Stochastic Approximation Proximal Method of Multipliers for Convex Stochastic Programming^{*}

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Abstract. This paper considers the problem of minimizing a convex expectation function over a closed convex set, coupled with a set of inequality convex expectation constraints. We present a new stochastic approximation type algorithm, namely the stochastic approximation proximal method of multipliers (PMMSopt) to solve this convex stochastic optimization problem. We analyze regrets of a stochastic approximation proximal method of multipliers for solving convex stochastic optimization problems. Under mild conditions, we show that this algorithm exhibits $O(T^{-1/2})$ regret for both objective reduction and constraint violation if parameters in the algorithm are properly chosen, when the objective and constraint functions are generally convex, where T denotes the number of iterations. Moreover, we show that, with at least $1 - e^{-T^{1/4}}$ probability, the algorithm has no more than $O(T^{-1/4})$ objective regret and no more than $O(T^{-1/8})$ constraint violation regret. To the best of our knowledge, this is the first time that such a proximal point method for solving expectation constrained stochastic optimization is presented in the literature.

Key words. stochastic approximation, proximal method of multipliers, objective regret, constraint violation regret, high probability regret bound, convex stochastic optimization. **AMS Subject Classifications(2000):** 90C30.

1 Introduction

In this paper, we consider the following stochastic optimization problem

$$\min_{\substack{x \in X_0 \\ \text{s.t.} }} f(x) = \mathbb{E}[F(x,\xi)]$$
s.t.
$$g_i(x) = \mathbb{E}[G_i(x,\xi)] \le 0, i = 1, \dots, p.$$
(1.1)

Here $X_0 \subset \Re^n$ is a nonempty bounded closed convex set, ξ is a random vector whose probability distribution P is supported on set $\Xi \subseteq \Re^q$ and $F : \mathcal{O}_0 \times \Xi \to \Re, G_i : \mathcal{O}_0 \times \Xi \to \Re$,

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i = 1, ..., p, where $\mathcal{O}_0 \subset \Re^n$ is an open bounded convex set containing X_0 . Let Φ be the feasible region of Problem (1.1):

$$\Phi = \{x \in X_0 : g_i(x) \le 0, i = 1, \dots, p\}.$$

We assume that expectations

$$\mathbb{E}[F(x,\xi)] = \int_{\Xi} F(x,\xi) dP(\xi), \ \mathbb{E}[G_i(x,\xi)] = \int_{\Xi} G_i(x,\xi) dP(\xi), i = 1, \dots, p$$

are well defined and finite valued for every $x \in \mathcal{O}_0$. Moreover, we assume that the functions $F(\cdot,\xi)$ and $G_i(\cdot,\xi)$, $i = 1, \ldots, p$ are continuous and convex on \mathcal{O}_0 . Denote $G(x,\xi) = (G_1(x,\xi),\ldots,G_p(x,\xi))^T$ and $g(x) = (g_1(x),\ldots,g_p(x))^T$, then

$$g(x) = \int_{\Xi} G(x,\xi) dP(\xi).$$

It is well-known that a computational difficulty of solving stochastic optimization problem (1.1) is that expectation is an multidimensional integral and it cannot be computed with a high accuracy for large dimension q. The aim of this paper is to construct a stochastic approximation proximal method of multipliers for solving Problem (1.1). To this end we make the following assumptions.

- (A1) It is possible to generate an i.i.d. sample ξ_1, ξ_2, \ldots , of realizations of random vector ξ .
- (A2) There is an oracle, which, for any point $(x,\xi) \in \mathcal{O}_0 \times \Xi$ returns stochastic subgradients $v_0(x,\xi), v_1(x,\xi), \ldots, v_p(x,\xi)$ of $F(x,\xi), G_1(x,\xi), \ldots, G_p(x,\xi)$ such that

$$v_i(x) = \mathbb{E}[v_i(x,\xi)], i = 0, 1, \dots, p$$

are well defined and are subgradients of $f(\cdot), g_1(\cdot), \ldots, g_p(\cdot)$ at x, respectively, i.e., $v_0(x) \in \partial f(x), v_i(x) \in \partial g_i(x), i = 1, \ldots, p$.

(A3) Let $D_0 > 0$, $\nu_f > 0$ and $\nu_g > 0$ such that

$$||x' - x''|| \le D_0, \forall x', x'' \in X_0$$

and

$$F(x',\xi) - F(x'',\xi) \le \nu_f, \|G(x,\xi)\| \le \nu_g, \forall x', x'' \in \mathcal{O}_0, \xi \in \Xi.$$

(A4) Let $\kappa_f > 0$ and $\kappa_g > 0$ such that

$$\|v_0(x,\xi)\| \le \kappa_f, \ \|v_i(x,\xi)\| \le \kappa_g, i = 1, \dots, p, \forall (x,\xi) \in \mathcal{O}_0 \times \Xi$$

where $v_0(x,\xi)$ is a stochastic subgradient of $F(x,\xi)$ and $v_i(x,\xi)$ is a stochastic subgradient of $G_i(x,\xi), i = 1, ..., p, (x,\xi) \in \mathcal{O}_0 \times \Xi$.

