Computing Approximate Equilibria in Weighted Congestion Games via Best-Responses^{*}

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Abstract

We present a deterministic polynomial-time algorithm for computing $d^{d+o(d)}$ -approximate (pure) Nash equilibria in (proportional sharing) weighted congestion games with polynomial cost functions of degree at most d. This is an exponential improvement of the approximation factor with respect to the previously best deterministic algorithm. An appealing additional feature of the algorithm is that it only uses best-improvement steps in the actual game, as opposed to the previously best algorithms, that first had to transform the game itself. Our algorithm is an adaptation of the seminal algorithm by Caragiannis et al. [8, 9], but we utilize an approximate potential function directly on the original game instead of an exact one on a modified game.

A critical component of our analysis, which is of independent interest, is the derivation of a novel bound of $[d/\mathcal{W}(d/\rho)]^{d+1}$ for the Price of Anarchy (PoA) of ρ -approximate equilibria in weighted congestion games, where \mathcal{W} is the Lambert-W function. More specifically, we show that this PoA is *exactly* equal to $\Phi_{d,\rho}^{d+1}$, where $\Phi_{d,\rho}$ is the unique positive solution of the equation $\rho(x+1)^d = x^{d+1}$. Our upper bound is derived via a smoothness-like argument, and thus holds even for mixed Nash and correlated equilibria, while our lower bound is simple enough to apply even to singleton congestion games.

1 Introduction

Congestion games constitute one of the most important and well-studied class of games in the field of algorithmic game theory [34, 36, 38]. These games are tailored to model settings where selfish players compete over sets of common resources. Prominent examples include traffic routing in networks and load balancing games (see, e.g., [34, Chapters 18 and 20]). In these games' most general form, known as *weighted* congestion games, each player has her own (positive) weight and the cost of a resource is a nondecreasing function of the total weight of players using it. An important special case is that of unweighted games, where all players have the same weight. The cost of a resource then depends only on the number of players using it.

Players are selfish and each one chooses a set of resources that minimizes her own cost. On the other hand, a central authority would aim at minimizing *social cost*, that is, the sum of

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players' costs. It is well known that these two objectives do not, in general, align: due to the selfish behaviour of players, the game may reach a stable state (i.e., a Nash equilibrium [28, 33]) that is suboptimal in terms of social cost. This gap is formally captured by the fundamental notion of *Price of Anarchy* (PoA) [30], defined as the ratio between the social cost of the worst equilibrium and that of an optimal solution enforced by a centralized authority.

From the seminal work of Rosenthal [35], we know that unweighted congestion games always have (pure Nash) equilibria¹. This is a direct consequence of the fact that they are potential games [32]. However, finding such a stable state is, in general, computationally hard [1, 19]. An important question then is whether one can efficiently compute approximate equilibria [8, 39]. These are states of the game where no player can unilaterally deviate and improve her cost by more than a factor of $\rho \geq 1$; (exact) equilibria correspond to the special case where $\rho = 1$.

The situation becomes even more challenging in the general setting of weighted congestion games [9], where exact equilibria may not even exist [25]. Consequently, weighted congestion games do not generally admit a potential function. Thus, in this setting one needs more sophisticated approaches and approximation tools to establish computability of approximate equilibria. This is precisely the problem we study in the present paper: the efficient computation of ρ approximate equilibria in weighted congestion games, with ρ as small as possible. We focus on resource cost functions that are polynomials (with nonnegative coefficients), parametrized by their degree d; this is a common assumption in the literature of congestion games (see, e.g., [38, 2, 12, 14, 25]).

1.1 Related Work

The potential function approach has long become a central tool for obtaining results about existence and computability of approximate equilibria in weighted congestion games. The concept of a potential function for unweighted congestion games was proposed by Rosenthal [35], who used it to prove existence of pure Nash equilibria in such games. Later, Monderer and Shapley [32] formally introduced and studied extensively the class of potential games. As it turns out, weighted congestion games do not admit a potential function in general, even for "well-behaved" instances [31, 25, 22, 27]. Exceptions include games with linear and exponential resource cost functions [27]. However, recently, Christodoulou et al. [14] showed that polynomial weighted congestion games have an approximate analogue of a potential function, which they called Faulhaber's potential. An exact potential function decreases whenever a player improves her cost, and even by the same amount. In contrast, the approximate potential of [14] is only guaranteed to decrease when a player deviates and improves her cost by a factor of at least α . Factor α is greater than 1, but at most d+1, where d is the degree of the game. We will use this approximate potential function in the analysis of our algorithm. Other approximate potential functions have been successfully used before to establish the existence of approximate equilibria in congestion games; see [10, 16, 26]. The current state-of-the-art is that congestion games of degree d always admit d-approximate equilibria, but there exist examples of such games that do not have $\Theta(\sqrt{d})$ -approximate equilibria; these upper and lower bounds are from the very recent work of Caragiannis and Fanelli [7] and Christodoulou et al. [13], respectively.

On the algorithmic side, there have been many negative results concerning exact equilibria in various classes of congestion games. Fabrikant et al. [19] showed that even in the unweighted case of a network congestion game, computing equilibria is PLS-complete. Dunkel and Schulz [18] showed that it is strongly NP-complete to determine whether an equilibrium exists in a given weighted congestion game.² As a further negative result, Ackermann et al. [1] proved that

¹In this paper we focus exclusively on *pure* Nash equilibria; this is standard in the congestion games literature. ²Christodoulou et al. [13] showed that this remains hard even for approximate equilibria and polynomial

it is PLS-complete to compute equilibria even in the linear unweighted case. These hardness results motivated the search for polynomial-time methods to compute *approximate* equilibria. In general, this remains a computationally hard problem; Skopalik and Vöcking [39] showed that for any polynomially computable ρ , finding a ρ -approximate equilibrium in a congestion game is a PLS-complete problem. The focus of research thus shifted towards searching for positive algorithmic results for ρ -approximate equilibria of various special classes of weighted congestion games. The first such result was obtained by Chien and Sinclair [11], who showed convergence of the best-response dynamics, in symmetric unweighted congestion games with "well-behaved" cost functions, to $(1 + \varepsilon)$ -approximate equilibria in polynomially many steps with respect to ε^{-1} and the number of players.

The next significant positive result of this kind, and of great importance for our work, was obtained for approximate equilibria in polynomial unweighted congestion games by Caragiannis et al. [8]. They designed a seminal deterministic algorithm that efficiently computes $d^{O(d)}$ -approximate equilibria in such games. Subsequently, Caragiannis et al. [9] extended this algorithm to handle the weighted case, achieving an approximation factor of $d^{2d+o(d)}$. In the present paper, we reduce this factor to $d^{d+o(d)}$. The algorithm in [9] first transforms the original game into an *approximating game* (called Ψ -game) defined over the same players and states. Then it finds and returns a state which is a $d^{d+o(d)}$ -approximate equilibrium of this new game. Caragiannis et al. [9] show that when translated back to the original game, the approximation guarantee can only deteriorate by a factor of d!, giving their $d^{2d+o(d)}$ -approximation result.

It is important to note here that the sequence of moves chosen by Algorithm 1 in [9] need not be a "real" best-response sequence when projected back into the original game. As a matter of fact, it may contain moves that increase the deviating player's individual cost. In an attempt to address this, Caragiannis et al. [9] themselves present also a modification of Algorithm 1. Their Algorithm 2 [9, Section 5] actually runs in the original game but unfortunately offers a significantly worse approximation guarantee of $d^{O(d^2)}$.

Very relevant to ours is also the work of Feldotto et al. [20] who describe a randomized variant of [9]'s algorithm which is able to compute $d^{d+o(d)}$ -approximate equilibria in weighted congestion games (but only with high probability); this is of the same order as the approximation ratio of our deterministic algorithm.³ Similar to [9] though, their algorithm does not perform actual best-improvement moves in the original game: it runs in a modified instance where the Shapley value rule of [29] is used to share the total cost of a resource among the players occupying it. Under this cost sharing rule, however, it is computationally hard to identify players' costs. Instead, one has to estimate them using a sampling approach, which is exactly the source of randomization in [20]'s algorithm. Another subtle limitation of the algorithm in [20] is that its running time is polynomial in the size of the strategy sets. Although this is absolutely fine when the input game is given explicitly, it can have serious implications for games that are succinctly representable but have exponentially many strategies; an important canonical example are network congestion games.

In another paper, Feldotto et al. [21] explored PoA-like bounds on the potential function in unweighted games. This enabled them to bound from above the approximation factor that the algorithmic framework of [8] yields when applied to unweighted games with general cost functions.

The study of the Price of Anarchy (PoA) was initiated by Koutsoupias and Papadimitriou [30]. One of the first significant results concerning tight bounds on the PoA of atomic

congestion games.

³This is not a mere coincidence: both in [20] and our paper (but, as a matter of fact, in [9] as well), it turns out that the "bottleneck" in the approximation ratio bound is essentially given by PoA-style bounds (see, e.g., our " ρ -PoA" bound in Theorem 1 and the" ρ -stretch" notion in [20, 9]) which are roughly of the same d^d order.

congestion games was obtained by Awerbuch et al. [3] and Christodoulou and Koutsoupias [15]. They proved the tight bound of 5/2 on the exact PoA of linear unweighted congestion games. In the next few years, several PoA results were obtained. For example, Gairing and Schoppmann [23] provided various upper and lower bounds for the exact PoA of singleton⁴ unweighted congestion games. Subsequently, Aland et al. [2] introduced a systematic approach to upperbounding the exact PoA of polynomial weighted congestion games, which was later extended to general classes of cost functions and named *smoothness framework* in [4, 37]. Aland et al. [2] gave the tight bound of Φ_d^{d+1} on the PoA of exact pure NE in polynomial weighted games, where Φ_d is the unique root of the equation $(x + 1)^d = x^{d+1}$. Based on the same technique, Christodoulou et al. [16] provided a tight bound on the PoA of ρ -approximate equilibria in unweighted congestion games. It turned out to be equal to $\frac{\rho((z+1)^{2d+1}-z^{d+1}(z+2)^d)}{(z+1)^{d+1}-z^{d+1}-\rho((z+2)^d-(z+1)^d)}$, where z is the maximum integer that satisfies $\frac{z^{d+1}}{(z+1)^d} < \rho$. In our notation, this is equivalent to $z = |\Phi_{d,\rho}|$.

Since the development of the smoothness method, other approaches to finding tight bounds on the PoA have been investigated. Recently, Bilò [5] was able to rederive, through the use of a primal-dual framework, the upper bound on the PoA of linear unweighted games from [15]. He also provided a simplified lower bound instance. Furthermore, he was able to show the upper bound of $\left(\frac{\rho+\sqrt{\rho^2+4\rho}}{2}\right)^2$ on the PoA of ρ -approximate equilibria for the special case of linear weighted games. It turns out to be equal to $\Phi_{1,\rho}^2$, the special case for d = 1 of the general tight bound $\Phi_{d,\rho}^{d+1}$ that we present in this paper. Moreover, he provided matching instances with PoA equal to $\Phi_{1,\rho}^2$, for ρ in a certain subset of $[1,\infty)$.

1.2 Our Results and Techniques

We study approximate (pure Nash) equilibria in polynomial weighted congestion games of degree d. Our main result is a polynomial-time deterministic algorithm for computing $d^{d+o(d)}$ approximate equilibria in such games. Our algorithm runs in polynomial time in the description of the game, even for succinctly representable games with exponentially large strategy spaces; in particular, it is applicable to *network* congestion games.⁵ This result improves upon the $d^{2d+o(d)}$ -approximation factor of the seminal algorithm of Caragiannis et al. [9, 8]. Interestingly, our algorithm can also be readily used to *deterministically* compute $d^{d+o(d)}$ -approximate equilibria in weighted congestion games under *Shapley cost sharing*; this is of the same order as the *randomized* $d^{d+o(d)}$ guarantee of Feldotto et al. [20].⁶

Our algorithm, as well as the outline of its analysis, is clearly based on the ideas in [9]. However, there is a fundamental difference between our approach and previous ones. Our algorithm builds a polynomially-long sequence of best-response moves (leading from any state to an approximate equilibrium) in the actual game; we then utilize, in the absence of an exact potential function for the game, an approximate potential introduced by Christodoulou et al. [14] in order to analyse the runtime and approximation guarantee. By contrast, the original $d^{2d+o(d)}$ -approximation algorithm of Caragiannis et al. [9], as well as the randomized $d^{d+o(d)}$ -approximation algorithm of Feldotto et al. [20], first transforms the input game into an exact-potential auxiliary game and performs best-responses in that modified game; the outcome is then projected back into the original game, at the expense of a certain increase in the approx-

 $^{^{4}}$ See Footnote 11 for a formal definition.

 $^{{}^{5}}$ We want to note here that the algorithms of Caragiannis et al. [9] can also handle network games without computational issues. However, as we already mentioned above, that is not the case for the algorithm in [20].

⁶This is an immediate consequence of our deterministic $d^{d+o(d)}$ guarantee for the proportional sharing setting studied here, paired with Lemma 8 in [20].

imation factor.⁷ A noteworthy exception is Algorithm 2 of [9, Section 5], which also creates a best-response sequence in the original game; this, however, is achieved at the expense of a significant deterioration of the approximation guarantee of the computed equilibrium, which becomes $d^{O(d^2)}$. Thus, when compared to Algorithm 2 of [9], the improvement of the approximation factor provided by our $d^{d+o(d)}$ -approximate algorithm is even more significant.

