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Coordination Games on Weighted Directed Graphs

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Abstract. We study strategic games on weighted directed graphs, where each player's payoff is defined as the sum of the weights on the edges from players who chose the same strategy, augmented by a fixed nonnegative integer bonus for picking a given strategy. These games capture the idea of coordination in the absence of globally common strategies. We identify natural classes of graphs for which finite improvement or coalition-improvement paths of polynomial length always exist, and consequently a (pure) Nash equilibrium or a strong equilibrium can be found in polynomial time. The considered classes of graphs are typical in network topologies: simple cycles correspond to the token ring local area networks, whereas open chains of simple cycles correspond to multiple independent rings topology from the recommendation G.8032v2 on Ethernet ring protection switching. For simple cycles, these results are optimal in the sense that without the imposed conditions on the weights and bonuses, a Nash equilibrium may not even exist. Finally, we prove that determining the existence of a Nash equilibrium or of a strong equilibrium is NP-complete already for unweighted graphs, with no bonuses assumed. This implies that the same problems for polymatrix games are strongly NP-hard.

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Keywords: noncooperative games • coordination games • Nash equilibrium • strong equilibrium • computational complexity

1. Introduction

1.1. Background

This paper is concerned with pure Nash equilibria in a natural subclass of strategic form games. Recall that a pure Nash equilibrium of a strategic game is a joint strategy in which each player plays a best response. It is a natural solution concept which has been widely used to reason about strategic interaction between rational agents. Although Nash's theorem guarantees existence of a mixed strategy Nash equilibrium for all finite games, pure Nash equilibria need not always exist. In various games, for instance Cournot competition games or congestion games, pure Nash equilibria (from now, just Nash equilibria) do exist and correspond to natural outcomes.

In many scenarios of strategic interaction, apart from the question of the existence of Nash equilibria, an important concern is whether an equilibrium can be efficiently computed. In this context the concept of an *improvement path* is relevant. These are maximal paths constructed by starting at an arbitrary joint strategy and allowing a single player who does not hold a best response to switch to a better strategy at each stage. By definition, every finite improvement path terminates in a Nash equilibrium.

In a seminal paper, Monderer and Shapley [40] identified the class of finite games in which every improvement path is guaranteed to be finite, and coined this property as the *finite improvement property* (*FIP*). These are games with which one can associate a *generalised ordinal potential*, a function on the set of joint strategies that properly tracks the qualitative change in players' payoffs resulting from a strategy change. Thus, the FIP not only guarantees the existence of Nash equilibria but also ensures that it is possible to reach it from any initial joint strategy by a simple update dynamics amounting to a *local search*. This makes the FIP a desirable property. An important class of games that have the FIP are the *congestion games* that, as already noted in Rosenthal [45], actually have an *exact potential*, a function that exactly tracks the quantitative difference in players' payoffs.

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However, the requirement that *every* improvement path is finite is very strong and only a few classes of games have this property. Young [55] proposed a weakening of the FIP that stipulates that from any initial joint strategy only *some* improvement path is finite. Games for which this property holds are called *weakly acyclic games*. So in weakly acyclic games, Nash equilibria can be reached through an appropriately chosen sequence of unilateral deviations of players, irrespective of the starting joint strategy.

Although the existence of a finite improvement path guarantees the existence of a Nash equilibrium, it does not necessarily result in an efficient algorithm to compute it. In fact, in various games, improvement paths can be exponentially long. Fabrikant et al. [23] showed that computing a Nash equilibrium in congestion games is complete for the complexity class Polynomial Local Search (PLS-complete). Even in the class of symmetric network congestion games, for which it is known that a Nash equilibrium can be efficiently computed (Fabrikant et al. [23]), there are games in which some best response improvement paths are exponentially long (Ackermann et al. [1]). Thus, identifying natural classes of games in which starting from any joint strategy a Nash equilibrium can be reached by an efficiently generated improvement path of polynomial length is of obvious interest and is the focus of this paper.

1.2. Motivation

In game theory, coordination games are often used to model situations of cooperation, where players can increase their payoffs by coordinating on certain strategies. For two player games, this implies that coordinating strategies constitute Nash equilibria. The main characteristic of coordination is that players find it advantageous that other players follow their choice. In this paper, we study a simple class of multiplayer coordination games, in which each player can choose to coordinate his actions within a certain neighbourhood. The neighbourhood structure is specified by a weighted directed graph, the nodes of which are identified with the players.

Henceforth, we will refer to any strategy as a *colour*. The sets of colours available to players are usually not mutually disjoint, as otherwise players would not be able to coordinate on the same action. Given a joint strategy, the payoff for a player is defined as the sum of the weights of the incoming edges from other players who choose the same colour plus a fixed bonus for picking this particular colour. We refer to this subclass of strategic games as coordination games on weighted directed graphs, in short, just *coordination games*. Coordination games capture the following key characteristics:

- *Join-the-crowd property*: the payoff of each player weakly increases when more players choose his strategy (this is because the weights are assumed to be positive).
- *Local dependency*: the payoff of each player depends only on the choices made by a certain group of players (namely the neighbours in the given weighted directed graph).
- *Heterogeneous strategy sets*: players may have different strategy sets.
- *Individual preferences*: the (positive) bonuses express players' private preferences.

Coordination games constitute a formal model to analyse strategic interaction in situations where agents' benefit from aligning their choices with other agents in their neighbourhood. Such circumstances arise in various natural situations, for instance, when clients have to choose between multiple competing (for instance, mobile phone) providers offering similar services. It is often beneficial to choose the same service provider as the one chosen by friends or relatives. Thus, the join-the-crowd property and local dependency naturally hold. It is also natural to envisage that a provider imposes some bounds on incentives that are provided. For instance, a mobile phone operator might impose a cap on the number of free calls and/or on the number of people with whom calls are free using its network. Thus, weighted edges in the neighbourhood structure that capture the quantitative "influence," in general, need not be symmetric. Weighted directed edges are therefore appropriate to model this general situation.

In this paper, we focus on the existence and efficient computation of Nash equilibria in coordination games on specific directed graphs. Given that players can try to coordinate their choice within a group, it is also natural to consider a notion of equilibrium that takes into account deviations by subsets of players. We therefore also study the existence of strong equilibria, which are joint strategies from which no subset of players can profitably deviate. We consider whether strong equilibria can be efficiently computed by means of short improvement paths in which at each stage all players in a group can profitably deviate. We call such paths *coalitional improvement paths*, in short, *c-improvement paths*.

The coordination games studied here generalise the model introduced in Apt et al. [7] and further studied in Apt et al. [8]. In these works the neighbourhood structure is represented by an unweighted and *undirected* graph. A switch to *directed* graphs turns out to be a major shift and leads to fundamentally different results. For example, in the case of undirected graphs, Nash equilibria always exist (in fact, these are exact potential games), whereas even for simple directed graphs Nash equilibria do not exist. As a result both the structural results as well as the techniques used here significantly differ from the ones in Apt et al. [8].

A natural application of coordination games is in the analysis of strategic behaviour in social networks. The threshold model (Apt and Markakis [2], Granovetter [27]), in which members of the network are viewed as nodes in a weighted graph, is one of the prevalent models used to reason about social networks. Each node is associated with a threshold and a node adopts an "item" (which can be a disease, trend, or a specific product) when the total weight of incoming edges (or influence) from the nodes that have already adopted this item exceeds its threshold. The existence of directed edges is natural in such a scenario, because the "strength of influence" captured by a quantitative value need not always be symmetric between members in a social network. When we omit bonuses, our coordination games become special cases of the *social network games* introduced and analysed in Simon and Apt [48], provided one allows thresholds to be equal to zero.

1.3. Related Work

The class of games that have the FIP, introduced in Monderer and Shapley [40], was a subject of extensive research. Prominent examples of such games are congestion games. Weakly acyclic games have received less attention, but the interest in them is growing. Milchtaich [38] showed that although congestion games with player specific payoff functions do not have the FIP, they are weakly acyclic. Brokkelkamp and Vries [15] improved upon this result by showing that a specific scheduling of players is sufficient to construct a finite improvement path beginning at an arbitrary starting point. According to this scheduling the players are free to choose their best response when updating their strategies.

Weak acyclicity of a game also ensures that certain modifications of the traditional no-regret algorithm yield an almost sure convergence to a Nash equilibrium (Marden et al. [36]). Engelberg and Schapira [18, 19] show that specific internet routing games are weakly acyclic. In turn, Kawald and Lenzner [32] established that certain classes of network creation games are weakly acyclic and moreover that a specific scheduling of players can ensure that the resulting improvement path converges to a Nash equilibrium in $\mathcal{O}(n \log n)$ steps. Furthermore, Meir et al. [37] propose the use of weakly acyclic games as a tool to analyse some iterative voting procedures.

Some structural results also exist. Fabrikant et al. [22] proved that the existence of a unique Nash equilibrium in every subgame implies that the game is weakly acyclic. A comprehensive classification of weakly acyclic games in terms of schedulers is provided in Apt and Simon [4] and more extensively in Apt and Simon [5], where it was also shown that games solvable by means of iterated elimination of never best responses to pure strategies are weakly acyclic. Finally, Milchtaich [39] provided a characterisation of weakly acyclic games in terms of a weak potential and showed that every finite extensive form game with perfect information is weakly acyclic.

As already mentioned, coordination games on unweighted and undirected graphs were introduced and studied in Apt et al. [8]. It was shown there that the improvement paths are guaranteed to converge in polynomial number of steps. Given this result, the study focused on the analysis of strong equilibria and its variants. The authors also provided bounds on the inefficiency of strong equilibria and identified restrictions on the neighbourhood structure that ensure efficient computation of strong equilibria. These coordination games were augmented in Rahn and Schäfer [44] by bonuses (which the authors call *individual preferences*). The authors studied the existence of α-approximate k-equilibria and their inefficiency with respect to (w.r.t.) social optima. These equilibria are outcomes in which no group of at most k players can deviate in such a way that each member increases his payoff by at least a factor α.

The games we study here are related to various well-studied classes of strategic form games. In particular, coordination games on graphs form a natural subclass of *polymatrix games* (Yanovskaya [54]). These are multiplayer games where the players' utilities are pairwise separable. Polymatrix games are well-studied and they include classes of strategic form games with good computational properties like the two-player zero-sum games. Simon and Wojtczak [50] studied the computational complexity of checking for the existence of constrained pure Nash equilibria in a subclass of polymatrix games defined on weighted directed graphs. Hoefer [30] studied clustering games that are also polymatrix games based on undirected graphs. In this setup, each player has the same set of strategies and as a result these games have, in contrast to ours, the FIP. A special class of polymatrix games was considered in Cai and Daskalakis [16], which coincide with the coordination games on undirected weighted graphs without bonuses. The authors showed that these games have an exact potential and that finding a pure Nash equilibrium is PLS-complete. However, the proof of the latter result crucially exploits the fact that the edge weights can be negative (which captures anticoordination behaviour). In Apt and Shoja [3] it was shown how coordination and anticoordination on simple cycles can be used to model and reason about the concept of self-stabilisation introduced in Dijkstra [17], one of the main approaches to fault-tolerant computing.

When the graph is undirected and complete, coordination games on graphs are special cases of the monotone increasing congestion games that were studied in Rozenfeld and Tennenholtz [46].

Another generalisation concerns distributed coalition formation (Hajdukova [28]), where players have preferences over members of the same coalition. Such a generalisation of polymatrix game over subsets of players, called hypergraphical games, was introduced in Papadimitriou and Roughgarden [42]. Analysis of coalition formation games in the presence of constraints on the number of coalitions that can be formed was investigated in Sless et al. [52]. Simon and Wojtczak [51] studied a subclass of hypergraphical games where the underlying group interactions are restricted to coordination and anticoordination. In this model, players' utilities depend not just on the groups that are formed by the strategic interaction, but also on the choice of action that the members of the group decide to coordinate on. It is shown that such games have a Nash equilibrium, which can be computed in pseudopolynomial time. Moreover, in the pure coordination setting, when the game possesses a certain acyclic structure, strong equilibria exist and can be computed in polynomial time.

Coordination games on graphs are also related to *additively separable hedonic games* (Banerjee et al [13], Bogomolnaia and Jackson [14]), which were originally proposed in a cooperative game theory setting. In these games players are the nodes of a weighted graph and can form coalitions. The payoff of a node is defined as the total weight of all edges to neighbors that are in the same coalition. The work on these games mostly focused on computational issues (see, e.g., Aziz and Brandl [10], Aziz et al. [11, 12], Gairing and Savani [24]).

In Apt et al. [8], we also mentioned related work on strategic games that involve colouring of the vertices of an undirected graph, in relation to the vertex colouring problem. In these games, the players are nodes in a graph that choose colours. However, the payoff function differs from the one we consider here: it is zero if a neighbour chooses the same colour and the number of nodes that chose the same colour otherwise. The reason is that these games are motivated by the question of finding the chromatic number of a graph. Representative references are Panagopoulou and Spirakis [41], where it is shown that an efficient local search algorithm can be used to compute a good vertex colouring, and Escoffier et al. [20], where this work is extended by analysing socially optimal outcomes and strong equilibria. Furthermore, strong and k-equilibria in strategic games on graphs were also studied in Gourvès and Monnot [25, 26]. These games are related to, respectively, the Max-Cut and Max-k-Cut problems. These classes of games do not satisfy the join-the-crowd property, so these results are not comparable with ours.

1.4. Our Contributions

In this paper, we identify various natural classes of weighted directed graphs for which the resulting games, possibly with bonuses, are weakly acyclic. Moreover, we prove that in these games, starting from any arbitrary joint strategy, improvement paths of polynomial length can be effectively constructed. So not only do these games have Nash equilibria, but they can also be efficiently computed by a simple form of local search. Because coordination games on graphs are polymatrix games, our results identify natural classes of polymatrix games in which Nash equilibria are guaranteed to exist and can be computed efficiently.

We first analyse coordination games on simple cycles. Even in this limited setting, improvement paths of infinite length may exist. However, we show that finite improvement paths always exist when at most two nodes have bonuses or at most two edges have weights. We also show that without these restrictions, Nash equilibria may not exist, so these results are optimal. We then extend this setting to *open chains* of simple cycles, that is, simple cycles that form a chain and show the existence of finite improvement paths.

Most of our constructions involve a common, though increasingly more complex, proof technique. In each case we identify a scheduling of players that is easy to compute and such that, when combined with an appropriate scheme to update strategies, guarantees that starting from an arbitrary initial joint strategy, in the resulting improvement path, a Nash equilibrium is reached in a polynomial number of steps.

We also study strong equilibria. In the restricted case of weighted directed acyclic graphs (DAGs), we show that strong equilibria can be found along every coalitional improvement path. We also show that when only two colours are used, the coordination games do not necessarily have the FIP, but both Nash and strong equilibria can always be reached starting from an arbitrary initial joint strategy by, respectively, an improvement or a c-improvement path.

To deal with simple cycles, we show that any finite improvement path can be extended by just one profitable coalitional deviation to reach a strong equilibrium. This allows us to strengthen the results on the existence of Nash equilibria to the case of strong equilibria. We also prove the existence of strong equilibria when the graphs are open chains of cycles. Finally, we show that in some coordination games, strong equilibria exist but cannot be reached from some initial joint strategies by any c-improvement path.

Building upon these results, we study the complexity of finding and determining the existence of Nash equilibria and strong equilibria. In particular, we show that strong equilibrium in a coordination game on a simple cycle can be computed in linear time. However, determining the existence of a Nash equilibrium even for games on unweighted graphs and without bonuses turns out to be NP-complete.

Table 1 summarises our main results concerning the complexity of finding Nash and strong equilibria. For the complexity results, we assume that all edge weights are natural numbers. We list, respectively, the length of the

Table 1. Bounds on the length of the shortest improvement and c-improvement (c-impr.) paths for a given class of graphs or colouring and on the complexity of finding Nash and strong equilibria. All edges are unweighted, and there are no bonuses unless stated otherwise.