(A5) There exist $\epsilon_0 > 0$ and $\hat{x} \in X_0$ such that

$$g_i(\widehat{x}) \le -\epsilon_0, \ i = 1, \dots, p$$

The stochastic approximation (SA) technique is going back to the pioneering paper by Robbins and Monro [16]. Since then SA algorithms, due to low demand for computer memory and cheap computation cost at every iteration, became widely used in stochastic optimization and online optimization, see, e.g. [11], [19] and [18]. In the classical analysis of the SA algorithm, initiated from the works [2] and [17], it is assumed that $f(\cdot)$ is twice continuously differentiable and strongly convex and in the case when the minimizer of fbelongs to the interior of Φ , exhibits asymptotically optimal rate of convergence $\mathbb{E}[f(x^t) - f^*] = O(t^{-1})$, where x^t is t-th iterate and f^* is the minimal value of f(x) over $x \in \Phi$. This algorithm, however, is very sensitive to a choice of the respective stepsizes. For overcoming this drawback, an important improvement of the SA method was developed by Polyak [14] and Polyak and Juditsky [15], where longer stepsizes were suggested with consequent averaging of the obtained iterates. Adopting the averaging technique to iterates generated by (under our notations)

$$x^{j+1} = \Pi_{\Phi}(x^j - \gamma_j v_0(x^j, \xi_j)), \tag{1.2}$$

Nemirovski, Juditsky, Lan, and Shapiro [10] shows that, without assuming smoothness and strong convexity of the objective function, the convergence rate is $O(t^{-1/2})$. This paper also demonstrates that a properly modified SA approach can be competitive and even significantly outperform the SAA method for a certain class of convex stochastic problems. After the seminal work of [10], there are many significant results appeared, even for nonconvex stochastic optimization problems, see [8],[7],[3],[4], [5] and [6]. Among all mentioned works, the feasible region set is an abstract closed convex set, none of these SA algorithms are applicable to expectation constrained problems. The computation of projection Π_{Φ} is quite easy only when Φ is of a simple structure. However, when Φ is defined by (1.1), the computation of projection Π_{Φ} is prohibitive, this is a difficult work. Therefore, it is quite important to obtain a numerical method with lower iteration complexity of both objective and constraint violation.

For stochastic optimization problems with expectation constraints, Yu et. al [20] proposed an algorithm that achieves $O(T^{-1/2})$ expected regret and constraint violations and $O(T^{-1/2} \log T)$ high probability regret and constraint violations. Lan and Zhou [9] proposed a cooperative stochastic approximation type algorithm inspired by Polayk's subgradient method, which exhibits $O(T^{-1/2})$ convergences in terms of objective value and constraint violation. In [1], the authors presented a novel Constraint Extrapolation method for solving convex functional constrained problems which includes the stochastic optimization problem (1.1) as a special case. The method is a single-loop primal-dual type method, which utilizes linear approximations of the constraint functions to define the extrapolation step and also exhibits $O(T^{-1/2})$ convergence results.

A natural way to handle constraints for constrained optimization problems is to use augmented Lagrangian, which results in proximal point methods. It is well-known that Rockafellar [13] proposed three proximal point methods for convex programming, namely the proximal point method developed in [12] applied to maximum monotone inclusions of the primal optimality, the dual optimality and the saddle point optimality. The augmented Lagrange method, just the proximal point method applied to the dual optimality, has been studied deeply not only for convex optimization but also for non-convex optimization. And the proximal point method for the primal optimality has been extensively implemented for solving various structured convex optimization problems. However, the proximal point method for the saddle point optimality, the so-called proximal method of multipliers by Rockafellar [13], has not been paid much attention.

In this paper, we study the proximal method of multipliers for stochastic convex problem, and analyze its regret bounds as well as probability guarantee for both objective reduction and constraint violation.

For i.i.d. sample ξ_1, ξ_2, \ldots of realizations of random vector ξ . Consider the following convex optimization problem

min
$$F(x, \xi_t)$$

s.t. $G_i(x, \xi_t) \le 0, i = 1, \dots, p.$ (1.3)

The augmented Lagrangian function is defined by

$$\mathcal{L}^{t}_{\sigma}(x,\lambda) := F(x,\xi_{t}) + \frac{1}{2\sigma} \left[\|\Pi_{\Re^{p}_{+}}(\lambda + \sigma G(x,\xi_{t}))\|^{2} - \|\lambda\|^{2} \right], \ \forall (x,\lambda) \in \Re^{n} \times \Re^{p}.$$
(1.4)

Then PMMSopt for Problem (1.1) may be described as follows.

PMMSopt: A proximal method of multipliers for solving Problem (1.1).

Step 0 Input $\lambda^0 = 0 \in \Re^p$ and $x^0 \in \Re^n$. Set t := 0.

Step 1 Set

$$x^{t+1} = \arg\min\left\{\mathcal{L}_{\sigma}^{t}(x,\lambda^{t}) + \frac{\alpha}{2} \|x - x^{t}\|^{2}, x \in X_{0}\right\}$$

$$\lambda^{t+1} = [\lambda^{t} + \sigma G(x^{t+1},\xi_{t})]_{+}.$$
(1.5)

Step 2 Set t := t + 1 and go to Step 1.