We believe that our approach offers a number of advantages compared to the previously best algorithms (namely, Algorithm 1 of [9] and the algorithm of [20]), at the cost of our proofs being arguably more involved due to the use of an approximate potential to the main game, instead of an exact potential to an auxiliary game. First, similarly to Algorithm 2 of [9], since our algorithm runs on the actual game, it can be interpreted as a more natural *learning* process via "real" Nash dynamics. An additional implication from a computational complexity perspective is that our algorithm can be applied even on games for which we are only given a best-response oracle. Moreover, since also our analysis is performed in a direct way on the original game (in contrast to the analyses of Algorithms 1 and 2 of [9] and the analysis of [20], each of which considers an auxiliary game), we believe that it provides a more transparent understanding of the inner workings of the original algorithmic paradigm of Caragiannis et al. [9].

As a necessary tool for proving the approximation guarantee of the algorithm and as a result of independent interest, we obtain a tight bound on the PoA of ρ -approximate equilibria, denoted by $\operatorname{PoA}_d(\rho)$, of polynomial weighted congestion games of degree d, for any $\rho \geq 1$ and degree $d \geq 1$. It turns out to be equal to $\Phi_{d,\rho}^{d+1}$, where $\Phi_{d,\rho}$ is the unique positive solution of the equation $\rho(x+1)^d = x^{d+1}$. This bound generalizes the following results: the tight bound of Φ_d^{d+1} on the PoA (of exact equilibria) of weighted congestion games [2]; the tight bound on the ρ -approximate PoA of *unweighted* congestion games [16]; and the upper bound on the ρ -approximate PoA of *linear* weighted congestion games [5]. Our matching lower bound proof extends an example from [23] that bounds the PoA of singleton unweighted congestion games. As such, our lower bound is easily verified to hold for singleton and network weighted congestion games.⁸ To prove the upper bound, we essentially utilize the *smoothness method* developed in [2, 4, 37]. The smoothness approach automatically extends the validity of our tight PoA bound from pure Nash to mixed Nash and correlated equilibria as well [4, 37].

One further contribution is an analytic upper bound $\operatorname{PoA}_d(\rho) \leq \left[\frac{d}{W(d/\rho)}\right]^{d+1}$, involving the Lambert-W function. This bound adds to the understanding of the different asymptotic behaviour of $\operatorname{PoA}_d(\rho)$ with respect to each of the two parameters, d and ρ , and plays an important role in deriving the desired approximation factor of $d^{d+o(d)}$ in the analysis of the main algorithm of our paper. It is interesting to note here that this bound also generalizes, in a smooth way with respect to ρ , a similar result presented in [14] for the special case of exact equilibria (i.e., $\rho = 1$).

All proofs omitted from the main text can be found in the Appendix.

2 Model and Notation

We denote by \mathbb{R} and $\mathbb{R}_{\geq 0}$ the set of real and nonnegative real numbers, respectively, and by \mathfrak{P}_d the class of polynomials of degree at most d with nonnegative coefficients⁹. A well-established notation in the literature of congestion games is that of Φ_d , for d a positive integer, as the unique

⁷This increase is by a factor of d! for Algorithm 1 in [9], and $O(d^2)$ for [20].

⁸The fact that the worst-case PoA can be realized at such simple singleton games should come as no surprise, due to the work of Bilò and Vinci [6, Theorem 1]. Our contribution here lies in determining the actual value of the PoA.

⁹Formally, $\mathfrak{P}_d = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f(x) = \sum_{i=0}^d a_i x^i, \text{ where } a_i \in \mathbb{R}_{\geq 0} \text{ for all } i\}.$

positive root of the equation $(x+1)^d = x^{d+1}$. Notice how, the special case of d = 1 corresponds to the golden ratio constant $\phi \approx 1.618$. In this paper, we introduce a further generalization by defining, for all $\rho \geq 1$, $\Phi_{d,\rho}$ to be the unique positive root of the equation $\rho(x+1)^d = x^{d+1}$. Also, we shall make use of (the principal real branch of) the classical function known as the Lambert-W function [17]: for $\tau \geq 0$, $W(\tau)$ is defined to be the unique solution to the equation $x \cdot e^x = \tau$.

A polynomial¹⁰ (weighted) congestion game of degree d, with d a positive integer, is a tuple $\Gamma = (N, E, (w_u)_{u \in N}, (S_u)_{u \in N}, (c_e)_{e \in E})$. Here, N is a (finite) set of |N| = n players and E is a finite set of resources. Each resource $e \in E$ has a polynomial cost function $c_e \in \mathfrak{P}_d$. Every player $u \in N$ has a set of strategies $S_u \subseteq 2^E$ and each vector $\mathbf{s} \in S := \times_{u \in N} S_u$ will be called a state (or strategy profile) of the game Γ . Following standard game-theoretic notation, for any $\mathbf{s} \in S$ and $u \in N$, we denote by \mathbf{s}_{-u} the profile of strategies of all players if we remove the strategy s_u of player u; in this way, we have $\mathbf{s} = (s_u, \mathbf{s}_{-u})$. Finally, each player $u \in N$ has a real positive weight $w_u > 0$. However, we may henceforth assume that $w_u \geq 1$ for all $u \in N$, as we can without loss of generality appropriately scale player weights and cost functions, without affecting our results in this paper.

Given $\mathbf{s} \in S$, we let $x_e(\mathbf{s}) := \sum_{u:e \in s_u} w_e$ denote the total weight of players using resource e in state \mathbf{s} . Generalizing this definition to any group of players $R \subseteq N$, we let $x_{R,e}(\mathbf{s}) := \sum_{u \in R: e \in s_u} w_e$. The cost of a player u at state \mathbf{s} is defined as

$$C_u(\mathbf{s}) := w_u \sum_{e \in s_u} c_e(x_e(\mathbf{s})).$$

Players are selfish and rational, and thus choose strategies as to minimize their own cost. Let $\mathcal{BR}_u(\mathbf{s}) = \mathcal{BR}_u(\mathbf{s}_{-u})$ be a *best-response* strategy of player u to the strategies \mathbf{s}_{-u} of the other players, that is, $\mathcal{BR}_u(\mathbf{s}) \in \operatorname{argmin}_{s'_u \in S_u} C_u(\mathbf{s}_{-u}, s'_u)$ (in case of ties, we make an arbitrary selection). A state \mathbf{s} of the game is a *(pure Nash) equilibrium*, if all players are already playing best-responses, that is, no player can unilaterally improve her costs; formally, $C_u(\mathbf{s}) \leq C_u(s'_u, \mathbf{s}_{-u})$ for all $u \in N$ and $s'_u \in S_u$.

For a real parameter $\rho \geq 1$, a unilateral deviation of player u to strategy s'_u from state **s** is called a ρ -move if $C_u(\mathbf{s}) > \rho \cdot C_u(\mathbf{s}_{-u}, s'_u)$. Extending the notion of an equilibrium in two directions, we call a state **s** a ρ -approximate equilibrium (or simply a ρ -equilibrium) for a given group of players $R \subseteq N$, if none of the players in R has a ρ -move; formally, $C_u(\mathbf{s}) \leq \rho C_u(s'_u, \mathbf{s}_{-u})$ for all $u \in R$ and $s'_u \in S_u$. If this holds for R = N, then we simply refer to **s** as a ρ -equilibrium of our game. We use $\mathcal{Q}^{\rho}_{\rho}$ to denote the set of all ρ -equilibria of game Γ .

Ideally, our objective is to find states that induce low total cost in our game; we capture this notion by defining the *social cost* $C(\mathbf{s})$ of a state \mathbf{s} to be the sum of the players' costs, i.e., $C(\mathbf{s}) := \sum_{u \in N} C_u(\mathbf{s})$. Extending this to any subset of players $R \subseteq N$, we also denote

$$C_{R}(\mathbf{s}) := \sum_{u \in R} C_{u}(\mathbf{s}) = \sum_{u \in R} w_{u} \sum_{e \in s_{u}} c_{e}(x_{e}(\mathbf{s})) = \sum_{e \in E} x_{R,e}(\mathbf{s})c_{e}(x_{e}(\mathbf{s})).$$

Clearly, $C_N(\mathbf{s}) = C(\mathbf{s})$.

The standard way to quantify the inefficiency due to selfish behaviour, is to study the worst-case ratio between any equilibrium and the optimal solution, quantified by the notion of the Price of Anarchy (PoA). Formally, given a game Γ and a parameter $\rho \geq 1$, the PoA of ρ -equilibria (or simply the ρ -PoA) of Γ is PoA(Γ) := $\max_{\mathbf{s} \in \mathcal{Q}_{\rho}^{\Gamma}} \frac{C(\mathbf{s})}{C(\mathbf{s}^{*})}$, where $\mathbf{s}^{*} \in \operatorname{argmin}_{\mathbf{s}} C(\mathbf{s})$.

 $^{^{10}}$ We shall usually omit the word "polynomial" and refer to these games as weighted congestion games of degree d, or simply as congestion games, when this causes no confusion.

Finally, taking the worst case over all polynomial congestion games of degree d, we can define the ρ -PoA of degree d as

$$\operatorname{PoA}_{d}(\rho) := \sup_{\Gamma} \operatorname{PoA}(\Gamma) = \sup_{\Gamma} \max_{\mathbf{s} \in \mathcal{Q}_{\rho}^{\Gamma}} \frac{C(\mathbf{s})}{C(\mathbf{s}^{*})}$$

3 The Price of Anarchy

In this section we present our tight bound on the PoA of ρ -approximate equilibria for (weighted) congestion games. We first extend the smoothness method of Aland et al. [2] to obtain the upper bound on the PoA (Theorem 1), and then explicitly construct an example that extends a result of Gairing and Schoppmann [23] and provides the matching lower bound on the PoA (Theorem 2). We note here that there is a specific reason that this section precedes Section 4, where our algorithm for computing approximate pure Nash equilibria is presented. The estimation of the approximation guarantee of the algorithm requires the use of the closed-form bound on the PoA_d(ρ) we provide in Theorem 1 and, furthermore, the "Key Property" of our algorithm (Theorem 3) rests critically on an application of Lemma 1 below.

3.1 Upper Bound

We formulate our upper bound on the PoA as the following theorem:

Theorem 1. The Price of Anarchy of ρ -approximate equilibria in (weighted) polynomial congestion games of degree d, is at most $\Phi_{d,\rho}^{d+1}$, where $\Phi_{d,\rho}$ is the unique positive root of the equation $\rho(x+1)^d = x^{d+1}$. In particular,

$$\operatorname{PoA}_{d}(\rho) \leq \left[\frac{d}{\mathcal{W}(d/\rho)}\right]^{d+1}$$

where $W(\cdot)$ denotes the Lambert-W function.

Theorem 1 is a direct consequence of the following Lemma 1, applied with R = N, and Lemma 3. The reason we are proving a more general version of Lemma 1 than what's needed for just establishing our PoA upper bound of Theorem 1, is that we will actually need it for the analysis of our main algorithm in Section 4.3.

Lemma 1. For any group of players $R \subseteq N$, let \mathbf{s} and \mathbf{s}^* be states such that \mathbf{s} is a ρ -equilibrium for group R and every player in $N \setminus R$ uses the same strategy in both \mathbf{s} and \mathbf{s}^* . Then, the social cost ratio of the two states is bounded by $\frac{C_R(\mathbf{s})}{C_R(\mathbf{s}^*)} \leq \Phi_{d,\rho}^{d+1}$.

The proof of Lemma 1 will essentially follow the smoothness technique [37] (see, e.g., [38, Theorem 14.6]). However, special care still needs to be taken related to the fact that only a subset R of players is deviating between the two states \mathbf{s} and \mathbf{s}^* . In particular, the key step in the smoothness derivation is captured by the following lemma (proved in Appendix A) that quantifies the PoA bound:

Lemma 2. For any constant $\rho \geq 1$ and positive integer d,

$$B := \inf_{\substack{\lambda \in \mathbb{R} \\ \mu \in (0, \frac{1}{\rho})}} \left\{ \frac{\lambda \rho}{1 - \mu \rho} \right| \forall x, y, z \ge 0, f \in \mathfrak{P}_d : yf(z + x + y) \le \lambda yf(z + y) + \mu xf(z + x) \right\} = \Phi_{d, \rho}^{d+1}.$$

The constraint that Lemma 2 imposes on parameters λ and μ is slightly more general than the analogous lemma in the smoothness derivation of Aland et al. [2]; namely, our condition contains an extra variable z. This is a consequence of exactly the aforementioned fact that Lemma 2 is tailored to upper bounding a generalization of the PoA for groups of players.