Graph/bonus/colouring	Improvement path	NE	c-impr. path	SE
Weighted simple cycles with ≤ 1 node with bonuses	$2n-1$ (Theorem 1)	$\mathcal{O}(nl)$ (Theorem 11)	$2n$ (Corollary 1i)	$\mathcal{O}(nl)$ (Theorem 11)
Simple cycles with bonuses with ≤ 1 nontrivial weight	$3n-1$ (Theorem 2)	$\mathcal{O}(nl)$ (Theorem 11)	$3n$ (Corollary 1ii)	$\mathcal{O}(nl)$ (Theorem 11)
Weighted simple cycles with more than two nodes with bonuses		Nash equilibrium may not exist (Example 4)		
Weighted simple cycles with two nodes with bonuses	$3n$ (Theorem 3)	$\mathcal{O}(nl)$ (Theorem 11)	$3n$ (Corollary 1iii)	$\mathcal{O}(nl)$ (Theorem 11)
Simple cycles with bonuses and two nontrivial weights	$4n-1$ (Theorem 4)	$\mathcal{O}(nl)$ (Theorem 11)	$4n$ (Corollary 1iiii)	$\mathcal{O}(nl)$ (Theorem 11)
Open chains of cycles	$3vm^3$ (Theorem 5)	$\mathcal{O}(vm^3l)$ (Theorem 12)	$4vm^4$ (Theorem 8)	$\mathcal{O}(v^2m^5l)$ (Theorem 13)
Weighted DAGs with bonuses	$n-1$ (Theorem 6)	$\mathcal{O}(nl+\|E\|)$ (Theorem 14)	$n-1$ (Theorem 6)	$\mathcal{O}(nl+\|E\|)$ (Theorem 14)
Two colours	$2n$ (Theorem 9)	$\mathcal{O}(n+\|E\|)$ (Theorem 15)	$2n$ (Theorem 10)	$\mathcal{O}(n^2+n\|E\|)$ (Theorem 15)

shortest improvement path from an arbitrary initial joint strategy, the complexity of finding a Nash equilibrium (abbreviated to NE), the length of the shortest c-improvement path starting from an arbitrary initial joint strategy, and the complexity of finding a strong equilibrium (abbreviated to SE). Here, n is the number of nodes, $|E|$ the number of edges, and l the number of colours. In the case of open chain of cycles, m denotes the number of simple cycles in the chain and v the number of nodes in a simple cycle.

Most, though not all, results of this paper were reported earlier in shortened versions, as two conference papers (Apt et al. [6], Simon and Wojtczak [49]). Some of these results, notably on bounds on the length of (c-)improvement paths, were improved.

1.5. Potential Applications

Coordination games constitute a natural and well-studied model that represents various practical situations. The class of games we study in this paper models an extension of the coordination concept to a network setting, where the network is represented as a weighted directed graph, and where common strategies are not guaranteed to exist, and the payoffs functions take care of individual preferences.

The classes of graphs that we consider are frequently used as network topologies. For example, the token ring local area networks are organised in directed simple cycles, whereas the open chains of simple cycles are supported by the recommendation G.8032v2 on Ethernet ring protection switching.[1]

The basic technique that we use to show finite convergence to Nash equilibria is based on finite improvement paths of polynomial length. The concept of an improvement path is fundamental in the study of games but it also can be used to explain and analyse various real world applications. One such example is the border gateway protocol (BGP), the purpose of which is to assign routes to the nodes of the internet and to use them for routing packets. Over the years, there has been extensive research in the network communications literature on how stable routing states are achieved and maintained in BGP in spite of strategic concerns. Fabrikant and Papadimitriou [21] and, independently, Levin et al. [35] observed that the operation of the BGP can be viewed as a best response dynamics in a natural class of routing games and finite improvement paths that terminate in Nash equilibria essentially translate to stable routing states. Following this observation, Engelberg and Schapira [19] presented a game-theoretic analysis of routing on the internet in presence of "misbehaving players" or backup edges.

Finally, coordination games on graphs are also relevant to cluster analysis. Its main objective is to organise a set of naturally related objects into groups according to some similarity measure. When adopting the game-theoretic perspective, one can view possible cluster names as strategies and a satisfactory clustering of the considered graph as an equilibrium in the coordination game associated with the considered graph. Clustering from a game-theoretic perspective (using evolutionary games) was among others applied to car and pedestrian detection in images and face recognition (see Pelillo and Buló [43]). This approach was shown to perform very well against the state of the art.

1.6. Structure of This Paper

In the next section, we recall the relevant game-theoretic concepts and the notions of (c-)improvement paths, Nash and strong equilibria on which we focus. In Section 3, we introduce the class of games that forms the

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subject of this paper. The technical presentation starts in Section 4, in which we analyse the games the underlying graphs of which are (possibly weighted) simple cycles. In Section 5, we study open chains of simple cycles.

Then, in Section 6 we consider the problem of the existence of strong equilibria. Next, in Section 7, we study the complexity of finding and of determining the existence of Nash equilibria and strong equilibria. We conclude by summarising in Section 8 the results and stating a natural open problem.

2. Preliminaries

Throughout this paper, $n > 1$ denotes the number of players. A *strategic game* $\mathcal{G} = (S_1, \dots, S_n,\ p_1, \dots, p_n)$ for n players consists of a nonempty set S_i of *strategies* and a *payoff function* $p_i : S_1 \times \cdots \times S_n \rightarrow \mathbb{R}$, for each player i. We denote $S_1 \times \cdots \times S_n$ by S, call each element $s \in S$ a *joint strategy*, and abbreviate the sequence $(s_j)_{j \neq i}$ to s_{-i}. Occasionally we write (s_i, s_{-i}) instead of s. We call a strategy s_i of player i a *best response* to a joint strategy s_{-i} of his opponents if for all $s'_i \in S_i$, $p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i})$. A joint strategy s is called a *Nash equilibrium* if each s_i is a best response to s_{-i}.

Fix a strategic game $\mathcal{G}$. We say that $\mathcal{G}$ satisfies the *positive population monotonicity* (*PPM*; see Konishi et al. [33]) if for all joint strategies s and players i and j, $p_i(s) \leq p_i(s_i, s_{-j})$. (Note that (s_i, s_{-j}) refers to the joint strategy in which player j chooses s_i.) So if player j switches to player i's strategy and the remaining players do not change their strategies, then i's payoff weakly increases.

Next, by a *polymatrix game* (see Yanovskaya [54]) we mean a game $(S_1, \dots, S_n, p_1, \dots, p_n)$ in which for all pairs of players i and j there exists a *partial* payoff function a^{ij} such that for any joint strategy $s = (s_1, \dots, s_n)$, the payoff of player i is given by $p_i(s) := \sum_{j \neq i} a^{ij}(s_i, s_j)$. So polymatrix games are strategic games in which the influence of a strategy selected by a player on the payoff of another player is always the same, regardless of what strategies other players select.

We call a nonempty subset $K := \{k_1, \dots, k_m\}$ of the set of players $N := \{1, \dots, n\}$ a *coalition*. Given a joint strategy s, we abbreviate the sequence $(s_{k_1}, \dots, s_{k_m})$ of strategies to s_K and $S_{k_1} \times \cdots \times S_{k_m}$ to S_K. We occasionally write (s_K, s_{-K}) instead of s.

Given two joint strategies s' and s and a coalition K, we say that s' is a *deviation of the players in* K from s if $K = \{i \in N \mid s_i \neq s'_i\}$. We denote this by $s \xrightarrow{K} s'$ and drop K if it is a singleton. If in, addition $p_i(s') > p_i(s)$ holds for all $i \in K$, we say that the deviation s' from s is *profitable* and say that $s \xrightarrow{K} s'$ is a *c-improvement step*. Furthermore, we say that a coalition K *can profitably deviate from* s if there exists a profitable deviation of the players in K from s. Next, we call a joint strategy s a *k-equilibrium*, where $k \in \{1, \dots, n\}$, if no coalition of at most k players can profitably deviate from s. Using this definition, a *Nash equilibrium* is a 1-equilibrium and a *strong equilibrium* (see Aumann [9]) is an n-equilibrium.

A *coalitional improvement path*, in short, a *c-improvement path*, is a possibly infinite sequence $\rho = (s^1, s^2, \dots)$ of joint strategies such that for every $k \geq 1$ there is a coalition K such that $s^k \xrightarrow{K} s^{k+1}$ is a profitable deviation of the players in K, with the property that if it is finite, then it cannot be extended. So if ρ is finite, then there is no profitable deviation from the last element of the sequence, which we denote by *last*(ρ). Clearly, if a c-improvement path is finite, its last element is a strong equilibrium.

We say that $\mathcal{G}$ has the *finite c-improvement property* (*c-FIP*) if every c-improvement path is finite. Furthermore, we say that the function $P : S \rightarrow A$, where A is a set, is a *generalised ordinal c-potential*, also called *generalised strong potential*, for $\mathcal{G}$ (see Harks et al. [29], Holzman and Law-Yone [31]) if for some strict partial ordering $(P(S), \succ)$ the fact that s' is a profitable deviation of the players in some coalition from s implies that $P(s') \succ P(s)$. If a finite game admits a generalised ordinal c-potential, then it has the c-FIP. The converse also holds (see, e.g., Apt et al. [8]).

We say that $\mathcal{G}$ is *c-weakly acyclic* if for every joint strategy there exists a finite c-improvement path that starts at it. Thus, games that are c-weakly acyclic have a strong equilibrium. We call a c-improvement path an *improvement path* if each deviating coalition consists of one player. The notion of a game having the *FIP* or being *weakly acyclic* is then defined by referring to the improvement paths instead of c-improvement paths.

In this paper, we are interested in determining the existence of "short" improvement and c-improvement paths starting from *any* initial joint strategy. This motivates the following concept that we shall extensively use. We say that a game *ensures improvement paths of length* X (where X can also be expressed using the $\mathcal{O}(\cdot)$ function) if for each joint strategy there exists an improvement path that starts at it and is of length (at most) X. We use an analogous notion for the c-improvement paths.

To find such short (c-)improvement paths starting from an arbitrary initial joint strategy, we need to select the players in the right order. This motivates the following notion. By a *schedule* we mean a finite or infinite sequence, each element of which is a player. Let ϵ denote the empty sequence and $seq : i$ the finite sequence seq extended by i. Given an initial joint strategy s, a schedule generates an (not necessarily unique) initial fragment of an improvement path, defined inductively as follows:

$$path(s,\epsilon) := s,$$
$$path(s,seq:i) := \begin{cases} path(s,seq) & \text{if } i \text{ holds a best response in the last element of } path(s,seq), \\ path(s,seq) \rightarrow s' & \text{otherwise,} \end{cases}$$

where s' is the result of updating the strategy of player i in the last element of $path(s,seq)$ to a best response.

Sometimes we additionally specify how players update their strategies to best responses, but even then the generated improvement paths do not need to be unique. The process of selecting a strategy is always linear in the number of strategies. To show that a game ensures short improvement paths, we provide in each case an appropriate schedule. Note that an infinite schedule can generate a finite improvement path, which is the case when the last element of $path(s,seq)$ is a Nash equilibrium.

In the proofs, we always mention the bounds on the improvement paths, but actually these are bounds on the relevant prefixes of the defined schedules, which are always longer or of the same length.

3. Coordination Games on Directed Graphs

We now define the class of games we are interested in. Fix a finite set M of l colours. A *weighted directed graph* (G, w) is a pair, where $G=(V,E)$ is a directed graph without self loops and parallel edges over the set of vertices $V=\{1,\dots,n\}$, and w is a function that associates with each edge $e\in E$ a positive weight w_e. We say that a weight is *nontrivial* if it is different from one.

Furthermore, we say that a node j is an *in-neighbour* (from now on a *neighbour*) of the node i if there is an edge $j\rightarrow i$ in E. We denote by N_i the set of all neighbours of node i in the graph G. A *colour assignment* is a function $C: V\rightarrow \mathcal{P}(M)$ that assigns to each node of G a nonempty set of colours.

We also introduce the concept of a *bonus*, which is a function β that assigns to each node i and colour $c\in M$ a nonnegative integer $\beta(i,c)$. When stating our results, bonuses are assumed to be not present (or, equivalently, are assumed to be all equal to zero) unless explicitly stated otherwise. We say that a bonus is *nontrivial* if it is different from the constant function zero.

Given a weighted graph (G, w), a colour assignment C and a bonus function β, a strategic game $\mathcal{G}(G,w,C,\beta)$ is defined as follows:

- the players are the nodes,
- the set of strategies of player (node) i is the set of colours $C(i)$ (we occasionally refer to the strategies as *colours*),
- the payoff function for player i is $p_i(s)=\sum_{j\in N_i, s_i=s_j} w_{j\rightarrow i}+\beta(i,s_i)$.

So each node simultaneously chooses a colour, and the payoff to the node is the sum of the weights of the edges from its neighbours that chose its colour augmented by the bonus to the node for choosing its colour. We call these games *coordination games on weighted directed graphs* (hereafter, just *coordination games*).

Note that because the weights are nonnegative, each coordination game satisfies the PPM. When the weights of all the edges are one, we are dealing with a coordination game whose underlying graph is unweighted. In this case, we simply drop the function w from the description of the game and drop the qualification "unweighted" when referring to the graph.

Similarly, when all the bonuses are zero, we obtain a coordination game without bonuses. Likewise, in the description of such a game, we omit the function β. In a coordination game without bonuses when the underlying graph is unweighted, each payoff function is simply defined by $p_i(s) := |\{j\in N_i \mid s_i=s_j\}|$. Here is an example of such a game.

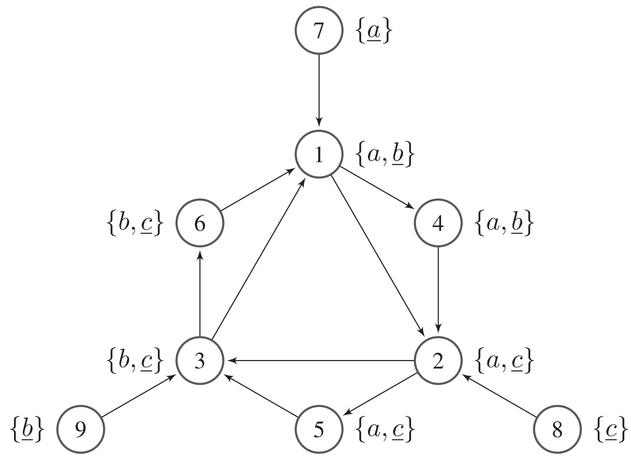
Example 1. Consider the directed graph and the colour assignment depicted in Figure 1. Take in the corresponding coordination game the joint strategy that consists of the underlined colours. Then the payoffs are as follows:

- zero for nodes 1, 7, 8, and 9,
- one for nodes 2, 4, 5, and 6,
- two for node 3.

Note that this joint strategy is not a Nash equilibrium. In fact, this game has no Nash equilibrium. To see this, observe that we only need to consider the strategies selected by nodes 1, 2, and 3, because each of the nodes 4, 5, and 6 always plays a best response by selecting the strategy of its only predecessor, and each of the nodes 7, 8, and 9 has just one strategy.

We now list all joint strategies for nodes 1, 2, and 3 and in each of them underline a strategy that is not a best response to the choice of the other players: $(\underline{a},a,b)$, $(a,a,\underline{c})$, $(a,c,\underline{b})$, $(a,\underline{c},c)$, $(b,\underline{a},b)$, $(\underline{b},a,c)$, $(b,c,\underline{b})$, and $(\underline{b},c,c)$. □

Figure 1. A coordination game with a selected joint strategy.



In the above game, no bonuses are used, and the edges in the underlying graph are unweighted. In Example 4, we exhibit a coordination game with bonuses which has a much simpler underlying graph with weighted edges and in which no Nash equilibrium exists. The above example of course raises several questions, for instance, are there restricted classes of coordination games where a Nash equilibrium always exists? Is the above example minimal in the number of colours? Does there exists coordination games that have a Nash equilibrium but are not weakly acyclic? How difficult is it to determine whether a Nash equilibrium exists? We shall address these and other questions in the rest of this paper.

4. Simple Cycles

Given that coordination games need not always have a Nash equilibrium, we consider special graph structures to identify classes of games where a Nash equilibrium is guaranteed to exist. In this section, we focus on simple cycles. To fix the notation, suppose that the considered directed graph is $1 \to 2 \to \ldots \to n \to 1$. We begin with the following simple example showing that the coordination games on a simple cycle do not have the FIP. Here and elsewhere, to increase readability, when presenting profitable deviations, we underline the strategies that were modified.

Example 2. Suppose $n \geq 3$. Consider a coordination game on a simple cycle where the nodes share at least two colours, say, a and b. Take the joint strategy $(a, b, \ldots, b)$. Then both $(a, \underline{b}, b, \ldots, b) \to (a, a, b, \ldots, b)$ and $(\underline{a}, a, b, \ldots, b) \to (b, a, b, \ldots, b)$ are profitable deviations. After these two steps, we obtain a joint strategy $(b, a, b, \ldots, b)$ that is a rotation of the initial one. Iterating, we obtain an infinite improvement path.

On the other hand, a weaker result holds.

Theorem 1. *Every coordination game on a weighted simple cycle in which at most one node has bonuses ensures improvement paths of length* $\leq 2n - 1$.

Proof. First, assume that no node has bonuses. Fix an initial joint strategy. We construct the desired improvement path by scheduling the players in the round robin fashion, starting with player 1. We prove that after at most two rounds we reach a Nash equilibrium.