In the above algorithm, $[y]_+ = \prod_{\Re_+^p} [y]$ denotes the projection of y on to \Re_+^p for any $y \in \Re^p$. Note that the iterations $x^t = x^t(\xi_{[t-1]})$ and $\lambda^t = \lambda^t(\xi_{[t-1]})$ are mappings of the history $\xi_{[t-1]} = (\xi_1, \ldots, \xi_{t-1})$ of the generated random process and hence are random.

As far as we are concerned, the main contributions of this paper can be summarized as follows.

- (1) When $\sigma = T^{-1/2}$, $\alpha = T^{1/2}$, under mild assumptions, it is proved that the regret of objective function of T iterations is of order $O(T^{-1/2})$, and the regret of constraint violation of T iterations is of order $O(T^{-1/2})$.
- (2) When $\sigma = T^{-1/2}$, $\alpha = T^{1/2}$, under mild assumptions, it is proved that the following probability guarantees hold:

$$\Pr\left[\frac{1}{T}\sum_{t=0}^{T-1}G_i(x^t,\xi_t) \le \omega_c(T)\right] \ge 1 - e^{-T^{1/4}}$$
(1.6)

for all $i \in \{1, \ldots, p\}$, with

$$\omega_c(T) = \left[8 \left(1 + 2\sqrt{p}\kappa_g^2 \right) \frac{\nu_g^2}{\epsilon_0} \right] T^{-1/4} + o(T^{-1/4});$$

and

$$\Pr\left[\frac{1}{T}\sum_{t=0}^{T-1}F(x^t,\xi_t) \le \frac{1}{T}\sum_{t=0}^{T-1}F(\widehat{x},\xi_t) + \omega_o(T)\right] \ge 1 - e^{-T^{1/4}},\tag{1.7}$$

with

$$\omega_o(T) = \frac{8\sqrt{2}\nu_g^3}{\epsilon_0} T^{-1/8} + o(T^{-1/8})$$

The remaining parts of this paper are organized as follows. In Section 2, we develop properties of PMMSopt, which play a key role in the analysis for objective regret, constraint violation regret. In Section 3, we establish bounds of objective regret, constraint violation regret of PMMSopt for Problem (1.1). In Section 4, we develop probability guarantees for objective reduction and constraint violation of PMMSopt. We draw a conclusion and give some discussions in Section 5.

2 Properties of PMMSopt

In this section, we develop properties of PMMSopt, which will be used in the analysis for objective regret and constraint violation regret.

Lemma 2.1 Let (x^t, λ^t) be generated by PMMSopt and Assumption A(1)–Assumption (A3) be satisfied. Then

$$\|\lambda^{t+1}\|^{2} \leq \|\lambda^{t}\|^{2} + 2\sigma\langle\lambda^{t}, G(x^{t+1}, \xi_{t})\rangle + \sigma^{2}\nu_{g}^{2},$$
(2.1)

and

$$\|\lambda^t\| - \sigma\nu_g \le \|\lambda^{t+1}\| \le \|\lambda^t\| + \sigma\nu_g.$$
(2.2)

Proof. Noting that for any $a \in \Re$, $[a]^2_+ \leq a^2$, we have

$$\begin{aligned} \|\lambda^{t+1}\|^2 &= \sum_{i=1}^p [\lambda_i^t + \sigma G_i(x^{t+1}, \xi_t)]_+^2 \\ &\leq \sum_{i=1}^p [\lambda_i^t + \sigma G_i(x^{t+1}, \xi_t)]^2 \\ &= \sum_{i=1}^p \left([\lambda_i^t]^2 + 2\lambda_i^t \sigma G_i(x^{t+1}, \xi_t) + \sigma^2 G_i(x^{t+1}, \xi_t)^2 \right) \\ &\leq \|\lambda^t\|^2 + 2\sigma \langle \lambda^t, G(x^{t+1}, \xi_t) \rangle + \sigma^2 \nu_g^2. \end{aligned}$$

It follows from the nonexpansion property of the projection $\Pi_{\Re^p_+}(\cdot)$, we have

$$\|\lambda^{t+1} - \lambda^t\| = \|[\lambda^t + \sigma G(x^{t+1}, \xi_t)]_+ - [\lambda^t]_+\| \le \sigma \|G(x^{t+1}, \xi_t)\|_+$$

which implies (2.2). The proof is completed.

