathcal{S})$. The next lemma takes care of this problem.

LEMMA B.1. *Let $\mathcal{X}$ be a connected topological space, let $u : \mathcal{X} \longrightarrow \mathbb{R}$ be a continuous function representing the synchronic preference order $\succeq_{\rm sy}$, and let $\mathcal{A} \subseteq \mathcal{C}_{\rm b}(\mathcal{T}, \mathcal{X})$. If $\mathcal{A}$ satisfies* (LV) *relative to $\succeq_{\rm sy}$, then for any $\alpha \in \mathcal{A}$, the function $u\circ\alpha : \mathcal{T} \longrightarrow \mathbb{R}$ is comeasurable.*

Proof. Let $\mathcal{U} := u(\mathcal{X})$; then $\mathcal{U}$ is an interval, because $\mathcal{X}$ is connected and u is continuous. Let $r \in \mathbb{R}$. We must show that $\mathrm{int}\,((u\circ\alpha)^{-1}(-\infty, r]) \in \mathfrak{S}$. There are three cases. If $r < u(x)$ for all $x \in \mathcal{X}$, then $u^{-1}(-\infty, r] = \emptyset$. But $\mathrm{int}\,(\emptyset) = \emptyset$, which is an element of $\mathfrak{S}$. On the other hand, if $r > u(x)$ for all $x \in \mathcal{X}$, then $u^{-1}(-\infty, r] = \mathcal{X}$. But $\mathrm{int}\,(\mathcal{X}) = \mathcal{X}$, which is an element of $\mathfrak{S}$. Finally, suppose $r \in \mathcal{U}$. Then there exists $x \in \mathcal{X}$ such that $r := u(x)$. Then $u^{-1}(-\infty, r] = \mathcal{X}(\preceq_{\rm sy} x)$, because u represents $\succeq_{\rm sy}$. Thus, $(u\circ\alpha)^{-1}(-\infty, r] = \alpha^{-1}\,(u^{-1}(-\infty, r]) = \alpha^{-1}\,[\mathcal{X}(\preceq_{\rm sy} x)]$. Thus, $\mathrm{int}\,((u\circ\alpha)^{-1}(-\infty, r]) = \mathrm{int}\,(\alpha^{-1}\,[\mathcal{X}(\preceq_{\rm sy} x)])$, which is an element of $\mathfrak{S}$ by (LV), because $\alpha \in \mathcal{A}$. In all three cases, $\mathrm{int}\,((u\circ\alpha)^{-1}(-\infty, r]) \in \mathfrak{S}$. By a similar argument, $\mathrm{int}\,((u\circ\alpha)^{-1}[r, \infty)) \in \mathfrak{S}$. This holds for all $r \in \mathbb{R}$. Thus, $u\circ\alpha$ is comeasurable. □

Another companion paper [33] considers decisions under uncertainty with imperfect perception. The main result of [33] is an axiomatic characterization of subjective expected utility representations based on content-utility representations like (B6). The next theorem is an adaptation of this characterization theorem to the setting of intertemporal choice. It can also be seen as a "content" version of the DUI representations for intertemporal preferences that appear in the body of the present paper. It is a key result, from which we will derive all of our main results.

THEOREM B.1. *Let $\mathcal{T} \subseteq \mathbb{R}$ be any union of open and/or closed intervals with nonempty interiors. Let $\mathcal{X}$ be a connected topological space, and let $\mathcal{A} \subseteq \mathcal{C}_{\rm b}(\mathcal{T}, \mathcal{X})$. Let $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ be an intertemporal preference structure on $\mathcal{A}$ that satisfies conditions* (R*) *and* (LV).

(a) $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ *satisfies axioms* (ISep), (LSep), (StEq), (C), (Dom), (CIP), *and* (TC) *if and only if it has a content-utility representation* (u,μ), *where* u *is a continuous function on* $\mathcal{X}$ *and* μ *is a content on* $\mathfrak{S}$ *with full support.*

(b) $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ *satisfies the seven axioms in part* (a) *and also* (Eph) *if and only if it has a content-utility representation, and furthermore,* μ *is nonatomic.*

(c) *Suppose that* $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ *and* $\mathcal{A}$ *also satisfy condition* (Core*). *Then* $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ *satisfies the seven axioms in part* (a) *and also* (Stat) *if and only if it has a content-utility representation, and furthermore,* μ *is exponential.*

In all three cases, μ *is unique, and* u *is unique up to positive affine transformation.*

Proof. (a) Let $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ be the intermittent preference structure induced by $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ through axiom (LSep*). The proof strategy is almost verbatim identical to the proof of Theorem 1 in [33], but with $\mathcal{T}$ and $\mathfrak{S}$ playing the role of the state space $\mathcal{S}$ and the Boolean algebra $\mathfrak{B}$ in that paper. (Hereafter we will simply call this "Theorem 1".) What is called an "intermittent preference structure" in this paper is called a "conditional preference structure" in Theorem 1, and the "synchronic" preference order $\succeq_{\rm sy}$ in this paper is called an "ex post" preference order $\succeq_{\rm xp}$ in Theorem 1. What is called a "content" in this paper is called a "credence" in Theorem 1, since it represents the "beliefs" of a subjective expected utility maximizer.

Theorem 1 requires the condition (Rch) introduced at the end of Appendix A. But (Rch) follows from condition (R*), by Lemma A.8. Theorem 1 also uses the axiom (Sep) that appears in Section 3. But this is implied by the conjunction of axioms (ISep*) and (LSep*), by Proposition A.1. Axiom (StEq) in this paper corresponds to axiom (CEq) in Theorem 1. All the other axioms (C), (Dom), (CIP), and (TC) are identical.

It remains to resolve two discrepancies between the framework of the present paper and that of [33]. Theorem 1 makes a key structural assumption:

(CM) There is a Boolean algebra $\mathfrak{D}$ of regular open subsets of $\mathcal{X}$ such that all the functions in $\mathcal{A}$ are "comeasurable" with respect to $\mathfrak{S}$ and $\mathfrak{D}$.

(The definition of "comeasurable" is not important here.) Theorem 1 also uses one other axiom:

(M) For any $x\in\mathcal{X}$, the sets $\{y\in\mathcal{X};\ y\succ_{\rm sy} x\}$ and $\{y\in\mathcal{X};\ y\prec_{\rm sy} x\}$ are in $\mathfrak{D}$.

Assuming the structural conditions (Rch) and (CM), Theorem 1 shows that a conditional preference structure $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ satisfies axioms (StEq), (C), (Dom), (Sep), (CIP), (TC) and (M) if and only if it admits a content-utility representation (B6), and furthermore, the function u is "measurable" with respect to $\mathfrak{D}$. However, in Theorem 1, the $\mathfrak{D}$-measurability of the utility function u only serves to obtain axiom (M), and not the other axioms. To prove the necessity of the other axioms, we can therefore proceed here exactly as in the proof of Theorem 1.

Conversely, in the proof of Theorem 1, axiom (M) only serves to obtain the $\mathfrak{D}$-measurability of u. (See Claim 6 in the proof of Proposition A.3 in [33] for details.) Meanwhile, assumption (CM) and the $\mathfrak{D}$-measurability of u are only used to make sure that $u\circ\alpha$ is $\mathfrak{S}$-comeasurable for any $\alpha\in\mathcal{A}$. But given structural condition (LV), Lemma B.1 *already* guarantees that $u\circ\alpha$ is $\mathfrak{S}$-comeasurable for any $\alpha\in\mathcal{A}$. Thus, we do not need either structural condition (CM) or axiom (M) to prove the sufficiency of the axioms for the representation. Finally, the uniqueness of the representation can be obtained exactly as in Theorem 1, since the argument invoked there uses neither condition (CM), nor axiom (M) nor $\mathfrak{D}$-measurability.

(b) "$\Longrightarrow$" Suppose $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfies the seven axioms in part (a) and also (Eph). Let (μ,u) be the content-utility representation obtained in part (a). We must show that μ is non-atomic. Suppose $\mathcal{T}=[0,T]$. (The same argument works if $\mathcal{T}=[0,\infty)$ but is even simpler in that case.) By contradiction, suppose there exists $t\in\mathcal{T}$ and $\epsilon>0$, such that $\mu[\mathcal{S}]\geq\epsilon$ for every $\mathcal{S}\in\mathfrak{S}$ with

$t \in \mathcal{S}$. Let $w, x \in \mathcal{X}$, with $w \prec_{\text{sy}} x$. Thus, $u(w) < u(x)$, because u represents $\succeq_{\text{sy}}$. Without loss of generality (applying an affine transform to u if necessary) assume $u(x) = 1$ and $u(w) = 0$. Since u is continuous and $\mathcal{X}$ is connected, the Intermediate Value Theorem yields some $y \in \mathcal{X}$ with $u(y) = \epsilon$; thus, $y \succ_{\text{sy}} w$. Find $z \in \mathcal{X}$ with $z \approx w$; then $u(z) = 0$ and $y \succ_{\text{sy}} z$.

Let $\mathcal{S} \in \mathfrak{S}$ and let $\mathcal{R} := \neg\mathcal{S}$. Then $(x_{\mathcal{S}} z)_{\upharpoonright\mathcal{S}} \approx_{\mathcal{S}} \kappa^x$, and thus, the content-utility representation (B6) yields $\mathbb{I}^\mu_{\mathcal{S}}[u \circ (x_{\mathcal{S}} z)_{\upharpoonright\mathcal{S}}] = \mathbb{I}^\mu_{\mathcal{S}}[u \circ \kappa^x] = \mathbb{I}^\mu_{\mathcal{S}}[1] = \mu[\mathcal{S}]$. Meanwhile, $(x_{\mathcal{S}} z)_{\upharpoonright\mathcal{R}} \approx_{\mathcal{R}} \kappa^z$, and thus, the content-utility representation (B6) yields $\mathbb{I}^\mu_{\mathcal{R}}[u \circ (x_{\mathcal{S}} z)_{\upharpoonright\mathcal{R}}] = \mathbb{I}^\mu_{\mathcal{R}}[u \circ \kappa^z] = \mathbb{I}^\mu_{\mathcal{S}}[0] = 0$. Thus, since $\mathcal{T} = \mathcal{S} \vee \mathcal{R}$, equation (B3) implies that

$$\begin{aligned} \mathbb{I}^\mu[u \circ (x_{\mathcal{S}} z)] &= \mathbb{I}^\mu_{\mathcal{S}}[u \circ (x_{\mathcal{S}} z)_{\upharpoonright\mathcal{S}}] + \mathbb{I}^\mu_{\mathcal{R}}[u \circ (x_{\mathcal{S}} z)_{\upharpoonright\mathcal{R}}] \\ &= \mu[\mathcal{S}] + 0 \quad = \quad \mu[\mathcal{S}]. \end{aligned} \tag{B7}$$