Phase 1. This phase lasts at most $n - 1$ steps. Each time we select a player who does not hold a best response and update his strategy to a best response. Such a modification affects only the payoff of the successor player, so after we considered player $n - 1$, in the current joint strategy s, each of the players $1, 2, \ldots, n - 1$ holds a best response.
If at this moment the current strategy of player n is also a best response, then s is a Nash equilibrium, and the improvement path terminates. Otherwise we move to the next phase.

Phase 2. We repeat the same process as in Phase 1, but starting with s and player n.
By the definition of the game, the property that at least $n - 1$ players hold a best response continues to hold for all consecutive joint strategies, and a Nash equilibrium is reached when the selected player holds a best response.

Suppose player n switches to a strategy c. Recall that $C(i)$ is the set of colours available to player i. Let

$$n_0 := \begin{cases} n-1 & \text{if } \forall i \in \{1,\dots,n-1\} : c \in C(i) \text{ and } s_i \neq c, \\ \min\,\{i \in \{1,\dots,n-1\} \mid c \notin C(i) \text{ or } s_i = c\} - 1 & \text{otherwise.} \end{cases}$$

The improvement path terminates after the players $1,\dots,n_0$ successively switched to c, as at this moment player n_0+1 holds a best response.

Suppose that a node has bonuses. Then we rename the nodes so that this is node n. Then the argument used in reasoning about Phase 2 remains correct. □

As a side remark, note that the renaming of the players used at the end of the above proof is necessary, as otherwise the used schedule can generate improvement paths that are longer than $2n-1$.

Example 3. Suppose that $n \geq 5$ and that the simple cycle is unweighted. Assume that there are four colours, a,b,c,d, and consider the following colour assignment:

$$C(1) = \dots = C(n-3) = C(n) = \{a,b,c,d\},\ C(n-2) = \{a,\bar{c}\}, C(n-1) = \{c,d\},$$

where the overline indicates the only positive bonus in the game.

Consider the joint strategy $(b,\dots,b,a,d,a)$. If we follow the clockwise schedule starting with player 1, there is only one improvement path, namely,

$$(\underline{b},\dots,b,a,d,a) \to^* (a,\dots,a,a,d,\underline{a}) \to$$
$$(\underline{a},\dots,a,a,d,d) \to^* (d,\dots,d,\underline{a},d,d) \to (d,\dots,d,c,\underline{d},d) \to (d,\dots,d,c,c,\underline{d}) \to$$
$$(\underline{d},\dots,d,c,c,c) \to^* (c,\dots,c,c,c,c).$$

In each joint strategy, we underlined the strategy of the scheduled player from which he profitably deviates and each $\to^*$ refers to a sequence of $n-3$ profitable deviations. So this improvement path is of length $3n-5$, and thus longer than $2n-1$ because $n \geq 5$.

Furthermore, the following result holds.

Theorem 2. *Every coordination game with bonuses on a simple cycle in which at most one edge has a nontrivial weight ensures improvement paths of length* $\leq 3n-1$.

Proof. We first assume that no edge has a nontrivial weight. As in the proof of Theorem 1, we schedule the players clockwise starting with player 1. However, we are now more specific about the strategies to which the players switch. Let $MB(i)$ be the set of available colours to player i with the maximal bonus, that is,

$$MB(i) := \{c \in C(i) \mid \beta(i,c) = \max_{d \in C(i)} \beta(i,d)\}.$$

Below we stipulate that whenever the selected player i updates his strategy to a best response, he always selects a strategy from $MB(i)$. Note that this is always possible, because the bonuses are nonnegative integers. Indeed, suppose that the strategy of player's i predecessor is c. If $c \in MB(i)$, then player i selects c, and otherwise he can select an arbitrary strategy from $MB(i)$. Fix an initial joint strategy.

Phase 1. This phase is the same as in the proof of Theorem 1, except for the above proviso. So when this phase ends, the players $1,\dots,n-1$ hold a best response. If at this moment the current joint strategy s is a Nash equilibrium, the improvement path terminates. Otherwise we move to the next phase.

Phase 2. We repeat the same process as in Phase 1, but starting with s and player n and proceeding at most n steps. From now on, at each step, at least $n-1$ players have a best response strategy. So if at a certain moment the scheduled player holds a best response, the improvement path terminates. Otherwise, the players $n,1,\dots,n-1$ successively update their strategies, and after n steps, we move to the final phase.

Phase 3. We repeat the same process as in Phase 2, again starting with player n. In the previous phase, each player updated his strategy, so now, in the initial joint strategy, each player i holds a strategy from $MB(i)$. Hence, each player can improve his payoff only if he switches to the strategy selected by his predecessor that also has the maximal bonus. Let c be the strategy to which player n switches, and let

$$n_0 := \begin{cases} n-1 & \text{if } \forall i \in \{1,\dots,n-1\} : c \in MB(i) \text{ and } s_i \neq c, \\ \min\,\{i \in \{1,\dots,n-1\} \mid c \notin MB(i) \text{ or } s_i = c\} - 1 & \text{otherwise.} \end{cases}$$

The improvement path terminates after the players $1, \dots, n_0$ have successively switched to c, as at this moment player n_0+1 holds a best response.

If some edge has a nontrivial weight, then we rename the players so that this edge is into the node n. Notice that now we cannot require that player n selects a best response from $MB(n)$, because the colour of his predecessor can yield a higher payoff because of the presence of the weight. So we drop this requirement for node n but maintain it for the other nodes.

Then, at the beginning of Phase 3, we can claim only that each player $i \neq n$ holds a strategy from $MB(i)$, but this is sufficient for the remainder of the proof. □

We would like to generalise the above two results to coordination games with bonuses on arbitrary weighted simple cycles. However, the following example shows that if we allow in a simple cycle nontrivial weights on three edges and associate bonuses with three nodes, then some coordination games have no Nash equilibrium.

Example 4. Consider the weighted simple cycle and the colour assignment depicted in Figure 2, where the overlined colours have bonus one. The resulting coordination game does not have a Nash equilibrium. The list of joint strategies, each of them with an underlined strategy that is not a best response to the choice of other players, is the same as in Example 1: $(\underline{a}, a, b)$, $(a, a, \underline{c})$, $(a, c, \underline{b})$, $(a, \underline{c}, c)$, $(b, \underline{a}, b)$, $(\underline{b}, a, c)$, $(b, c, \underline{b})$, and $(\underline{b}, c, c)$. In fact, the game considered in that example simulates this game.

In what follows, we show that this counterexample is minimal in the sense that if in a weighted simple cycle with bonuses at most two nodes have bonuses or at most two edges have nontrivial weights, then the coordination game has a Nash equilibrium. More precisely, we establish the following two results.

Theorem 3. *Every coordination game on a weighted simple cycle in which two nodes have bonuses ensures improvement paths of length* $\leq 3n$.

Proof. Relabel the nodes if necessary so that one of the nodes that has bonuses is node 1. Let k be the second node that has bonuses. Fix an initial joint strategy. We schedule, as before, the players clockwise, starting with player 1.

Phase 1. This phase lasts at most n steps. We repeatedly select the first player who does not hold a best response and update his strategy to a best response. A best response can be either the colour of the predecessor or, in the case of nodes 1 and k only, a colour with the maximal bonus. In the case of equal payoffs of these two options, we give a preference to the former. As in the previous proofs, a strategy update of a given node affects only the payoff of the successor node. If at the end of Phase 1 the current strategy of player 1 is also a best response, then we reached a Nash equilibrium and the improvement path terminates. Otherwise, we move on to the next phase.

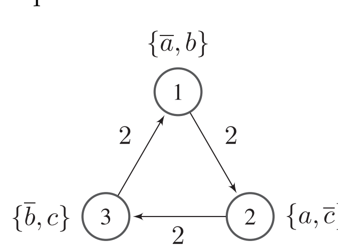
Phase 2. In this phase, we perform at most two rounds of clockwise updates of all the nodes, starting at player 1. We explicitly distinguish 10 scenarios, which are defined as follows. (They also play an important role in the proof of Theorem 5 in Section 5.) We focus on two types of strategy updates by the nodes with bonuses:

- an update to an inner colour (recorded as **i**), that is, the colour of its predecessor, or
- an update to an outer colour (recorded as **o**), that is, one of the colours with a maximal bonus.

If a colour is both inner and outer, then we record it as **i**. An *update scenario* is now a sequence of recordings of consecutive updates by the nodes with bonuses that is generated during the above two phases.

One possible update scenario is **[iooi]**, which takes place when player 1 first adopts the colour of its predecessor (**i**) and this colour then propagates until player k is reached. At this point, player k adopts a different colour with the maximal bonus (**o**), and this colour propagates further until player 1 is reached again. Player 1 then adopts a different colour with the maximal bonus (**o**), which then propagates and is also adopted by player k (**i**). This propagation stops at a node j lying between the nodes k and 1. At this point, a Nash equilibrium is reached because player j holds a best response and hence all players hold a best response.

Figure 2. A coordination game without a Nash equilibrium.



In general, an update scenario has to stop after an **oi** or **ii** is recorded, because then the same colour is propagated throughout the whole cycle and no new colour is introduced. Moreover, an update string cannot contain **ooo** as a subsequence, because then the third update to an outer colour would yield the same payoff as the first one, so it cannot be improving the payoff. It is now easy to enumerate all update scenarios satisfying these two constraints, and these are as follows: **[o]**, **[oi]**, **[oo]**, **[ooi]**, **[i]**, **[ii]**, **[io]**, **[ioi]**, **[ioo]**, **[iooi]**. The only one of length four is the already considered update scenario **[iooi]**, which yields the longest sequence of profitable deviations in Phase 2, which is $2n$. □

Now consider coordination games on simple cycles with bonuses in which two edges have nontrivial weights. The following example shows that if we follow the clockwise schedule starting with player 1, then the bound $3n$ given by Theorem 3 does not need to hold.

Example 5. Suppose that $n \geq 5$, the weights of the edges $n-3 \to n-2$ and $n-1 \to n$ are two, and the weights of the other edges are one. Let $C = \{a,b,c,d,e,f,g,h,i\}$. Define the colour and the bonus assignment as follows, where the overlined colours have bonus one:

$$
\begin{aligned}
&C(1) = C\backslash\{e\}; \bar{f}, \bar{g}, \bar{i},\\
&C(2) = C\backslash\{d\}; \bar{e}, \bar{f}, \bar{g}, \bar{i},\\
&C(3) = \ldots = C(n-3) = C,\\
&C(n-2) = C\backslash\{g,i\}; \bar{h},\\
&C(n-1) = C\backslash\{f\}; \bar{g}, \bar{h},\\
&C(n) = C\backslash\{h\}; \bar{i},
\end{aligned}
$$

Consider now the joint strategy $(a,b,\ldots,b,c,c,d)$. If we follow the clockwise schedule starting at player 1, we can generate the following improvement path in which each player $i \neq n-2, n$ always switches to a colour from $MB(i)$ (we cannot require it from players $n-2$ and n because the weights equal two):

$$
\begin{aligned}
&(\underline{a},b,\ldots,b,c,c,d) \to (d,\underline{b},\ldots,b,c,c,d) \to (d,\bar{e},\underline{b},\ldots,b,c,c,d) \to (d,e,e,\underline{b},\ldots,b,c,c,d) \to^{*}\\
&(\underline{d},e,\ldots,e,e,e) \to (\bar{f},\underline{e},\ldots,e,e,e) \to^{*} (f,\ldots,f,\underline{e},e) \to (f,\ldots,f,\bar{g},\underline{e}) \to\\
&(\underline{f},\ldots,f,g,g) \to^{*} (g,\ldots,g,\underline{f},g,g) \to (g,\ldots,g,\bar{h},\underline{g},g) \to (g,\ldots,g,h,h,\underline{g}) \to\\
&(\underline{g},\ldots,g,h,h,\bar{i}) \to^{*} (i,\ldots,i,h,h,i).
\end{aligned}
$$

In each joint strategy, we underlined the strategy of the scheduled player from which he profitably deviates and overlined the first occurrences of the newly introduced strategies. Each $\to *$ refers to a sequence of $n-3$ profitable deviations. So this improvement path is of length $4n-3 > 3n-1$.

However, a slightly larger bound can be established.

Theorem 4. *Every coordination game on a simple cycle with bonuses in which two edges have nontrivial weights ensures improvement paths of length* $\leq 4n-1$.

Proof. Rename the nodes so that the edges with a nontrivial weight are into the nodes k and n. We stipulate that each player $i \neq k, n$ always selects a best response from the set $MB(i)$ of available colours to player i with the maximal bonus. This is always possible for the reasons given in the proof of Theorem 2. As in the earlier proofs, we construct the desired improvement path by scheduling the players clockwise, starting with player 1.

Phase 1. This phase lasts at most $2n-1$ steps. If this way we do not reach a Nash equilibrium we move to the next phase.

Phase 2. In this phase, we continue the clockwise strategy updates for all the nodes starting with player n. We show that this can continue for at most two rounds.

In the second round of the previous phase, each player $i \neq n$ updated his strategy, so at the beginning of this phase, each player $i \neq k, n$ holds a strategy from $MB(i)$.

We focus on the strategy updates by the nodes k and n. To this end we reuse the reasoning used in the proof of Theorem 3 that involves the analysis of the update scenarios. So, as before, we distinguish between the updates of the nodes k and n to an inner colour (recorded as **i**) or to an outer colour (recorded as **o**) and consider the resulting update scenarios, so sequences of **i** and **o**.

For the same reasons as before, an update scenario has to stop after an **oi** or **ii** is recorded, and it cannot contain

ooo as a subsequence, as also here updates of a node to an outer colour yield the same payoff. Therefore, the same argument shows that the longest possible sequence of updates in this phase is $2n$. □

5. Open Chains of Simple Cycles

In this section, we study directed graphs that consist of an open chain of $m \geq 2$ simple cycles. For simplicity, we assume that all cycles have the same number of nodes denoted by v. The results we show hold for arbitrary cycles as long as each cycle has at least three nodes. Formally, for $j \in \{1, 2, \ldots, m\}$, let $\mathcal{C}_j$ be the cycle $[j,1] \to [j,2] \to \ldots \to [j,v] \to [j,1]$. An *open chain of cycles* $\mathcal{C}_1, \ldots, \mathcal{C}_m$ is a directed graph in which for all $j \in \{1, \ldots, m-1\}$, we have $[j,1] = [j+1,k]$ for some $k \in \{2, \ldots, v\}$. In other words, it consists of a sequence of m cycles such that any two consecutive cycles have exactly one node in common.

Any node that connects two cycles is called a *link node*. The node that connects $\mathcal{C}_j$ with $\mathcal{C}_{j+1}$, so $[j,1]$, which is also $[j+1,k]$, is called an *up-link node* in $\mathcal{C}_j$ and, at the same time, a *down-link node* in $\mathcal{C}_{j+1}$. The total number of nodes in such a graph is $n = vm - (m-1)$. Figure 3 depicts an example of an open chain.

Throughout this section, we assume a fixed coordination game on an open chain of cycles $\mathcal{C}_1, \ldots, \mathcal{C}_m$. We prove that such a game ensures improvement paths of polynomial length. The main idea of our construction is to build an improvement path by composing in an appropriate way the improvement paths for the simple cycles that form the open chain.

This is possible because, given a joint strategy, each cycle in the open chain can be viewed as a single cycle with at most two bonuses for which we know that an improvement path of length at most $3v$ exists because of Theorems 1 and 3. This is because the only nodes that have in-degree two are the link nodes, and given a joint strategy, the edge to a link node u from another cycle can be regarded as a bonus of one for the colour of the predecessor of u in another cycle. More formally, for a given joint strategy s and a cycle $\mathcal{C}_j$, we define the bonus function $\beta_j^s(u,c)$ as follows:

$$\beta_j^s(u,c) := \begin{cases} 1 & \text{if } u \text{ is a link node and } c = s(v), \\ & \text{where the node } v \text{ belongs to } \mathcal{C}_{j-1} \text{ or to } \mathcal{C}_{j+1} \text{ and } v \to u \text{ is an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

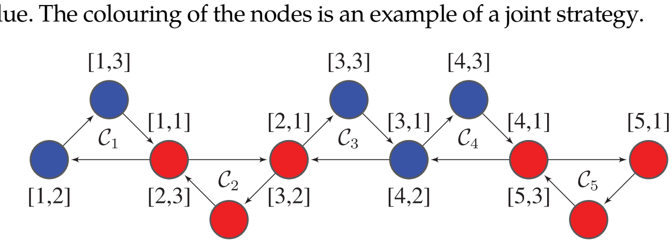
Furthermore, to each improvement path χ in the coordination game on $\mathcal{C}_j$ with the bonus function β_j^s there corresponds a unique initial segment $\bar{\chi}$ of an improvement path in the coordination game on the open chain $\mathcal{C}_1, \ldots, \mathcal{C}_m$. The following lemma will be useful a number of times.