Lemma 2.2 Let (x^t, λ^t) be generated by PMMSopt and Assumption (A1)–Assumption (A4) be satisfied. Then, for any stochastic subgradient $v_i(x^t, \xi_t)$ of $G_i(\cdot, \xi_t)$ at x^t , i = 1, ..., p,

$$\sum_{t=0}^{T-1} G_i(x^t, \xi_t) \leq \frac{1}{\sigma} \lambda_i^T + \sum_{t=0}^{T-1} \|v_i(x^t, \xi_t)\| \|x^{t+1} - x^t\|$$
(2.3)

and

$$\mathbb{E}\left[\sum_{t=0}^{T-1} G_i(x^t, \xi_t)\right] \le \frac{1}{\sigma} \mathbb{E}[\lambda_i^T] + \kappa_g \sum_{t=0}^{T-1} \mathbb{E}[\|x^{t+1} - x^t\|].$$
(2.4)

Proof. From the definition $\lambda_i^{t+1} = [\lambda_i^t + \sigma G_i(x^{t+1}, \xi_t)]_+$, we have from the convexity of $G_i(\cdot, \xi_t)$, for any stochastic subgradient $v_i(x^t, \xi_t)$ of $G_i(\cdot, \xi_t)$ at x^t that

$$\begin{split} \lambda_i^{t+1} &\geq \lambda_i^t + \sigma G_i(x^{t+1}, \xi_t) \\ &\geq \lambda_i^t + \sigma (G_i(x^t, \xi_t) + \langle v_i(x^t, \xi_t), x^{t+1} - x^t \rangle) \\ &\geq \lambda_i^t + \sigma (G_i(x^t, \xi_t) - \|v_i(x^t, \xi_t)\| \|x^{t+1} - x^t\|), \end{split}$$

which, from $\lambda^0 = 0$, implies that

$$\sum_{t=0}^{T-1} G_i(x^t, \xi_t) \leq \frac{1}{\sigma} \lambda_i^T + \sum_{t=0}^{T-1} \|v_i(x^t, \xi_t)\| \|x^{t+1} - x^t\|$$
$$\leq \frac{1}{\sigma} \lambda_i^T + \kappa_g \sum_{t=0}^{T-1} \|x^{t+1} - x^t\|.$$

Taking expectation operation in the both sides of the above inequality and using Assumption (A4), we obtain (2.4).

Lemma 2.3 Let (x^t, λ^t) be generated by PMMSopt and Assumption (A1)–Assumption (A3) be satisfied. Then for any $x \in X_0$,

$$F(x^{t+1},\xi_t) + \frac{1}{2\sigma} \|\lambda^{t+1}\|^2 + \frac{\alpha}{2} \|x^{t+1} - x^t\|^2$$

$$\leq F(x,\xi_t) + \frac{1}{2\sigma} \|[\lambda^t + \sigma G(x,\xi_t)]_+\|^2 + \frac{\alpha}{2} [\|x - x^t\|^2 - \|x - x^{t+1}\|^2].$$
(2.5)

Proof. Since (2.5) is equivalent to the following inequality

$$\mathcal{L}_{\sigma}^{t}(x^{t+1},\xi_{t}) + \frac{\alpha}{2} \|x^{t+1} - x^{t}\|^{2} \le \mathcal{L}_{\sigma}^{t}(x,\xi_{t}) + \frac{\alpha}{2} [\|x - x^{t}\|^{2} - \|x - x^{t+1}\|^{2}], \ \forall x \in X_{0}, \quad (2.6)$$

we only need to prove (2.6). Define

$$\psi^t(x) = \delta_{X_0}(x) + \mathcal{L}^t_{\sigma}(x,\xi_t) + \frac{\alpha}{2} ||x - x^t||^2$$

and

$$q^{t}(x) = \mathcal{L}_{\sigma}^{t}(x,\xi_{t}) - \mathcal{L}_{\sigma}^{t}(x^{t+1},\xi_{t}) - \langle \nabla_{x}\mathcal{L}_{\sigma}^{t}(x^{t+1},\xi_{t}), x - x^{t+1} \rangle.$$

Then

$$\psi^{t}(x) = \left[\mathcal{L}_{\sigma}^{t}(x^{t+1},\xi_{t}) - \langle \nabla_{x}\mathcal{L}_{\sigma}^{t}(x^{t+1},\xi_{t}), x - x^{t+1} \rangle \right] + \frac{\alpha}{2} \|x - x^{t}\|^{2} + q^{t}(x) + \delta_{X_{0}}(x).$$

Let

$$q_0^t(x) = \delta_{X_0}(x) + q^t(x) + \langle \nabla_x \mathcal{L}_{\sigma}^t(x^{t+1})(x - x^{t+1}) \rangle + \frac{1}{2} \|x - x^t\|^2 - \frac{1}{2} \|x - x^{t+1}\|^2.$$

Then q_0^t is convex and

$$\psi^t(x) = \mathcal{L}^t_{\sigma}(x^{t+1}, \xi_t) + \frac{1}{2} \|x - x^{t+1}\|^2 + q_0^t(x).$$

Since $0 \in \partial \psi^t(x^{t+1})$, we have $0 \in \partial q_0^t(x^{t+1})$ and q_0^t arrives its minimum value at x^{t+1} . Then for every $x \in \Re^n$, $q_0^t(x) \ge q_0^t(x^{t+1}) = \delta_{X_0}(x^{t+1}) + \frac{\alpha}{2} ||x^{t+1} - x^t||^2$, which is equivalent to

$$\delta_{X_0}(x) + \mathcal{L}_{\sigma}^t(x,\xi_t) + \frac{\alpha}{2} \|x - x^t\|^2 - \frac{\alpha}{2} \|x - x^{t+1}\|^2 \ge \delta_{X_0}(x^{t+1}) + \mathcal{L}_{\sigma}^t(x^{t+1},\xi_t) + \frac{\alpha}{2} \|x^{t+1} - x^t\|^2.$$

The proof is completed.