By a very similar argument, $\mathbb{I}^\mu_{\mathcal{S}}[u \circ (w_{\mathcal{S}} y)_{\upharpoonright\mathcal{S}}] = \mathbb{I}^\mu_{\mathcal{R}}[0] = 0$ and $\mathbb{I}^\mu_{\mathcal{R}}[u \circ (w_{\mathcal{S}} y)_{\upharpoonright\mathcal{R}}] = \mathbb{I}^\mu_{\mathcal{R}}[\epsilon] = \epsilon\, \mu[\mathcal{R}]$; thus, equation (B3) implies that

$$\mathbb{I}^\mu[u \circ (w_{\mathcal{S}} y)] \quad = \quad 0 + \mu[\mathcal{R}] \cdot \epsilon \quad = \quad \epsilon \cdot (1 - \mu[\mathcal{S}]). \tag{B8}$$

Now, if $t \in \mathcal{S}$, then $\mu[\mathcal{S}] > \epsilon$ (by hypothesis). Thus, (B7) yields $\mathbb{I}^\mu[u \circ (x_{\mathcal{S}} z)] > \epsilon$, while (B8) yields $\mathbb{I}^\mu[u \circ (w_{\mathcal{S}} y)] < \epsilon - \epsilon^2 < \epsilon$. Thus, $\mathbb{I}^\mu[u \circ (w_{\mathcal{S}} y)] < \mathbb{I}^\mu[u \circ (x_{\mathcal{S}} z)]$, and thus, $(w_{\mathcal{S}} y) \prec_{\mathcal{T}} (x_{\mathcal{S}} z)$ by the content-utility representation (B6). This holds for any $\mathcal{S} \in \mathfrak{S}$ with $t \in \mathcal{S}$, contradicting axiom (Eph). By contradiction, if $\{\succeq_{\mathcal{I}}\}_{\mathcal{I} \in \mathfrak{I}}$ satisfies (Eph), then μ must be nonatomic.

"$\Longleftarrow$" Suppose $\{\succeq_{\mathcal{I}}\}_{\mathcal{I} \in \mathfrak{I}}$ admits a content-utility representation where μ is nonatomic. From part (a), we already know that $\{\succeq_{\mathcal{S}}\}_{\mathcal{S} \in \mathfrak{S}}$ satisfies the axioms (LSep), (ISep), (StEq), (C), (Dom), (CIP), and (TC). We must show that $\{\succeq_{\mathcal{I}}\}_{\mathcal{I} \in \mathfrak{I}}$ also satisfies (Eph).

Let $t \in \mathcal{T}$, and let $w, x, y, z \in \mathcal{X}$ with $y \succ_{\text{sy}} z$. We must find some $\mathcal{S} \in \mathfrak{S}$ with $t \in \mathcal{S}$ such that $(w_{\mathcal{S}} y) \succ (x_{\mathcal{S}} z)$. If $w \succeq_{\text{sy}} x$, then this is automatically true by axioms (Dom) and (Sep) — (Sep) itself being a consequence of (LSep) and (ISep) via Proposition A.1. So, suppose $w \prec_{\text{sy}} x$. Given the content-utility representation (B6), it is sufficient to obtain $\mathbb{I}^\mu[u \circ (w_{\mathcal{S}} y)] > \mathbb{I}^\mu[u \circ (x_{\mathcal{S}} z)]$. Let $A := u(y) - u(z)$ and $B := u(x) - u(w)$. Then A and B are positive. For any $\mathcal{S} \in \mathfrak{S}$, straightforward computations similar to (B7) yield

$$\begin{aligned} \mathbb{I}^\mu[u \circ (w_{\mathcal{S}} y)] &= u(y) + (u(w) - u(y))\, \mu[\mathcal{S}] \\ \text{and} \quad \mathbb{I}^\mu[u \circ (x_{\mathcal{S}} z)] &= u(z) + (u(x) - u(z))\, \mu[\mathcal{S}], \\ \text{thus} \quad \mathbb{I}^\mu[u \circ (w_{\mathcal{S}} y)] - \mathbb{I}^\mu[u \circ (x_{\mathcal{S}} z)] &= u(y) - u(z) + (u(w) - u(y) - u(x) + u(z))\, \mu[\mathcal{S}] \\ &= A - (A + B)\, \mu[\mathcal{S}]. \end{aligned} \tag{B9}$$

Since μ is nonatomic, there exists some $\mathcal{S} \in \mathfrak{S}$ containing t such that $\mu[\mathcal{S}] < A/(A+B)$. From this, it follows that $A - (A+B)\,\mu[\mathcal{S}] > 0$. Thus, (B9) yields $\mathbb{I}^\mu[u \circ (w_{\mathcal{S}} y)] > \mathbb{I}^\mu[u \circ (x_{\mathcal{S}} z)]$, and hence $(w_{\mathcal{S}} y) \succ (x_{\mathcal{S}} z)$, as desired.

(c) "$\Longrightarrow$" Suppose $\{\succeq_{\mathcal{I}}\}_{\mathcal{I} \in \mathfrak{I}}$ satisfies the axioms in part (a) and also (Stat). Let (μ, u) be the content-utility representation from part (a). We must show that μ is exponential.

Let $\mathcal{S} \in \mathfrak{S}$, and let $t \in [0, \infty)$ be such that $\mathcal{S}^{+t} \subseteq \mathcal{T}$. Let $\mathfrak{R} := \{\mathcal{R} \in \mathfrak{S};\ \ \mathcal{R} \subseteq \mathcal{S}\}$. Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be the set of trajectories identified by the (Core*) condition, let $\mathcal{A}' := \mathcal{A}_0(\mathcal{S})$, and consider the "restricted" intertemporal preference structure $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}}$ defined on $\mathcal{A}'$ (with $\mathcal{S}$ playing the role of $\mathcal{T}$). It is easily verified that this preference structure satisfies (LV), (StEq), (C), (Dom), (Sep), (CIP), and (TC). By hypothesis, it satisfies (R*). Thus, part (a) yields a content μ' on $\mathfrak{R}$ and a continuous function $u' : \mathcal{X} \longrightarrow \mathbb{R}$ that give a content-utility representation (B6) for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}}$ on $\mathcal{A}'$. Furthermore, μ' is unique, and u' is unique up to positive affine transformation. (Here, it is important that part (a) was formulated to allow $\mathcal{T}$ to be *any* union of open intervals and closed intervals in $\mathbb{R}$ —not necessarily a single *closed* interval, as in our main results.)

Now, define the content μ_1 on $\mathfrak{R}$ by setting $\mu_1(\mathcal{R}) := \mu(\mathcal{R})/\mu(\mathcal{S})$ for all $\mathcal{R}\in\mathfrak{R}$. Then it is easily verified that (u,μ_1) is *also* a content-utility representation for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}}$ on $\mathcal{A}'$, because of the fact that (u,μ) is a content-utility representation for $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ on $\mathcal{A}$. Thus, by the aforementioned uniqueness, we must have $\mu_1=\mu'$.

Finally, define the content μ_2 on $\mathfrak{R}$ by setting $\mu_2(\mathcal{R}) := \mu(\mathcal{R}^{+t})/\mu(\mathcal{S}^{+t})$ for all $\mathcal{R}\in\mathfrak{R}$. Then (u,μ_2) is another a content-utility representation for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}}$ on $\mathcal{A}'$. To see this, let $\alpha,\beta\in\mathcal{A}'$; then for any $\mathcal{R}\in\mathfrak{R}$,

$$\Big(\alpha_{\upharpoonright\mathcal{R}}\succeq_{\mathcal{R}}\beta_{\upharpoonright\mathcal{R}}\Big) \underset{(*)}{\Longleftrightarrow} \Big(\alpha^{+t}_{\upharpoonright\mathcal{R}^{+t}}\succeq_{\mathcal{R}^{+t}}\beta^{+t}_{\upharpoonright\mathcal{R}^{+t}}\Big) \underset{(\dagger)}{\Longleftrightarrow} \Big(\mathbb{I}^{\mu}_{\mathcal{R}^{+t}}[u\circ\alpha^{+t}]\geq\mathbb{I}^{\mu}_{\mathcal{R}^{+t}}[u\circ\beta^{+t}]\Big)$$
$$\underset{(\diamond)}{\Longleftrightarrow} \Big(\mathbb{I}^{\mu_2}_{\mathcal{R}}[u\circ\alpha]\geq\mathbb{I}^{\mu_2}_{\mathcal{R}}[u\circ\beta]\Big), \quad \text{as desired.}$$

Here $(*)$ is by axiom (Stat), and uses the fact that $\alpha^{+t}_{\upharpoonright\mathcal{R}^{+t}},\beta^{+t}_{\upharpoonright\mathcal{R}^{+t}}\in\mathcal{A}(\mathcal{R}^{+t})$ because $\alpha_{\upharpoonright\mathcal{R}},\beta_{\upharpoonright\mathcal{R}}\in\mathcal{A}_0(\mathcal{R})$ because $\alpha,\beta\in\mathcal{A}'=\mathcal{A}_0(\mathcal{S})$. Next, $(\dagger)$ is because (u,μ) is a content-utility representation for $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$, and $(\diamond)$ is by the definition of μ_2 and Proposition PV-5.3(b) (a "change of variables" theorem for integration with respect to contents).

Thus, (u,μ_2) is a *another* content-utility representation for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}}$; therefore, by the aforementioned uniqueness, we must also have $\mu_2=\mu'$, which means that $\mu_2=\mu_1$. But then, for any $\mathcal{R}\in\mathfrak{R}$, we have:

$$\frac{\mu[\mathcal{R}]}{\mu[\mathcal{S}]} \quad=\quad \mu_1[\mathcal{R}] \quad=\quad \mu_2[\mathcal{R}] \quad=\quad \frac{\mu[\mathcal{R}^{+t}]}{\mu[\mathcal{S}^{+t}]},$$

so equation (B2) is satisfied. This works for any $\mathcal{S}$, $\mathcal{R}$ and t; thus, μ is exponential.