Lemma 1. *Consider a coordination game on an open chain and a joint strategy s. Each node with payoff ≥ 1 in s plays a best response. This also holds for coordination games on a simple cycle in which each node has at most one bonus equal to one and all other bonuses are zero.*

Proof. The claim obviously holds for all nodes with the maximum possible payoff. Note that in the graphs considered here, for each node there are at most two colours that can give a payoff of one. These are the colours of the predecessors of a link node in an open chain, and the node's predecessor and the unique colour with bonus equal one in a simple cycle. The only possibility for such nodes to get payoff two is if both of these colours coincide, which depends only on the colour(s) selected by its predecessor(s). Therefore, it is not possible for a node with a payoff of one to unilaterally improve its payoff further. □

We claim that Algorithm 1 below finds an improvement path of polynomial length. It repeatedly tries to correct the cycle with the least index in which some node does not play a best response.

Figure 3. (Color online) An open chain consisting of five cycles. Four nodes have double labels as they are link nodes. Each node can select either red or blue. The colouring of the nodes is an example of a joint strategy.



To express this procedure, we use the constructions explained in the proofs of Theorems 1 and 3. Furthermore, for a joint strategy s that is not a Nash equilibrium, we denote by $NBR(s)$ the least $j \in \{1,\ldots,m\}$ such that some node in $\mathcal{C}_j$ does not play a best response in s. In the example given in Figure 3, we have $NBR(s) = 1$.

The execution of this algorithm, when dealing with a cycle $\mathcal{C}_j$, may "destabilise" some lower cycles, and hence may require going back and forth along the sequence of cycles. In other words, the value of j may fluctuate. However, we can identify the minimum value below which j cannot drop.

Algorithm 1

Input: A coordination game on an open chain of cycles $\mathcal{C}_1,\ldots,\mathcal{C}_m$ and an initial joint strategy s_0.
Output: A finite improvement path starting at s_0.

1 $\rho := s_0$;
2 $s := last(\rho)$;
3 **while** *s is not a Nash equilibrium* **do**
4 $\quad j := NBR(s)$;
5 $\quad \hat{s} :=$ the restriction of s to the nodes of $\mathcal{C}_j$;
6 $\quad \chi :=$ the improvement path constructed in the proof of Theorem 1 or 3 for the coordination game on $\mathcal{C}_j$ with the bonus function β_j^s, starting at $\hat{s}$;
7 $\quad \rho := \rho\bar{\chi}$;
8 $\quad s := last(\rho)$
9 **return** ρ.

To see this we introduce the following notion. Given a joint strategy s we assign to every cycle $\mathcal{C}_j$ one out of five possible *grades*, U+, +, U−, −, and ?, as follows:

$$grade^s(\mathcal{C}_j) := \begin{cases} \text{U+} & \text{if all its nodes play their best response in } s \text{ and } s([j,v]) = s([j,1]),\\ + & \text{if all its nodes play their best response in } s \text{ and } s([j,v]) \neq s([j,1]),\\ \text{U}- & \text{if } [j,2] \text{ is the only node that does not play a best response in } s,\\ & \text{and } s([j,v]) = s([j,1])\\ - & \text{if } [j,2] \text{ is the only node that does not play a best response in } s,\\ & \text{and } s([j,v]) \neq s([j,1])\\ ? & \text{otherwise.} \end{cases}$$

Thus the grade ? means that for some $k \neq 2$, the node $[j,k]$ does not play a best response in s.

The following observation clarifies the relevance of the grade U+ and is useful for the subsequent considerations.

Lemma 2. *Suppose that after Line 4 of Algorithm 1 the grade of a cycle $\mathcal{C}_i$ given s is* U+ *and $j > i$. Then, from that moment on, $j > i$ remains true, and the grade of $\mathcal{C}_i$ remains* U+.

Proof. During each **while** loop iteration, j can drop at most by 1, so the grade of $\mathcal{C}_i$ could be modified only if eventually after Line 4 $j = i+1$ holds. The initial grade U+ of $\mathcal{C}_i$ implies that initially the colours of the nodes $[i,1]$ and $[i,v]$ are the same, and consequently the payoff for the node $[i,1]$ is ≥ 1, and it remains so whenever its other predecessor, belonging to $\mathcal{C}_j$, switches to another colour.

But $[i,1]$ is also the down-link node $[j,k]$ of $\mathcal{C}_j$. Hence, by Lemma 1, the improvement path constructed in Line 6 does not modify the colour of $[j,k]$, that is, of the node $[i,1]$. So the grade of $\mathcal{C}_i$ remains U+, and hence if the **while** loop does not terminate right away, j increases after Line 4. □

Furthermore, let $grade(s)$ be the sequence of grades given s assigned to each cycle, that is,

$$grade(s) := (grade^s(\mathcal{C}_1),\ldots,grade^s(\mathcal{C}_m)).$$

For instance, $grade(s) = (-, \text{U+}, ?, +, \text{U+})$ for the game and joint strategy s presented in Figure 3.

Suppose that Algorithm 1 selects j in Line 4. It then constructs in Line 6 the improvement path that starts in $\hat{s}$ defined in Line 5, for the coordination game with bonuses on the cycle $\mathcal{C}_j$, as described in the proofs of Theorems 1 and 3. We now explain how this can change $grade(s)$. Note that only the grades of $\mathcal{C}_j$ and its adjacent cycles $\mathcal{C}_{j-1}$ and $\mathcal{C}_{j+1}$ (if they exist) can be affected.

Lemma 3. *The improvement path constructed in Line 6 of Algorithm 1 modifies the grades of $\mathcal{C}_j$ and its adjacent cycles $\mathcal{C}_{j-1}$ and $\mathcal{C}_{j+1}$, if they exist, as explained in Figures 4, 5, 6, 7, and 8.*

Figure 4. Possible changes of the grades of $\mathcal{C}_j$ and $\mathcal{C}_{j+1}$ when $j = 1$.

−	x
+/U+	x

U−	x
U+	x

?	x
+/U+	any

Proof. We begin with some remarks and explanations. For a joint strategy s, $NBR(s)$ returns the least index j of a cycle with a node that does not play a best response. So the initial grade of the cycle $\mathcal{C}_{j-1}$, if it exists, is + or U+ , and the initial grade of the cycle $\mathcal{C}_j$ is U−, −, or ?. Moreover, the grade of $\mathcal{C}_j$ can only change to + or U+, because after Line 6, all nodes in $\mathcal{C}_j$ play a best response. These observations allow us to limit the number of considered cases.

In Figures 4–8, we list above the horizontal bar the initial situation for the discussed cycles, and under the bar one or more outcomes that can arise. Furthermore, the initial grade of $\mathcal{C}_{j+1}$ is a parameter x. If there are several options for the new grade of a given cycle, these are separated by slashes. Finally "any" is an abbreviation for U+/+/U−/−/?.

Figure 4 corresponds to the case when $j = 1$. In turn, Figures 5, 6, and 7 correspond to the cases when $1 < j < m$ and initially the grade of $\mathcal{C}_j$ is U−, −, or ?, respectively. Finally, Figure 8 corresponds to the case when $j = m$.

The cases considered in Figures 5 and 6 refer to the update scenarios defined in Phase 2 in the proof of Theorem 3. They are concerned with the relation of the colour of the up-link node in the cycle $\mathcal{C}_j$ to the colour of its predecessor in this cycle.

The justifications of these changes of the grades are lengthy and are provided in the appendix. □

Next, we introduce a progress measure μ defined on the current joint strategy that increases according to the lexicographic order each time the joint strategy s is modified in Line 8. In effect, μ is a weak potential in the sense of Milchtaich [39]. For a joint strategy s, $\mu(s)$ is a quadruple, the definition of which uses the function NBR and two other functions that we now define.

Let $guard(s)$ be the largest $j \in \{1, \dots, m\}$ such that given s the grade of $\mathcal{C}_j$ is U+ and the grade of each cycle $\mathcal{C}_1, \dots, \mathcal{C}_{j-1}$ is either + or U+. If no such j exists, as is the case in the example given in Figure 3, then we let $guard(s) = 0$.

Furthermore, let $prefix(s)$ be the longest prefix of $grade(s)$ such that at most one of the grades it contains is −, U−, or ?. Moreover, this prefix stops after a cycle with grade ?. For the example given in Figure 3, we have $prefix(s) = (-, \text{U}+)$.

Here is an example illustrating the introduced notions to which we shall return shortly.

Example 6. Suppose that $grade(s_1) := (+, \text{U}+, \text{U}+, +, -, \text{U}-, ?)$ for a joint strategy s_1. Then $NBR(s_1) = 5$, $guard(s_1) = 3$, and $prefix(s_1) = (+, \text{U}+, \text{U}+, +, -)$. Suppose that $grade(s_2) := (+, \text{U}+, -, +, \text{U}+, \text{U}-, \text{U}-, ?)$ for a joint strategy s_2. Then $NBR(s_2) = 3$, $guard(s_2) = 2$, and $prefix(s_2) = (+, \text{U}+, -, +, \text{U}+)$.

We can now define $\mu(s)$. First, we set $\mu(s) = (m+1, 0, 0, 0)$ if s is a Nash equilibrium. Otherwise

$$\mu(s) := \begin{cases} (guard(s), 1, 0, -NBR(s)) & \text{if } prefix(s) \text{contains } \text{U}- \text{ or if it contains } \text{U}+, \\ & \text{some where after } -, \\ (guard(s), 0, |prefix(s)|, -NBR(s)) & \text{otherwise.} \end{cases}$$

For example, for the joint strategies s_1 and s_2 used in Example 6, we have $\mu(s_1) = (3, 0, 5, -5)$ and $\mu(s_2) = (2, 1, 0, -3)$, respectively.

Figure 5. Possible changes of the grades of $\mathcal{C}_{j-1}$, $\mathcal{C}_j$, and $\mathcal{C}_{j+1}$ when $1 < j < m$ and the grade of $\mathcal{C}_j$ is U−.

case	+	U−	x	U+	U−	x
[i]	+	U+	x	U+	U+	x
[ii]	+/−/U+/U−	U+	x	impossible		
[io]	U−/U+	+/U+	x	impossible		
[ioi]	U−/U+	U+	any	impossible		
[ioo]	U−/U+	+	any	impossible		
[iooi]	impossible			impossible		

Figure 6. Possible changes of the grades of C_{j-1}, C_j, and C_{j+1} when $1 < j < m$ and the grade of C_j is –.

case	+	–	x	U+ – x
[o]	+	+	x	U+ + x
[oi]	+/–/U+/U–	+/U+	x	impossible
[oo]	U–/U+	+	x	impossible
[ooi]	U–/U+	U+	any	impossible

To see the evolution of the progress measure $\mu(s)$, we present in Figure 9 an example run of Algorithm 1 on an open chain of eight cycles by recording at each step the corresponding changes of the grades and of the progress measure. It illustrates the fact that during the execution of the algorithm, the index of the first cycle with no Nash equilibrium, that is, the value of $NBR(s)$, can arbitrarily decrease.

The following lemma explains the relevance of μ.

Lemma 4. *The progress measure $\mu(s)$ increases w.r.t. the lexicographic ordering $<_{lex}$ each time one of the updates presented in Figures 4, 5, 6, 7, and 8 takes place.*

As in the case of Lemma 3, the proof is lengthy and proceeds by a detailed case analysis. It can be found in the appendix.

We are now in position to prove the appropriate result concerning open chains of cycles.

Theorem 5. *Every coordination game on an open chain of m cycles, each with v nodes, ensures improvement paths of length $\leq 3vm^3$.*

Proof. Let s_0 be an arbitrary initial joint strategy in this coordination game. We argue that starting at s_0, Algorithm 1 computes a finite improvement path ρ of length at most $3vm^3$. By Lemma 4, $\mu(s)$ increases according to the lexicographic order each time the joint strategy s is modified in Line 8.

We now estimate the number of different values the progress measure μ can take. If s is a Nash equilibrium, then $\mu(s) = (m+1, 0, 0, 0)$, which accounts for one value. Otherwise $guard(s) \in \{0, \ldots, m-1\}$ and $guard(s) + 1 \leq NBR(s) \leq |prefix(s)| \leq m$, because by definition, the index $NBR(s)$ cannot be smaller than $guard(s) + 1$, and the grade of the cycle with this index belongs to $prefix(s)$. Therefore, the number of values μ can take is

$$1 + \sum_{g=0}^{m-1} \sum_{p=g+1}^{m} (p-g) + \sum_{g=0}^{m-1}(m-g) = 1 + \sum_{g=0}^{m-1} \frac{(m-g)(1+m-g)}{2} + \frac{m(1+m)}{2} =$$

$$1 + \sum_{x=1}^{m} \frac{x(1+x)}{2} + \frac{m(1+m)}{2} = 1 + \frac{m(m+1)(m+2)}{6} + \frac{m(1+m)}{2} =$$

$$1 + \frac{m(m+1)(m+5)}{6} \leq m^3 \text{ for m} \geq 2.$$

As a result, the length of the improvement path constructed by Algorithm 1 is at most $3vm^3$, because by Theorems 1 and 3, the improvement path in Line 6 takes at most $3v$ improvement steps. □

Finally, so far we assumed that we know the decomposition of the game graph into a chain of cycles in advance. In general, the input may be an arbitrary graph, and we would need to find this decomposition first. Fortunately, this can be done in linear time as the following result shows.

Proposition 1. *Checking whether a given graph G is an open chain of cycles, and if so partitioning G into simple cycles $C_1, \ldots, C_m$, can be done in $\mathcal{O}(|G|)$ time.*

Figure 7. Possible changes of the grades of C_{j-1}, C_j, and C_{j+1} when $1 < j < m$ and the grade of C_j is ?

+	?	x	U+	?	x
+/–/U+/U–	+/U+	any	U+	+/U+	any

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Figure 8. Possible changes of the grades of $\mathcal{C}_{j-1}$ and $\mathcal{C}_j$ when $j = m$.

case	+	U−	U+	U−	**case**	+	−	U+	−
[i]	+	U+	U+	U+	**[o]**	+	+	U+	+
[ii]	+/−/U+/U−	U+	impossible		**[oi]**	+/−/U+/U−	+/U+	impossible	
[io]	U−/U+	+/U+	impossible		**[oo]**	U−/U+	+	impossible	
[ioi]	U−/U+	U+	impossible		**[ooi]**	U−/U+	U+	impossible	

+	?
+/−/U+/U−	+/U+

U+	?
U+	+/U+

Proof. First note that if G is an open chain of cycles, then there are no bidirectional edges, and each of its nodes has either out- and in-degree values both equal to one or both equal to two. These two conditions can be easily checked in linear time by simply going through all the nodes and their edges in G.

Assume that the above two conditions hold. Let A be the set of all nodes in G that we already identified to have out- and in-degrees both equal to two. We first build a new directed graph G' whose set of nodes is A, and there is an edge from $u \in A$ to $v \in A$ iff v is reachable from u by traversing only nodes with out- and in-degree both equal to one. We illustrate this construction in Figure 10.

Such a graph can be built using a single run of the depth first search algorithm starting from any node in A. Now note that the original graph G is an open chain of cycles iff this graph G' is a simple path whose two ends have a self-loop and all edges are bidirectional.

This condition can also be checked in linear time, by simply following all edges of G' in one direction. To partition G into simple cycles, we label one of the end nodes of G' as $[1, 1]$. Its only adjacent node we label as $[2, 1]$, the other adjacent node of $[2, 1]$ as $[3, 1]$, and so on until the node at the other end is of G' labeled as $[m-1, 1]$. These are the labels of the link nodes. The labels of the remaining nodes in each cycle C_j for $j \in \{1, \ldots, m\}$ can then be simply inferred by following the edges in the original graph G. □

6. Strong Equilibria

In this section, we study the existence of strong equilibria and the existence of finite c-improvement paths. To start with, we establish two results about the games that have the strongest possible property, the c-FIP.

First, we establish a structural property of a coalitional deviation from a Nash equilibrium in our coordination games. It will be used to prove c-weak acyclicity for a class of games on the basis of their weak acyclicity. Note that such a result cannot hold for all classes of graphs because there exists a coordination game on an undirected graph which is weakly acyclic but has no strong equilibrium (see Apt et al. [8]).

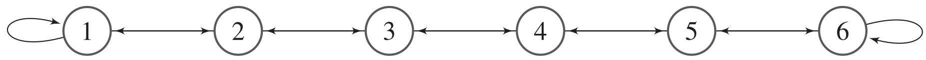
Lemma 5. *Consider a coordination game. Any node involved in a profitable coalitional deviation from a Nash equilibrium belongs to a directed simple cycle that deviated to the same colour.*

Proof. Suppose that s' is profitable deviation of a coalition K from a Nash equilibrium s. It suffices to show that each node in K has a neighbour in K deviating to the same colour. Assume that for some player $i \in K$ it is not the case. Then

Figure 9. The evolution of $grade(s)$ and $\mu(s)$ during an example run of Algorithm 1.