Proof of Lemma 1. Assume that $\lambda \in \mathbb{R}$ and $\mu \in (0, \frac{1}{\rho})$ are parameters such that, for any polynomial f of degree d with nonnegative coefficients and for any $x, y, z \ge 0$, it is

$$yf(z+x+y) \le \lambda yf(z+y) + \mu xf(z+x)$$

Applying this for the cost function $c_e(\cdot)$ of any resource e, and replacing $x \leftarrow x_{R,e}(\mathbf{s}), y \leftarrow x_{R,e}(\mathbf{s}^*)$, and $z \leftarrow x_{N\setminus R,e}(\mathbf{s}) = x_{N\setminus R,e}(\mathbf{s}^*)$ (the last equality holding due to the fact that every player in $N \setminus R$ uses the same strategy in \mathbf{s} and \mathbf{s}^*) we have that

$$x_{R,e}(\mathbf{s}^*) c_e(x_e(\mathbf{s}) + x_{R,e}(\mathbf{s}^*)) \le \lambda x_{R,e}(\mathbf{s}^*) c_e(x_e(\mathbf{s}^*)) + \mu x_{R,e}(\mathbf{s}) c_e(x_e(\mathbf{s})).$$
(1)

Here we also used that $z + x + y = x_e(\mathbf{s}) + x_{R,e}(\mathbf{s}^*)$, $z + y = x_e(\mathbf{s}^*)$, and $z + x = x_e(\mathbf{s})$. Summing (1) over all resources e, we obtain the following inequality:

$$\sum_{e \in E} x_{R,e}(\mathbf{s}^*) c_e(x_e(\mathbf{s}) + x_{R,e}(\mathbf{s}^*)) \le \lambda C_R(\mathbf{s}^*) + \mu C_R(\mathbf{s}).$$
(2)

Next, using the fact that **s** is a ρ -equilibrium we can upper-bound the social cost of the players in R by

$$C_{R}(\mathbf{s}) = \sum_{u \in R} C_{u}(\mathbf{s}) \le \rho \sum_{u \in R} C_{u}(\mathbf{s}_{-u}, s_{u}^{*}) = \rho \sum_{u \in R} w_{u} \sum_{e \in s_{u}^{*}} c_{e}(x_{e}(\mathbf{s}_{-u}, s_{u}^{*})).$$
(3)

Now, observe that for any player $u \in R$ and any resource $e \in s_u^*$ that player u uses in profile s^* , it is

$$x_e(\mathbf{s}_{-u}, s_u^*) \le x_e(\mathbf{s}) + w_u \le x_e(\mathbf{s}) + x_{R,e}(\mathbf{s}^*).$$

The first inequality holds because player u is the only one deviating between states \mathbf{s} and (\mathbf{s}_{-u}, s_u^*) , while the second one because u definitely uses resource e in profile s^* . Using the above, due to the monotonicity of the cost functions c_e , the bound in (3) can be further developed to give us

$$C_R(\mathbf{s}) \le \rho \sum_{u \in R} w_u \sum_{e \in s_u^*} c_e(x_e(\mathbf{s}) + x_{R,e}(\mathbf{s}^*)) = \rho \sum_{e \in E} x_{R,e}(\mathbf{s}^*) c_e(x_e(\mathbf{s}) + x_{R,e}(\mathbf{s}^*))$$

and thus, deploying the bound from (2), we finally arrive at

$$C_R(\mathbf{s}) \le \rho \lambda \, C_R(\mathbf{s}^*) + \rho \mu \, C_R(\mathbf{s}),$$

which is equivalent to

$$\frac{C_R(\mathbf{s})}{C_R(\mathbf{s}^*)} \le \frac{\lambda\rho}{1-\mu\rho}$$

Taking the infimum of the right-hand side, over the set of all feasible parameters $\lambda \in \mathbb{R}$ and $\mu \in (0, \frac{1}{\rho})$, Lemma 2 gives us desired upper bound of $\frac{C_R(\mathbf{s})}{C_R(\mathbf{s}^*)} \leq \Phi_{d,\rho}^{d+1}$.

We conclude this section by presenting the following useful bound on the generalized golden ratio $\Phi_{d,\rho}$, which is used in Theorem 1 to get the corresponding analytic expression for our PoA bound. As discussed in the introduction of the current section, we will use it in the proof of Theorem 6 in Section 4, for deriving the improved approximation guarantee of our algorithm. **Lemma 3.** For any $\rho \geq 1$ and any positive integer d,

$$\Phi_{d,\rho} \le \frac{d}{\mathcal{W}\left(d/\rho\right)},$$

where $\mathcal{W}(\cdot)$ is the Lambert-W function.

Proof. Recall that $\Phi_{d,\rho}$ is the solution of equation $\rho(x+1)^d = x^{d+1}$, which can be rewritten as

$$\frac{d(x+1)^d}{x^{d+1}} = \frac{d}{\rho}.$$

By the proof of Lemma 11 (see function h), this equation has a unique positive root and the left side is monotonically decreasing as a function of x; thus, to conclude the proof of our lemma, it suffices to prove that

$$\frac{d(\tilde{x}+1)^d}{\tilde{x}^{d+1}} \le \frac{d}{\rho} \qquad \text{for} \quad \tilde{x} := \frac{d}{\mathcal{W}\left(\frac{d}{\rho}\right)}.$$
(4)

Indeed, substituting for convenience $\tilde{y} := \frac{d}{\tilde{x}} = \mathcal{W}\left(\frac{d}{\rho}\right)$, we have that

$$\frac{d(\tilde{x}+1)^d}{\tilde{x}^{d+1}} = \tilde{y}\left(1+\frac{\tilde{y}}{d}\right)^d = \tilde{y}\left[\left(1+\frac{\tilde{y}}{d}\right)^{\frac{d}{\tilde{y}}}\right]^{\tilde{y}} \le \tilde{y}e^{\tilde{y}} = \mathcal{W}\left(\frac{d}{\rho}\right)e^{\mathcal{W}\left(\frac{d}{\rho}\right)} = \frac{d}{\rho}$$

For the inequality we used the fact that $\left(1+\frac{1}{t}\right)^t < e$ for all t > 0. The last equality is a direct consequence of the definition of the Lambert-W function.

3.2 Lower Bound

To prove a matching PoA lower bound to the upper bound of Section 3.1, we consider a simple instance involving n players and n + 1 resources. Each player has just 2 strategies, and each strategy consists of a single resource; letting $n \to \infty$, we obtain the desired lower bound of $\Phi_{d,\rho}^{d+1}$. This bound extends smoothly the lower bound of Φ_d^{d+1} for the PoA of exact ($\rho = 1$) equilibria by Gairing and Schoppmann [23, Theorem 4]. We also want to mention here that the construction used in the proof of Theorem 2 below can be extended to apply to network congestion games (see, e.g., [14, Proposition 3.4]).

Theorem 2. Let d be a positive integer and $\rho \geq 1$. For every $\varepsilon > 0$, there exists a (singleton¹¹) weighted polynomial congestion game of degree d, whose ρ -approximate PoA is at least $\Phi_{d,\rho}^{d+1} - \varepsilon$, where $\Phi_{d,\rho}$ is the unique positive root of the equation $\rho(x+1)^d = x^{d+1}$.

Proof. Consider the following congestion game, with n players $N = \{1, ..., n\}$ and n + 1 resources $E = \{1, ..., n + 1\}$. Each player i has a weight of $w_i = w^i$, where $w := \frac{1}{\Phi_{d,\rho}}$. Resources have cost functions

$$c_j(t) = \begin{cases} \frac{1}{\rho} \Phi_{d,\rho}^{d+2}, & j = 1, \\ \\ \Phi_{d,\rho}^{(d+1)j} t^d, & j = 2, \dots, n+1 \end{cases}$$

¹¹In singleton congestion games the strategies of all players consist of a single resource. Formally $|s_u| = 1$, for any $u \in N$ and all $s_u \in S_u$.

Each player *i* has only two available strategies, denoted by s_i^* and s_i : either use only resource *i*, or only resource *i* + 1. Formally, $S_i = \{s_i^*, s_i\}$, where $s_i^* = \{i\}$ and $s_i = \{i+1\}$. The social cost of profile $\mathbf{s} = (s_1, \ldots, s_n)$, where every player *i* uses the (i + 1)-th resource, is

$$C(\mathbf{s}) = \sum_{i=1}^{n} C_i(\mathbf{s}) = \sum_{i=1}^{n} w_i c_{i+1}(w_i)$$

= $\sum_{i=1}^{n} w^i \Phi_{d,\rho}^{(d+1)(i+1)} w^{id} = \sum_{i=1}^{n} \left(\Phi_{d,\rho}^{i+1} w^i \right)^{d+1}$
= $\sum_{i=1}^{n} \Phi_{d,\rho}^{d+1} = n \Phi_{d,\rho}^{d+1}.$

while that of $\mathbf{s}^* = (s_1^*, \dots, s_n^*)$, where player *i* uses the *i*-th resource is

$$C(\mathbf{s}^*) = \sum_{i=1}^n w_i c_i(w_i) = w \frac{1}{\rho} \Phi_{d,\rho}^{d+2} + \sum_{i=2}^n w^i \Phi_{d,\rho}^{(d+1)i} w^{id} = \frac{1}{\rho} \Phi_{d,\rho}^{d+1} + \sum_{i=2}^n 1 = \frac{1}{\rho} \Phi_{d,\rho}^{d+1} + n - 1.$$

We now claim that profile s is a ρ -equilibrium. Indeed, for player i = 1,

$$\frac{C_1(\mathbf{s})}{C_1(\mathbf{s}_{-1}, s_1^*)} = \frac{c_2(w_1)}{c_1(w_1)} = \frac{\Phi_{d,\rho}^{2(d+1)} w^d}{\frac{1}{\rho} \Phi_{d,\rho}^{d+2}} = \rho \frac{\Phi_{d,\rho}^{2(d+1)-d}}{\Phi_{d,\rho}^{d+2}} = \rho,$$

and for all players $i = 2, \ldots, n$,

$$\frac{C_i(\mathbf{s})}{C_i(\mathbf{s}_{-u_i}, s_i^*)} = \frac{c_{i+1}(w_i)}{c_i(w_i + w_{i-1})} = \frac{\Phi_{d,\rho}^{(d+1)(i+1)}(w^i)^d}{\Phi_{d,\rho}^{(d+1)i}(w^i + w^{i-1})^d} = \Phi_{d,\rho}^{d+1} \left(1 + \frac{1}{w}\right)^{-d} = \Phi_{d,\rho}^{d+1} (\Phi_{d,\rho} + 1)^{-d} = \rho_{d,\rho}^{d+1} \left(1 + \frac{1}{w}\right)^{-d}$$

the last equality coming for the definition of the generalized golden ratio $\Phi_{d,\rho}$; thus, no player can unilaterally deviate from **s** and gain (strictly) more than a factor of ρ .

Since **s** is a ρ -equilibrium, the ρ -PoA of our game is at least

$$\frac{C(\mathbf{s})}{\min_{\mathbf{s}'} C(\mathbf{s}')} \ge \frac{C(\mathbf{s})}{C(\mathbf{s}^*)} = \frac{n\Phi_{d,\rho}^{d+1}}{n-1+\frac{1}{\rho}\Phi_{d,\rho}^{d+1}} \to \Phi_{d,\rho}^{d+1}$$

as n grows arbitrarily large. This concludes our proof.

4 The Algorithm

In this section we describe and study our algorithm for computing $d^{d+o(d)}$ -approximate equilibria in weighted congestion games of degree d. The algorithm, as well as the general outline of its analysis, are inspired by the work of Caragiannis et al. [9]. However, as discussed in Section 1.2, here we are using the approximate potential function of Christodoulou et al. [14]. This will be crucial in proving that the algorithm is indeed poly-time and, more importantly, has the improved approximation guarantee of $d^{d+o(d)}$.

In Section 4.1, we introduce the aforementioned potential function from [14], along with some natural extensions that will be useful for our analysis — *partial* and *subgame* potentials. This is complemented by a set of technical lemmas through which our use of this potential function will be instantiated in the rest of the paper. In Section 4.2, we describe our approximation algorithm and next, in Section 4.3, we show that it indeed runs in polynomial time; the critical step in achieving this is proving the "Key Property" (Theorem 3) of our algorithm, which appropriately bounds the potential of certain groups of players throughout its execution. Finally, in Section 4.4 we establish the desired approximation guarantee of the algorithm.

4.1 The Potential Function Technique

Our subsequent proofs regarding both the runtime and the approximation guarantee of our algorithm, will rely heavily on the use of an approximate potential function. In particular, we will use a straightforward variant of the *Faulhaber potential* function, introduced by Christodoulou et al. [14]. Consider a polynomial weighted congestion game of degree d. For every resource e with cost function $c_e(x) = \sum_{\nu=0}^d a_{e,\nu} x^{\nu}$, we set

$$\phi_e(x) := a_{e,0} x + \sum_{\nu=1}^d a_{e,\nu} \left(x^{\nu+1} + \frac{\nu+1}{2} x^{\nu} \right).$$
(5)

The potential of any state \mathbf{s} of the game is then defined as

$$\Phi(\mathbf{s}) := \sum_{e \in E} \phi_e(x_e(\mathbf{s})).$$

The potential function introduced above satisfies a crucial property which is given in the following lemma and which will be extensively used in the rest of our paper. To state it, we will first define¹² an auxiliary constant

$$\alpha := d + 1. \tag{6}$$

Lemma 4. For any resource $e, x \ge 0$ and $w \ge 1$:

$$w \cdot c_e(x+w) \le \phi_e(x+w) - \phi_e(x) \le \alpha \cdot w \cdot c_e(x+w),$$

where function ϕ_e is defined given in (5) and parameter α in (6).