"$\Longleftarrow$" Suppose $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ admits a content-utility representation (B6), where μ is exponential. From part (a), we already know that $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfies the axioms (ISep), (LSep), (StEq), (C), (Dom), (CIP), and (TC). We must show that $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ also satisfies (Stat). To see this, let $\mathcal{S}\in\mathfrak{S}$ and let $t\in\mathcal{T}$ be such that $\mathcal{S}^{+t}\in\mathfrak{S}$. Let $C:=\mu[\mathcal{S}^{+t}]/\mu[\mathcal{S}]$. Then equation (B2) says $\mu[\mathcal{R}^{+t}]=C\,\mu[\mathcal{R}]$ for all $\mathcal{R}\in\mathfrak{S}$ with $\mathcal{R}\subseteq\mathcal{S}$ (because μ is exponential). Thus, for any $\alpha,\beta\in\mathcal{A}_0(\mathcal{S})$, an application of Proposition PV-5.3(b) yields

$$\mathbb{I}^{\mu}_{\mathcal{S}^{+t}}[u\circ\alpha^{+t}] \;=\; C\,\mathbb{I}^{\mu}_{\mathcal{S}}[u\circ\alpha] \quad\text{and}\quad \mathbb{I}^{\mu}_{\mathcal{S}^{+t}}[u\circ\beta^{+t}] \;=\; C\,\mathbb{I}^{\mu}_{\mathcal{S}}[u\circ\beta]. \tag{B10}$$

Thus,

$$\Big(\alpha\succeq_{\mathcal{S}}\beta\Big) \underset{(*)}{\Longleftrightarrow} \Big(\mathbb{I}^{\mu}_{\mathcal{S}}[u\circ\alpha]\geq\mathbb{I}^{\mu}_{\mathcal{S}}[u\circ\beta]\Big),$$
$$\underset{(\dagger)}{\Longleftrightarrow} \Big(\mathbb{I}^{\mu}_{\mathcal{S}^{+t}}[u\circ\alpha^{+t}]\geq\mathbb{I}^{\mu}_{\mathcal{S}^{+t}}[u\circ\beta^{+t}]\Big) \quad \underset{(*)}{\Longleftrightarrow} \quad \Big(\alpha^{+t}\succeq_{\mathcal{S}^{+t}}\beta^{+t}\Big),$$

as desired. Here, both $(*)$ are by the content-utility representation (B6), and $(\dagger)$ is by (B10). □

Let ν be a Borel measure on $\mathbb{R}$. Say ν is ***nonatomic*** if $\nu[\{t\}]=0$ for all $t\in\mathbb{R}$. It follows that $\nu[\mathcal{Z}]=0$ for any countable subset $\mathcal{Z}\subset\mathbb{R}$. Say ν has ***full support*** if $\nu[\mathcal{O}]>0$ for any open subset of $\mathbb{R}$. The following results will be used repeatedly in Appendices C and D.

Lemma B.2. *Let $\mathcal{T}\subseteq\mathbb{R}$ be a closed interval.*

(a) *Let $\rho:\mathcal{T}\longrightarrow[0,1]$ be a nondecreasing function, and let ν be a normal Borel probability measure on $\mathcal{T}$. The following are equivalent:*

(a1) *For any Borel measurable $\mathcal{S}\subseteq\mathcal{T}$, we have $\nu[\mathcal{S}]=\int_{\mathcal{S}}1\ \mathrm{d}\rho$* (where this is a Lebesgue-Stieltjes integral). *Furthermore, ν is nonatomic and has full support.*

(a2) *$\rho(t)=\nu[(-\infty,t)]$ for all $t\in\mathcal{T}$. Furthermore, $\rho:\mathcal{T}\longrightarrow[0,1]$ is continuous and strictly increasing, with $\inf(\rho(\mathcal{T}))=0$ and $\sup(\rho(\mathcal{T}))=1$.*

If these statements are true, then for any $\mathcal{S} \in \mathfrak{S}$ *and any* $g \in \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathbb{R})$*, we have*

$$\int_{\mathcal{S}} g \; \mathrm{d}\nu \quad = \quad \int_{\mathcal{S}} g \; \mathrm{d}\rho, \tag{B11}$$

where the left side is a Lebesgue integral, and the right side is a Riemann-Stieltjes integral.
(b) *Suppose* ν *is a nonatomic Borel measure on* $\mathcal{T}$*. For all* $\mathcal{S} \in \mathfrak{S}$*, define* $\mu[\mathcal{S}] = \nu[\mathcal{S}]$*. Then* μ *is a nonatomic content on* $\mathfrak{S}$*. If* ν *has full support, then so does* μ*. For any* $\mathcal{S} \in \mathfrak{S}$ *and* $g \in \mathcal{G}_{\mathfrak{S}}(\mathcal{S})$*,*

$$\mathbb{I}^{\mu}_{\mathcal{S}}[g] \quad = \quad \int_{\mathcal{S}} g \; \mathrm{d}\nu. \tag{B12}$$

(c) *If* ν_1 *and* ν_2 *are two nonatomic Borel measures on* $\mathcal{T}$*, and* $\nu_1(\mathcal{I}) = \nu_2(\mathcal{I})$ *for all open intervals* $\mathcal{I} \subseteq \mathcal{T}$ (in particular, if $\nu_1[\mathcal{S}] = \nu_2[\mathcal{S}]$ for all $\mathcal{S} \in \mathfrak{S}$)*, then* $\nu_1 = \nu_2$*.*

Proof. (a) See [11, Theorem 1.16, p.34] or [1, Theorems 10.48 and 10.49, pp. 393-394].[12]

(c) Let $\mathfrak{A}$ be the Boolean algebra consisting of all finite disjoint unions of open intervals in $\mathbb{R}$. (So $\mathfrak{S} \subsetneq \mathfrak{A}$, but $\mathfrak{A}$ also includes unions of the form $(a,b) \sqcup (b,c)$, for example.) Observe that $\mathfrak{A}$ generates the Borel sigma algebra on $\mathbb{R}$. Thus, any finite premeasure on $\mathfrak{A}$ extends to a unique Borel measure on $\mathbb{R}$; see [11, Theorem 1.14, p.30] or [1, Theorem 10.10, p.377]. But ν_1 and ν_2 define identical premeasures when restricted to $\mathfrak{A}$, because they agree on all open intervals. Thus, $\nu_1 = \nu_2$.

(b) Let $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{S}$ be disjoint. Then $\mathcal{S}_1 \vee \mathcal{S}_2 = \mathcal{S}_1 \sqcup \mathcal{S}_2 \sqcup \mathcal{Z}$, where $\mathcal{Z} \subseteq (\partial \mathcal{S}_1) \cup (\partial \mathcal{S}_2)$. But $\partial \mathcal{S}_1$ and $\partial \mathcal{S}_2$ are countable, by the definition of $\mathfrak{S}$. Thus, $\nu[\partial \mathcal{S}_1] = \nu[\partial \mathcal{S}_2] = 0$, because ν is nonatomic. Thus, $\nu[\mathcal{Z}] = 0$. Thus, $\nu[\mathcal{S}_1 \vee \mathcal{S}_2] = \nu[\mathcal{S}_1] + \nu[\mathcal{S}_2]$. This shows that μ is a content. Furthermore, μ is nonatomic, because ν is a nonatomic, normal Borel measure. Finally, μ has full support if ν does.

To verify equation (B12), let $\epsilon > 0$. Then Proposition PV-4.3 yields a simple function $f : \mathcal{S} \longrightarrow \mathbb{R}$ such that $g(s) - \epsilon < f(s) \leq g(s)$ for all $s \in \mathcal{S}$. Thus,

$$\left| \int_{\mathcal{S}} g \; \mathrm{d}\nu - \int_{\mathcal{S}} f \; \mathrm{d}\nu \right| \quad \leq \quad \epsilon \cdot \nu[\mathcal{S}]. \tag{B13}$$

Suppose f is subordinate to the $\mathfrak{S}$-partition $\{\mathcal{S}_1, \ldots, \mathcal{S}_N\}$ of $\mathcal{S}$, let $r_1, \ldots, r_N$ be the constant values that f takes on $\mathcal{S}_1, \ldots, \mathcal{S}_N$. Let $\mathcal{Z} := \mathcal{S} \setminus (\mathcal{S}_1 \sqcup \cdots \sqcup \mathcal{S}_N)$; then $\mathcal{Z} \subseteq (\partial \mathcal{S}_1) \cup \cdots \cup (\partial \mathcal{S}_N)$, so that $\mathcal{Z}$ is countable, and hence $\nu[\mathcal{Z}] = 0$. Thus,

$$\int_{\mathcal{S}} f \; \mathrm{d}\nu \;=\; \int_{\mathcal{Z}} f \; \mathrm{d}\nu + \sum_{n=1}^{N} \int_{\mathcal{S}_N} f \; \mathrm{d}\nu \;=\; 0 + \sum_{n=1}^{N} r_n \, \nu[\mathcal{S}_n] \underset{(*)}{=} \sum_{n=1}^{N} r_n \, \mu[\mathcal{S}_n] \underset{(\dagger)}{=} \int_{\mathcal{S}}^{\diamond} f \; \mathrm{d}\mu. \tag{B14}$$

where $(*)$ is by the definition of μ, and $(\dagger)$ is by (B4). Finally, by (B5), we can choose f so that

$$\left| \mathbb{I}^{\mu}_{\mathcal{S}}[g] - \int_{\mathcal{S}}^{\diamond} f \; \mathrm{d}\mu \right| < \epsilon. \tag{B15}$$

Combining formulae (B13)-(B15), we deduce that

$$\left| \int_{\mathcal{S}} g \; \mathrm{d}\nu - \mathbb{I}^{\mu}_{\mathcal{S}}[g] \right| \quad \leq \quad \epsilon \cdot (1 + \nu[\mathcal{S}]).$$

This holds for any $\epsilon > 0$, so we deduce that $\int_{\mathcal{S}} g \; \mathrm{d}\nu = \mathbb{I}^{\mu}_{\mathcal{S}}[g]$, as claimed. □

[12] Strictly speaking, (a1) should say "$\nu(s,t] = \rho(t) - \rho(s)$ for all $s, t \in \mathcal{T}$", while (a2) should say "$\rho(t) = \nu[(-\infty, t]]$ for all $t \in \mathcal{T}$". But including the endpoint t makes no difference, because ν is nonatomic —equivalently, ρ is continuous.

Appendix C: Proofs from Section 4

Proof of Theorem 1. "$\Longleftarrow$" Suppose that ρ and $u:\mathcal{X}\longrightarrow\mathbb{R}$ provide a DUI representation of $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ as in equation (8). Define the Borel measure ν from ρ as in Lemma B.2(a1), and then for any $\mathcal{S}\in\mathfrak{S}$, define $\mu[\mathcal{S}]:=\nu[\mathcal{S}]$. Lemma B.2(b) says that μ is a nonatomic content with full support on $\mathfrak{S}$, and for any $\mathcal{S}\in\mathfrak{S}$, $\mathbb{I}^{\mu}_{\mathcal{S}}[u\circ\alpha]=\int_{\mathcal{S}}u\circ\alpha\ \mathrm{d}\nu=\int_{\mathcal{S}}u\circ\alpha\ \mathrm{d}\rho$; thus, (u,μ) provides a content-utility representation (B6). Thus, $\{\succeq_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ must satisfy all of (ISep), (LSep), (StEq), (C), (Dom), (CIP), (TC) and (Eph), by Theorem B.1(b).