$grade(s)$								$\mu(s)$
+	+	+	+	?	U+	U+	−	$(0, 0, 5, -5)$
+	+	+	−	+	?	U+	−	$(0, 0, 5, -4)$
+	+	−	+	+	?	U+	−	$(0, 0, 5, -3)$
+	U−	U+	?	+	?	U+	−	$(0, 1, 0, -2)$
U−	+	?	?	+	?	U+	−	$(0, 1, 0, -1)$
U+	+	?	?	+	?	U+	−	$(1, 0, 3, -3)$
U+	+	U+	?	+	?	U+	−	$(3, 0, 4, -4)$
U+	+	U+	+	+	?	U+	−	$(3, 0, 6, -6)$
U+	+	U+	+	+	U+	U+	−	$(7, 0, 8, -8)$
U+	+	U+	+	+	U+	U+	+	$(9, 0, 0, 0)$

Figure 10. Graph G' corresponding to an open chain G with seven cycles (and six link nodes).



$$
\begin{aligned}
& p_i(s) < p_i(s'_K, s_{-K}) \\
& = \sum\nolimits_{j \in N_j \cap K : s'_j = s'_i} w_{j \to i} + \sum\nolimits_{j \in N_j \setminus K : s_j = s'_i} w_{j \to i} + \beta(i, s'_i) \\
& \leq 0 + \sum\nolimits_{j \in N_i : s_j = s'_i} w_{j \to i} + \beta(i, s'_i) = p_i(s'_i, s_{-i}),
\end{aligned}
$$

which contradicts the fact that s is a Nash equilibrium. □

Theorem 6. *Every coordination game with bonuses on a weighted DAG has the c-FIP and a fortiori a strong equilibrium. Furthermore, every Nash equilibrium is a strong equilibrium. Finally, the game ensures both improvement paths and c-improvement paths of length $\leq n-1$, where, recall, n is the number of nodes.*

Proof. Given a weighted DAG (V, E) on n nodes, denote these nodes by $1, \ldots, n$ in such a way that for all $i, j \in \{1, \ldots, n\}$,

$$\text{if } i < j \text{ then } (j \to i) \notin E. \tag{1}$$

So if $i < j$, then the payoff of the node i does not depend on the strategy selected by the node j.

Then given a coordination game whose underlying directed graph is the above weighted DAG and a joint strategy s we abbreviate the sequence $p_1(s), \ldots, p_n(s)$ to $p(s)$. We now claim that $p : S \to \mathbb{R}^n$ is a generalised ordinal c-potential when we take for the partial ordering $\succ$ on $p(S)$ the lexicographic ordering $>_{lex}$ on the sequences of reals.

So suppose that some coalition K profitably deviates from the joint strategy s to s'. Choose the smallest $j \in K$. Then $p_j(s') > p_j(s)$, and by (1), $p_i(s') = p_i(s)$ for $i < j$. By the definition of $>_{lex}$, this implies $p(s') >_{lex} p(s)$, as desired. Hence, the game has the c-FIP.

The second claim is a direct consequence of Lemma 5 that implies that no coalition deviations are possible from a Nash equilibrium for DAGs.

Finally, to prove the last claim, given an initial joint strategy, schedule the players in the order $1, \ldots, n$ and repeatedly update the strategy of each selected player to a best response. By (1) this yields an improvement path of length $\leq n-1$. By the second claim, this path is also a c-improvement path. □

Example 2 shows that it is difficult to come up with other classes of directed graphs for which the coordination game has the FIP, let alone the c-FIP. However, the weaker property of c-weak acyclicity holds for the games on simple cycles considered in Section 4. Below we put $i \ominus 1 = i - 1$ if $i > 1$ and $1 \ominus 1 = n$.

Theorem 7. *Consider a coordination game with bonuses on a weighted simple cycle. Any finite improvement path is a finite c-improvement path or can be extended to it by a single profitable deviation of all players.*

Proof. Take a finite improvement path and denote by s the Nash equilibrium it reaches. If s is a strong equilibrium, then we are done. Otherwise, there exists a coalition K with a profitable deviation from s. By Lemma 5, the coalition K consists of all players and all of them switch to the same colour.

Let C be the set of common colours c such that a switching by all players to c is a profitable deviation from s. We just showed that C is nonempty. Select an arbitrary player i_0 and choose a colour from C for which player i_0 has a maximal bonus. Let s' be the resulting joint strategy.

We first claim that s' is a Nash equilibrium. Otherwise, some player i can profitably deviate from s'_i to a colour c. Then we have $s'_{i \ominus 1} \neq c$, because all players hold the same colour in s'. So we have $p_i(s) < p_i(s') < p_i(c, s'_{-i}) = \beta(i, c) \leq p_i(c, s_{-i})$, which is a contradiction because s is a Nash equilibrium.

Next, we claim that s' is a strong equilibrium. Otherwise, by the initial observation there is a profitable deviation of all players from s' to some joint strategy s'' in which all players switch to the same colour. So $p_{i_0}(s') < p_{i_0}(s'')$. Moreover, this profitable deviation is also a profitable deviation of all players from s, which contradicts the choice of i_0. □

The above result directly leads to the following conclusions.

Corollary 1.

i. *Every coordination game on a weighted simple cycle in which at most one node has bonuses ensures c-improvement paths of length $\leq 2n$.*

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ii. *Every coordination game with bonuses on a simple cycle in which at most one edge has a nontrivial weight ensures c-improvement paths of length* $\leq 3n$.

iii. *Every coordination game on a weighted simple cycle in which two nodes have bonuses ensures c-improvement paths of length* $\leq 3n+1$.

iv. *Every coordination game on a simple cycle with bonuses in which two edges have nontrivial weights ensures c-improvement paths of length* $\leq 4n$.

Proof. The corollary follows from Theorems 1, 2, 3, 4, and 7. □

We conclude this analysis of coordination games on simple cycles by the following observation that sheds light on Theorem 7 and is of independent interest.

Proposition 2. *Consider a coordination game with bonuses on a simple cycle with n nodes. Then every Nash equilibrium is an* $(n-1)$*-equilibrium.*

Proof. Take a Nash equilibrium s. It suffices to prove that it is an $(n-1)$-equilibrium. Suppose otherwise. Then for some coalition K of size $\leq n-1$ and a joint strategy s', $s \xrightarrow{K} s'$ is a profitable deviation.

Take some $i \in K$ such that $i \ominus 1 \notin K$. We have $p_i(s') > p_i(s)$. Also $p_i(s'_i, s_{-i}) = p_i(s')$, because $s_{i\ominus 1} = s'_{i\ominus 1}$. So $p_i(s'_i, s_{-i}) > p_i(s)$, which contradicts the fact that s is a Nash equilibrium. □

From the definition of an $(n-1)$-equilibrium and Proposition 2, it follows that for a coordination game with bonuses on a simple cycle with n nodes, every Nash equilibrium is a k-equilibrium for all $k \in \{1, \ldots, n-1\}$. We now show that, as in the case of simple cycles, coordination games on open chains of cycles are c-weakly acyclic, so a fortiori have strong equilibria.

Theorem 8. *Every coordination game on an open chain of cycles of m simple cycles, each with v nodes, ensures c-improvement paths of length* $4vm^4$.

Proof. Assume the considered open chain of cycles $\mathcal{C}$ consists of the simple cycles $\mathcal{C}_j$, where $j \in \{1, 2, \ldots, m\}$. We begin with the following useful fact.

Lemma 6. *Suppose that in a joint strategy s for the coordination game on* $\mathcal{C}$*, a simple cycle* $\mathcal{C}_i$ *is unicoloured. Then in any profitable deviation from s, the colours of the nodes in* $\mathcal{C}_i$ *do not change.*

Proof. The payoff of each node of the cycle $\mathcal{C}_i$ in s is ≥ 1. For the nonlink nodes, the payoff is then maximal, so none of these nodes can be a member of a coalition that profitably deviates. This implies that a link node cannot be a member of a coalition that profitably deviates either. Indeed, otherwise its payoff increases to two, and hence, in the new joint strategy, its colour is the same as the colour of its predecessor j in the cycle $\mathcal{C}_i$, which is not the case, because we just explained that the colour of j does not change. □

We now construct the desired c-improvement path ξ as an alternation of an improvement path guaranteed by Theorem 5 and a single profitable deviation by a coalition. Each time such a profitable coalitional deviation takes place, by Lemma 5, the deviating coalition includes a simple cycle $\mathcal{C}_i$, all nodes of which switch to the same colour. By Lemma 6, each time this is a different cycle, which is moreover disjoint from the previous cycles. This implies that the number of such profitable deviations in ξ is at most $\lceil m/2 \rceil$.

So ξ is finite, and by Theorem 5, its length is at most $(\lceil m/2 \rceil + 1) \cdot 3vm^3 + \lceil m/2 \rceil$, where the first term counts the total length of at most $\lceil m/2 \rceil + 1$ improvement paths that separate at most $\lceil m/2 \rceil$ coalitional deviations, which is the second term of this expression. But $\lceil m/2 \rceil + 1 \leq m$ for $m \geq 2$, so $(\lceil m/2 \rceil + 1) \cdot 3vm^3 + \lceil m/2 \rceil \leq 3vm^4 + \lceil m/2 \rceil \leq 4vm^4$. □

Example 2 shows that even when only two colours are used, coordination games need not have the FIP. This is in contrast to the case of undirected graphs for which we proved in Apt et al. [8] that the corresponding class of coordination game does have the FIP. On the other hand, a weaker property does hold.

Theorem 9. *Every coordination game in which only two colours are used ensures improvement paths of length* $\leq 2n$.

Proof. We prove the result for a more general class of games, namely, the ones that satisfy the PPM (the property defined in Section 2). Call the colours blue and red. When a node holds the blue colour we refer to it as a blue node, and the likewise for the red colour. Take a joint strategy s.

Phase 1. We consider a maximal sequence ξ of profitable deviations starting in s in which each node can only switch to blue. At each step the number of blue nodes increases, so ξ is of length at most n. Let s^1 be the last joint strategy in ξ. If s^1 is a Nash equilibrium, then ξ is the desired finite improvement path. Otherwise, we move to the next phase.

Phase 2. We consider a maximal sequence χ of profitable deviations starting in s^1 in which each node can only switch to red. Also χ is of length at most n. Let s^2 be the last joint strategy in χ.

We claim that s^2 is a Nash equilibrium. Suppose otherwise. Then some node, say, i, can profitably switch in s^2 to blue. Suppose that node i is red in s^1. In s^1, there are weakly more blue nodes than in s^2, so by the PPM also in s^1 node i can profitably switch to blue. This contradicts the choice of s^1.

Hence, node i is blue in s^1, whereas it is red in s^2. So in some joint strategy s^3 from χ node i profitably switched to red. Then $s^3 = (i : b, s^3_{-i})$ and $p_i(i : b, s^3_{-i}) < p_i(i : r, s^3_{-i}) \le p_i(i : r, s^2_{-i}) < p_i(i : b, s^2_{-i})$, where the weak inequality holds because of the PPM. But in s^3 there are weakly more blue nodes than in s^2, so by the PPM, $p_i(i : b, s^2_{-i}) \le p_i(i : b, s^3_{-i})$. This yields a contradiction. □

The following simple example shows that in the coordination games in which only two colours are used, Nash equilibria do not need to be strong equilibria. □

Example 7. Consider a bidirectional cycle $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 1$ in which each node has two colours, a and b. Then (a, a, b, b) is a Nash equilibrium, but it is not a strong equilibrium because of the profitable deviation to (a, a, a, a), which is a strong equilibrium.

On the other hand the following counterpart of the above result holds for the c-improvement paths.

Theorem 10. *Every coordination game in which only two colours are used ensures c-improvement paths of length* $\le 2n$.

Proof. As in the above proof, we establish the result for the games that satisfy the PPM. We retain the terminology of blue and red colours, which we abbreviate to b and r. Take a joint strategy s.

Phase 1. We consider a maximal sequence ξ of profitable deviations of the coalitions starting in s in which the nodes can only switch to blue. At each step, the number of blue nodes increases, so ξ is of length at most n. Let s^1 be the last joint strategy in ξ. If s^1 is a strong equilibrium, then ξ is the desired finite c-improvement path. Otherwise we move to the next phase.

Phase 2. We consider a maximal sequence χ of profitable deviations of the coalitions starting in s^1 in which the nodes can only switch to red. Also χ is of length at most n. Let s^2 be the last joint strategy in χ.

We claim that s^2 is a strong equilibrium. Suppose otherwise. Then for some joint strategy s', $s^2 \xrightarrow{K} s'$ is a profitable deviation of some coalition K. Let L be the set of nodes from K that switched in this deviation to blue. By the definition of s^2 the set L is nonempty.

Given a set of nodes M and a joint strategy s, we denote by $(M : b, s_{-M})$ the joint strategy obtained from s by letting the nodes in M to select blue, and similarly for the red colour. Also it should be clear which joint strategy we denote by $(M : b, P \backslash M : r, s_{-P})$, where $M \subseteq P$.

We claim that $s^2 \xrightarrow{L} (L : b, s^2_{-L})$ is a profitable deviation of the players in L. Indeed, we have, for all $i \in L$,

$$p_i(s^2) < p_i(L : b, s^2_{-L}), \tag{2}$$

because by the assumption $p_i(s^2) < p_i(s')$ and by the PPM, $p_i(s') \le p_i(L : b, s^2_{-L})$.

Let M be the set of nodes from L that are red in s^1. Suppose that M is nonempty. We show that then for all $i \in M$,

$$p_i(M : r, L \backslash M : b, s^1_{-L}) < p_i(M : b, L \backslash M : b, s^1_{-L}). \tag{3}$$

Indeed, we have, for all $i \in M$,

$$\begin{aligned} & p_i(M : r, L \backslash M : b, s^1_{-L}) \le p_i(M : r, L \backslash M : b, s^2_{-L}) \\ \le \; & p_i(M : r, L \backslash M : r, s^2_{-L}) < p_i(M : b, L \backslash M : b, s^2_{-L}) \\ \le \; & p_i(M : b, L \backslash M : b, s^1_{-L}), \end{aligned}$$

where the weak inequalities hold due to the PPM and the strict inequality holds by the definition of L.

But $s^1 = (M : r, L \backslash M : b, s^1_{-L})$, so (3) contradicts the definition of s^1. Thus, M is empty, that is, all nodes from L are blue in s^1.

Let i be a node from L that as first turns red in χ. So in some joint strategy s^3 from χ node i profitably switched to red in a profitable deviation to a joint strategy s^4. Then $s^3 = (L : b, s^3_{-L})$, $s^4 = (i : r, s^4_{-i})$ and

$$p_i(L : b, s^3_{-L}) < p_i(i : r, s^4_{-i}) \le p_i(s^2) < p_i(L : b, s^2_{-L}),$$

where the weak inequality holds due to the PPM and the strict inequalities hold by the definition of i and (2). But in $(L : b, s^3_{-L})$, there are weakly more blue nodes than in $(L : b, s^2_{-L})$, so by the PPM, $p_i(L : b, s^2_{-L}) \le p_i(L : b, s^3_{-L})$.

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This yields a contradiction. (The final step in this proof in Apt et al. [6] contained a bug that is now corrected.) □

When the underlying graph is symmetric and the set of strategies for every node is the same, the existence of strong equilibrium for coordination games with two colours follows from proposition 2.2 in Konishi et al. [34]. Theorem 10 shows a stronger result, namely, that these games are c-weakly acyclic. Example 1 shows that when three colours are used, Nash equilibria, so a fortiori, strong equilibria, do not need to exist. Finally, note that sometimes strong equilibria exist even though the coordination game is not c-weakly acyclic.

Example 8. Consider the coordination game depicted in Figure 11. Note that the underlying graph is strongly connected and that all edges except $1 \to 2$, $2 \to 3$, and $3 \to 1$ are bidirectional. Although the graph is weighted, the weighted edges can be replaced by unweighted ones by adding auxiliary nodes without affecting the strong connectedness of the graph. The behaviour of the game on this new unweighted graph will be analogous to the one considered.

Let us analyse the initial joint strategy s that consists of the underlined colours in Figure 11. We argue that the only nodes that can profitably switch colours (possibly in a coalition) are the nodes 1, 2, and 3 and that this is the case independently of their strategies.

First consider the nodes A, B, and C. They have the maximum possible payoff of five, independently of the strategies of the nodes 1, 2, and 3, so none of them can be a member of a profitably deviating coalition.