Proof. From the proof¹³ of Theorem 4.4 of Christodoulou et al. [14] we know that, if for resource e with cost function $c_e(x) = \sum_{\nu=0}^d a_{e,\nu} x^{\nu}$ we define

$$\hat{\phi}_e(x) = \sum_{\nu=1}^d a_{e,\nu} S_\nu(x), \quad \text{where } S_\nu(x) = \begin{cases} \frac{x^{\nu+1}}{\nu+1} + \frac{x^{\nu}}{2}, & \nu = 1, \dots, d, \\ x, & \nu = 0, \end{cases}$$
(7)

then

$$\frac{1}{d+1} \cdot w \cdot c_e(x+w) \le \hat{\phi}_e(x+w) - \hat{\phi}_e(x) \le w \cdot c_e(x+w).$$

We now multiply each ν -th term of the sum for $\hat{\phi}_e(x)$ in (7) by $\nu + 1$, defining

$$\phi_e(x) = \sum_{\nu=1}^d a_{e,\nu}(\nu+1)S_{\nu}(x)$$

which coincides with our definition of the potential in (5). It is not difficult to see (by following the same proof of [14, Thereom 4.4]) that the above translates into essentially scaling the entire potential by d + 1, resulting in the desired bound in the statement of Lemma 4.

¹²The reason for introducing extra notation here, and not making the arguably simpler choice to directly use the actual value of d + 1 instead of variable α , for the rest of the paper, is that we want to assist readability: by using α , we make clear where exactly this factor from our approximate potential comes into play.

¹³In particular, see Eq. (4.10) in the proof of [14, Claim 4.7] and the related [14, Lemma 4.3], both applied for the special case of $\gamma = 1$ here.

The notions of *partial* and *subgame* potential were introduced in [9], and we now adapt them for the setting of our new approximate potential. The *subgame potential* with respect to a group of players $R \subseteq N$ of a state **s** is defined as:

$$\Phi^R(\mathbf{s}) := \sum_{e \in E} \phi_e(x_{R,e}(\mathbf{s})).$$

The partial potential with respect to a group of players $R \subseteq N$ is then defined as:

$$\Phi_R(\mathbf{s}) := \Phi(\mathbf{s}) - \Phi^{N \setminus R}(\mathbf{s}) = \sum_{e \in E} \left[\phi_e(x_e(\mathbf{s})) - \phi_e(x_{N \setminus R, e}(\mathbf{s})) \right].$$
(8)

In the original work by Christodoulou et al. [14], the variable w in Lemma 4 was interpreted as a single player's weight, and x as the total remaining weight on resource e. In our analysis, however, x and w will each play the role of the total weight of some group's players who are using resource e. Lemma 4 then becomes a very powerful algebraic tool that relates social cost of groups of players and partial potentials in a variety of settings, and plays an important role in many of our proofs. In many cases, it will be present in such proofs implicitly, through the following corollary (Lemma 5), which says that the chosen potential function is cost-revealing in a strong sense. That is, for any group of players, the ratio of the partial potential and the social cost of that group is bounded from both above and below.

Lemma 5. For any group of players $R \subseteq N$,

$$\sum_{u \in R} C_u(\mathbf{s}) \le \Phi_R(\mathbf{s}) \le \alpha \sum_{u \in R} C_u(\mathbf{s}).$$

Proof. First consider each resource e separately. Setting $x = x_{N \setminus R, e}(\mathbf{s})$ and $w = x_{R, e}(\mathbf{s})$ in Lemma 4, we obtain

$$\begin{aligned} x_{R,e}(\mathbf{s}) \cdot c_e(x_{N \setminus R,e}(\mathbf{s}) + x_{R,e}(\mathbf{s})) &\leq \phi_e(x_{N \setminus R,e}(\mathbf{s}) + x_{R,e}(\mathbf{s})) - \phi_e(x_{N \setminus R,e}(\mathbf{s})) \\ &\leq \alpha \cdot x_{R,e}(\mathbf{s}) \cdot c_e(x_{N \setminus R,e}(\mathbf{s}) + x_{R,e}(\mathbf{s})). \end{aligned}$$

Summing over all resources, we get that

$$\sum_{u \in R} C_u(\mathbf{s}) \le \Phi(\mathbf{s}) - \Phi^{N \setminus R}(\mathbf{s}) \le \alpha \sum_{u \in R} C_u(\mathbf{s}).$$

By definition (8), $\Phi_R(\mathbf{s}) = \Phi(\mathbf{s}) - \Phi^{N \setminus R}(\mathbf{s})$, concluding the proof.

The following lemma provides a relation between the change in potential due to a single player's deviation and a linear combination of that player's old and new costs. For an exact potential function, the latter would simply be the difference in the cost experienced by that player. However, in our case, the player's cost in one of the two states is weighted by an additional factor of α compared to her cost in the other state. Thus, Lemma 6 only implies a decrease in the potential function if the deviating player has improved her cost by at least a factor α .

Lemma 6. Let u be a player, R be an arbitrary subset of players with $u \in R$, and s and s' be two states that differ only in the strategy of player u. Then,

$$\Phi_R(\mathbf{s}) - \Phi_R(\mathbf{s}') \ge C_u(\mathbf{s}) - \alpha C_u(\mathbf{s}').$$

Proof. First observe that, since each player in $N \setminus R$ plays the same strategy in both profiles **s** and **s'**, it must be that $x_{N \setminus R, e}(\mathbf{s}) = x_{N \setminus R, e}(\mathbf{s}')$. Using this, we can derive that

$$\Phi^{N\setminus R}(\mathbf{s}) = \sum_{e\in E} \phi_e(x_{N\setminus R, e}(\mathbf{s})) = \sum_{e\in E} \phi_e(x_{N\setminus R, e}(\mathbf{s}')) = \Phi^{N\setminus R}(\mathbf{s}')$$

and thus

$$\Phi_R(\mathbf{s}) - \Phi_R(\mathbf{s}') = \left[\Phi(\mathbf{s}) - \Phi^{N \setminus R}(\mathbf{s})\right] - \left[\Phi(\mathbf{s}') - \Phi^{N \setminus R}(\mathbf{s}')\right] = \Phi(\mathbf{s}) - \Phi(\mathbf{s}'),$$

where the first equality is due to the definition of partial potentials (8).

Finally, due to Lemma 4 (via Lemma 4.1 of Christodoulou et al. [14]) we know that Φ is indeed an α -approximate potential, that is,

$$\Phi(\mathbf{s}) - \Phi(\mathbf{s}') \ge C_u(\mathbf{s}) - \alpha C_u(\mathbf{s}')$$

given that \mathbf{s} and \mathbf{s}' differ only on the strategy of player u.

4.2 Description of the Algorithm

We shall now describe our algorithm for finding $d^{d+o(d)}$ -approximate equilibria in weighted congestion games of degree d (see Algorithm 1). We remark, once again, that it is inspired by a similar algorithm by Caragiannis et al. [9]. However, a critical difference is that our algorithm runs directly in the *actual* game (using the original cost functions, and thus, players' deviations that are best-responses with respect to the actual game); as a result, we also need to appropriately calibrate the original parameters from Caragiannis et al. [9]. First, we fix the following constant that essentially captures our *target approximation factor*:

$$p := (2d+3)(d+1)(4d)^{d+1} = d^{d+o(d)}.$$
(9)

The following lemma captures a critical property of the above parameter, which is the one that will essentially give rise to the specific approximation factor of our algorithm (see Theorem 6).

Lemma 7. For any positive integer d, the parameter p defined in (9) satisfies the following property:

$$p \ge (2\alpha + 1)\alpha \cdot \operatorname{PoA}_d\left(\alpha + \frac{1}{p}\right) = (2\alpha + 1)\alpha \Phi_{d,\alpha + \frac{1}{p}}^{d+1}$$

where α is given in (6).

Proof. First, recall from (6) that $\alpha = d + 1$. Next, we note that the Lambert-W function is increasing on the positive reals (see Corless et al. [17]) and so, for any $d \ge 1$,

$$\mathcal{W}\left(\frac{d}{d+2}\right) \ge \mathcal{W}\left(\frac{1}{3}\right) \approx 0.258 > \frac{1}{4}.$$

Furthermore, the ρ -approximate Price of Anarchy $\text{PoA}_d(\rho)$ is also nondecreasing with respect to the approximation parameter $\rho \geq 1$ (since the set of allowable approximate equilibria gets larger; see Section 2). Thus, from Theorem 1 we can see that

$$\operatorname{PoA}_d\left(\alpha + \frac{1}{p}\right) \le \operatorname{PoA}_d\left(d+2\right) \le \left[\frac{d}{\mathcal{W}\left(\frac{d}{d+2}\right)}\right]^{d+1} \le (4d)^{d+1}.$$

Algorithm 1 Computing $d^{d+o(d)}$ -approximate equilibria in weighted polynomial congestion games of degree d

INPUT: A polynomial weighted congestion game Γ of degree d; A state \mathbf{s}_{init} of Γ **OUTPUT:** A $d^{d+o(d)}$ -equilibrium state \mathbf{s}

1: $c_{\max} \leftarrow \max_{u \in N} C_u(\mathbf{s}_{\min}), c_{\min} \leftarrow \min_{u \in N} C_u(\mathbf{0}_{-u}, \mathcal{BR}_u(\mathbf{0})), m \leftarrow \log \frac{c_{\max}}{c_{\min}}$ 2: $g \leftarrow np^3(1+m(1+p))^d d^d + 1$ $\triangleright p$ is defined in (9) 3: $b_i \leftarrow g^{-i} c_{\max}$ for all $i = 1, \dots, m$ 4: $\mathbf{s} \leftarrow \mathbf{s}_{\text{init}}$ 5: $N_{\text{fixed}} \leftarrow \emptyset$ 6: while $\exists u \in N : C_u(\mathbf{s}) \geq b_1 \wedge u$ has an $(\alpha + \frac{1}{n})$ -move do $\mathbf{s} \leftarrow (\mathbf{s}_{-u}, \mathcal{BR}_u(\mathbf{s}))$ 7: 8: for i = 1 to m - 1 do while $\exists u \in N \setminus N_{\text{fixed}}$: 9: $\left[C_u(\mathbf{s}) \in [b_{i+1}, b_i) \land u \text{ has an } (\alpha + \frac{1}{p}) \text{-move}\right] \bigvee \left[C_u(\mathbf{s}) \ge b_i \land u \text{ has a } p \text{-move}\right] \mathbf{do}$ $\mathbf{s} \leftarrow^{\mathsf{L}}(\mathbf{s}_{-u}, \mathcal{BR}_u(\mathbf{s}))$ 10: $N_{\text{fixed}} \leftarrow N_{\text{fixed}} \cup \{ u \in N \setminus N_{\text{fixed}} : C_u(\mathbf{s}) \ge b_i \}$ 11: 12: $N_{\text{fixed}} \leftarrow N_{\text{fixed}} \cup \{u \in N \setminus N_{\text{fixed}} : C_u(\mathbf{s}) \ge b_m\}$

From this, we can finally get that indeed

$$(2d+3)(d+1)\operatorname{PoA}_d\left(\alpha+\frac{1}{p}\right) \le (2d+3)(d+1)(4d)^{d+1} = p.$$

The equality in the statement of our lemma is just a consequence of the fact that the ρ -PoA is exactly equal to $\Phi_{d,\rho}^{d+1}$ (from Section 3; see Theorem 1 and Theorem 2).

The input to the algorithm consists of the description of a weighted polynomial congestion game Γ of fixed degree d and an (arbitrary) initial state \mathbf{s}_{init} of Γ . We now let $c_{max} := \max_{u \in N} C_u(\mathbf{s}_{init})$ and $c_{\min} := \min_{u \in N} C_u(\mathbf{0}_{-u}, \mathcal{BR}_u(\mathbf{0}))$. Here, $\mathbf{0}$ denotes the "empty state", that is, a fictitious state in which all players' strategies are empty sets. Recall also (see Section 2) that $\mathcal{BR}_u(\cdot)$ returns a best-response move of player u at a given state. Observe that c_{\max} is the maximum cost of a player in state \mathbf{s}_{init} , and c_{\min} can be used as a lower bound on the cost of any player in any state of our game. We also define parameter $m = \log \frac{c_{\max}}{c_{\min}}$. Notice that m is polynomial on the input (that is, the description of our game Γ). Next, we introduce a factor g (see Line 2 of Algorithm 1), that depends (polynomially) on both the aforementioned parameter m of our game and the approximation factor p. Using this, we set m+1 "boundaries" $c_{\max} = b_0, b_1, \ldots, b_m$ so that $b_i = g^{-i}c_{\max}$ for the player costs (see discussion below).

The algorithm runs in *m* phases, indexed by i = 0, 1, ..., m - 1. It is helpful to introduce here the following notation. We denote by s^i the state of the game immediately after the end of phase *i*. Furthermore, let R_i be the set of players who were at least once selected by the algorithm to make an improvement move during phase *i*.

Phase i = 0 is "preparatory": we repeatedly select a player whose cost exceeds the largest (non-trivial) boundary b_1 and who has an $(\alpha + p^{-1})$ -move, and allow her to make a best-response move. This phase ends when there are no such players left.

Next follow the "main" m-1 phases. The algorithm constructs the final state s^{m-1} , which is the sought $d^{d+o(d)}$ -approximate equilibrium, step by step. That is, it finalizes players' strategies in "packets" rather than one-by-one. In particular, during each phase it fixes the strategies of a certain group of players, and never changes them again. The *i*-th phase itself, consists of a sequence of best-responses. We allow players (who are not "fixed" yet) to repeatedly make best-response moves, according to a certain rule: if a player's current cost is "large", i.e. lies in $[b_i, \infty)$, she is allowed to best-respond only if she can improve at least by a factor of p (that is, she has a p-move); if, however, her cost is "small", i.e. in $[b_{i+1}, b_i)$, she can play even if she only has an $(\alpha + p^{-1})$ -move.¹⁴ The phase ends when there are no players left to make a deviation according to the above rule. At this point, the algorithm fixes the strategies of all players whose *current* cost lies in $[b_i, \infty)$, and adds them to the set N_{fixed} .