"$\Longrightarrow$" Recall that $\mathcal{T}=[0,T]$. If $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfies Axioms (ISep), (LSep), (StEq), (C), (Dom), (CIP), (TC), and (Eph) then Theorem B.1(b) says it has a content-utility representation (B6), given by a nonatomic content μ on $\mathfrak{S}$ with full support, and a continuous utility function $u:\mathcal{X}\longrightarrow\mathbb{R}$. The Horn-Tarski Extension Theorem [17, Thm. 1.22] yields a content μ_0 on $\mathfrak{RO}(\mathcal{T})$ that extends μ. Next, since $\mathcal{T}$ is compact, Corollary PV-6.7 yields a normal Borel measure ν on $\mathcal{T}$, together with a collection of non-negative, Borel-measurable functions $\{\phi_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{RO}(\mathcal{T})}$ on $\mathcal{T}$ (a "liminal density structure"), such that for any $\mathcal{R}\in\mathfrak{RO}(\mathcal{T})$,

$$\mu_0[\mathcal{R}] \quad = \quad \nu[\mathcal{R}]+\int_{\partial\mathcal{R}}\phi_{\mathcal{R}}\ \mathrm{d}\nu. \tag{C1}$$

and for any $f\in\mathcal{C}(\mathcal{T},\mathbb{R})$,

$$\mathbb{I}^{\mu_0}_{\mathcal{R}}[f] \quad = \quad \int_{\mathcal{R}}f\ \mathrm{d}\nu+\int_{\partial\mathcal{R}}f\cdot\phi_{\mathcal{R}}\ \mathrm{d}\nu. \tag{C2}$$

Claim 1: *ν is nonatomic.*

Proof. We must show that $\nu[\{t\}]=0$ for all $t\in\mathcal{T}$. First, suppose $t\in(0,T)$. Recall that μ is nonatomic; thus, for any $\epsilon>0$, there is some $\delta>0$ such that $\mu[\mathcal{R}]<\epsilon$, where $\mathcal{R}:=(t-\delta,t+\delta)$. Thus, $\mu_0[\mathcal{R}]<\epsilon$ because μ_0 is an extension of μ. Thus, equation (C1) implies that $\nu[\mathcal{R}]<\epsilon$ (because $\phi_{\mathcal{R}}$ is non-negative). Thus, $\nu[\{t\}]<\epsilon$. This holds for all $\epsilon>0$; thus, $\nu[\{t\}]=0$.

The same proof works if $t=0$ or $t=T$; simply replace "$(t-\delta,t+\delta)$" with "$[0,\delta)$" or "$(T-\delta,T]$" throughout the previous paragraph. ◇ `Claim 1`

Claim 2: *ν has full support.*

Proof. Suppose $\mathcal{R}:=(a,b)$ for some $a<b<\infty$. Then $\partial\mathcal{R}=\{a,b\}$ so that $\nu[\partial\mathcal{R}]=0$ by Claim 1. Thus, (C1) simplifies to $\mu_0[\mathcal{R}]=\nu[\mathcal{R}]$. But $\mu_0[\mathcal{R}]=\mu[\mathcal{R}]>0$ for all nonempty $\mathcal{R}\in\mathfrak{S}$, because μ has full support. Thus, ν also has full support. ◇ `Claim 2`

Define $\rho:[0,T]\longrightarrow[0,1]$ from the measure ν as in Lemma B.2(a2). Then ρ is a continuous and strictly increasing, because ν is a nonatomic probability measure with full support. Also, ρ achieves its infimum 0 and supremum 1 on $[0,T]$, because $[0,T]$ is compact. Thus, ρ is a bijection from $[0,T]$ to $[0,1]$. If $\mathcal{S}\in\mathfrak{S}$, then for any $f\in\mathcal{C}(\mathcal{T},\mathbb{R})$,

$$\mathbb{I}^{\mu}_{\mathcal{S}}[f] \underset{(*)}{=\!=} \mathbb{I}^{\mu_0}_{\mathcal{S}}[f] \underset{(\dagger)}{=\!=} \int_{\mathcal{S}}f\ \mathrm{d}\nu \underset{(\diamond)}{=\!=} \int_{\mathcal{S}}f\ \mathrm{d}\rho. \tag{C3}$$

Here $(*)$ is because μ_0 is an extension of μ, and $(\dagger)$ is by equation (C2), because the ν-integral over $\partial\mathcal{S}$ is zero because $\nu(\partial\mathcal{S})=0$, because $\partial\mathcal{S}$ is finite (by the definition of $\mathfrak{S}$) and ν is nonatomic (by Claim 1). Finally, $(\diamond)$ is by equation (B11) in Lemma B.2(a). Now combine (C3) with the content-utility representation (B6) to obtain the discounted utility integral representation (8).

Uniqueness. Suppose that both (u,ρ) and (u',ρ') provide DUI representations of $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$. Let ν and ν' be the Borel probability measures obtained from ρ and ρ' to $\mathfrak{S}$ via Lemma B.2(a1). Let μ and μ' be the contents obtained by restricting ν and ν' to $\mathfrak{S}$, as in Lemma B.2(b). Then (u,μ) and

(u',μ') both provide content-utility representations as in formula (B6), as explained in the proof of "$\Longleftarrow$". By uniqueness in Theorem B.1(a), u and u' are positive affine transformations of each other, while $\mu=\mu'$. Thus, ν and ν' agree on $\mathfrak{S}$. Thus, $\nu=\nu'$, by Lemma B.2(c). Thus, $\rho=\rho'$. □

Proof of Theorem 2. "$\Longleftarrow$" Suppose that δ and $u:\mathcal{X}\longrightarrow\mathbb{R}$ provide an exponentially discounted utility integral representation of $\{\succeq_\mathcal{S}\}_{\mathcal{S}\in\mathfrak{S}}$ as in equation (9). Let λ be the Lebesgue measure on $[0,T]$, and define the nonatomic, normal Borel measure ν by setting $\nu[\mathcal{B}]:=\int_\mathcal{B}\delta^t\ \mathrm{d}\lambda[t]$ for all Borel-measurable subsets $\mathcal{B}\subseteq[0,T]$. Now, for any $\mathcal{S}\in\mathfrak{S}$, define $\mu[\mathcal{S}]:=\nu[\mathcal{S}]$. Lemma B.2(b) says that μ is a nonatomic content with full support on $\mathfrak{S}$, and for any $\mathcal{S}\in\mathfrak{S}$,

$$\mathbb{I}^\mu_\mathcal{S}[u\circ\alpha] \quad = \quad \int_\mathcal{S} u\circ\alpha\ \mathrm{d}\nu \quad = \quad \int_\mathcal{S}\delta^t\, u\circ\alpha(t)\ \mathrm{d}\lambda[t]. \tag{C4}$$

Now combine representation (9) with equation (C4) to obtain a content-utility representation (B6). It is easily verified that the content μ is exponential. Thus, $\{\succeq_\mathcal{S}\}_{\mathcal{S}\in\mathfrak{S}}$ must satisfy all of axioms (ISep), (LSep), (StEq), (C), (Dom), (CIP), (TC) and (Stat), by Theorem B.1(c).

"$\Longrightarrow$" If $\{\succeq_\mathcal{I}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfies Axioms (ISep), (LSep), (StEq), (C), (Dom), (CIP), (TC), and (Stat) then Theorem B.1(c) says it has an content-utility representation (B6), given by a continuous utility function $u:\mathcal{X}\longrightarrow\mathbb{R}$ and a content μ on $\mathfrak{S}$ with full support, which is exponential in the sense of equation (B2).

Claim 1:

(a) *For any $t\in(-T,T)$, there exists $C(t)>0$ such that, for all $\mathcal{R}\in\mathfrak{S}$ with $\mathcal{R}^{+t}\in\mathfrak{S}$ we have $\mu[\mathcal{R}^{+t}]=C(t)\cdot\mu[\mathcal{R}]$.*

(b) *For all $t_1,t_2\in(-T,T)$ such that $t_1+t_2\in(-T,T)$ also, we have $C(t_1+t_2)=C(t_1)\cdot C(t_2)$.*

(c) $\lim\limits_{\epsilon\to 0}\ \sup\limits_{s\in(\epsilon,T-\epsilon)}\ \mu[(s-\epsilon,s+\epsilon)]=0.$

(d) *C is continuous.*

(e) *μ is nonatomic.*

(f) *There is some $\delta>0$ such that $C(t)=\delta^t$ for all $t\in(-T,T)$.*

Proof. (a) First suppose $t\in(0,T)$. Let $\mathcal{S}:=[0,T-t)$; then $\mathcal{S}^{+t}=(t,T]$. For any $\mathcal{R}\in\mathfrak{S}$, we have $\mathcal{R}^{+t}\in\mathfrak{S}$ if and only if $\mathcal{R}\subseteq\mathcal{S}$. In this case, $\mathcal{R}^{+t}\subseteq\mathcal{S}^{+t}$, and

$$\frac{\mu[\mathcal{R}]}{\mu[\mathcal{S}]} \quad = \quad \frac{\mu[\mathcal{R}^{+t}]}{\mu[\mathcal{S}^{+t}]}, \tag{C5}$$

because μ is exponential. Let $C(t):=\mu[\mathcal{S}^{+t}]/\mu[\mathcal{S}]$. Then equation (C5) says that $\mu[\mathcal{R}^{+t}]:=C(t)\,\mu[\mathcal{R}]$ for all $\mathcal{R}\subseteq\mathcal{S}$. If $t\in(-T,0)$, then the argument is similar, only with $\mathcal{S}:=(-t,T]$ so that $\mathcal{S}^{+t}=[0,T+t)$. Finally, it is obvious that $C(0)=0$.

(b) Let $t_1,t_2\in(-T,T)$, and suppose $t=t_1+t_2\in(-T,T)$ also. Let $\mathcal{R}\in\mathfrak{S}$ be any time span such that $\mathcal{R}^{+t}\in\mathfrak{S}$ and $\mathcal{R}^{+t_1}\in\mathfrak{S}$. Then $\mathcal{R}^{+t}=(\mathcal{R}^{+t_1})^{+t_2}$. Thus, we have

$$C(t)\,\mu[\mathcal{R}] \;=\; \mu[\mathcal{R}^{+t}] \;=\; \mu[(\mathcal{R}^{+t_1})^{+t_2}] \;=\; C(t_2)\,\mu[\mathcal{R}^{+t_1}] \;=\; C(t_2)\,C(t_1)\,\mu[\mathcal{R}]. \tag{C6}$$

Now, $\mu[\mathcal{R}]>0$ because μ has full support. So cancel $\mu[\mathcal{R}]$ from (C6) to get $C(t)=C(t_1)\cdot C(t_2)$.