The proof is completed.

In order to give a bound for $\sum_{t=0}^{T-1} G_i(x^t, \xi_t)$ in (2.3), we need to estimate an upper bound of $\sum_{t=0}^{T-1} ||x^{t+1} - x^t||$, which is given in the following lemma.

Lemma 2.4 Let (x^t, λ^t) be generated by PMMSopt and Assumption (A1)–Assumption (A4) be satisfied. Then for $2\alpha - p\kappa_g^2\sigma > 0$,

$$\|x^{t+1} - x^t\| \le \frac{1}{2\alpha - p\kappa_g^2 \sigma} [2\kappa_f + \sqrt{p}\kappa_g \|\lambda^t\| + \nu_g \sqrt{p}\kappa_g \sigma]$$
(2.7)

Proof. Let us denote

$$B = \lambda^t + \sigma G(x^t, \xi_t), A = B + \sigma V(x^t, \xi_t)^T (x^{t+1} - x^t)$$

where $V(x^t, \xi_t) = (v_1(x^t, \xi_t), \dots, v_p(x^t, \xi_t))$. Choosing $x = x^t$ in Lemma 2.3, we obtain

$$\alpha \|x^{t+1} - x^t\|^2 \leq F(x^t, \xi_t) - F(x^{t+1}, \xi_t) + \frac{1}{2\sigma} \|[\lambda^t + \sigma G(x^t, \xi_t)]_+\|^2 - \frac{1}{2\sigma} \|[\lambda^t + \sigma G(x^{t+1}, \xi_t)]_+\|^2.$$

$$(2.8)$$

Noting that

$$G(x^{t+1},\xi_t) \ge G(x^t,\xi_t) + V(x^t,\xi_t)^T (x^{t+1} - x^t).$$

we have

$$[\lambda^t + \sigma G(x^{t+1}, \xi_t)]_+ \ge [\lambda^t + \sigma [G(x^t, \xi_t) + V(x^t, \xi_t)^T (x^{t+1} - x^t)]]_+ = A_+$$

and in turn,

$$|[\lambda^{t} + \sigma G(x^{t+1}, \xi_{t})]_{+}||^{2} \ge ||A_{+}||^{2}.$$
(2.9)

Therefore, we get from (2.8) that

$$\begin{split} \alpha \|x^{t+1} - x^t\|^2 &\leq F(x^t, \xi_t) - F(x^{t+1}, \xi_t) + \frac{1}{2\sigma} \|[\lambda^t + \sigma G(x^t, \xi_t)]_+\|^2 \\ &\quad -\frac{1}{2\sigma} \|[\lambda^t + \sigma G(x^{t+1}, \xi_t)]_+\|^2 \\ &\leq \langle v_0(x^t, \xi_t), x^t - x^{t+1} \rangle + \frac{1}{2\sigma} \|B_+\|^2 - \frac{1}{2\sigma} \|A_+\|^2 \\ &= \langle v_0(x^t, \xi_t), x^t - x^{t+1} \rangle + \frac{1}{2\sigma} [\|B_+\| + \|A_+\|] [\|B_+\| - \|A_+\|] \\ &\leq \kappa_f \|x^t - x^{t+1}\| + \frac{1}{2\sigma} [\|B_+\| + \|A_+\|] [\|B_- A_+\|] \\ &\leq \kappa_f \|x^t - x^{t+1}\| + \frac{1}{2\sigma} [\|B\| + \|A\|] [\|B - A\|] \\ &\leq \kappa_f \|x^t - x^{t+1}\| + \frac{1}{2\sigma} [\|\lambda^t\| + \sigma \|G(x^t, \xi_t)\| + \|B - A\|] [\|B - A\|] \\ &\leq \kappa_f \|x^t - x^{t+1}\| + \frac{1}{2\sigma} [\|\lambda^t\| + \sigma \|G(x^t, \xi_t)\| \\ &\quad + \sigma \|V(x^t, \xi_t)^T(x^{t+1} - x^t)\|] [\sigma \|V(x^t, \xi_t)^T(x^{t+1} - x^t)\|] \\ &\leq \kappa_f \|x^t - x^{t+1}\| + \frac{1}{2} \|\lambda^t\| \|V(x^t, \xi_t)^T(x^{t+1} - x^t)\| \\ &\quad + [\frac{1}{2} \|G(x^t, \xi^t)\| + \frac{1}{2} \|V(x^t, \xi_t)^T(x^{t+1} - x^t)\|] [\sigma \|V(x^t, \xi_t)^T(x^{t+1} - x^t)\|] \\ &\leq \kappa_f \|x^t - x^{t+1}\| + \frac{1}{2} \sqrt{\rho} \kappa_g \|\lambda^t\| \|x^{t+1} - x^t\| \\ &\quad + [\frac{1}{2} \nu_g + \frac{1}{2} \sqrt{\rho} \kappa_g \|x^{t+1} - x^t\|] [\sigma \sqrt{\rho} \kappa_g \|x^{t+1} - x^t\|], \end{split}$$

which implies

$$\|x^{t+1} - x^t\| \le \frac{1}{2\alpha - p\kappa_g^2 \sigma} [2\kappa_f + \sqrt{p}\kappa_g \|\lambda^t\| + \nu_g \sqrt{p}\kappa_g \sigma]$$

The proof is completed.