Immediately after the end of phase m-1, the algorithm returns the current state of the game and terminates. Observe that after the final phase of the algorithm, all players have been included in N_{fixed} : using the definitions of m and b_m , it is not difficult to see that $b_m \leq c_{\min}$.

4.3 Running Time

In this section, we prove the "Key Property" of our algorithm (Theorem 3), and then establish Theorem 4 which guarantees that the runtime of the algorithm is polynomial. We remark here that we assume, for the runtime analysis, that best-response strategies are efficiently computable.¹⁵ One should also keep in mind that the degree d of the game is considered a *constant*. However, with respect to the analysis of the approximation guarantee of the algorithm (see the following Section 4.4), it will actually be treated as a parameter: the approximation factor will be given as a function of d. We want to emphasize here that all these assumptions are standard in the literature of algorithms for approximate equilibria in congestion games (see e.g. [8, 9, 21, 20]).

Recall that R_i is the set of players that made at least one improvement move during the *i*-th phase of the algorithm. In the following it will be useful to denote by $C_u(i)$, for any $u \in R_i$, the cost of player u immediately after her last move within phase i. Then we can prove the following upper bound on the partial potential of R_i immediately after phase i:

Lemma 8. For every phase $i, \Phi_{R_i}(\mathbf{s}^i) \leq \alpha \sum_{u \in R_i} C_u(i)$.

Proof. Rename players in R_i as $u_1, ..., u_{|R_i|}$ in increasing order of time of their last move during phase *i*. Call $\mathbf{s}^{i,j}$ the state immediately after player u_j made her last move during phase *i*. Define $R_i^j := \{u_{j+1}, ..., u_{|R_i|}\}$, the set of players who moved during phase *i* after state $\mathbf{s}^{i,j}$. Then, using the definition of the partial potential (8), we can obtain the telescoping sum

$$\Phi_{R_i}(\mathbf{s}^i) = \Phi(\mathbf{s}^i) - \Phi^{N \setminus R_i}(\mathbf{s}^i) = \sum_{j=1}^{|R_i|} \left[\Phi^{N \setminus R_i^j}(\mathbf{s}^i) - \Phi^{N \setminus R_i^{j-1}}(\mathbf{s}^i) \right]$$
(10)

We now consider each term of the above sum separately. For any $j = 1, \ldots, |R_i|$ and any resource e, we have $x_{N \setminus R_i^j, e}(\mathbf{s}^{i,j}) = x_{N \setminus R_i^j, e}(\mathbf{s}^i)$ and $x_{N \setminus R_i^{j-1}, e}(\mathbf{s}^{i,j}) = x_{N \setminus R_i^{j-1}, e}(\mathbf{s}^i)$. Indeed, only players in R_i^j made moves between state $\mathbf{s}^{i,j}$ and the end of phase i, and consequently,

¹⁴One can think of this second type of moves as "preparatory" for the next phase. Indeed, they ensure that immediately before the beginning of the (i + 1)-th phase, players who are not yet fixed and whose costs exceed b_{i+1} are in an $(\alpha + p^{-1})$ -equilibrium. We can expect such players to perform relatively few *p*-moves during the (i + 1)-th phase, since the approximation factor $\alpha + p^{-1}$ is significantly smaller than *p* (see (9)).

¹⁵Formally, we want function $\mathcal{BR}_u(\mathbf{s}_{-u})$ to be computable in polynomial time, for any player u and any strategies \mathbf{s}_{-u} of the other players. This assumption is necessary in general, since the number of strategies of a player can, in principle, be exponential in the input size. Think, for example, of source-sink paths in network congestion games. Of course, if the strategy sets of the players have polynomial size in the first place, then the time-efficiency of best-responses comes for free anyway.

players in $N \setminus R_i^j$ and in $N \setminus R_i^{j-1} \subseteq N \setminus R_i^j$ never changed their strategies between states $\mathbf{s}^{i,j}$ and \mathbf{s}^i . Thus,

$$\begin{split} \varPhi^{N \setminus R_i^j}(\mathbf{s}^i) - \varPhi^{N \setminus R_i^{j-1}}(\mathbf{s}^i) &= \sum_{e \in E} \left[\phi_e(x_{N \setminus R_i^j, e}(\mathbf{s}^i)) - \phi_e(x_{N \setminus R_i^{j-1}, e}(\mathbf{s}^i)) \right] \\ &= \sum_{e \in E} \left[\phi_e(x_{N \setminus R_i^j, e}(\mathbf{s}^{i,j})) - \phi_e(x_{N \setminus R_i^{j-1}, e}(\mathbf{s}^{i,j})) \right] \end{split}$$

At this point, we use that $N \setminus R_i^j = (N \setminus R_i^{j-1}) \cup \{u_j\}$, paired with Lemma 4 and the monotonicity of c_e , to further obtain

$$\begin{split} \Phi^{N\setminus R_i^j}(\mathbf{s}^i) - \Phi^{N\setminus R_i^{j-1}}(\mathbf{s}^i) &= \sum_{e\in E} \left[\phi_e(x_{N\setminus R_i^{j-1}, e}(\mathbf{s}^{i,j}) + x_{u_j, e}(\mathbf{s}^{i,j})) - \phi_e(x_{N\setminus R_i^{j-1}, e}(\mathbf{s}^{i,j})) \right] \\ &\leq \sum_{e\in E} \alpha x_{u_j, e}(\mathbf{s}^{i,j}) \cdot c_e(x_{N\setminus R_i^{j-1}, e}(\mathbf{s}^{i,j}) + x_{u_j, e}(\mathbf{s}^{i,j})) \\ &\leq \alpha \sum_{e\in E} x_{u_j, e}(\mathbf{s}^{i,j}) \cdot c_e(x_e(\mathbf{s}^{i,j})) \\ &= \alpha C_{u_i}(\mathbf{s}^{i,j}). \end{split}$$

Applying this bound to each term in the sum in (10), we finally get the desired inequality

$$\Phi_{R_i}(\mathbf{s}^i) \le \sum_{j=1}^{|R_i|} \alpha C_{u_j}(\mathbf{s}^{i,j}) = \alpha \sum_{u \in R_i} C_u(i),$$

where for the last equality we used the definition of $C_u(i)$.

The following result is necessary to prove Theorem 3 and is essentially the cornerstone of the approach to computing approximate equilibria that our algorithm takes. Namely, Lemma 9 considers a group of players in an approximate equilibrium at some state \mathbf{s} and shows that the potential of \mathbf{s} is at most a fixed factor away from the potential of any other state that only differs from \mathbf{s} in the strategies of players in that group. In particular, this factor turns out to be of the order of the ρ -PoA (see also Theorem 1).

Lemma 9. For any group of players $R \subseteq N$, consider any two states \mathbf{s} and \mathbf{s}' such that \mathbf{s} is a ρ -equilibrium state for the players in R and every player in $N \setminus R$ uses the same strategy in \mathbf{s} and \mathbf{s}' . Then $\Phi_R(\mathbf{s}) \leq \alpha \Phi_{d,\rho}^{d+1} \Phi_R(\mathbf{s}')$.

Proof. Utilizing Lemma 5 we can bound the ratio of the partial potentials by

$$\frac{\varPhi_R(\mathbf{s})}{\varPhi_R(\mathbf{s}')} \leq \frac{\alpha \sum_{u \in R} C_u(\mathbf{s})}{\sum_{u \in R} C_u(\mathbf{s}')} \leq \alpha \, \Phi^{d+1}_{d,\rho},$$

where the last inequality holds from by Lemma 1.

Finally, the following theorem establishes that the partial potential of the players R_i , who are going to move during the *i*-th phase, is linearly-bounded by the *i*-th cost boundary b_i at the beginning of that phase. This result will be used in two ways: on one hand, it is the basis of the proof of Theorem 4, which guarantees polynomial runtime of the algorithm; on the other hand, Theorem 3 will be used in Section 4.4 as well, to bound the approximation factor of the algorithm (Theorem 5 in particular).

Theorem 3 (Key Property). For every phase $i \ge 1$ of the algorithm, $\Phi_{R_i}(\mathbf{s}^{i-1}) \le npb_i$.

Proof. We split $R_i = P_i \cup Q_i$, where P_i are players in R_i whose last move during phase i was a p-move, and Q_i those whose last move during phase i was an $(\alpha + p^{-1})$ -move. Now suppose there were τ best-response moves executed during the course of phase i (see Line 10 in Algorithm 1), and denote the corresponding states in the game by $\mathbf{s}^{i-1} = \mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_{\tau} = \mathbf{s}^i$. Then

$$\Phi_{R_i}(\mathbf{s}^{i-1}) - \Phi_{R_i}(\mathbf{s}^i) = \sum_{j=1}^{\tau} \left[\Phi_{R_i}(\mathbf{t}_{j-1}) - \Phi_{R_i}(\mathbf{t}_j) \right].$$
(11)

Notice here that, due to Lemma 6 and the fact that $p, \alpha + p^{-1} > \alpha$, i.e., all best-responses in the above sum are better than α -moves, all terms of the above sum are positive. Thus, keeping only those terms j for which $\mathbf{t}_{j-1} \to \mathbf{t}_j$ corresponds to the last move during phase i of some player in P_i , we get that

$$\Phi_{R_i}(\mathbf{s}^{i-1}) - \Phi_{R_i}(\mathbf{s}^i) \ge \sum_{u \in P_i} (p - \alpha) C_u(i), \tag{12}$$

where recall that $C_u(i)$ denotes the cost of player u after her last move within phase i.

By Lemma 8, we also have that

$$\sum_{u \in P_i} C_u(i) = \sum_{u \in R_i} C_u(i) - \sum_{u \in Q_i} C_u(i) \ge \frac{1}{\alpha} \Phi_{R_i}(\mathbf{s}^i) - \sum_{u \in Q_i} C_u(i) \ge \frac{1}{\alpha} \Phi_{R_i}(\mathbf{s}^i) - nb_i,$$

the last inequality holding due to the fact that all players in Q_i have cost at most b_i at the end of phase *i*. Combining this with (12), and solving with respect to $\Phi_{R_i}(\mathbf{s}^i)$, we finally get that

$$\Phi_{R_i}(\mathbf{s}^i) \le \frac{\alpha}{p} \left[\Phi_{R_i}(\mathbf{s}^{i-1}) + (p-\alpha)nb_i \right] \le \frac{\alpha}{p} \left[\Phi_{R_i}(\mathbf{s}^{i-1}) + pnb_i \right].$$

To arrive at a contradiction, and complete the proof of our theorem, from now on assume that $\Phi_{R_i}(\mathbf{s}^{i-1}) > nb_i p$. Then, the above inequality becomes

$$\Phi_{R_i}(\mathbf{s}^i) < \frac{2\alpha}{p} \Phi_{R_i}(\mathbf{s}^{i-1}).$$
(13)

We now introduce some additional notation for the current proof. First, partition R_i into X_i and Y_i , where X_i contains the players in R_i whose cost is at least b_i at state \mathbf{s}^{i-1} , and Y_i those whose cost is strictly less than b_i . Secondly, consider a helper game¹⁶ $\overline{\Gamma}$ with the same set of players $\overline{N} = N$ and resources $\overline{E} = E \cup \{e_u : u \in Y_i\}$. The new, extra resources have costs

$$\bar{c}_{e_u}(t) = \frac{b_i}{w_u} \quad \text{for all } u \in Y_i,$$

while the old ones have the same costs $\bar{c}_e(t) = c_e(t)$ for all $e \in E$. All players in $N \setminus Y_i$, have the same strategies in $\bar{\Gamma}$ as before in Γ . However, for the rest we set

$$\bar{S}_u = \{s_u\} \cup \{s'_u \cup \{e_u\} \mid s'_u \in S_u \setminus \{s_u\}\} \quad \text{for all } u \in Y_i,$$

where s_u is the strategy that u uses in state \mathbf{s}^{i-1} of the original game Γ . Denote the potential in Γ' by $\bar{\Phi}$.

We define two special strategy profiles, namely $\bar{\mathbf{s}}^{i-1}$ and $\bar{\mathbf{s}}^i$, of our new game $\bar{\Gamma}$. In state $\bar{\mathbf{s}}^{i-1}$, every player uses the same strategy as in state \mathbf{s}^{i-1} of the original game Γ . In state $\bar{\mathbf{s}}^i$, again players use the same strategies as in state \mathbf{s}^i of Γ , except from those players in Y_i that

¹⁶This is exactly the same construction as in the proof of Caragiannis et al. [9, Lemma 4.3].

happen to have different strategies between \mathbf{s}^{i-1} and \mathbf{s}^i . In particular, if a player $u \in Y_i$ uses strategy s_u under profile \mathbf{s}^{i-1} and strategy $s'_u \neq s_u$ at profile \mathbf{s}^i , then we set her strategy at $\bar{\mathbf{s}}^i$ to be $s'_u \cup \{e_u\}$.