Furthermore, each node from the set $\{4, \ldots, 9\}$ has two neighbours, each with the same weight. One of them is from the set {A, B, C} with whom it shares the same colour, which results in the payoff of two. So for each node from $\{4, \ldots, 9\}$ a possible profitable coalitional deviation has to involve a neighbour from {A, B, C}.

Therefore, the only nodes that can profitably deviate are nodes 1, 2, and 3. Moreover, this will continue to be the case in any joint strategy resulting from a sequence of profitable coalitional deviations starting from s. (Another way to look at it is by arguing that the restriction of s to the nodes $\{A, B, C, 4, \ldots, 9\}$ is a strong equilibrium in the game on these nodes in which we add to the nodes from $\{4, 6, 8\}$ bonuses two and to the nodes from $\{5, 7, 9\}$ bonuses three.)

So it suffices to analyse the weighted simple cycle and the colour assignment depicted in Figure 12, with the nontrivial bonuses mentioned above the colours.

However, the resulting coordination game does not have a Nash equilibrium and a fortiori no strong equilibrium. To see it, first notice that each of the nodes can secure a payoff of at least three, whereas, selecting a colour with a trivial bonus, it can secure a payoff of at most two. So we do not need to analyse joint strategies in which a node selects a colour with a trivial bonus. This leaves us with the following list of joint strategies: $(\underline{a}, a, b)$, $(a, a, \underline{c})$, $(a, c, \underline{b})$, $(a, \underline{c}, c)$, $(b, \underline{a}, b)$, $(\underline{b}, a, c)$, $(b, c, \underline{b})$, and $(\underline{b}, c, c)$. In each of them, as in Examples 1 and 4, we underlined a strategy that is not a best response to the choice of other players. This means that no c-improvement path in this game terminates.

Figure 11. A coordination game with strong equilibria unreachable from a given initial joint strategy.

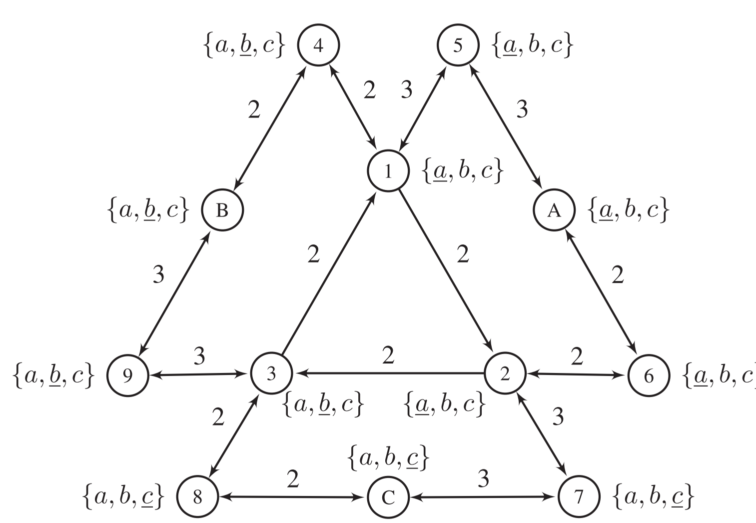
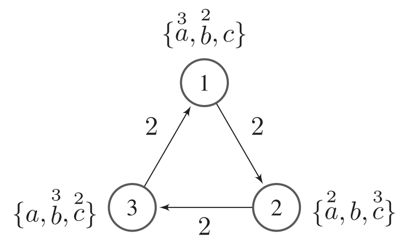


Figure 12. A coordination game without a Nash equilibrium.



Consequently no c-improvement path in the original game that starts with s terminates. Therefore, the original game is neither weakly acyclic nor c-weakly acyclic. On the other hand, it has three trivial strong equilibria in which all players pick the same colour.

Note that in the game considered in this example all players have the same sets of strategies. We can summarise this example informally as follows. There exists a graph with the same set of alternatives (called colours) for all nodes and an initial situation (modelled by a colour assignment) starting from which no stable outcome (modelled as a Nash equlibrium) can be achieved even if forming coalitions is allowed.

7. Complexity Issues

Finally, we study the complexity of finding Nash equilibria and strong equilibria, and of determining their existence. The results obtained so far provide bounds on the length of short (c-)improvement paths. But in each proof we actually provide bounds on the length of the corresponding schedule, a notion defined in Section 2. This allows us to determine in each case the complexity of finding a Nash equilibrium or a strong equilibrium, by analysing the cost of finding a profitable deviation from a given joint strategy. For the case of weighted graphs, we assume all weights to be natural numbers.

We assume that the colour assignment C is given as a $\{0, 1\}$-matrix of size $V \times M$, such that (i, c) entry is one iff colour c is available to node i. The bonus function β, if present, is represented by another matrix of size $V \times M$, where the (i, c) entry holds the value of $\beta(i, c)$. The game graph is represented using adjacency lists, where for each node we keep a list of all outgoing and incoming edges and, if the graph is weighted, their weights are represented in binary. As usual, we provide the time complexity in terms of the number of arithmetic operations performed. All our algorithms operate only on numbers that are linear in the size of the input, so the actual number of bit operations is at most polylogarithmically higher.

Below, as in Table 1, n is the number of nodes, $|E|$ is the number of edges, l is the number of colours, and in the case of the open chains of cycles, m is the number of simple cycles in a chain and v is the number of nodes in each cycle. We first determine complexity of finding a best response.

Lemma 7. *Consider a coordination game. Given a joint strategy, a best response for a player i can be computed in time $\mathcal{O}(l + e_i)$, where e_i is the number of incoming edges to node i.*

Proof. We first calculate for each colour the sum of the weights on all edges from neighbors of player i with that colour. This can be done by simply iterating over all e_i incoming edges. We then iterate over all of these l values to select any colour with the highest such a value. □

When we only care about the current payoff of player i, then there is no need to iterate over all l colours, and we get the following.

Lemma 8. *Consider a coordination game. Given a joint strategy, the payoff of player i can be computed in time $\mathcal{O}(1 + e_i)$, where e_i is the number of incoming edges to node i.*

Proof. It suffices to iterate over all e_i incoming edges the sum the weights of all edges from neighbors of player i with the same colour. The term 1 is needed to cover the case of nodes with no neighbours. □

We can now deal with the complexity of finding a Nash equilibrium and a strong equilibrium for the coordination games on simple cycles that we considered in Section 4.

Theorem 11. *Consider a coordination game on a simple cycle that is either weighted with at most two nodes with bonuses or with bonuses with at most two edges having nontrivial weights. Both a Nash equilibrium and a strong equilibrium can be computed in time $\mathcal{O}(nl)$.*

Proof. In both cases, because of Theorems 1, 2, 3, and 4, to compute a Nash equilibrium, it suffices to follow a schedule of length $\mathcal{O}(n)$. At each step of this schedule, it suffices to consider only the deviations to a colour with the maximal bonus. We can find such colours in time $\mathcal{O}(l)$ and then simply follow the $\mathcal{O}(l)$ procedure given in Lemma 7 for finding a best response within this narrowed down set. We conclude that computing a Nash equilibrium can be done in time $\mathcal{O}(nl)$.

Finally, to compute a strong equilibrium, we first compute a Nash equilibrium and subsequently check whether there is a profitable deviation of all nodes to a single colour. By Theorem 7, one of these two joint strategies is a strong equilibrium.

The latter step involves iterating over all l colours and computing for each of them the payoff of all nodes when they all hold this single colour, assuming such a colour is shared by all nodes. Each iteration takes $\mathcal{O}(n)$ time, which results in total $\mathcal{O}(nl)$ time as well. □

The complexity of computing a Nash equilibrium for the coordination games on an open chain of cycles can be easily established, as most of the work was done in the proof of Theorem 5, which in turn built upon Theorems 1 and 3.

Theorem 12. *Consider a coordination game on an open chain of cycles. A Nash equilibrium can be computed in time* $\mathcal{O}(vm^3 l)$.

Proof. From Theorem 5, it follows that for an open chain of cycles there exists an improvement path of length at most $3vm^3$. Because of Lemma 7, computing each best response can be done in time $\mathcal{O}(l)$. It follows that a Nash equilibrium can be computed in time $\mathcal{O}(vm^3 l)$. □

To analyse the complexity of computing a strong equilibrium for the coordination games on an open chain of cycles, we make use of Algorithm 2.

Algorithm 2 **Input**: A strategic game $(S_1, \dots, S_n, p_1, \dots, p_n)$ that satisfies the PPM property, a joint strategy s, and a strategy c.

Output: A maximal coalition that can profitably deviate to c, if there exists one, and otherwise the empty set.

1 $A := \{i \in \{1, \dots, n\} \mid c \in S_i\}$; (i.e., A is the set of players that can select c)
2 **while** $A \neq \emptyset$ and $s \xrightarrow{A} s'$, where $s'_i = c$ for $i \in A$, is not a profitable deviation **do**
3   choose some $a \in A$ such that $p_a(s) \geq p_a(s')$;
4     $A := A \setminus \{a\}$;
5 **return** A

The following lemma establishes the correctness of Algorithm 2.

Lemma 9. *Consider a strategic game that satisfies the PPM property, a joint strategy s, and a strategy c. Algorithm 2 computes a maximal coalition that can profitably deviate from s to c, if there exists one, and otherwise returns the empty set.*

Proof. First note that because of Line 4, the algorithm always terminates. Suppose that A^* is a maximal coalition that can profitably deviate to c. So $s \xrightarrow{A^*} s^*$, where $s^*_i = c$ for $i \in A^*$. Consider the execution of the above algorithm. Then $A^* \subseteq A$ after Line 1. By the PPM property, no player from A^* can be removed in Line 4, because otherwise it could not profit from the deviation $s \xrightarrow{A^*} s^*$ either. So the coalition A the algorithm returns contains A^* and a fortiori is nonempty. Hence, the **while** loop was exited because $s \xrightarrow{A} s'$, where $s'_i = c$ for $i \in A$, is a profitable deviation. By the maximality of A^*, we get $A = A^*$.

If no coalition can profitably deviate to c, then the **while** loop is exited because $A = \emptyset$ and the algorithm returns the empty set. □

This lemma and Theorem 8 allow us to derive the following result.

Theorem 13. *Consider a coordination game on an open chain of cycles. A strong equilibrium can be computed in time* $\mathcal{O}(vm^4 l)$.

Proof. By Theorem 8, it follows that for an open chain of cycles, there exists a c-improvement path of length at most $4vm^4$. Moreover, such a path consists of $\mathcal{O}(vm^4)$ single-player improvement steps and $\mathcal{O}(m)$ of c-improvement steps. By Lemma 7, executing the former steps can be done in time $\mathcal{O}(vm^4 l)$. It remains to estimate the latter.

All considered c-improvement steps are from a Nash equilibrium. So by Lemma 5, any node involved in a c-improvement step belongs to a directed simple cycle that deviated to the same colour. It follows that in any c-improvement step, nodes that deviate to two different colours cannot be adjacent to each other and so do not influence each other payoffs. Therefore, any multicolour c-improvement step can be split into a sequence of unicolour c-improvement steps (one for each deviating colour).

Consider now a Nash equilibrium s that is not a strong equilibrium. Each coordination game satisfies the PPM property, so Lemma 9 implies that by executing Algorithm 2 for each colour c, in turn we eventually find a maximal coalition that can profitably deviate from s to the same colour or determine that no such coalition exists.

Let us now estimate the time complexity of executing Algorithm 2. Executing the assignment in Line 1 can be done in $\mathcal{O}(vm)$ time. Computing the payoffs of every node in s and s' in Line 2 can be done in $\mathcal{O}(vm)$ time due to Lemma 8. The **while** loop can be reentered at most vm times, because there are at most vm nodes in A. Furthermore, because we are dealing with an open chain of cycles, each removal of a node from A affects the payoff of at most two other players. So updating the payoffs of all players in s' can be done in $\mathcal{O}(1)$ time. Therefore executing the **while** loop takes in total $\mathcal{O}(vm)$ time. This is also the time complexity of executing the algorithm, because Line 5 takes only $\mathcal{O}(1)$ time.

To find a unicolour profitable deviation from a Nash equilibrium that is not a strong equilibrium, in the worst case, Algorithm 2 has to be executed for each colour. So each such c-improvement step takes in total $\mathcal{O}(vml)$ time. As there are $\mathcal{O}(m)$ of these c-improvement steps, their execution takes in total $\mathcal{O}(vm^2l)$ time. So the execution of these steps is dominated by the executions of the already considered single-player improvement steps that take in total $\mathcal{O}(vm^4l)$ time, which is then also the time bound for computing a strong equilibrium. □

Finally, we deal with the cases of weighted DAGs and games with two colours.

Theorem 14. *Consider a coordination game on a weighted DAG. Both a Nash equilibrium and a strong equilibrium can be computed in time $\mathcal{O}(nl + |E|)$.*

Proof. Consider a weighted DAG (V, E). The procedure given in Theorem 6 first relabels the nodes using $\{1,\dots,n\}$ in such a way that for all $i,j \in \{1,\dots,n\}$ if $i < j$, then $(j \to i) \notin E$. Such a relabelling can be done in time $\mathcal{O}(n + |E|)$ by means of a topological sort of nodes using a depth first search algorithm. Next, the schedule that we will use is simply $1,\dots,n$. Because of Lemma 7, given a joint strategy, the best response for a player i can be computed in $\mathcal{O}(l + e_i)$ time, where e_i is the number of incoming edges to node i.

Thus, a Nash equilibrium can be constructed in time $\mathcal{O}(\sum_{i\in V}(l + e_i)) = \mathcal{O}(nl + |E|)$. By Theorem 6, every Nash equilibrium is also a strong equilibrium. □

Theorem 15. *Consider a coordination game on a graph (V, E) in which only two colours are used. Then we have the following:*

i. *A Nash equilibrium can be computed in time $\mathcal{O}(n + |E|)$.*
ii. *A strong equilibrium can be computed in time $\mathcal{O}(n^2 + n|E|)$.*

Proof. Given node i, we denote by e_i the number of incoming edges to i, and by e'_i the number of outgoing edges from i.

Part i. The proof of Theorem 9 provides an algorithm that follows two phases to construct a Nash equilibrium. In the first phase, it constructs a maximal sequence of profitable deviations to the first colour (called blue). And in the second phase, it does the same for the second colour (called red). Note that by Lemma 8, given a joint strategy, the payoff of player i can be computed in $\mathcal{O}(1 + e_i)$ time. Therefore, a profitable deviation from any joint strategy (if it exists) can be found in time $\sum_{i\in V}\mathcal{O}(1 + e_i) = \mathcal{O}(n + |E|)$.
This yields time complexity of $\mathcal{O}(n^2 + n|E|)$ for both the first and the second phase, because each phase consists of at most n profitable deviations. We can reduce this to $\mathcal{O}(n + |E|)$ by precomputing for every player his payoff for selecting each colour and then updating these values as players switch strategies. Formally, we proceed as follows.
For each player i, given a joint strategy of its opponents, let (r_i, b_i) be its payoffs for selecting, respectively, red and blue colours. By Lemma 8, given an initial joint strategy, these pairs of payoffs for all players can be calculated in time $\sum_{i\in V}\mathcal{O}(1 + e_i) = \mathcal{O}(n + |E|)$. In the first phase, where players switch colour from red to blue only, we simultaneously create a list L of all players i whose current colour is red and $r_i < b_i$ holds.

We then repeatedly remove a player i from L and switch its colour to blue. This change affects the payoffs of e'_i other players. More precisely, $e'_i = |\{j \in V \mid i \in N_j\}|$, and for any j such that $i \in N_j$, the pair (r_j, b_j) is updated to $(r_j - w_{i\to j}, b_j + w_{j\to i})$. If after this change $r_j < b_j$ holds and player j holds colour red, then we add player j to the list L. Note that no player has be removed from L as a result of the deviation of player i because of the PPM property of our games. Therefore, after a deviation of player i, the time needed to update all values of (r_j, b_j) and the list L is $\mathcal{O}(1 + e'_i)$.

The first phase ends when L becomes empty. Then we rebuild the list by switching the role of the colours and proceed in the analogous way. In particular, from that moment on we add a player i to the list if $b_i < r_i$.

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In each phase, each player can switch its colour at most once, so the complexity of each phase, as well as both of them, is $\sum_{i\in V}\mathcal{O}(1+e'_i)=\mathcal{O}(n+|E|)$.

Part ii. The existence of c-improvement paths of length at most $2n$ is guaranteed by Theorem 10. The algorithm follows two phases to construct a strong equilibrium. In the first phase, it constructs a maximal sequence, ξ, of profitable coalition deviations to the first colour (called blue). And in the second phase, it does the same for the second colour (called red) to construct a sequence χ. It now suffices to estimate the time complexity of computing a single c-improvement step in the sequences ξ and χ.