(c) (by contradiction) Let $T_0\in(0,T)$, and let $\alpha:=C(T_0)$. So $\alpha>0$. For any $N\in\mathbb{N}$, part (b) says $C(T_0/2N)=\alpha^{1/2N}$. Thus, for any $n\in[0\ldots N]$, part (b) says $C\left(\frac{n}{2N}T_0\right)=\alpha^{n/2N}$. If $\alpha\neq 1$, then

$$\sum_{n=0}^{N-1} C\left(\frac{n}{2N}T_0\right)=\sum_{n=0}^{N-1}\alpha^{n/2N} \quad = \quad \frac{1-\alpha^{N/2N}}{1-\alpha^{1/2N}} \quad = \quad \frac{1-\sqrt{\alpha}}{1-\alpha^{1/2N}}.$$

$$\text{Thus,}\quad \lim_{N\to\infty}\sum_{n=0}^{N-1} C\left(\frac{n}{2N}T_0\right)=\lim_{N\to\infty}\frac{1-\sqrt{\alpha}}{1-\alpha^{1/2N}} \quad = \quad \infty, \tag{C7}$$

because the denominator converges to zero. This holds whether $\alpha > 1$ or $\alpha \in (0,1)$. Through an almost identical computation, we obtain:

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} C\left(\frac{-n}{2N} T_0\right) \quad = \quad \lim_{N\to\infty} \frac{1-\sqrt{\alpha^{-1}}}{1-\alpha^{-1/2N}} \quad = \quad \infty. \tag{C8}$$

Finally, note that if $\alpha = 1$, then $C(\frac{n}{M} T_0) = 1$ for all n and M, and thus once again

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} C\left(\frac{n}{2N} T_0\right) \;=\; \lim_{N\to\infty} N \;=\; \infty \quad \text{and} \quad \lim_{N\to\infty} \sum_{n=0}^{N-1} C\left(\frac{-n}{2N} T_0\right) \;=\; \infty. \tag{C9}$$

To obtain a contradiction suppose there exists $c > 0$ such that, for all $\epsilon > 0$, there is $s \in (\epsilon, T-\epsilon)$ with $\mu[(s-\epsilon, s+\epsilon)] > c$. Let $N \in \mathbb{N}$, and let $\epsilon := T_0/2N$. Find $s \in (\frac{\epsilon}{2}, T - \frac{\epsilon}{2})$ such that $\mu[\mathcal{Q}] > c$, where $\mathcal{Q} := (s - \frac{\epsilon}{2}, s + \frac{\epsilon}{2})$. The intervals $\mathcal{Q}, \mathcal{Q}^{+\epsilon}, \mathcal{Q}^{+2\,\epsilon}, \ldots, \mathcal{Q}^{+N\,\epsilon}$ are all disjoint from one another, and from $\mathcal{Q}^{+(-\epsilon)}, \mathcal{Q}^{+(-2\,\epsilon)}, \ldots, \mathcal{Q}^{+(-N\,\epsilon)}$. For all $n \in [-N \ldots N]$, if $\mathcal{Q}^{+n\epsilon} \subseteq [0,T]$, then

$$\mu[\mathcal{Q}^{+n\epsilon}] \underset{(*)}{=\!=} C(n\,\epsilon)\,\mu[\mathcal{Q}] \quad > \quad C(n\,\epsilon)\,c \underset{(\dagger)}{=\!=} C\left(\frac{n}{2N} T_0\right) c, \tag{C10}$$

where $(*)$ is by part (a) and $(\dagger)$ is by the definition of ϵ. Now, if $s \in [0, \frac{T}{2}]$, then $\mathcal{Q}, \mathcal{Q}^{+\epsilon}, \mathcal{Q}^{+2\,\epsilon}, \ldots, \mathcal{Q}^{+(N-1)\,\epsilon}$ are all contained in $[0,T]$. Thus,

$$\mu[[0,T]] \quad \geq \quad \mu\left[\bigvee_{n=0}^{N-1} \mathcal{Q}^{+n\,\epsilon}\right] \underset{(*)}{=\!=} \sum_{n=0}^{N-1} \mu\left[\mathcal{Q}^{+n\,\epsilon}\right] \underset{(\dagger)}{>} \sum_{n=0}^{N-1} C\left(\frac{n}{2N} T_0\right) c, \tag{C11}$$

where $(*)$ is by equation (B1) and $(\dagger)$ is by inequality (C10). On the other hand, if $s \in [\frac{T}{2}, T]$, then $\mathcal{Q}^{+(-\epsilon)}, \mathcal{Q}^{+(-2\,\epsilon)}, \ldots, \mathcal{Q}^{+(-(N-1)\,\epsilon)} \subseteq [0,T]$, so a computation very similar to (C11) yields

$$\mu[[0,T]] \quad \geq \quad \mu\left[\bigvee_{n=0}^{N-1} \mathcal{Q}^{+(-n\,\epsilon)}\right] \quad > \quad \sum_{n=0}^{N-1} C\left(\frac{-n}{2N} T_0\right) c. \tag{C12}$$

For any $N \in \mathbb{N}$, we get either inequality (C11) or inequality (C12). Combining these with (C7), (C8) and (C9), we conclude that $\mu[[0,T]] = \infty$. This contradicts the fact that μ is a content.

(d) Let $\mathcal{R} \in \mathfrak{S}$. Then $C(t) = \mu[\mathcal{R}^{+t}]/\mu[\mathcal{R}]$ for all t with $\mathcal{R}^{+t} \in \mathfrak{S}$. To show C is continuous, it suffices to show the function $t \mapsto \mu[\mathcal{R}^{+t}]$ is continuous for some $\mathcal{R} \in \mathfrak{R}$. To see this, let $\mathcal{R} = (r,s)$ for some $0 < r < s < T$, and let $0 \leq t_1 < t_2 < (s-r)$. So $\mathcal{R}^{+t_1} = (r+t_1, s+t_1)$ and $\mathcal{R}^{+t_2} = (r+t_2, s+t_2)$. Let $\mathcal{Q} := \mathcal{R}^{+t_1} \cap \mathcal{R}^{+t_2} = (r+t_2, s+t_1)$. Let $\mathcal{Q}_1 := (r+t_1, r+t_2)$ and let $\mathcal{Q}_2 := (s+t_1, s+t_2)$. Then $\mathcal{R}^{+t_1} = \mathcal{Q}_1 \vee \mathcal{Q}$ and $\mathcal{R}^{+t_2} = \mathcal{Q} \vee \mathcal{Q}_2$, so $\mu[\mathcal{R}^{+t_1}] = \mu[\mathcal{Q}_1] + \mu[\mathcal{Q}]$ and $\mu[\mathcal{R}^{+t_2}] = \mu[\mathcal{Q}] + \mu[\mathcal{Q}_2]$. So $\mu[\mathcal{R}^{+t_1}] - \mu[\mathcal{R}^{+t_2}] = \mu[\mathcal{Q}_1] - \mu[\mathcal{Q}_2] = \mu[(r+t_1, r+t_2)] - \mu[(s+t_1, s+t_2)]$. Thus,

$$\lim_{|t_1-t_2|\to 0} \left(\mu[\mathcal{R}^{+t_1}] - \mu[\mathcal{R}^{+t_2}]\right) \;=\; \lim_{|t_1-t_2|\to 0} \mu[(r+t_1, r+t_2)] - \lim_{|t_1-t_2|\to 0} \mu[(s+t_1, s+t_2)] \underset{(*)}{=\!=} 0,$$

as desired, where $(*)$ is by part (c). Thus, C is continuous on $[0,T)$. For all $t \in [0,T)$, part (b) implies that $C(-t) = 1/C(t)$; thus, C is also continuous on $(-T, 0]$.

(e) This follows immediately from part (c).

(f) Define $F(t) := \log[C(t)]$ for all $t \in (-T,T)$. Then part (b) implies that F satisfies the Cauchy functional equation: $F(t_1+t_2) = F(t_1) + F(t_2)$ for all $t_1, t_2 \in (-T,T)$ such that these values are defined. Meanwhile part (d) implies that F is continuous. Thus Corollary 3 of [34] yields a linear function $L : \mathbb{R} \longrightarrow \mathbb{R}$ such that $F(t) = L(t)$ for all $t \in (-T,T)$. Suppose $L(t) = c\,t$ for all $t \in \mathbb{R}$. Let $\delta := \exp(c)$. Then $C(t) = \exp[F(t)] = \exp[c\,t] = \delta^t$ for all $t \in (-T,T)$, as claimed. $\diamond$ `Claim 1`

Claim 1(e) says that μ is nonatomic. Now follow the proof of Theorem B.1(b) "$\Longleftarrow$" to deduce that $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfies Axiom (Eph). Now that we know that $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfies Axioms (ISep), (LSep), (StEq), (C), (Dom), (CIP), (TC), and (Eph), Theorem 1 implies that $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ has a DUI representation (8) for some (u',ρ). It remains to show that $\mathrm{d}\rho$ is an exponential function on $[0,T]$.

Recall from the proof of Theorem 1 that the function ρ was obtained from a Borel probability measure ν via Lemma B.2(a); meanwhile, ν was obtained from a content μ' that was part of a content-utility representation (μ',u') for $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ from Theorem B.1(b). But we already had a content-utility representation (μ,u) from Theorem B.1(c). The "uniqueness" part of Theorem B.1 implies that $\mu'=\mu$. Thus, for any $r,s\in(0,T)$ with $r<s$, we have $\mu[(r,s)]=\nu[(r,s)]=\rho(s)-\rho(r)$. Meanwhile, let $\delta>0$ be the constant from Claim 1(a,f); thus, $\mu[\mathcal{S}^{+t}]=\delta^t\,\mu[\mathcal{S}]$ for all $\mathcal{S}\in\mathfrak{S}$ and $t\in[0,T)$. For any $t\in[0,T)$, define $\rho_t:[0,T-t)\longrightarrow[0,1)$ by setting $\rho_t(s):=\rho(s+t)$ for all $s\in[0,T-t)$.

Claim 2: *For any $t\in[0,T)$, there is a constant k_t such that $\rho_t(s)=\delta^t\,\rho(s)+k_t$ for all $s\in[0,T-t)$.*

Proof. Fix $r\in(0,T-t)$, and let $k_t:=\rho(r+t)-\delta^t\,\rho(r)$. For any $s\in(r,T-t)$, we have $\rho(s+t)-\rho(r+t)=\nu[(r+t,s+t)]=\mu[(r+t,s+t)]\underset{(*)}{=}\delta^t\,\mu[(r,s)]=\delta^t\,\nu[(r,s)]=\delta^t\,\Big(\rho(s)-\rho(r)\Big)$, where $(*)$ is by Claim 1(a,f). Thus,

$$\rho_t(s)\quad=\quad\rho(s+t)\quad\underset{(*)}{=}\quad\delta^t\,\Big(\rho(s)-\rho(r)\Big)+\rho(r+t)\quad=\quad\delta^t\,\rho(s)+k_t, \tag{C13}$$

for all $s>r$. Note that the left-hand side of (C13) does not depend on r. Thus, k_t must also be independent of r. Thus, we can repeat this argument while letting $r\searrow 0$. Thus, equation (C13) holds for all $s\in(0,T-t)$. The case when $s=0$ follows from the continuity of ρ and ρ_t. $\diamondsuit$ `Claim 2`

Claim 3: *ρ is differentiable everywhere on $(0,T)$, and there is a constant $K>0$ such that $\rho'(t)=K\,\delta^t$ for all $t\in(0,T)$.*

Proof. Since ρ is continuous and strictly increasing, it is differentiable almost everywhere on $[0,T)$ [20, Thm.6, §31.2, p.321]. Let $q\in(0,T)$ be such that ρ is differentiable at q. For any $t\in(0,q)$ and any small $\epsilon>0$, we have $\rho_t(q-t+\epsilon)=\rho(q+\epsilon)$; thus, ρ_t must be differentiable at $q-t$, with

$$\rho_t'(q-t)\quad=\quad\rho'(q). \tag{C14}$$

But Claim 2 says that $\rho_t(s)=\delta^t\,\rho(s)+k_t$ for all $s\in[0,T)$. Thus, if ρ_t is differentiable at $q-t$, then ρ must also be differentiable at $q-t$, with

$$\rho_t'(q-t)\quad=\quad\delta^t\,\rho'(q-t). \tag{C15}$$

Combining (C14) and (C15), we obtain

$$\rho'(q)\quad=\quad\delta^t\,\rho'(q-t). \tag{C16}$$

Let $s:=q-t$, and define $K:=\rho'(q)/\delta^q$. Then

$$\rho'(s)\quad=\quad\rho'(q-t)\quad\underset{(*)}{=}\quad\frac{\rho'(q)}{\delta^t}\quad=\quad\frac{\rho'(q)}{\delta^q}\,\delta^{q-t}\quad=\quad K\,\delta^s, \tag{C17}$$

where $(*)$ is obtained by rearranging equation (C16). This argument works for any $t\in(0,q)$. Thus, we conclude that ρ is differentiable everywhere on the interval $(0,q)$, and equation (C17) holds for all $s\in(0,q)$. Note that the left-hand side of (C17) is independent of q. Thus, the constant K must also be independent of q.