Then, one can show (see Appendix B) the following about these states: first, $\bar{\mathbf{s}}^{i-1}$ is an $(\alpha + p^{-1})$ -equilibrium (of game $\bar{\Gamma}$) for all players in R_i ; and secondly,

$$\bar{\Phi}_{R_i}(\bar{\mathbf{s}}^i) \le \Phi_{R_i}(\mathbf{s}^i) + nb_i.$$

Using (13) and recalling our assumption that $\Phi_{R_i}(\mathbf{s}^{i-1}) > nb_i p$, the above inequality gives us that

$$\bar{\Phi}_{R_i}(\bar{\mathbf{s}}^i) < \frac{2\alpha + 1}{p} \Phi_{R_i}(\mathbf{s}^{i-1}) = \frac{2\alpha + 1}{p} \bar{\Phi}_{R_i}(\bar{\mathbf{s}}^{i-1}),$$

where the last equality is a result of the fact that all players use the same strategies in $\bar{\mathbf{s}}^{i-1}$ and \mathbf{s}^{i-1} , which consist only of edges that have exactly the same cost functions at Γ and $\bar{\Gamma}$. From Lemma 7, this becomes

$$\bar{\varPhi}_{R_i}(\bar{\mathbf{s}}^{i-1}) > \alpha \Phi^{d+1}_{d,\alpha+\frac{1}{p}} \bar{\varPhi}_{R_i}(\bar{\mathbf{s}}^i),$$

which contradicts Lemma 9, since all players in R_i are in an $(\alpha + p^{-1})$ -equilibrium in $\bar{\mathbf{s}}^{i-1}$, and every player in $N \setminus R_i$ has the same strategy in $\bar{\mathbf{s}}^{i-1}$ and $\bar{\mathbf{s}}^i$. This contradiction concludes our proof.

Now we are able to prove the main theorem of this section:

Theorem 4. Algorithm 1 runs in time polynomial in the number of bits of the representation of its input game Γ .

Proof. It suffices to show that every phase i = 0, ..., m-1 runs in poly-time, since the number of phases $m = \log \frac{c_{\max}}{c_{\min}}$ is polynomial in the representation of the game.

First, fix such a phase *i* and suppose there were τ single-player best-response moves (see Line 10 of Algorithm 1) during the course of this phase. For this proof only, set $\mathbf{s}^{-1} := \mathbf{s}_{\text{init}}$. Denote all intermediate states that correspond to these deviations by $\mathbf{s}^{i-1} = \mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_{\tau} = \mathbf{s}^i$ in chronological order, and call u_j the player who makes the best-response move corresponding to the change from state \mathbf{t}_{i-1} to \mathbf{t}_i . Then

$$\Phi(\mathbf{s}^{i-1}) - \Phi(\mathbf{s}^{i}) = \Phi_{R_i}(\mathbf{s}^{i-1}) - \Phi_{R_i}(\mathbf{s}^{i}) = \sum_{j=1}^{\tau} \left[\Phi_{R_i}(\mathbf{t}_{j-1}) - \Phi_{R_i}(\mathbf{t}_j) \right].$$
(14)

The first equality holds using the definition of the partial potential (8) and the fact that players in $N \setminus R_i$ do not move during phase i, so $\Phi^{N \setminus R_i}(\mathbf{s}^{i-1}) = \Phi^{N \setminus R_i}(\mathbf{s}^i)$. The second uses a telescoping sum.

Now observe that during the phase, players selected to make a move either make an $(\alpha + p^{-1})$ move or a *p*-move (for the i = 0, only the former). Hence, for any intermediate state *j*, it must be that

$$C_{u_j}(\mathbf{t}_{j-1}) \ge \min\{p, a+p^{-1}\} \cdot C_{u_j}(\mathbf{t}_j) = (a+p^{-1})C_{u_j}(\mathbf{t}_j)$$

Applying Lemma 6, we then obtain

$$\Phi_{R_i}(\mathbf{t}_{j-1}) - \Phi_{R_i}(\mathbf{t}_j) \ge C_{u_j}(\mathbf{t}_{j-1}) - \frac{\alpha}{\alpha + p^{-1}} C_{u_j}(\mathbf{t}_{j-1}) = \frac{1}{\alpha p + 1} C_{u_j}(\mathbf{t}_{j-1}).$$
(15)

By definition of the algorithm (see Algorithm 1), player u_j has to have cost $C_{u_j}(\mathbf{t}_{j-1}) \geq b_{i+1}$, otherwise she wouldn't have made any moves during phase *i*. Recalling that $b_{i+1} = b_i g^{-1}$, inequality (15) now tells us that, for all $j = 1, \ldots, \tau$,

$$\varPhi_{R_i}(\mathbf{t}_{j-1}) - \varPhi_{R_i}(\mathbf{t}_j) \geq \frac{b_i g^{-1}}{\alpha p + 1}$$

and so applying (14) we get that

$$\Phi_{R_i}(\mathbf{s}^{i-1}) - \Phi_{R_i}(\mathbf{s}^i) \ge \tau \frac{b_i g^{-1}}{\alpha p + 1}$$

If $i \ge 1$, this can be rewritten as

$$\tau \le g(\alpha p+1)b_i^{-1} \cdot \left[\Phi_{R_i}(\mathbf{s}^{i-1}) - \Phi_{R_i}(\mathbf{s}^i) \right] \le g(\alpha p+1) \cdot b_i^{-1} \Phi_{R_i}(\mathbf{s}^{i-1}) \le n \cdot g(\alpha p+1)p_i$$

where the last inequality holds due to Theorem 3.

For the remaining, special case of i = 0, since $b_0 = c_{\text{max}}$ and $\mathbf{s}^{-1} = \mathbf{s}_{\text{init}}$, we similarly get that

$$\tau \leq g(\alpha p+1) \cdot c_{\max}^{-1} \Phi(\mathbf{s}_{\text{init}}) \leq n \cdot \alpha g(\alpha p+1),$$

where the last inequality is a consequence of applying Lemma 5 (with $\mathbf{s} = \mathbf{s}_{\text{init}}$ and R = N):

$$\Phi(\mathbf{s}_{\text{init}}) \le \alpha \sum_{u \in N} C_u(\mathbf{s}_{init}) \le \alpha n c_{\max}.$$

In any case, we showed that indeed τ is polynomially bounded in the size of the input. \Box

4.4 Approximation Factor

In this section, we establish the desired approximation guarantee of our algorithm (Theorem 6). Our high-level approach is to examine the amount by which the cost of a player can change after she is "fixed" by the algorithm, and in particular, how the changes in other players' strategies during the following phases affect the beneficial deviating moves that she might have at the final state. We formalize this in Theorem 5, which is the backbone of proving Theorem 6, together with the Key Property (Lemma 8) of the previous section.

The following theorem establishes two very important properties. First, after the strategy of a player u becomes fixed, immediately after some phase j, and until the end of the algorithm (phase \mathbf{s}^{m-1}), her cost will not increase by more than a small factor; and furthermore, no deviation s'_u of player u can be much more profitable at the end of the algorithm than it was immediately after the fixing phase j:

Theorem 5. Suppose the strategy of player u was fixed at the end of phase j. Then,

$$C_u(\mathbf{s}^{m-1}) \le (1+3p^{-1})C_u(\mathbf{s}^j) \tag{16}$$

and

$$C_u(\mathbf{s}_{-u}^{m-1}, s'_u) \ge (1 - 2p^{-1})C_u(\mathbf{s}_{-u}^j, s'_u) \qquad \text{for all } s'_u \in S_u.$$
(17)

To prove our theorem, we will need to capture the relation of the cost of player u between state \mathbf{s}^{j} (when she was fixed) and \mathbf{s}^{m-1} (end state of the algorithm), in a more refined way. In particular, we want to see how the cost of player u can change during all intermediate phases $i = j + 1, \ldots, m - 1$. This is exactly captured by the following lemma, proven in Appendix C:

Lemma 10. Fix any $\varepsilon > 0$ and let $\xi_{\varepsilon} := \left(1 + \frac{1}{\varepsilon}\right)^d d^d$. Consider any player u whose strategy was fixed immediately after phase j. Then, for any phase i > j,

$$C_u(\mathbf{s}^i) \le (1+\varepsilon)C_u(\mathbf{s}^{i-1}) + \xi_\varepsilon \Phi_{R_i}(\mathbf{s}^{i-1})$$
(18)

and

$$C_u(\mathbf{s}_{-u}^i, s_u') \ge \frac{1}{1+\varepsilon} C_u(\mathbf{s}_{-u}^{i-1}, s_u') - \frac{\xi_{\varepsilon}}{1+\varepsilon} \Phi_{R_i}(\mathbf{s}^{i-1}) \quad \text{for all } s_u' \in S_u.$$
(19)

Lemma 10 states that the cost of player u cannot increase by too much during any phase i > j and, at the same time, the cost player u would get by deviating cannot decrease by too much during any phase i > j. We remark here that the upper bound provided by Lemma 10 will itself be bounded by the guarantee provided by Theorem 3. Now, by appropriately applying Lemma 10 in an iterative way, we can prove our theorem:

Proof of Theorem 5. First, fix an $\epsilon > 0$ such that

$$(1+\varepsilon)^m = 1 + \frac{1}{p}.$$
(20)

Then, by Lemma 13 it must be that $m(1+p) \ge \varepsilon^{-1}$, and so for the parameter g of our algorithm (see Line 2 of Algorithm 1) we have the bound

$$g - 1 = np^{3}(1 + m(1 + p))^{d}d^{d} \ge np^{3}\xi_{\varepsilon},$$
(21)

where ξ_{ε} is defined as in Lemma 10.

Also, observe that, since j is the phase at which player u is fixed, it must be that

$$C_u(\mathbf{s}^j) \ge b_j$$
 and $C_u(\mathbf{s}^j) \le p \cdot C_u(\mathbf{s}_{-u}^j, s_u^\prime),$ (22)

for any deviation $s'_u \in S_u$ of player u. These are both consequences of the definition of our algorithm: the first inequality comes immediately from Line 11 of Algorithm 1, while the second one is due to the fact that player u has no p-move left at the end of phase j (see Line 9 of Algorithm 1).

Next, for proving (16), apply (18) recursively, backwards for $i = m - 1, \ldots, j + 1$, to get that

$$C_{u}(\mathbf{s}^{m-1}) \leq (1+\varepsilon)^{m-1-j} C_{u}(\mathbf{s}^{j}) + \xi_{\varepsilon} \sum_{i=j+1}^{m-1} (1+\varepsilon)^{m-1-i} \Phi_{R_{i}}(\mathbf{s}^{i-1})$$
$$\leq (1+\varepsilon)^{m} \left[C_{u}(\mathbf{s}^{j}) + \xi_{\varepsilon} \sum_{i=j+1}^{m-1} \Phi_{R_{i}}(\mathbf{s}^{i-1}) \right]$$
$$= (1+p^{-1}) \left[C_{u}(\mathbf{s}^{j}) + \xi_{\varepsilon} \sum_{i=j+1}^{m-1} \Phi_{R_{i}}(\mathbf{s}^{i-1}) \right], \qquad (23)$$

where for the second inequality we have used the fact that $m-1-j, m-1-i \leq m$, and for the last equality we used (20). By using Theorem 3, we can now bound the second term of (23) by

$$\xi_{\varepsilon} \sum_{i=j+1}^{m-1} \Phi_{R_i}(\mathbf{s}^{i-1}) \le \xi_{\varepsilon} np \sum_{i=j+1}^{m-1} b_i = \xi_{\varepsilon} np b_j \sum_{i=1}^{m-1-j} g^{-i} \le \frac{\xi_{\varepsilon} np}{g-1} b_j \le p^{-2} b_j \le p^{-2} C_u(\mathbf{s}^j).$$
(24)

where the last two inequalities are due to (21) and (22), respectively.

Combining (23) with (24), we finally get

$$C_u(\mathbf{s}^{m-1}) \le (1+p^{-1})(1+p^{-2})C_u(\mathbf{s}^j) = (1+p^{-1}+p^{-2}+p^{-3})C_u(\mathbf{s}^j),$$

which establishes (16).

Moving to (17) now, and applying (19) recursively, backwards for $i = m - 1, \ldots, j + 1$, we get that

$$C_{u}(\mathbf{s}_{-u}^{m-1}, s_{u}') \geq (1+\varepsilon)^{j+1-m} C_{u}(\mathbf{s}_{-u}^{j}, s_{u}') - \xi_{\varepsilon} \sum_{i=j+1}^{m-1} (1+\varepsilon)^{i-m} \Phi_{R_{i}}(\mathbf{s}^{i-1})$$

$$\geq (1+\varepsilon)^{-m} C_{u}(\mathbf{s}_{-u}^{j}, s_{u}') - \xi_{\varepsilon} \sum_{i=j+1}^{m-1} \Phi_{R_{i}}(\mathbf{s}^{i-1})$$

$$= (1+p^{-1})^{-1} C_{u}(\mathbf{s}_{-u}^{j}, s_{u}') - \xi_{\varepsilon} \sum_{i=j+1}^{m-1} \Phi_{R_{i}}(\mathbf{s}^{i-1}), \qquad (25)$$

the second inequality coming from observing that $j + 1 - m \ge m$, $i - m \le 0$ and the last one by using (20). Using (24) and (22), we can bound the second term in (25) by

$$\xi_{\varepsilon} \sum_{i=j+1}^{m-1} \Phi_{R_i}(\mathbf{s}^{i-1}) \le p^{-2} C_u(\mathbf{s}^j) \le p^{-1} C_u(\mathbf{s}_{-u}^j, s_u'),$$

and thus (25) finally becomes

$$C_u(\mathbf{s}_{-u}^{m-1}, s'_u) \ge \left[(1+p^{-1})^{-1} - p^{-1} \right] C_u(\mathbf{s}_{-u}^j, s'_u) \ge (1-2p^{-1}) C_u(\mathbf{s}_{-u}^j, s'_u),$$

where for the last inequality we used the fact that, for any real x > 0, $(1 + x)^{-1} \ge 1 - x$. This establishes (17), concluding the proof of the theorem.