In each such step, a coalition is selected that deviates profitably to a single colour, blue or red; the joint strategy is modified; and the payoffs of the players are appropriately modified. Without loss of generality, we can assume that each time a maximal coalition is selected. By Lemma 9, such a coalition can be computed using Algorithm 2. So it suffices to determine the complexity of Algorithm 2 and of the computation of the new joint strategy and the modified payoffs in case of coordination games with two colours.

The complexity of executing the assignment in Line 1 is $\mathcal{O}(n)$. To evaluate the condition of the **while** loop in Line 2, we first calculate $p_i(s)$ and $p_i(s')$ for every player i. By Lemma 8, all these values and the set of players $A' := \{i\in A\,|\,p_i(s)\geq p_i(s')\}$, for which the deviation to s' is not profitable, can be calculated in time $\sum_{i\in V}\mathcal{O}(1+e_i)=\mathcal{O}(n+|E|)$. Note that the body of the **while** loop is executed as long as $A'\neq\emptyset$.

After each removal of a node $a\in A'$ from A in Line 4 (and as a result from A'), the payoffs $p_i(s')$ of at most e_a other players are affected, and by Lemma 8, updating them takes time $\mathcal{O}(1+e_a)$. At the same time, if for any of these e_a players, the deviation to s' is not longer profitable, that is, $p_i(s)\geq p_i(s')$ holds, then we add him to A'. Note that no player has to be removed from A' after a deviation of player a because of the PPM property of our games.

Now, each player is removed in Line 4 at most once, so the total time needed to execute this **while** loop is $\sum_{i\in V}\mathcal{O}(1+e'_i)=\mathcal{O}(n+|E|)$ time. Finally, Line 5 takes $\mathcal{O}(1)$ time. So for both colours, the execution of Algorithm 2 takes $\mathcal{O}(n+|E|)$ time. Once the algorithm returns the empty set, we switch the colours and move to the second phase. This phase ends when the algorithm returns the empty set. By Theorem 10, it follows that a strong equilibrium can be computed in time $\mathcal{O}(n^2+n|E|)$. □

Finally, we study the complexity of determining the existence of Nash equilibria and of strong equilibria. We already noticed in Example 1 that some coordination games have no Nash equilibria. In general, the following holds.

Theorem 16. *The Nash equilibrium existence problem in coordination games without bonuses (on unweighted graphs) is NP-complete.*

Proof. The problem is in NP, because we can simply guess a colour assignment and check whether it is a Nash equilibrium can be done in polynomial time.

To prove NP-hardness, we first provide a reduction from the 3-SAT problem, which is NP-complete, to coordination games on directed graphs with natural number weights. Assume we are given a 3-SAT formula $\phi=(a_1\vee b_1\vee c_1)\wedge(a_2\vee b_2\vee c_2)\wedge\ldots\wedge(a_k\vee b_k\vee c_k)$ with k clauses and n propositional variables $x_1,\ldots,x_n$, where each a_i,b_i,c_i is a literal equal to x_j or $\neg x_j$ for some j. We will construct a coordination game $\mathcal{G}_\phi$ of size $\mathcal{O}(k)$ with natural number weights such that $\mathcal{G}_\phi$ has a Nash equilibrium iff ϕ is satisfiable.

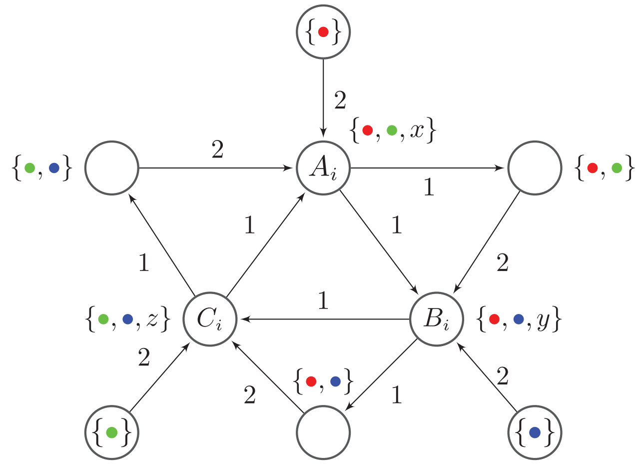
First, for every propositional variable x_i, we have a corresponding node X_i in $\mathcal{G}_\phi$ with two possible colours $\top$ and $\perp$. Intuitively, for a given truth assignment, if x_i is true, then $\top$ should be chosen for X_i, and otherwise $\perp$ should be chosen. In our construction, we make use of a gadget, denoted by $D_i(x,y,z)$, with three parameters $x,y,z\in\{\top,\perp\}$ and i used just for labelling purposes, and presented in Figure 13. This gadget behaves similarly to the game without Nash equilibrium analysed in Example 1.

What is important is that for all possible parameters values, the gadget $D_i(x,y,z)$ does not have a Nash equilibrium. Indeed, each of the nodes A_i, B_i, and C_i can always secure a payoff of two, so selecting $\top$ or $\perp$ is never a best response, and hence in no Nash equilibrium does a node choose $\top$ or $\perp$. The rest of the reasoning is as in Example 1. For any literal l, let

$$\mathrm{pos}(l) := \begin{cases}\top & \text{if } l \text{ is a positive literal,}\\ \perp & \text{otherwise.}\end{cases}$$

For every clause $(a_i\vee b_i\vee c_i)$ in ϕ, we add to the game graph $\mathcal{G}_\phi$ the $D_i(\mathrm{pos}(a_i),\mathrm{pos}(b_i),\mathrm{pos}(c_i))$ instance of the gadget. Finally, for every literal a_i, b_i, or c_i in ϕ, which is equal to x_j or $\neg x_j$ for some j, we add an edge from X_j to

Figure 13. (Color online) Gadget D_i with three parameters $x, y, z \in \{\top, \bot\}$ and three distinguished nodes A_i, B_i, C_i.



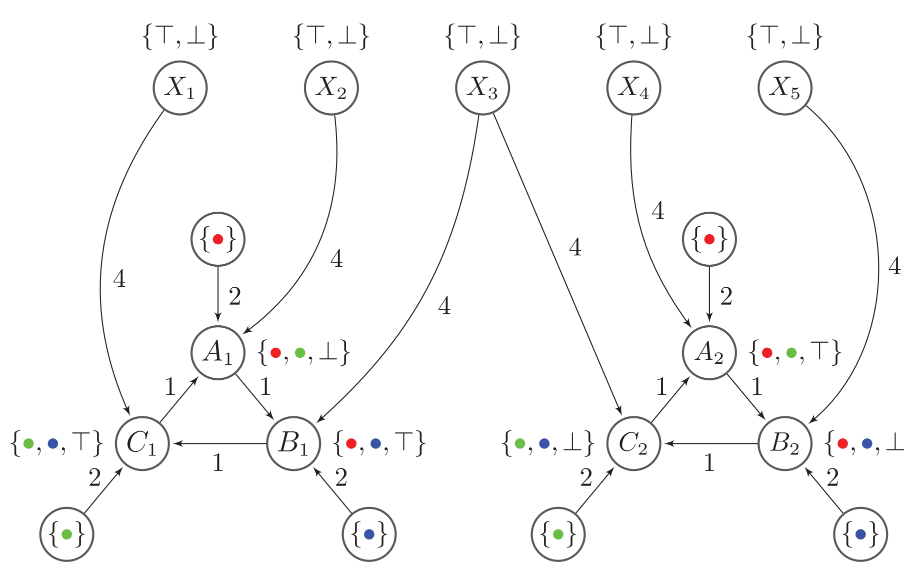
A_i, B_i, or C_i, respectively, with weight four. We depict an example game $\mathcal{G}_\phi$ in Figure 14. (This figure corrects the corresponding figure in Apt et al. [6]). We claim that $\mathcal{G}_\phi$ has a Nash equilibrium iff ϕ is satisfiable.

(⇒). Assume there is a Nash equilibrium s in the game $\mathcal{G}_\phi$. We claim that the truth assignment $\nu : \{x_1, \ldots, x_n\} \to \{\top, \bot\}$ that assigns to each x_j the colour selected by the node X_j in s makes ϕ true. Fix $i \in \{1, \ldots, k\}$. We need to show that ν makes one of the literals a_i, b_i, or c_i of the clause $(a_i \vee b_i \vee c_i)$ true.

From the above observation about the gadgets, it follows that at least one of the nodes A_i, B_i, or C_i selected in s the same colour as its neighbour X_j. Without loss of generality, suppose it is A_i. The only colour these two nodes, A_i and X_j, have in common is pos(a_i). So X_j selected in s pos(a_i), which, by the definition of ν, equals $\nu(x_j)$. Moreover, by construction x_j is the variable of the literal a_i. But $\nu(x_j) = \text{pos}(a_i)$ implies that ν makes a_i true.

(⇐). Assume ϕ is satisfiable. Take a truth assignment $\nu : \{x_1, \ldots, x_n\} \to \{\top, \bot\}$ that makes ϕ true. For all j, we assign the colour $\nu(x_j)$ to the node X_j. We claim that this assignment can be extended to a Nash equilibrium in $\mathcal{G}_\phi$.

Figure 14. (Color online) The game $\mathcal{G}_\phi$ corresponding to the formula $\phi = (\neg x_2 \vee x_3 \vee x_1) \wedge (x_4 \vee \neg x_5 \vee \neg x_3)$, where in each gadget the nodes of in-degree one are omitted.



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Fix $i \in \{1, \ldots, k\}$ and consider the $D_i(\text{pos}(a_i), \text{pos}(b_i), \text{pos}(c_i))$ instance of the gadget. The truth assignment ν makes the clause $(a_i \vee b_i \vee c_i)$ true. Suppose without loss of generality that ν makes a_i true. We claim that then it is always a unique best response for the node A_i to select the colour $\text{pos}(a_i)$.

Indeed, let j be such that $a_i = x_j$ or $a_i = \neg x_j$. Notice that the fact that ν makes a_i true implies that $\nu(x_j) = \text{pos}(a_i)$. So when node A_i selects $\text{pos}(a_i)$, the colour assigned to X_j, its payoff is four.

This partial assignment of colours can be completed to a Nash equilibrium. Indeed, remove from the directed graph of $\mathcal{G}_\phi$ all X_j nodes and the nodes that secured the payoff four, together with the edges that use any of these nodes. The resulting graph has no cycles, so by Theorem 6, the corresponding coordination game has a Nash equilibrium. Combining both assignments of colours, we obtain a Nash equilibrium in $\mathcal{G}_\phi$.

To conclude the result for coordination games without weights, notice that an edge with a natural number weight w can be simulated by adding w extra players to the game. More precisely, an edge $(i \to j)$ with the weight w can be simulated by the extra set of players $\{i_1, \ldots, i_w\}$ and the following $2 \cdot w$ unweighted edges: $\{(i \to i_1), (i \to i_2), \ldots, (i \to i_w), (i_1 \to j), (i_2 \to j), \ldots, (i_w \to j)\}$. Given a colour assignment in the original game with the weighted edges, we then assign to each of the new nodes $i_1, \ldots, i_w$ the colour set of the node i. Then the initial coordination game has a Nash equilibrium iff the new one, without weights, has one. Furthermore, the new game can be constructed in linear time. □

Corollary 2. *The strong equilibrium existence problem in coordination games without bonuses (on unweighted graphs) is NP-complete.*

Proof. It suffices to note that in the above proof, the (⇒) implication holds for a strong equilibrium as well, whereas in the proof of the (⇐) implication, by virtue of Theorem 6, actually a strong equilibrium is constructed. □

An interesting application of Theorem 16 is in the context of polymatrix games introduced in Section 2. It was shown in Simon and Apt [48] that deciding whether a polymatrix game has a Nash equilibrium is NP-complete. We can strengthen this result by showing that the problem is strongly NP-hard, that is, NP-hard even if all input numbers are bounded by a polynomial in the size of the input.

Theorem 17. *Deciding whether a polymatrix game has a Nash equilibrium is strongly NP-complete.*

Proof. Any coordination game $\mathcal{G} = (G, C)$ on an unweighted graph $G = (V, E)$ can be viewed as a polymatrix game $\mathcal{P}$ whose values of all partial payoffs functions are equal to either zero or one. Specifically, the set of players in $\mathcal{P}$ is the same as in $\mathcal{G}$, that is, V. The strategy set S_i of player i is simply $C(i)$. We define

$$a^{ij}(s_i, s_j) := \begin{cases} 1 & \text{if } j \in N_i \text{ and } s_i = s_j, \\ 0 & \text{otherwise,} \end{cases}$$

where, as before, N_i is the set of neighbours of node i in the assumed directed graph G. Notice that the payoffs in both games are the same because for any joint strategy $s = (s_1, \ldots, s_n)$, $p_i^{\mathcal{P}}(s) = \sum_{j \neq i} a^{ij}(s_i, s_j) = |\{j \in N_i | s_i = s_j\}| = p_i^{\mathcal{G}}(s)$. NP-hardness follows, because this problem was shown to be NP-hard for coordination games on unweighted graphs in Theorem 16. As all numerical inputs are assumed to be zero or one, they are obviously bounded by a polynomial in the size of the input. So strong NP-hardness follows. As shown in Simon and Apt [48], deciding whether a given polymatrix game has a Nash equilibrium is in NP, which, together with the above, implies strong NP-completeness of this problem. □

8. Conclusions

In this paper, we studied natural coordination games on weighted directed graphs in the presence of bonuses representing individual preferences. In our presentation, we focused on the existence of Nash and strong equilibria and on ways of computing them efficiently if they exist. To this end, we extensively used improvement and coalitional improvement (in short c-improvement) paths that can be seen as an instance of a local search.

We identified natural classes of graphs for which coordination games have improvement or c-improvement paths of polynomial length. For simple cycles, these results are optimal in the sense that lifting any of the imposed restrictions may result in a coordination game without a Nash equilibrium.

In proving our results, we used increasingly more complex ways of constructing (c-)improvement paths of polynomial length. In particular, the construction in the proof of Theorem 5 relied on the constructions considered in the proofs of Theorems 1 and 3.

For the class of graphs we considered, local search in the form of the (c-)improvement paths turns out to be an efficient way of computing a Nash equilibrium or a strong equilibrium. But this is not true in general. In fact, Example 8 shows that this form of local search does not guarantee that a Nash equilibrium or a strong equilibrium can be found, even when the underlying graph is strongly connected and all nodes have the same set of colours. We also showed that the existence problem both for Nash and strong equilibria is NP-complete even for the coordination games on unweighted graphs and without bonuses.

There are other directed graphs than the ones we considered here, for which the coordination games are weakly or c-weakly acyclic. For example, we proved in Apt et al. [8] that the coordination games on complete graphs have the c-FIP and the proof carries through to the complete directed graphs. In turn, in Apt et al. [6], we showed that that every coordination game on a directed graph in which all strongly connected components are simple cycles is c-weakly acyclic. Furthermore, in Simon and Wojtczak [49], weighted open chains of cycles, closed chains of cycles, and simple cycles with appropriate cross-edges were considered.

For some of these classes of graphs some problems remain open, for instance, the existence of finite c-improvement paths for weighted open chains of cycles. A rigorous presentation of the proofs of weak acyclicity and c-weak acyclicity for the corresponding coordination games is lengthy and quite involved. We plan to present them in a sequel paper. Finally, we believe that the following generalisation of several of our results is true.

Conjecture 1. *Coordination games on graphs with all nodes of in-degree ≤ 2 are c-weakly acyclic.*

Extensive computer simulations seem to support this conjecture. However, our techniques do not seem to adapt easily to this bigger class of graphs.

Next, by Nash's theorem, a mixed strategy Nash equilibrium always exists in coordination games irrespective of the underlying graph structure. However, the complexity of finding one is an intriguing open problem. This problem is known to be hard for the complexity class Polynomial Parity Arguments on Directed graphs (PPAD-complete) PPAD-hard for various restricted classes of polymatrix games (Cai and Daskalakis [16], Rubinstein [47]; so it is unlikely to be solvable in polynomial time), but generalising this result to coordination games will be very challenging because of the special structure of players' payoffs. Still, we conjecture that this is indeed possible.

Conjecture 2. *Finding a mixed Nash equilibrium in coordination games is a PPAD-hard problem.*

Finally, note that in Section 7 we assumed that all weights of the graph edges are natural numbers. It is known that allowing weights to be rational may change the complexity of the studied computational problem; for example, the well-known knapsack and partition problems become strongly NP-complete (Wojtczak [53]). However, most computational problems for coordination games with rational weights can be reduced in polynomial time to the same problem for coordination games with integer weights by simply multiplying all the weights by the least common multiple of all the weights' denominators. This results in an exponential blowup of value of the numbers, but only in a polynomial increase in their size when they are represented in the standard binary notation.