In the above argument, q is any point where ρ is differentiable. But ρ is differentiable almost everywhere on $[0,T)$, so we can choose q to be arbitrarily large. By repeating the above argument for increasingly large values of q, we deduce that ρ is differentiable everywhere on $(0,T)$, and equation (C17) holds for all $s\in(0,T)$. $\diamondsuit$ `Claim 3`

For any $\mathcal{S} \in \mathfrak{S}$, and any function $f \in \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathbb{R})$, Claim 3 implies that

$$\int_{\mathcal{S}} f \, \mathrm{d}\rho \quad = \quad K \int_{\mathcal{S}} \delta^t f(t) \, \mathrm{d}t.$$

Substitute this into DUI representation (8) to obtain the exponential DUI representation (9). □

Proof of Theorem 3. $\mathcal{A}'$ contains all constant trajectories, so we can define a synchronous preference order $\succeq'_{\mathrm{sy}}$ on $\mathcal{X}$ by formula (7). But $\{\succeq'_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfies axiom (Dom) with respect to $\succeq_{\mathrm{sy}}$; thus, $\succeq'_{\mathrm{sy}}$ must be identical with $\succeq_{\mathrm{sy}}$.

Let $\mathcal{S} \in \mathfrak{S}$, and let $\alpha', \beta' \in \mathcal{A}'(\mathcal{S})$. We must prove the biconditional (8). Since $\mathcal{A}$ is uniformly $\succeq_{\mathrm{sy}}$-dense in $\mathcal{A}'$, there exist $\{\alpha_n\}_{n=1}^{\infty} \in \mathcal{A}(\mathcal{S})$ and $\{\beta_n\}_{n=1}^{\infty} \in \mathcal{A}(\mathcal{S})$ such that

$$\lim_{n\to\infty} d_{\mathcal{S}}(\alpha_n, \alpha') \ = \ 0 \text{ and } \lim_{n\to\infty} d_{\mathcal{S}}(\beta_n, \beta') \ = \ 0, \tag{C18}$$

$$\text{with} \quad \alpha_n(s) \ \succeq_{\mathrm{sy}} \ \alpha'(s) \text{ and } \beta'(s) \ \succeq_{\mathrm{sy}} \ \beta_n(s), \quad \text{for all } n \in \mathbb{N} \text{ and } s \in \mathcal{S}. \tag{C19}$$

"$\Longrightarrow$" Suppose $\alpha' \succeq'_{\mathcal{S}} \beta'$. Since $\{\succeq'_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ satisfies axiom (Dom), statement (C19) implies

$$\alpha_n \quad \succeq'_{\mathcal{S}} \quad \alpha' \quad \succeq'_{\mathcal{S}} \quad \beta' \quad \succeq'_{\mathcal{S}} \quad \beta_n,$$

and thus $\alpha_n \succeq_{\mathcal{S}} \beta_n$, for all $n \in \mathbb{N}$. Thus,

$$\int_{\mathcal{S}} u \circ \alpha_n \, \mathrm{d}\rho \quad \geq \quad \int_{\mathcal{S}} u \circ \beta_n \, \mathrm{d}\rho, \qquad \text{for all } n \in \mathbb{N}, \tag{C20}$$

by the assumed DUI representation (8) for $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$. Meanwhile, since u is uniformly continuous, the uniform convergence equations (C18) imply that

$$\lim_{n\to\infty} \left\| u \circ \alpha_n - u \circ \alpha' \right\|_{\infty} \ = \ 0 \quad \text{and} \quad \lim_{n\to\infty} \left\| u \circ \beta_n - u \circ \beta') \right\|_{\infty} \ = \ 0. \tag{C21}$$

Thus,

$$\int_{\mathcal{S}} u \circ \alpha' \, \mathrm{d}\rho \ \underset{(*)}{=} \ \lim_{n\to\infty} \int_{\mathcal{S}} u \circ \alpha_n \, \mathrm{d}\rho \ \underset{(\dagger)}{\geq} \ \lim_{n\to\infty} \int_{\mathcal{S}} u \circ \beta_n \, \mathrm{d}\rho \ \underset{(*)}{=} \ \int_{\mathcal{S}} u \circ \beta' \, \mathrm{d}\rho.$$

Here, both $(*)$ are by (C21), while $(\dagger)$ is by (C20).

"$\Longleftarrow$" Since $\succeq_{\mathrm{sy}}$ is the synchronic preference order associated with $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$, and u comes from a DUI representation for $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$, we must have

$$\Big(x \succeq_{\mathrm{sy}} y\Big) \quad \Longleftrightarrow \quad \Big(u(x) \geq u(y)\Big), \qquad \text{for all } x, y \in \mathcal{X}. \tag{C22}$$

Now, suppose that $\int_{\mathcal{S}} u \circ \alpha' \, \mathrm{d}\rho \geq \int_{\mathcal{S}} u \circ \beta' \, \mathrm{d}\rho$. For all $n, m \in \mathbb{N}$, statements (C19) and (C22) yields $u[\alpha_n(s)] \geq u[\alpha'(s)]$ and $u[\beta'(s)] \geq u[\beta_m(s)]$ for all $s \in \mathcal{S}$, and thus,

$$\int_{\mathcal{S}} u \circ \alpha_n \, \mathrm{d}\rho \quad \geq \quad \int_{\mathcal{S}} u \circ \alpha' \, \mathrm{d}\rho \quad \geq \quad \int_{\mathcal{S}} u \circ \beta' \, \mathrm{d}\rho \quad \geq \quad \int_{\mathcal{S}} u \circ \beta_m \, \mathrm{d}\rho.$$

Thus, $\alpha_n \succeq_{\mathcal{S}} \beta_m$ for all $n, m \in \mathbb{N}$, by the assumed DUI representation (8) for $\{\succeq_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$. Since $\{\succeq'_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$ is uniformly continuous, we can use the first limit in (C18) to deduce that $\alpha \succeq'_{\mathcal{S}} \beta_m$ for all $m \in \mathbb{N}$. Next, using the second limit in (C18) and the uniform continuity of $\{\succeq'_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$, we deduce that $\alpha \succeq'_{\mathcal{S}} \beta$, as claimed. □

Appendix D: Proofs from Section 5 Theorem 4 is actually a consequence of Theorem 6, so we will prove it last.

Proof of Theorem 5. "$\Longleftarrow$" The proof is similar to the proof of Theorem 1 "$\Longleftarrow$".

"$\Longrightarrow$" Let $[0,\infty]$ be the Alexandroff compactification of $[0,\infty)$. Any $\alpha \in \mathcal{C}_{\mathrm{L}}([0,\infty),\mathcal{X})$ has a unique extension $\overline{\alpha} \in \mathcal{C}([0,\infty],\mathcal{X})$ defined by $\overline{\alpha}_{\restriction[0,\infty)} := \alpha$ and $\overline{\alpha}(\infty) := \lim_{t\to\infty} \alpha(t)$.

For any time span $\mathcal{S} \in \mathfrak{S}$, we define the subset $\overline{\mathcal{S}} \subset [0,\infty]$ as follows: if $\mathcal{S}$ is bounded or perennial, then $\overline{\mathcal{S}} := \mathcal{S}$. If $\mathcal{S}$ is abiding, then $\mathcal{S} = \lfloor s_1,t_1) \sqcup (s_2,t_2) \sqcup \cdots \sqcup (s_N,\infty)$ for some $0 \leq s_1 < t_1 < \cdots < s_N$. In this case, we define $\overline{\mathcal{S}} := \lfloor s_1,t_1) \sqcup (s_2,t_2) \sqcup \cdots \sqcup (s_N,\infty]$. In either case, we define $\partial\overline{\mathcal{S}}$ to be the boundary of $\overline{\mathcal{S}}$ in $[0,\infty]$. There are two cases to consider. If $\mathcal{S}$ is bounded or abiding, then $\partial\overline{\mathcal{S}} = \partial\mathcal{S}$, a finite subset of $[0,\infty)$. Otherwise, if $\mathcal{S}$ is perennial, then $\partial\overline{\mathcal{S}} = \partial\mathcal{S} \sqcup \{\infty\}$, which is countable.

The proof strategy is now very similar to the proof of Theorem 1 "$\Longrightarrow$": first, obtain a content-utility representation from Theorem B.1(b) given by a nonatomic content μ with full support on $\mathfrak{S}$. Then extend μ to a content μ_0 on $\mathfrak{RO}(\mathcal{T})$, via the Horn-Tarski Extension Theorem [17, Thm. 1.22]. But now, instead of Corollary PV-6.7, we use Theorem PV-7.2 to obtain a normal Borel probability measure ν on $[0,\infty]$, together with a collection of Borel-measurable functions $\{\phi_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{RO}(\mathcal{T})}$ on $[0,\infty]$, such that for any $\mathcal{R} \in \mathfrak{RO}(\mathcal{T})$,

$$\mu_0[\mathcal{R}] \quad = \quad \nu[\overline{\mathcal{R}}] + \int_{\partial\overline{\mathcal{R}}} \phi_{\mathcal{R}} \, \mathrm{d}\nu. \tag{D1}$$

and for any $f \in \mathcal{C}_{\mathrm{L}}(\mathcal{T},\mathbb{R})$,

$$\mathbb{I}^{\mu_0}_{\mathcal{R}}[f] \quad = \quad \int_{\overline{\mathcal{R}}} \overline{f} \, \mathrm{d}\nu + \int_{\partial\overline{\mathcal{R}}} \overline{f}\cdot\phi_{\mathcal{R}} \, \mathrm{d}\nu. \tag{D2}$$

As in Claims 1 and 2 in the proof of Theorem 1, we establish that ν is nonatomic and has full support on $[0,\infty)$ —but now the argument uses equation (D1) rather than equation (C1). Let $M := \nu[\{\infty\}]$, and for all $\mathcal{S} \in \mathfrak{S}_p$, define $c_{\mathcal{S}} := \phi_{\mathcal{S}}(\infty)$. If $M > 0$, then for any perennial partition $\{\mathcal{S}_1,\ldots,\mathcal{S}_N\}$, Theorem PV-7.2, says that $\sum_{n=1}^N c_{\mathcal{S}_n} = 1$. Thus, the collection $\{c_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}}$ is a coeternity structure. Now there are three cases.