We are now ready to prove the main theorem of this section:

Theorem 6. At the end of Algorithm 1, all players are in a $d^{d+o(d)}$ -equilibrium.

Proof. Consider an arbitrary player u and any strategy s'_u she can deviate to from state s^{m-1} . Suppose her strategy was fixed right after state s^j . Then, by the definition of our algorithm (see Algorithm 1), she has no *p*-move in state s^j . Applying this along with Theorem 5, we see that

$$\frac{C_u(\mathbf{s}^{m-1})}{C_u(\mathbf{s}_{-u}^{m-1}, s'_u)} \le \frac{1+3p^{-1}}{1-2p^{-1}} \frac{C_u(\mathbf{s}^j)}{C_u(\mathbf{s}_{-u}^j, s'_u)} \le \frac{1+3p^{-1}}{1-2p^{-1}} \cdot p = d^{d+o(d)},$$

where the last step holds due to (9) and observing that, since $p \ge 160$ (by (9) and $d \ge 1$), it must be that $\frac{1+3p^{-1}}{1-2p^{-1}} \le 163/158 \approx 1.032$.

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Appendix

A Proof of Lemma 2

Our proof of Lemma 2 follows along the lines of Aland et al. [2, Section 5.3]. Still, care needs to be taken in order to incorporate correctly the added approximation factor ρ and, to a smaller degree, parameter z that essentially corresponds to the deviations of subsets of players in our upper bound result in Lemma 1.

We will first need a technical lemma, which is a straightforward generalization of [2, Lemma 5.2]:

Lemma 11. Fix any $\mu \ge 0$, $\rho, d > 0$ and define the function $g(x) = \rho(x+1)^d - \mu \cdot x^{d+1}$. Then g has exactly one local maximum in $\mathbb{R}_{\ge 0}$. More precisely, there exists a unique $\xi \in \mathbb{R}_{\ge 0}$ such that g is strictly increasing in $[0,\xi)$ and strictly decreasing in (ξ,∞) . *Proof.* For the derivative of g we have that, for any x > 0:

$$g'(x) = 0 \quad \iff \quad \rho d(x+1)^{d-1} - \mu (d+1)x^d = 0 \quad \iff \quad \frac{(x+1)^{d-1}}{x^d} = \frac{\mu (d+1)}{\rho d}.$$
 (26)

. .

Now define the function $h(x) = \frac{(x+1)^{d-1}}{x^d}$ and observe that $h'(x) = -\frac{(x+1)^{d-2}\cdot(x+d)}{x^{d+1}} < 0$ for all x > 0. Additionally, $\lim_{x\to 0^+} h(x) = \infty$ and $\lim_{x\to\infty} h(x) = 0$. Thus, since $c := \frac{\mu(d+1)}{\rho d} \ge 0$, there exists a *unique* $\xi \in [0,\infty)$ such that $h(\xi) = c$; furthermore, h(x) > c for $x \in [0,\xi)$ and h(x) < c for $x > \xi$. Given the derivation in (26), this is enough to conclude the proof of the lemma.

We are now ready to prove Lemma 2. First, examine the critical condition

$$y \cdot f(z + x + y) \le \lambda y \cdot f(z + y) + \mu x \cdot f(z + x) \qquad \forall x, y, z \ge 0, \ \forall f \in \mathfrak{P}_d$$

of the set whose infimum defines B in the statement of our lemma. Note that for any $z \ge 0$, if we define $\bar{f}(t) := f(z+t)$ for all $t \in \mathbb{R}$, then \bar{f} is again a polynomial with nonnegative coefficients. Therefore, the condition above is equivalent to the simpler one:

$$y \cdot f(x+y) \le \lambda y \cdot f(y) + \mu x \cdot f(x) \qquad \forall x, y, z \ge 0, \ \forall f \in \mathfrak{P}_d.$$

From now on, performing a consecutive transformation of this condition in exactly the same way as in the proof of [2, Lemma 5.5], we can eventually derive that

$$B = \inf_{\lambda \in \mathbb{R}, \, \mu \in (0, \frac{1}{\rho})} \left\{ \frac{\lambda \rho}{1 - \mu \rho} \right| \, \forall x \ge 0 : \, \lambda \ge (x + 1)^d - \mu x^{d + 1} \right\}.$$

Now, taking into consideration that $\rho, 1 - \mu \rho > 0$ and $\mu \rho \in (0, 1)$, we can finally do the following transformations:

$$B = \inf_{\lambda \in \mathbb{R}, \, \mu \in (0, \frac{1}{\rho})} \left\{ \frac{\lambda \rho}{1 - \mu \rho} \middle| \, \forall x \ge 0 : \, \frac{\lambda \rho}{1 - \mu \rho} \ge \frac{\rho(x + 1)^d - \mu \rho x^{d+1}}{1 - \mu \rho} \right\}$$
$$= \inf_{\mu \in (0, 1)} \left\{ \max_{x \ge 0} \left\{ \frac{\rho(x + 1)^d - \mu \rho x^{d+1}}{1 - \mu \rho} \right\} \right\}$$
$$= \inf_{\mu \in (0, 1)} \left\{ \max_{x \ge 0} \left\{ \frac{\rho(x + 1)^d - \mu x^{d+1}}{1 - \mu} \right\} \right\}$$
(27)

Using expression (27) for B, we will next show that indeed $B = \Phi_{d,\rho}^{d+1}$. First, for the $B \ge \Phi_{d,\rho}^{d+1}$ part, it is enough to observe that for any $\mu \in (0,1)$:

$$\max_{x \ge 0} \left\{ \frac{\rho(x+1)^d - \mu x^{d+1}}{1-\mu} \right\} \ge \frac{\rho(\Phi_{d,\rho} + 1)^d - \mu \Phi_{d,\rho}^{d+1}}{1-\mu} = \Phi_{d,\rho}^{d+1}$$

For the $B \leq \Phi_{d,\rho}^{d+1}$ part now, first define the constant

$$\hat{\mu} := \frac{\rho d (\Phi_{d,\rho} + 1)^{d-1}}{(d+1) \Phi_{d,\rho}^d}$$

Using the fundamental property of the generalized golden ratio that $\Phi_{d,\rho}^{d+1} = \rho(\Phi_{d,\rho} + 1)^d$, we can now see that $\hat{\mu}$ satisfies the following properties:

$$\hat{\mu} = \frac{d\Phi_{d,\rho}^{d+1}}{(d+1)(\Phi_{d,\rho}+1)\Phi_{d,\rho}^d} = \frac{d\Phi_{d,\rho}}{(d+1)(\Phi_{d,\rho}+1)} \in (0,1)$$

and

$$\frac{\rho(\Phi_{d,\rho}+1)^d - \hat{\mu}\Phi_{d,\rho}^{d+1}}{1 - \hat{\mu}} = \Phi_{d,\rho}^{d+1}$$

Additionally, notice that by Lemma 11 we thus know that quantity $\rho(x+1)^d - \hat{\mu}x^{d+1}$ is maximized for x being the *unique* solution of equation (26), which in our case is equivalent to

$$\frac{(x+1)^{d-1}}{x^d} = \frac{\hat{\mu}(d+1)}{\rho d} = \frac{(\Phi_{d,\rho}+1)^{d-1}}{\Phi_{d,\rho}^d} \quad \Longleftrightarrow \quad x = \Phi_{d,\rho}.$$

So, we can finally upper-bound B via (27) by

$$B \le \max_{x\ge 0} \left\{ \frac{\rho(x+1)^d - \hat{\mu}x^{d+1}}{1 - \hat{\mu}} \right\} = \frac{\rho(\Phi_{d,\rho} + 1)^d - \hat{\mu}\Phi_{d,\rho}^{d+1}}{1 - \hat{\mu}} = \Phi_{d,\rho}^{d+1}.$$

B Remaining Proof of Theorem 3

In this section we show that the profiles $\bar{\mathbf{s}}^{i-1}$ and $\bar{\mathbf{s}}^i$ of the game $\bar{\Gamma}$ defined in the proof of Theorem 3, indeed have the desired properties. Namely, $\bar{\mathbf{s}}^{i-1}$ is an $(\alpha + p^{-1})$ -equilibrium for all players in R_i , and furthermore,

$$\bar{\varPhi}_{R_i}(\bar{\mathbf{s}}^i) \le \varPhi_{R_i}(\mathbf{s}^i) + nb_i.$$
⁽²⁸⁾

Our analysis is essentially the same as in the proof of Caragiannis et al. [9, Lemma 4.3]; we still present our derivation below, for completeness.

First, we need to show that no player in $R_i = X_i \cup Y_i$ has an $(\alpha + p^{-1})$ -move at state $\bar{\mathbf{s}}^{i-1}$ of game $\bar{\Gamma}$. Recall that all players use the same strategies under both $\bar{\mathbf{s}}^{i-1}$ and \mathbf{s}^{i-1} and, furthermore, the cost functions of the resources in these strategies are the same between the two games $\bar{\Gamma}$ and Γ . Also, observe that the cost of all players in R_i at \mathbf{s}^{i-1} is strictly less than b_{i-1} , otherwise they would have been "fixed" at the end of the (i-1)-th phase (see Line 11 of Algorithm 1), contradicting the definition of R_i containing players that perform improving moves during the *i*-th phase.

Consider first the players in X_i , whose cost by definition is at least b_i at \mathbf{s}^{i-1} . By the analysis above, their costs actually have to lie within $[b_i, b_{i-1})$ at the end of the (i-1)-th phase. Furthermore, their strategy sets (by the definition of $\overline{\Gamma}$) are exactly the same in $\overline{\Gamma}$ and Γ , and thus, any $(\alpha + p^{-1})$ -move from $\overline{\mathbf{s}}^{i-1}$ in $\overline{\Gamma}$ would give rise to an $(\alpha + p^{-1})$ -move from \mathbf{s}^{i-1} in Γ ; such a move cannot exist, however, or otherwise, by the definition of our algorithm, there would be still $(\alpha + p^{-1})$ -moves left for the players in X_i at the end of phase (i + 1) (see Line 9 of Algorithm 1).

Consider now the remaining players in Y_i , whose cost is less than b_i . A unilateral deviation from $\bar{\mathbf{s}}^{i-1}$ of any such player $u \in Y_i$, currently playing s_u , would cause her to use resources $s'_u \cup \{e_u\}$ for some $s'_u \neq s_u$. This will result in her experiencing a cost of at least $w_u c_{e_u}(w_u) = w_u \cdot \frac{b_i}{w_u} = b_i$, which is strictly worse than her current cost.

Now we move on to prove (28). Recall that all players in $N \setminus Y_i$ use the same strategies in $\bar{\mathbf{s}}^i$ and \mathbf{s}^i . Also, each resource $e_u \in \bar{E} \setminus E$ can be used only by player $u \in Y_i$, and thus $x_{N\setminus R_i,e_u}(\bar{\mathbf{s}}^i) = 0$ and $x_{R_i,e_u}(\bar{\mathbf{s}}^i) \leq w_u$, and furthermore, from the definition of our potential in (5), it is $\phi_{e_u}(x) = \frac{b_i}{w_u}x$. Putting all the above together and using the definition of the partial potential (8), we can derive that

$$\begin{split} \bar{\varPhi}_{R_i}(\bar{\mathbf{s}}^i) &= \sum_{e \in E'} \left[\phi_e(x_{N \setminus R_i, e}(\bar{\mathbf{s}}^i) + x_{R_i, e}(\bar{\mathbf{s}}^i)) - \phi_e(x_{N \setminus R_i, e}(\bar{\mathbf{s}}^i)) \right] \\ &= \sum_{e \in E} \left[\phi_e(x_{N \setminus R_i, e}(\bar{\mathbf{s}}^i) + x_{R_i, e}(\bar{\mathbf{s}}^i)) - \phi_e(x_{N \setminus R_i, e}(\bar{\mathbf{s}}^i)) \right] \\ &+ \sum_{u \in Y_i} \left[\phi_{e_u}(x_{N \setminus R_i, e_u}(\bar{\mathbf{s}}^i) + x_{R_i, e_u}(\bar{\mathbf{s}}^i)) - \phi_{e_u}(x_{N \setminus R_i, e_u}(\bar{\mathbf{s}}^i)) \right] \\ &= \varPhi_{R_i}(\mathbf{s}^i) + \sum_{u \in Y_i} [\phi_{e_u}(x_{R_i, e_u}(\bar{\mathbf{s}}^i)) - \phi_{e_u}(0)] \\ &\leq \varPhi_{R_i}(\mathbf{s}^i) + \sum_{u \in Y_i} \phi_{e_u}(w_u) \\ &\leq \varPhi_{R_i}(\mathbf{s}^i) + nb_i. \end{split}$$

C Proof of Lemma 10

We will first need a series of technical algebraic lemmas.