It is easy to see that a joint strategy is a Nash equilibrium or a strong Nash equilibrium in the original game if and only if it is in the new game with the integer weights. So as long as such a transformation results in a coordination game of the type listed in Table 1, we get a polynomial time algorithm for finding a Nash equilibrium or a strong Nash equilibrium in the original game. In particular, these problems for the coordination games with only two colours or on DAGs can always be solved in polynomial time even when the weights are rational. Notice that the problem of checking for the existence of a Nash equilibrium in a coordination game with rational weights is still in NP (simply guess a joint strategy and check whether it is a Nash equilibrium) and at the same time it is NP-hard, as we already established it for the coordination games with the weights equal to zero or one. So this problem is strongly NP-complete.

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Appendix
We provide here proofs of Lemmas 3 and 4 (restated below).

Lemma 3. *The improvement path constructed in Line* 6 *of Algorithm* 1 *modifies the grades of $\mathcal{C}_j$ and its adjacent cycles $\mathcal{C}_{j-1}$ and $\mathcal{C}_{j+1}$, if they exist, as explained in Figures* 4, 5, 6, 7, *and* 8.

Remainder of the Proof of Lemma 3. To complete the proof of Lemma 3, we provide justification of the changes of the grades in Figures 4, 5, 6, 7, and 8.

Figure 4.

Case 1. The initial grade of $\mathcal{C}_j$ is –. This corresponds to the situation at the beginning of Phase 2 in the proof of Theorem 1 when exactly one node has a bonus. This phase starts with the node $[j,2]$ and ends after at most $n-1$ steps. So the colour of $[j,1]$ is not modified and consequently the payoff to the down-link node $[j+1,k]$ of $\mathcal{C}_{j+1}$ is not modified. Furthermore, the new grade of $\mathcal{C}_j$ can be either + or U+ depending whether at the end of this phase the colours of $[j,v]$ and $[j,1]$ differ.

Case 2. The initial grade of $\mathcal{C}_j$ is U−. The reasoning is the same as in **Case 1**. However, the colour of $[j,v]$ is now not modified. The reason is that the only colour that is propagated is that of $[j,1]$ and initially it is also the colour of $[j,v]$. So the new grade of $\mathcal{C}_j$ is now U+ .

Case 3. The initial grade of $\mathcal{C}_j$ is ?. This corresponds to the situation at the beginning of Phase 1 in the proof of Theorem 1 when exactly one node has a bonus. The constructed improvement path ends after at most $2n-1$ steps, so in the process the colour of $[j,1]$ can change. If it does, then the grade of the cycle $\mathcal{C}_{j+1}$ can change arbitrarily. In particular, it can become U+ or U− if the down-link node of $\mathcal{C}_{j+1}$ is $[j+1,v]$. Furthermore, the new grade of $\mathcal{C}_j$ can be either + or U+ , for the same reasons as in **Case 1**.

Figure 5.

The assumption that the grade of $\mathcal{C}_j$ is initially U− means that initially the colours of $[j,1]$ and its predecessor $[j,v]$ in this cycle are the same. Then the construction in Line 6 of the improvement path for the considered coordination game for $\mathcal{C}_j$ with bonuses for the link nodes corresponds to any update scenario presented in Phase 2 of the proof of Theorem 3 that starts with *i*. There are six such scenarios to consider.

Case [i]. This means that the propagation of the colour of the up-link node of $\mathcal{C}_j$ stops before the down-link node of $\mathcal{C}_j$ is reached. So the improvement path constructed in Line 6 does not change the colours of the link nodes of $\mathcal{C}_j$ and of the predecessor $[j,v]$ of the up-link node $[j,1]$. Hence, the grades of $\mathcal{C}_{j-1}$ and $\mathcal{C}_{j+1}$ remain unchanged, and the grade of $\mathcal{C}_j$ becomes U+.
The remaining cases consider the situations in which the down-link node of $\mathcal{C}_j$ switches to another colour. We now claim that in these cases the grade of $\mathcal{C}_{j-1}$ is initially +. Indeed, if this grade is initially U+, then the payoff to the up-link node $[j-1,1]$ of $\mathcal{C}_{j-1}$ is ≥ 1. But $[j-1,1]$ is also the down-link node of $\mathcal{C}_j$, so the claim follows by Lemma 1.

Case [ii]. This means that the propagation of the new colour of the up-link node of $\mathcal{C}_j$ stops between the down-link and up-link nodes of $\mathcal{C}_j$ and that the down-link node adopted the colour of the up-link node. So the improvement path constructed in Line 6 does not change the colours of $[j,1]$ and its predecessor $[j,v]$.
Hence, the grade of $\mathcal{C}_j$ becomes U+, and the grade of $\mathcal{C}_{j+1}$ remains unchanged. On the other hand, the grade of $\mathcal{C}_{j-1}$ can remain unchanged or change from + to –, U+, or U− because of the new colour of the up-link node $[j-1,1]$ of $\mathcal{C}_{j-1}$.

Case [io]. This means that the propagation of the colours stops between the down-link and up-link nodes of $\mathcal{C}_j$, but now the down-link node (so $[j-1,1]$) has adopted the colour of its predecessor $[j-1,v]$ in $\mathcal{C}_{j-1}$. So, as in the previous case, the grade of $\mathcal{C}_{j+1}$ remains unchanged.
However, the grade of $\mathcal{C}_j$ can now also become + if this propagation of the colours changes the colour of the predecessor $[j,v]$ of the up-link node $[j,1]$. Furthermore, the grade of $\mathcal{C}_{j-1}$ now changes from + to U− or U+ because the new colour of $[j-1,1]$ is now the colour of $[j-1,v]$ and as a result the node $[j-1,2]$ can now become the only node that does not play a best response.

Case [ioi]. This means that the propagation of the colours now stops between the up-link and down-link nodes of $\mathcal{C}_j$, but now the down-link node (so $[j-1,1]$) has adopted the colour of its predecessor $[j-1,v]$ in $\mathcal{C}_{j-1}$ and subsequently the up-link node $[j,1]$ of $\mathcal{C}_j$ adopted the colour of its predecessor $[j,v]$ in $\mathcal{C}_j$. So the grade of $\mathcal{C}_j$ now becomes U+.
Furthermore, the grade of $\mathcal{C}_{j-1}$ now changes from + to U− or U+ for the same reasons as in the previous case. Finally, the grade of $\mathcal{C}_{j+1}$ can now change arbitrarily for the same reasons as in Case 3 concerning Figure 4.

Case [ioo]. This case is similar to the previous one, with the difference that in the second round of the propagation of the colours the up-link node $[j,1]$ of $\mathcal{C}_j$ adopted the colour of its predecessor in $\mathcal{C}_{j+1}$ instead of the colour of its predecessor $[j,v]$ in $\mathcal{C}_j$. Consequently, the grade of $\mathcal{C}_j$ now becomes +. Furthermore, the grade of $\mathcal{C}_{j-1}$ can now change from + to U− or U+, whereas the grade of $\mathcal{C}_{j+1}$ can now change arbitrarily, both for the same reason as in the previous case.

Case [iooi]. This case cannot occur. Indeed, it would imply that the down-link node in $\mathcal{C}_j$ first switches to the colour of its predecessor in $\mathcal{C}_{j-1}$ and later switches to different colour. But the second switch is not possible because of Lemma 1.

Figure 6.

The assumption that the grade of $\mathcal{C}_j$ is initially – means that initially the colours of $[j,1]$ and its predecessor $[j,v]$ in this cycle differ. Then the construction in Line 6 of the improvement path for the considered coordination game for $\mathcal{C}_j$ with bonuses for the link nodes corresponds to any update scenario presented *in* Phase 2 of the proof of Theorem 3 that starts with *o*. There are four such scenarios to consider.

Case [o]. The reasoning is the same as in Case **[i]** above with the difference that the grade of $\mathcal{C}_j$ becomes now + as the colours of $[j,1]$ and $[j,v]$ do not change and hence remain different.
In the remaining cases the grade of $\mathcal{C}_{j-1}$ is initially + for the reasons given after Case **[i]** above.**Case [oi].** This case is analogous to Case **[ii]** above. In particular, the improvement path constructed in Line 6 does not change the colours of $[j,1]$ and its predecessor $[j,v]$. Hence, the grade of $\mathcal{C}_j$ becomes + and the grade of $\mathcal{C}_{j+1}$ remains unchanged, whereas the grade of $\mathcal{C}_{j-1}$ can remain unchanged or change from + to –, U+, or U−.

Case [oo]. This case is analogous to Case **[io]** above. So, as in that case, the grade of $\mathcal{C}_{j+1}$ remains unchanged, and the grade of $\mathcal{C}_{j-1}$ now changes from + to U− or U+. However, the grade of $\mathcal{C}_j$ can now also become U+ if this propagation of the colours changes the colour of $[j,v]$ to the colour of its successor $[j,1]$.

Case [ooi]. This case is analogous to Case **[ioi]** above. So, as in that case, the grade of $\mathcal{C}_j$ now becomes U+, the grade of $\mathcal{C}_{j-1}$ changes from + to U− or U+, and the grade of $\mathcal{C}_{j+1}$ can change arbitrarily.

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Figure 7.
This case corresponds to the situation at the beginning of Phase 1 in the proof of Theorem 3. The constructed improvement path ends after at most $3n$ steps, so in the process the colour of $[j, 1]$ can change. Therefore, as in Case 3 concerning Figure 4, the grade of the cycle $\mathcal{C}_{j+1}$ can change arbitrarily, whereas the grade of $\mathcal{C}_j$ can become either + or U+ .
Finally, if initially the grade of $\mathcal{C}_{j-1}$ is +, then, as in Case **[ii]**, its grade can remain unchanged or change to –, U+, or U–. Furthermore, if initially this grade is U+, then by the argument used in the proof of Lemma 2, the grade does not change.

Figure 8.
We reduce the analysis for this case to the previous three cases by extending the open chain with a new cycle $\mathcal{C}_{m+1}$ in which all new nodes have to their disposal colours that all differ from the colours available to the nodes of $\mathcal{C}_m$. Then, in Algorithm 1, the bonus function for the up-link node of $\mathcal{C}_m$ is always zero on the colours available to it, and consequently for $j = m$, the improvement path constructed in Line 6 of Algorithm 1 is the same as for the original open chain. So for the case when $j = m$, we can use Figures 5, 6, and 7 with the last columns always omitted. This yields Figure 8.
A perceptive reader can inquire why the row corresponding to the case **[ioo]** is missing. The reason is that it deals with the situation when the up-link node of $\mathcal{C}_j$ switches to an outer colour, that is, a colour of its predecessor in $\mathcal{C}_{j+1}$. But for $j = m$, this cannot happen by the choice of the colours for the new nodes.
We use below the following observation.

Claim A.1. *Let s and s' be two joint strategies such that $\mu(s) = (guard(s), 0, |prefix(s)|, -NBR(s))$, $guard(s) \le guard(s')$, and $|prefix(s)| < |prefix(s')|$. Then $\mu(s) <_{lex} \mu(s')$ holds.*

Proof. Either $\mu(s') = (guard(s'), 0, |prefix(s')|, -NBR(s'))$ or $\mu(s') = (guard(s'), 1, \dots)$ and in both cases $\mu(s) <_{lex} \mu(s')$ holds. □

Lemma 4. *The progress measure $\mu(s)$ increases w.r.t. the lexicographic ordering $<_{lex}$ each time one of the updates presented in Figures 4, 5, 6, 7, and 8 takes place.*

Proof. We check using Lemma 3 that $\mu(s)$ increases w.r.t. the lexicographic ordering $<_{lex}$ each time one of the updates presented in Figures 4, 5, 6, 7, and 8 takes place. So throughout the analysis, we assume that $j = NBR(s)$. Let s' denote the new joint strategy computed in Line 8 of the algorithm. Lemma 2 implies that $guard(s) \le guard(s')$. Furthermore, thanks to the definition of $\mu(s')$, we can assume that s' is not a Nash equilibrium. We consider each figure separately.

Figure 4.
Then $j = 1$ and $guard(s) = 0$.

Case 1. The new grade of $\mathcal{C}_j$ is U+. Then $guard(s) < guard(s')$, and hence $\mu(s) <_{lex} \mu(s')$.

Case 2. The new grade of $\mathcal{C}_j$ is +.

Subcase 1. $\mu(s) = (guard(s), 1, 0, -NBR(s))$.

Then the initial grade of $\mathcal{C}_j$ is – and $prefix(s)$ contains U+, say, at position h. Hence, $prefix(s')$ also contains U+ at position h, and consequently $h \le guard(s')$. But $guard(s) = 0$, so $\mu(s) <_{lex} \mu(s')$.

Subcase 2. $\mu(s) = (guard(s), 0, |prefix(s)|, -NBR(s))$.

If the initial grade of $\mathcal{C}_j$ is –, then $|prefix(s)| < |prefix(s')|$ because by assumption, s' is not a Nash equilibrium. Otherwise, the initial grade of $\mathcal{C}_j$ is ? and then $|prefix(s)| = 1$ by the definition of $prefix(s)$, whereas $1 < |prefix(s')|$. So in both cases, by Claim A.1, $\mu(s) <_{lex} \mu(s')$.

Figure 5.
By definition, $\mu(s) = (guard(s), 1, 0, -NBR(s))$.

Case 1. The new grade of $\mathcal{C}_{j-1}$ is +. Then the case **[i]** or **[ii]** applies and hence the new grade of $\mathcal{C}_j$ is U+. So $guard(s) < guard(s')$, and hence $\mu(s) <_{lex} \mu(s')$.

Case 2. The new grade of $\mathcal{C}_{j-1}$ is U+. Then $guard(s) < guard(s')$, and hence $\mu(s) <_{lex} \mu(s')$.

Case 3. The new grade of $\mathcal{C}_{j-1}$ is –. Then the case **[ii]** applies, and hence the new grade of $\mathcal{C}_j$ is U+. So $\mu(s') = (guard(s'), 1, 0, -NBR(s'))$. But $guard(s) \le guard(s')$ and $-NBR(s) < -NBR(s')$, so $\mu(s) <_{lex} \mu(s')$.

Case 4. The new grade of $\mathcal{C}_{j-1}$ is U–. Then $\mu(s') = (guard(s'), 1, 0, -NBR(s'))$ and $\mu(s) <_{lex} \mu(s')$ for the same reasons as in the previous case.

Figure 6.

Case 1. For the joint strategy s, we have $\mu(s) = (guard(s), 1, 0, -NBR(s))$ and $prefix(s)$ contains – at position j, so it contains U+ at some position $h > j$. Moreover, by the definition of $prefix(s)$, all positions in it between j and h are + or U+.
So if the new grade of $\mathcal{C}_{j-1}$ is + or U+, then $j < guard(s')$, and hence $guard(s) < guard(s')$ because $guard(s) < NBR(s) = j$. So $\mu(s) <_{lex} \mu(s')$. Otherwise, the new grade of $\mathcal{C}_{j-1}$ is – or U–. If it is –, then $prefix(s')$ contains U+ at the position $h > j - 1$. So in both cases, $\mu(s') = (guard(s'), 1, 0, -NBR(s'))$. But $-NBR(s) < -NBR(s')$, so $\mu(s) <_{lex} \mu(s')$.

Case 2. For the joint strategy s, we have $\mu(s) = (guard(s), 0, |prefix(s)|, -NBR(s))$. If the new grade of $\mathcal{C}_{j-1}$ is + or U+, then $|prefix(s)| < |prefix(s')|$ because we assumed that s' is not a Nash equilibrium. So, by Claim A.1, $\mu(s) <_{lex} \mu(s')$. If the new grade of $\mathcal{C}_{j-1}$ is − or U−, then $guard(s) = guard(s')$ and $|prefix(s)| = |prefix(s')|$, but $-NBR(s) < -NBR(s')$, so $\mu(s) <_{lex} \mu(s')$.

Figure 7.

By the definition $prefix(s)$ ends with ?, so $|prefix(s)| = j$ and $\mu(s) = (guard(s), 0, |prefix(s)|, -NBR(s))$.
If the new grade of $\mathcal{C}_{j-1}$ is + or U+, then $j < |prefix(s')|$, so by Claim A.1, $\mu(s) <_{lex} \mu(s')$. If the new grade of $\mathcal{C}_{j-1}$ is − or U−, then $guard(s) = guard(s')$, $|prefix(s)| \le |prefix(s')|$ and $-NBR(s) < -NBR(s')$, so $\mu(s) <_{lex} \mu(s')$.

Figure 8.

The arguments for each case coincide with the arguments given for the corresponding cases concerning Figures 5, 6, and 7.

Endnote

[1] See http://www.beldensolutions.com/en/Company/Press/PR103EN0609/index.phtml.

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