Corollary 2.1 Let (x^t, λ^t) be generated by PMMSopt and Assumption (A1)–Assumption (A4) be satisfied. Then for $2\alpha - p\kappa_g^2 \sigma > 0$,

$$\sum_{t=0}^{T-1} G_i(x^t, \xi_t) \leq \frac{1}{\sigma} \lambda_i^T + \frac{2\kappa_g \kappa_f}{2\alpha - p\kappa_g^2 \sigma} T + \frac{\sqrt{p}\kappa_g^2}{2\alpha - p\kappa_g^2 \sigma} \sum_{t=0}^{T-1} \|\lambda^t\| + \frac{\sqrt{p}\nu_g \kappa_g^2 \sigma}{2\alpha - p\kappa_g^2 \sigma} T.$$
(2.11)

Lemma 2.5 Let (x^t, λ^t) be generated by PMMSopt and Assumption (A1), Assumption (A2) and Assumption (A5) be satisfied. Then for any $t_2 \leq t_1 - 1$ where t_1 and t_2 are positive integers,

 $\mathbb{E}\left[\langle\lambda^{t_1}, G(\widehat{x}, \xi_{t_1})\rangle \,|\, \xi_{[t_2]}\right] \le -\epsilon_0 \mathbb{E}\left[\|\lambda^{t_1}\| \,|\, \xi_{[t_2]}\right]. \tag{2.12}$

Proof. To prove this lemma, we first show that

$$\mathbb{E}\left[\lambda_i^{t_1}G_i(\widehat{x},\xi_{t_1}) \,|\, \xi_{[t_2]}\right] \leq -\epsilon_0 \mathbb{E}\left[\lambda_i^{t_1} \,|\, \xi_{[t_2]}\right].$$

For $i \in \{1, \ldots, p\}$, note that $\lambda_i^{t_1} = \lambda_i^{t_1}(\xi_{[t_1-1]})$ and $G_i(\widehat{x}, \xi_{t_1})$ is independent of $\xi_{[t_1-1]}$ and $\xi_{[t_2]} \subseteq \xi_{[t_1-1]}$ for $t_2 \leq t_1 - 1$. We have

$$\mathbb{E}\left[\lambda_{i}^{t_{1}}G_{i}(\widehat{x},\xi_{t_{1}}) \mid \xi_{[t_{2}]}\right] = \mathbb{E}\left\{\mathbb{E}\left[\lambda_{i}^{t_{1}}G_{i}(\widehat{x},\xi_{t_{1}}) \mid \xi_{[t_{1}-1]}\right] \mid \xi_{[t_{2}]}\right\}$$
$$= \mathbb{E}\left\{\lambda_{i}^{t_{1}}\mathbb{E}\left[G_{i}(\widehat{x},\xi_{t_{1}})\right] \mid \xi_{[t_{2}]}\right\}$$
$$= \mathbb{E}\left[G_{i}(\widehat{x},\xi_{t_{1}})\right]\left[\lambda_{i}^{t_{1}} \mid \xi_{[t_{2}]}\right]$$
$$\leq -\epsilon_{0}\mathbb{E}\left[\lambda_{i}^{t_{1}} \mid \xi_{[t_{2}]}\right].$$

Making a sum over $i \in \{1, \ldots, p\}$ yields

$$\mathbb{E}\left[\langle \lambda^{t_1}, G(\widehat{x}, \xi_{t_1}) \rangle \,|\, \xi_{[t_2]}\right] \le -\epsilon_0 \mathbb{E}\left[\sum_{i=1}^p \lambda_i^{t_1} \,|\, \xi_{[t_2]}\right] \le -\epsilon_0 \mathbb{E}\left[\|\lambda^{t_1}\| \,|\, \xi_{[t_2]}\right].$$

The proof is completed.