Lemma 12. For any $\psi \in (0,1]$ and all x > 0:

$$(1+x)^{\psi} - 1 \ge \psi x (1+x)^{\psi-1}$$

Proof. Set $\alpha \leftarrow \psi$ and $z \leftarrow x + 1$ in [9, Claim 2.2].

Lemma 13. Consider reals $\varepsilon, p > 0$ and $m \ge 1$, such that $(1 + \varepsilon)^m = 1 + p^{-1}$. Then, $\varepsilon^{-1} \le m(1 + p)$.

Proof. First, rearranging our assumption that $(1 + \varepsilon)^m = 1 + p^{-1}$, we get that

$$\varepsilon = \left(1 + \frac{1}{p}\right)^{1/m} - 1.$$

Thus, applying Lemma 12 with $x \leftarrow \frac{1}{p}$ and $\psi \leftarrow \frac{1}{m}$ we can derive that:

$$\varepsilon \ge \frac{1}{pm} \left(1 + \frac{1}{p}\right)^{1/m-1} = \frac{1}{m(1+p)} \left(1 + \frac{1}{p}\right)^{1/m} \ge \frac{1}{m(1+p)},$$

which concludes the proof.

The following lemma builds the algebraic foundation of the derivation of Lemma 10.

Lemma 14. Fix a positive integer d. For any polynomial $f \in \mathfrak{P}_d$ and any $\varepsilon > 0$, it holds that

$$yf(z+x+y) \le (1+\varepsilon)yf(z+x'+y) + \left(1+\frac{1}{\varepsilon}\right)^d d^d x f(z+x+y')$$

for any $x, x', y, y', z \ge 0$.

Proof. Since f is a nondecreasing function, it is enough to show that (setting $x' \leftarrow 0, y' \leftarrow 0$)

$$yf(z+x+y) \le (1+\varepsilon)yf(z+y) + \left(1+\frac{1}{\varepsilon}\right)^d d^d x f(z+x)$$

and by performing the transformation $f(t) \leftarrow f(z+t)$ (which preserves the property that $f \in \mathfrak{P}_d$), it is actually enough to show that

$$yf(x+y) \le (1+\varepsilon)yf(y) + \left(1+\frac{1}{\varepsilon}\right)^d d^d x f(x).$$
 (29)

Looking at the structure of the above inequality, and taking into consideration that f is a linear combination (with nonnegative coefficients) of monomials of degree at most d, we can deduce that it is enough to show that (29) holds for any $f(t) = t^{\nu}$ with $\nu = 0, 1, \ldots, d$, that is,

$$y(x+y)^{\nu} \le (1+\varepsilon)y^{\nu+1} + \left(1+\frac{1}{\varepsilon}\right)^d d^d x^{\nu+1}.$$

Dividing by $x^{\nu+1}$ (if x = 0, the inequality holds trivially) and making a change of variables $z \leftarrow \frac{y}{x}$, this can be rewritten as

$$z(z+1)^{\nu} - (1+\varepsilon)z^{\nu+1} \le \left(1 + \frac{1}{\varepsilon}\right)^d d^d.$$

It is not difficult to check that the above inequality indeed holds for the extreme cases of z = 0 or $\nu = 0$, so from now on let's assume that z > 0 and ν is a positive integer. Now, taking into consideration that $\nu \leq d$ and that $z \leq z + 1$, we can finally derive that it is enough to show that

$$g(z) := (z+1)^{\nu+1} - (1+\varepsilon)z^{\nu+1} \le \left(1 + \frac{1}{\varepsilon}\right)^{\nu}\nu^{\nu}$$
(30)

for all z > 0, where ν is a positive integer.

The derivative of g defined in (30) is $g'(z) = (\nu+1)[(z+1)^{\nu} - (1+\varepsilon)z^{\nu}]$, and so $\lim_{z\to 0} g'(z) = \nu + 1 > 0$, $\lim_{z\to\infty} g'(z) = -\infty$ (because $\varepsilon > 0$) and

$$g'(z) = 0 \iff \frac{(z+1)^{\nu}}{z^{\nu}} = 1 + \varepsilon \iff z = \bar{z} := \frac{1}{(1+\varepsilon)^{1/\nu} - 1}.$$
(31)

Thus, the maximum of g in $(0, \infty)$ is attained at \bar{z} . Also, by applying Lemma 12 with $\psi = 1/\nu$ and $x = \varepsilon$, we get that $(1 + \varepsilon)^{1/\nu} - 1 \ge \frac{1}{\nu} \varepsilon (1 + \varepsilon)^{1/\nu - 1}$, and so we can bound \bar{z} in (31) by

$$\bar{z} \le \nu \frac{1}{\varepsilon} (1+\varepsilon)^{1-1/\nu}.$$

Using the above, together with the properties of \bar{z} from (31) we can finally bound

$$g(z) \leq (\bar{z}+1)^{\nu+1} - (1+\varepsilon)\bar{z}^{\nu+1}$$

= $(\bar{z}+1) \cdot (1+\varepsilon)\bar{z}^{\nu} - (1+\varepsilon)\bar{z}^{\nu+1}$
= $(1+\varepsilon)\bar{z}^{\nu}$
 $\leq (1+\varepsilon)\nu^{\nu}\frac{1}{\varepsilon^{\nu}}(1+\varepsilon)^{\nu-1}$
= $\left(1+\frac{1}{\varepsilon}\right)^{\nu}\nu^{\nu},$

proving (30) and concluding the proof of our lemma.

We are finally ready to prove Lemma 10. Throughout this proof we will denote $Q = R_i \cup \{u\}$. Then we can write

$$C_u(\mathbf{s}^i) = \sum_{e \in E} x_{u,e}(\mathbf{s}^i) \cdot c_e(x_{N \setminus Q,e}(\mathbf{s}^i) + x_{R_i,e}(\mathbf{s}^i) + x_{u,e}(\mathbf{s}^i))$$

and

$$C_u(\mathbf{s}^{i-1}) = \sum_{e \in E} x_{u,e}(\mathbf{s}^i) \cdot c_e(x_{N \setminus Q,e}(\mathbf{s}^i) + x_{R_i,e}(\mathbf{s}^{i-1}) + x_{u,e}(\mathbf{s}^i)),$$

where for the second equality we used that, for any resource e, it is $x_{N\setminus Q,e}(\mathbf{s}^i) = x_{N\setminus Q,e}(\mathbf{s}^{i-1})$ and $x_{u,e}(\mathbf{s}^i) = x_{u,e}(\mathbf{s}^{i-1})$; this is a consequence of the fact that only players in R_i move during phase i, i.e., between states \mathbf{s}^{i-1} and \mathbf{s}^i . This is also exactly the reason why the partial potential of R_i can only decrease during phase i (due to Lemma 6 and the fact that players perform at least α -improvements). Thus $\Phi_{R_i}(\mathbf{s}^{i-1}) \geq \Phi_{R_i}(\mathbf{s}^i) = \Phi(\mathbf{s}^i) - \Phi^{N\setminus R_i}(\mathbf{s}^i)$ and so we can also write

$$\Phi_{R_{i}}(\mathbf{s}^{i-1}) \geq \sum_{e \in E} \left[\phi_{e}(x_{N \setminus Q, e}(\mathbf{s}^{i}) + x_{R_{i}, e}(\mathbf{s}^{i}) + x_{u, e}(\mathbf{s}^{i})) - \phi_{e}(x_{N \setminus Q, e}(\mathbf{s}^{i}) + x_{u, e}(\mathbf{s}^{i})) \right] \\
\geq \sum_{e \in E} x_{R_{i}, e}(\mathbf{s}^{i}) \cdot c_{e}(x_{N \setminus Q, e}(\mathbf{s}^{i}) + x_{R_{i}, e}(\mathbf{s}^{i}) + x_{u, e}(\mathbf{s}^{i})),$$

the last inequality being due to Lemma 4.

Observing the above expressions, we see that in order to prove (18) it is enough to show that, for any resource e, it is

$$\begin{aligned} x_{u,e}(\mathbf{s}^{i})c_{e}(x_{N\setminus Q,e}(\mathbf{s}^{i}) + x_{R_{i},e}(\mathbf{s}^{i}) + x_{u,e}(\mathbf{s}^{i})) \\ &\leq (1+\varepsilon)x_{u,e}(\mathbf{s}^{i-1})c_{e}(x_{N\setminus Q,e}(\mathbf{s}^{i-1}) + x_{R_{i},e}(\mathbf{s}^{i-1}) + x_{u,e}(\mathbf{s}^{i-1})) \\ &+ \xi_{\varepsilon}x_{R_{i},e}(\mathbf{s}^{i})c_{e}(x_{N\setminus Q,e}(\mathbf{s}^{i}) + x_{R_{i},e}(\mathbf{s}^{i}) + x_{u,e}(\mathbf{s}^{i})). \end{aligned}$$

Substituting, for simplicity, $y \leftarrow x_{u,e}(\mathbf{s}^i)$, $x \leftarrow x_{R_i,e}(\mathbf{s}^i)$, $x' \leftarrow x_{R_i,e}(\mathbf{s}^{i-1})$ and $z \leftarrow x_{N\setminus Q,e}(\mathbf{s}^i)$, this becomes

$$yc_e(z+x+y) \le (1+\varepsilon)yc_e(z+x'+y) + \xi_\varepsilon xc_e(z+x+y),$$

which indeed holds, due to Lemma 14.

We now move to proving (19), which can be equivalently rewritten as

$$C_u(\mathbf{s}_{-u}^{i-1}, s'_u) \le (1+\varepsilon)C_u(\mathbf{s}_{-u}^i, s'_u) + \xi_\varepsilon \Phi_{R_i}(\mathbf{s}^{i-1}).$$

$$(32)$$

We have that

$$C_{u}(\mathbf{s}_{-u}^{i-1}, s_{u}') = \sum_{e \in E} x_{u,e}(\mathbf{s}_{-u}^{i-1}, s_{u}') \cdot c_{e}(x_{N \setminus Q,e}(\mathbf{s}_{-u}^{i-1}, s_{u}') + x_{R_{i},e}(\mathbf{s}_{-u}^{i-1}, s_{u}') + x_{u,e}(\mathbf{s}_{-u}^{i-1}, s_{u}'))$$
$$= \sum_{e \in E} x_{u,e}(\mathbf{s}_{-u}^{i}, s_{u}') \cdot c_{e}(x_{N \setminus Q,e}(\mathbf{s}_{-u}^{i-1}) + x_{R_{i},e}(\mathbf{s}_{-u}^{i-1}) + x_{u,e}(\mathbf{s}_{-u}^{i}, s_{u}'))$$

and

$$C_{u}(\mathbf{s}_{-u}^{i}, s_{u}^{i}) = \sum_{e \in E} x_{u,e}(\mathbf{s}_{-u}^{i}, s_{u}^{i}) \cdot c_{e}(x_{N \setminus Q,e}(\mathbf{s}_{-u}^{i}, s_{u}^{i}) + x_{R_{i},e}(\mathbf{s}_{-u}^{i}, s_{u}^{i}) + x_{u,e}(\mathbf{s}_{-u}^{i}, s_{u}^{i}))$$
$$= \sum_{e \in E} x_{u,e}(\mathbf{s}_{-u}^{i}, s_{u}^{i}) \cdot c_{e}(x_{N \setminus Q,e}(\mathbf{s}_{-u}^{i-1}) + x_{R_{i},e}(\mathbf{s}_{-u}^{i}, s_{u}^{i}) + x_{u,e}(\mathbf{s}_{-u}^{i}, s_{u}^{i})).$$

For the simplifications above we used the equalities

$$x_{u,e}(\mathbf{s}_{-u}^{i-1}, s_u') = x_{u,e}(\mathbf{s}_{-u}^i, s_u')$$

$$\begin{aligned} x_{R_{i,e}}(\mathbf{s}_{-u}^{i-1}, s_{u}') &= x_{R_{i,e}}(\mathbf{s}^{i-1}) \\ x_{N\setminus Q,e}(\mathbf{s}_{-u}^{i-1}, s_{u}') &= x_{N\setminus Q,e}(\mathbf{s}_{-u}^{i}, s_{u}') = x_{N\setminus Q,e}(\mathbf{s}^{i-1}), \end{aligned}$$

which are consequences of the fact that all players except those in R_i have the same strategies in $(\mathbf{s}_{-u}^i, s'_u)$ and $(\mathbf{s}_{-u}^{i-1}, s'_u)$, and all players but u have the same strategies in state \mathbf{s}^{i-1} and $(\mathbf{s}_{-u}^{i-1}, s'_u)$.

Thus, in an analogous way in which we proved (18) before, we can see that in order to prove (32) it is enough to show that for any resource e it is

$$yc_e(z+x+y) \le (1+\varepsilon)yc_e(z+x'+y) + \xi_{\varepsilon}xc_e(z+x+y'),$$

where we have used the following substitutions: $x \leftarrow x_{R_i,e}(\mathbf{s}^{i-1}), x' \leftarrow x_{R_i,e}(\mathbf{s}_{-u}^i,s'_u), y \leftarrow x_{u,e}(\mathbf{s}_{-u}^{i-1},s'_u), y' \leftarrow x_{u,e}(\mathbf{s}^{i-1})$ and $z \leftarrow x_{N\setminus Q,e}(\mathbf{s}^i)$. The above is indeed true, again due to Lemma 14.