Case 1. Suppose $\mathcal{S} \in \mathfrak{S}_b$. Then $\overline{\mathcal{S}} = \mathcal{S}$, so that $\partial\overline{\mathcal{S}} = \partial\mathcal{S}$ is a finite subset of $[0,\infty)$, which gets ν-measure 0. Thus, the second integral in (D2) is trivial, so that (D2) becomes

$$\mathbb{I}^{\mu}_{\mathcal{S}}[f] \quad = \quad \mathbb{I}^{\mu_0}_{\mathcal{S}}[f] \quad = \quad \int_{\overline{\mathcal{S}}} \overline{f} \, \mathrm{d}\nu \quad = \quad \int_{\mathcal{S}} f \, \mathrm{d}\nu. \tag{D3}$$

Case 2. Suppose $\mathcal{S} \in \mathfrak{S}_a$. Then $\overline{\mathcal{S}} = \mathcal{S} \sqcup \{\infty\}$ so that again $\partial\overline{\mathcal{S}} = \partial\mathcal{S}$ is a finite subset of $[0,\infty)$, which gets ν-measure 0. So again the second integral in (D2) is trivial. But in this case, (D2) becomes

$$\mathbb{I}^{\mu}_{\mathcal{S}}[f] \quad = \quad \int_{\overline{\mathcal{S}}} \overline{f} \, \mathrm{d}\nu \quad = \quad \int_{\mathcal{S}} f \, \mathrm{d}\nu + \overline{f}(\infty)\,\nu[\{\infty\}] \quad = \quad \int_{\mathcal{S}} f \, \mathrm{d}\nu + M \cdot \lim_{t\to\infty} f(t). \tag{D4}$$

Case 3. Suppose $\mathcal{S} \in \mathfrak{S}_p$. Then $\overline{\mathcal{S}} = \mathcal{S}$, and $\nu[[0,\infty) \cap \partial\overline{\mathcal{S}}] = 0$, but $\partial\overline{\mathcal{S}}$ also contains ∞, so that

$$\int_{\partial\overline{\mathcal{S}}} \overline{f}\cdot\phi_{\mathcal{S}} \, \mathrm{d}\nu \quad = \quad \overline{f}(\infty)\cdot\phi_{\mathcal{S}}(\infty)\cdot\nu[\{\infty\}] \quad = \quad \left(\lim_{t\to\infty} f(t)\right)\cdot c_{\mathcal{S}}\cdot M.$$

Thus, (D2) becomes:

$$\mathbb{I}^{\mu}_{\mathcal{S}}[f] \quad = \quad \int_{\mathcal{S}} \overline{f} \, \mathrm{d}\nu + \int_{\partial\overline{\mathcal{S}}} \overline{f}\cdot\phi_{\mathcal{S}} \, \mathrm{d}\nu \quad = \quad \int_{\mathcal{S}} f \, \mathrm{d}\nu + c_{\mathcal{S}}\, M \lim_{t\to\infty} f(t). \tag{D5}$$

Now define a function $\rho:[0,\infty)\longrightarrow[0,1)$ from ν as in Lemma B.2(a2). Then ρ is a continuous, strictly increasing bijection, because ν is a nonatomic probability measure with full support. By applying equation (B11) in Lemma B.2, we can convert all the Lebesgue integrals with respect to ν in equations (D3), (D4) and (D5) into Riemann-Stieltjes integrals with respect to $\mathrm{d}\rho$. Combining these modified versions of (D3), (D4) and (D5) with the content-utility representation (B6) obtained from Theorem B.1(b), we obtain the formulae (11), (12) and (13) which comprise the extended DUI representation. The proof of *Uniqueness* is similar to Theorem 1. □

Proof of Theorem 6. For any $\mathcal{S}\in\mathfrak{S}$, recall that $\widehat{\mathcal{S}}:=\mathrm{int}_{\widehat{\mathcal{T}}}[\mathrm{clos}_{\widehat{\mathcal{T}}}(\mathcal{S})]$ is an open subset of $\widehat{\mathcal{T}}$. Furthermore, $\widehat{\mathcal{S}}\cap[0,\infty)=\mathcal{S}$, by Lemma PV-7.4. If $\mathcal{S}\in\mathfrak{S}_p$, then $\widehat{\mathcal{S}}$ was already described in Section 5. If $\mathcal{S}\in\mathfrak{S}_b$, then $\widehat{\mathcal{S}}=\mathcal{S}$. Finally, if $\mathcal{S}\in\mathfrak{S}_a$, then $\widehat{\mathcal{S}}=\mathcal{S}\sqcup\Omega$. In any of these cases, let $\partial\widehat{\mathcal{S}}$ denote the boundary of $\widehat{\mathcal{S}}$ in $\widehat{\mathcal{T}}$.

At this point, the proof is very similar to the proof of Theorem 5. However, instead of using the Alexandroff compactification $[0,\infty]$, we now use the Stone-Čech compactification $\widehat{\mathcal{T}}$. First, obtain a content-utility representation from Theorem B.1(b) given by a nonatomic content μ with full support on $\mathfrak{S}$, by following the same argument as in the proof of Theorem 1. Extend μ to a content μ_0 on $\mathfrak{RO}(\mathcal{T})$, via the Horn-Tarski Extension Theorem [17, Thm. 1.22]. Use Theorem PV-7.2 to obtain a normal Borel probability measure η on $\widehat{\mathcal{T}}$, together with a collection of Borel-measurable functions $\{\phi_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{RO}(\mathcal{T})}$ on $\widehat{\mathcal{T}}$, such that for any $\mathcal{R}\in\mathfrak{RO}(\mathcal{T})$,

$$\mu_0[\mathcal{R}] \quad = \quad \eta[\widehat{\mathcal{R}}]+\int_{\partial\widehat{\mathcal{R}}}\phi_{\mathcal{R}}\ \mathrm{d}\eta, \tag{D6}$$

where $\widehat{\mathcal{R}}:=\mathrm{int}_{\widehat{\mathcal{T}}}[\mathrm{clos}_{\widehat{\mathcal{T}}}(\mathcal{R})]$ and $\partial\widehat{\mathcal{R}}$ is the boundary of $\widehat{\mathcal{R}}$ in $\widehat{\mathcal{T}}$. Meanwhile, for any $f\in\mathcal{C}_{\mathrm{b}}(\mathcal{T},\mathbb{R})$,

$$\mathbb{I}^{\mu_0}_{\mathcal{R}}[f] \quad = \quad \int_{\widehat{\mathcal{R}}}\widehat{f}\ \mathrm{d}\eta+\int_{\partial\widehat{\mathcal{R}}}\widehat{f}\cdot\phi_{\mathcal{R}}\ \mathrm{d}\eta, \tag{D7}$$

where $\widehat{f}\in\mathcal{C}(\widehat{\mathcal{T}},\mathbb{R})$ is the unique continuous extension of f defined in property (SČ2).

Let ν be the restriction of η to $[0,\infty)$. Then ν is a normal Borel measure (not necessarily a probability measure). As in Claims 1 and 2 in the proof of Theorem 1, we establish that ν is nonatomic and has full support on $[0,\infty)$ —but now the proof uses equation (D6) instead of equation (C1). At this point, there are three cases.

Case 1. Suppose $\mathcal{S}\in\mathfrak{S}_b$. Then $\widehat{\mathcal{S}}=\mathcal{S}$, so that $\partial\widehat{\mathcal{S}}=\partial\mathcal{S}$ is a finite subset of $[0,\infty)$, which gets ν-measure 0 (because ν is nonatomic) and hence, η-measure 0. Thus, the second integral in (D7) is trivial, so that (D7) becomes

$$\mathbb{I}^{\mu}_{\mathcal{S}}[f] \quad = \quad \int_{\widehat{\mathcal{S}}}\widehat{f}\ \mathrm{d}\eta \quad = \quad \int_{\mathcal{S}}f\ \mathrm{d}\eta \quad = \quad \int_{\mathcal{S}}f\ \mathrm{d}\nu. \tag{D8}$$

Case 2. Suppose $\mathcal{S}\in\mathfrak{S}_a$. Then $\widehat{\mathcal{S}}=\mathcal{S}\sqcup\Omega$ so that $\partial\widehat{\mathcal{S}}=\partial\mathcal{S}$ is again a finite subset of $[0,\infty)$, which again gets η-measure zero. So again the second integral in (D7) is trivial. But in this case, $\Omega\subset\widehat{\mathcal{S}}$, so that (D7) becomes

$$\mathbb{I}^{\mu}_{\mathcal{S}}[f] \quad = \quad \int_{\widehat{\mathcal{S}}}\widehat{f}\ \mathrm{d}\eta \quad = \quad \int_{\mathcal{S}}f\ \mathrm{d}\nu+\int_{\Omega}\widehat{f}\ \mathrm{d}\eta. \tag{D9}$$

Case 3. Suppose $\mathcal{S}\in\mathfrak{S}_p$. Define $\partial_\infty\mathcal{S}:=\Omega\cap\partial\widehat{\mathcal{S}}$, as in Section 5. Meanwhile, note that $[0,\infty)\cap\partial\widehat{\mathcal{S}}=\partial\mathcal{S}$. Thus, $\partial\widehat{\mathcal{S}}=(\partial\mathcal{S})\sqcup(\partial_\infty\mathcal{S})$, so that we can write the second integral in (D7) as

$$\int_{\partial\widehat{\mathcal{S}}}\widehat{f}\cdot\phi_{\mathcal{S}}\ \mathrm{d}\eta \quad = \quad \int_{\partial\mathcal{S}}\widehat{f}\cdot\phi_{\mathcal{S}}\ \mathrm{d}\eta+\int_{\partial_\infty\mathcal{S}}\widehat{f}\cdot\phi_{\mathcal{S}}\ \mathrm{d}\eta \quad \underset{(*)}{=} \quad \int_{\partial_\infty\mathcal{S}}\widehat{f}\cdot\phi_{\mathcal{S}}\ \mathrm{d}\eta. \tag{D10}$$

Here, $(*)$ is because ν is non-atomic and $\partial\mathcal{S}$ is a countable subset of $[0,\infty)$, so it gets ν-measure 0, and hence, η-measure zero.