Lemma 2.6 Let s > 0 be an arbitrary integer. Let Assumption (A1)–Assumption (A4) be satisfied. At each round $t \in \{1, 2, ...\}$ in PMMSopt. For

$$\vartheta(\sigma, \alpha, s) = \frac{\epsilon_0 \sigma s}{2} + \nu_g \sigma(s-1) + \frac{\alpha D_0^2}{\epsilon_0 s} + \frac{2\nu_f}{\epsilon_0} + \frac{\sigma \nu_g^2}{\epsilon_0}, \qquad (2.13)$$

the following holds

$$|\|\lambda^{t+1}\| - \|\lambda^t\|| \le \sigma \nu_g \tag{2.14}$$

and

$$\mathbb{E}\left[\|\lambda^{t+s}\| - \|\lambda^t\| \,|\, \xi_{[t-1]}\right] \le \begin{cases} s\sigma\nu_g & \text{if } \|\lambda^t\| < \vartheta(\sigma, \alpha, s), \\ -s\frac{\sigma\epsilon_0}{2} & \text{if } \|\lambda^t\| \ge \vartheta(\sigma, \alpha, s). \end{cases}$$
(2.15)

Proof. Inequality (2.14) follows from Lemma 2.1. We only need to establish (2.15). Since it is obvious that

$$\mathbb{E}\left[\|\lambda^{t+s}\| - \|\lambda^t\| \,|\, \xi_{[t-1]}\right] \le s\sigma\nu_g$$

when $\|\lambda^t\| < \vartheta(\sigma, \alpha, s)$, it remains to prove

$$\mathbb{E}\left[\left\|\lambda^{t+s}\right\| - \left\|\lambda^{t}\right\| \left|\xi_{[t-1]}\right] \le -s\frac{\sigma\epsilon_{0}}{2}$$

when $\|\lambda^t\| \ge \vartheta(\sigma, \alpha, s)$.

For given positive integer s, suppose $\|\lambda^t\| \ge \vartheta(\sigma, \alpha, s)$. For any $l \in \{t, t+1, \ldots, t+s-1\}$, one has

$$F(x^{l+1},\xi_l) + \frac{1}{2\sigma} \|\lambda^{l+1}\|^2 + \frac{\alpha}{2} \|x^{l+1} - x^l\|^2$$

$$\leq F(\widehat{x},\xi_l) + \frac{1}{2\sigma} \|[\lambda^l + \sigma G(\widehat{x},\xi_l)]_+\|^2 + \frac{\alpha}{2} \left[\|\widehat{x} - x^l\|^2 - \|\widehat{x} - x^{l+1}\|^2\right].$$

	. 1

Using Assumption (A3) and the following inequality

$$\|[\lambda^l + \sigma G(\widehat{x}, \xi_l)]_+\|^2 \le \|\lambda^l\|^2 + 2\sigma \langle \lambda^l, G(\widehat{x}, \xi_l) \rangle + \sigma^2 \|G(\widehat{x}, \xi_l)\|^2,$$

we obtain

$$\frac{1}{2\sigma} \left[\|\lambda^{l+1}\|^2 - \|\lambda^l\|^2 \right] \leq \left(F(\widehat{x}, \xi_l) - F(x^{l+1}, \xi_l) \right) \\
+ \frac{1}{2\sigma} \left[\|[\lambda^l + \sigma G(\widehat{x}, \xi_l)]_+\|^2 - \|\lambda^l\|^2 \right] \\
- \frac{\alpha}{2} \|x^{l+1} - x^l\|^2 + \frac{\alpha}{2} \left[\|\widehat{x} - x^l\|^2 - \|\widehat{x} - x^{l+1}\|^2 \right] \\
\leq \nu_f + \langle \lambda^l, G(\widehat{x}, \xi_l) \rangle + \frac{\sigma}{2} \|G(\widehat{x}, \xi_l)\|^2 \\
- \frac{\alpha}{2} \|x^{l+1} - x^l\|^2 + \frac{\alpha}{2} \left[\|\widehat{x} - x^l\|^2 - \|\widehat{x} - x^{l+1}\|^2 \right].$$
(2.16)

Making a summation of (2.16) over $\{t,t+1,t+s-1\}$ and taking conditional expectation on $\xi_{[t-1]}$, we obtain from Lemma 2.6 that

$$\frac{1}{2\sigma} \mathbb{E} \left[\|\lambda^{t+s}\|^{2} - \|\lambda^{t}\|^{2} |\xi_{[t-1]} \right] \\
\leq \nu_{f}s + \frac{\sigma}{2}\nu_{g}^{2}s + \sum_{l=t}^{t+s-1} \mathbb{E} \left[\langle \lambda^{l}, G(\hat{x}, \xi_{l}) \rangle |\xi_{[t-1]} \right] \\
+ \frac{\alpha}{2} \mathbb{E} \left[\left(\|\hat{x} - x^{t}\|^{2} - \|\hat{x} - x^{t+s}\|^{2} \right) |\xi_{[t-1]} \right] \\
\leq \nu_{f}s + \frac{\sigma}{2}\nu_{g}^{2}s - \epsilon_{0} \sum_{l=0}^{s-1} \mathbb{E} \left[\|\lambda^{t+l}\| |\xi_{[t-1]} \right] \\
+ \frac{\alpha}{2} \mathbb{E} \left[\left(\|\hat{x} - x^{t}\|^{2} - \|\hat{x} - x^{t+s}\|^{2} \right) |\xi_{[t-1]} \right] \\
\leq \nu_{f}s + \frac{\sigma}{2}\nu_{g}^{2}s - \epsilon_{0} \sum_{l=0}^{s-1} \mathbb{E} \left[\|\lambda^{t}\| - \sigma\nu_{g}l |\xi_{[t-1]} \right] \\
+ \frac{\alpha}{2} \mathbb{E} \left[\left(\|\hat{x} - x^{t}\|^{2} - \|\hat{x} - x^{t+s}\|^{2} \right) |\xi_{[t-1]} \right] \\
(from $\|\lambda^{t+1}\| \ge \|\lambda^{t}\| - \sigma\nu_{g}) \\
\leq \nu_{f}s + \frac{\sigma}{2}\nu_{g}^{2}s + \frac{\alpha}{2} \mathbb{E} \left[\left(\|\hat{x} - x^{t}\|^{2} - \|\hat{x} - x^{t+s}\|^{2} \right) |\xi_{[t-1]} \right] \\
+ \epsilon_{0}\sigma\nu_{g} \frac{s(s-1)}{2} - \epsilon_{0} \sum_{l=0}^{s-1} \mathbb{E} \left[\|\lambda^{t}\| |\xi_{[t-1]} \right]$$$