Meanwhile, define $\mathcal{S}_\infty := \Omega \cap \widehat{\mathcal{S}}$ as in Section 5. Then $\widehat{\mathcal{S}} = \mathcal{S} \sqcup \mathcal{S}_\infty$, so that we can write the first integral in (D7) as

$$\int_{\widehat{\mathcal{S}}} \widehat{f}\ \mathrm{d}\eta \quad = \quad \int_{\mathcal{S}} \widehat{f}\ \mathrm{d}\eta + \int_{\mathcal{S}_\infty} \widehat{f}\ \mathrm{d}\eta \quad = \quad \int_{\mathcal{S}} f\ \mathrm{d}\nu + \int_{\mathcal{S}_\infty} \widehat{f}\ \mathrm{d}\eta. \tag{D11}$$

Substituting (D10) and (D11) into (D7), we obtain

$$\mathbb{I}^\mu_{\mathcal{S}}[f] \quad = \quad \int_{\mathcal{S}} f\ \mathrm{d}\nu + \int_{\mathcal{S}_\infty} \widehat{f}\ \mathrm{d}\eta + \int_{\partial_\infty \mathcal{S}} \widehat{f} \cdot \phi_{\mathcal{S}}\ \mathrm{d}\eta. \tag{D12}$$

Define $\rho : [0,\infty) \longrightarrow [0,1)$ from ν as in Lemma B.2(a1). Then ρ is a continuous, strictly increasing bijection, because ν is a nonatomic probability measure with full support. By applying equation (B11) in Lemma B.2, we can convert all the Lebesgue integrals with respect to ν in equations (D8), (D9) and (D12) into Riemann-Stieltjes integrals with respect to to $\mathrm{d}\rho$. Combining these modified versions of (D8), (D9) and (D12) with the content-utility representation (B6) obtained from Theorem B.1(b), we obtain the formulae (14), (15) and (16) which comprise the Stone-Čech DUI representation. The proof of *Uniqueness* is similar to Theorem 1. □

Proof of Theorem 4. "$\Longleftarrow$" The proof is similar to the proof of Theorem 2 "$\Longleftarrow$".

"$\Longrightarrow$" The first part of the proof is identical to the proof of Theorem 2, except for two changes. First, replace "T" with "∞" everywhere (and likewise replace expressions like "$T-t$" with "∞"). Second, after Claim 1, instead of invoking Theorem 1 to obtain a DUI representation of the form (8) for some u and ρ, we invoke Theorem 6 to obtain a Stone-Čech DUI representation of the form (14)-(16) for some u, ρ, η, and $\{\phi_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}_p}$. The proofs of Claims 2 and 3 are as before (replacing "T" with "∞" everywhere), and yield some $\delta \in (0,1)$ such that $\rho'(t) = \delta^t$ for all $t \in [0,\infty)$.

It remains to show that $\eta = 0$. To see this, let $M := \eta[\Omega]$. Suppose $\alpha \in \mathcal{A}$ is such that $\lim_{t\to\infty} \alpha(t) = x$ for some $x \in \mathcal{X}$. Then $\widehat{\alpha}(\omega) = x$ for all $\omega \in \Omega$. Thus,

$$\int_\Omega u \circ \widehat{\alpha}\ \mathrm{d}\eta \quad = \quad \int_\Omega u(x)\ \mathrm{d}\eta \quad = \quad u(x) \cdot M. \tag{D13}$$

Let $\alpha, \beta \in \mathcal{A}_0$, and suppose that $\lim_{t\to\infty} \alpha(t)$ and $\lim_{t\to\infty} \beta(t)$ exist. For any $T > 0$,

$$\begin{aligned}
&\left(\int_0^\infty \delta^t\, u[\alpha(t)]\ \mathrm{d}t + M \lim_{t\to\infty} u[\alpha(t)] \geq \int_0^\infty \delta^t\, u[\beta(t)]\ \mathrm{d}t + M \lim_{t\to\infty} u[\beta(t)]\right) \\
\underset{(*)}{\Longleftrightarrow}\ &\left(\alpha \succeq \beta\right) \underset{(\diamond)}{\Longleftrightarrow} \left(\alpha^{+T} \succeq_{[T,\infty)} \beta^{+T}\right) \\
\underset{(*)}{\Longleftrightarrow}\ &\left(\int_T^\infty \delta^t\, u[\alpha^{+T}(t)]\ \mathrm{d}t + M \lim_{t\to\infty} u[\alpha(t)] \geq \int_T^\infty \delta^t\, u[\beta^{+T}(t)]\ \mathrm{d}t + M \lim_{t\to\infty} u[\beta(t)]\right) \\
\underset{(\dagger)}{\Longleftrightarrow}\ &\left(\delta^T \int_0^\infty \delta^t\, u[\alpha(t)]\ \mathrm{d}t + M \lim_{t\to\infty} u[\alpha(t)] \geq \delta^T \int_0^\infty \delta^t\, u[\beta(t)]\ \mathrm{d}t + M \lim_{t\to\infty} u[\beta(t)]\right)
\end{aligned} \tag{D14}$$

Here, both $(*)$ are by the Stone-Čech DUI representation (15), substituting $\mathrm{d}\rho[t] = \delta^t$ into the integrals over $[0,\infty)$, and using equation (D13) to simplify the integrals over Ω. Meanwhile, $(\diamond)$ is by (Stat), and $(\dagger)$ is because

$$\int_T^\infty \delta^t\, u[\alpha^{+T}(t)]\ \mathrm{d}t \quad = \quad \delta^T \int_T^\infty \delta^{t-T}\, u[\alpha(t-T)]\ \mathrm{d}t \quad \underset{(\sharp)}{=} \quad \delta^T \int_0^\infty \delta^s\, u[\alpha(s)]\ \mathrm{d}s,$$

where (♯) by the change of variables $s := t - T$, and likewise,

$$\int_T^\infty \delta^t\, u[\beta^{+T}(t)]\ \mathrm{d}t \quad = \quad \delta^T \int_0^\infty \delta^s\, u[\beta(s)]\ \mathrm{d}s.$$

Let

$$A \ := \ \int_0^\infty \delta^t\, u[\alpha(t)]\ \mathrm{d}t,\ \ a \ := \ \lim_{t\to\infty} u[\alpha(t)],\ \ B \ := \ \int_0^\infty \delta^t\, u[\beta(t)]\ \mathrm{d}t,\ \ \text{and}\ \ b \ := \ \lim_{t\to\infty} u[\beta(t)].$$

Then statement (D14) simplifies to

$$\Big(A + M\,a \geq B + M\,b\Big) \iff \Big(\delta^T A + M\,a \geq \delta^T B + M\,b\Big), \quad \text{for all } T > 0. \tag{D15}$$

By assuming $M > 0$, we will derive a contradiction from (D15). Let $\lambda := -\ln(\delta) > 0$. By applying an affine transformation to u if necessary, we can assume there exist $x, y, z \in \mathcal{X}$ such that $u(x) = 2\,M\,\lambda$, $u(y) = 0$, and $u(z) = 1$. Now (Core*) stipulates that $\mathcal{A}_0$ satisfies (R*). Thus, for any $L > 0$, there exist $\alpha \in \mathcal{A}_0$ such that $\alpha_{\upharpoonright[0,L]} \approx_{[0,L]} \kappa^x_{\upharpoonright[0,L]}$ while $\alpha_{\upharpoonright[L,\infty)} = \kappa^y_{\upharpoonright[L,\infty)}$. In other words, $u[\alpha(t)] = 0$ for all $t > L$. Thus, $a = 0$, while

$$\begin{aligned} A = \int_0^\infty \delta^t\, u[\alpha(t)]\ \mathrm{d}t \ &\underset{(*)}{=\!=} \ \int_0^L \delta^t\, u[\alpha(t)]\ \mathrm{d}t \ \underset{(\dagger)}{=\!=} \ \int_0^L \delta^t\, u[\kappa^x(t)]\ \mathrm{d}t \\ = \int_0^L \delta^t\, u(x)\ \mathrm{d}t \ &\underset{(\diamond)}{=\!=} \ \int_0^L 2\,M\,\lambda\,\delta^t\ \mathrm{d}t \ = \ 2\,M\,(1-\delta^L). \end{aligned} \tag{D16}$$

Here, $(*)$ is because $u[\alpha(t)] = 0$ for all $t > L$, $(\dagger)$ is by the DUI representation (14), because $\alpha_{\upharpoonright[0,L]} \approx_{[0,L]} \kappa^x_{\upharpoonright[0,L]}$, and $(\diamond)$ is because $u(x) = 2\,M\,\lambda$.

Likewise, for any $L > 0$, there exist $\beta \in \mathcal{A}_0$ such that $\beta_{\upharpoonright[0,L]} \approx_{[0,L]} \kappa^y_{\upharpoonright[0,L]}$ while $\beta_{\upharpoonright[L,\infty)} = \kappa^z_{\upharpoonright[L,\infty)}$. In other words, $u[\beta(t)] = 1$ for all $t > L$. Thus, $b = 1$, while

$$\begin{aligned} B = \int_0^\infty \delta^t\, u[\beta(t)]\ \mathrm{d}t \ &\underset{(*)}{=\!=} \ \int_0^L \delta^t\, u[\beta(t)]\ \mathrm{d}t + \int_L^\infty \delta^t\ \mathrm{d}t \\ \underset{(\dagger)}{=\!=} \int_0^L \delta^t\, u[\kappa^y(t)]\ \mathrm{d}t + \frac{\delta^L}{\lambda} \ &= \ \int_0^L \delta^t\, u(y)\ \mathrm{d}t + \frac{\delta^L}{\lambda} \ \underset{(\diamond)}{=\!=} \ \frac{\delta^L}{\lambda}. \end{aligned} \tag{D17}$$

Here, $(*)$ is because $u[\beta(t)] = 1$ for all $t > L$, $(\dagger)$ is by the DUI representation (14), because $\beta_{\upharpoonright[0,L]} \approx_{[0,L]} \kappa^y_{\upharpoonright[0,L]}$, and $(\diamond)$ is because $u(y) = 0$, so that the integral is zero.

Now, suppose $M > 0$. Recall that $0 < \delta < 1$. Thus, if L is sufficiently large, then (D16) yields $A \approx 2M$ while (D17) yields $B \approx 0$. Thus, $A + M\,a \approx 2M + 0 = 2M$ while $B + M\,b \approx 0 + M \cdot 1 = M$; hence $A + M\,a > B + M\,b$. Meanwhile, for any $T > 0$, $\delta^T A + M\,a \approx 2\,\delta^T M$ while $\delta^T B + M\,b \approx M$. But if T is sufficiently large, then $\delta^T \approx 0$, and then $\delta^T A + M\,a \approx 0 < M \approx \delta^T B + M\,b$. Thus, we have $A + M\,a > B + M\,b$ while $\delta^T A + M\,a < \delta^T B + M\,b$, contradicting statement (D15).

To avoid a contradiction, we need $M = 0$; hence $\eta = 0$. Thus, $\{\phi_{\mathcal{S}}\}_{\mathcal{S}\in\mathfrak{S}_p}$ is irrelevant, and the Stone-Čech DUI representations (14) - (16) reduce to the exponential DUI representation (9). □

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