From (2.17), we get from Assumption (A3) that

$$\mathbb{E}\left[\|\lambda^{t+s}\|^{2} |\xi_{[t-1]}\right] \leq \mathbb{E}\left[\|\lambda^{t}\|^{2} |\xi_{[t-1]}\right]
+ 2\sigma\nu_{f}s + \sigma^{2}\nu_{g}^{2}s + \alpha\sigma D_{0}^{2} + \epsilon_{0}\sigma^{2}\nu_{g}s(s-1) - 2\epsilon_{0}\sigma s\mathbb{E}\left[\|\lambda^{t}\| |\xi_{[t-1]}\right]
= \mathbb{E}\left[(\|\lambda^{t}\| - \frac{\epsilon_{0}\sigma}{2}s)^{2} |\xi_{[t-1]}\right] - \frac{\epsilon_{0}^{2}\sigma^{2}}{4}s^{2} + \epsilon_{0}\sigma^{2}\nu_{g}s(s-1)
+ \alpha\sigma D_{0}^{2} + 2\sigma\nu_{f}s + \sigma^{2}\nu_{g}^{2}s - \epsilon_{0}\sigma s\mathbb{E}\left[\|\lambda^{t}\| |\xi_{[t-1]}\right]
\leq \mathbb{E}\left[(\|\lambda^{t}\| - \frac{\epsilon_{0}\sigma}{2}s)^{2} |\xi_{[t-1]}\right] - \frac{3\epsilon_{0}^{2}\sigma^{2}}{4}s^{2}
+ \left[\epsilon_{0}\sigma^{2}\nu_{g}s(s-1) + \frac{\epsilon_{0}^{2}\sigma^{2}}{2}s^{2} + \alpha\sigma D_{0}^{2} + 2\sigma\nu_{f}s + \sigma^{2}\nu_{g}^{2}s - \epsilon_{0}\sigma s\vartheta(\sigma,\alpha,s) \right]
= \mathbb{E}\left[(\|\lambda^{t}\| - \frac{\epsilon_{0}\sigma}{2}s)^{2} |\xi_{[t-1]}\right] - \frac{3\epsilon_{0}^{2}\sigma^{2}}{4}s^{2}
\leq \mathbb{E}\left[(\|\lambda^{t}\| - \frac{\epsilon_{0}\sigma}{2}s)^{2} |\xi_{[t-1]}\right].$$
(2.18)

This implies that

$$\mathbb{E}\left[\left\|\lambda^{t+s}\right\| \left|\xi_{[t-1]}\right] \le \mathbb{E}\left[\left\|\lambda^{t}\right\| \left|\xi_{[t-1]}\right] - \frac{\epsilon_{0}\sigma}{2}s.\right]$$

The proof is completed.

The following lemmas come from Yu et.al [20], which can be used to deal with the random process $\{\|\lambda^t\|\}$ and probability analysis for objective regret and constraint violation regret, respectively.

Lemma 2.7 Let $\{Z(t), t \ge 0\}$ be a discrete time stochastic process adapted to a filtration $\{\mathcal{F}(t), t \ge 0\}$ with Z(0) = 0 and $\mathcal{F}(0) = \{\emptyset, \Omega\}$. Suppose there exists an integer $t_0 > 0$, real constants $\theta > 0$, $\delta_{\max} > 0$ and $0 < \zeta \le \delta_{\max}$ such that

$$|Z(t+1) - Z(t)| \leq \delta_{\max} \text{ and}$$

$$\mathbb{E}[Z(t+t_0) - Z(t) | \mathcal{F}(t)] \leq \begin{cases} t_0 \delta_{\max} & \text{if } Z(t) < \theta \\ -t_0 \zeta & \text{if } Z(t) \ge \theta \end{cases}$$
(2.19)

hold for all $t \in \{1, 2, ...\}$. Then the following properties are satisfied.

1. The following inequality holds

$$\mathbb{E}[Z(t)] \le \theta + t_0 \delta_{\max} + t_0 \frac{4\delta_{\max}^2}{\zeta} \log\left[\frac{8\delta_{\max}^2}{\zeta^2}\right], \forall t \in \{1, 2, \ldots\}.$$

2. For any constant $0 < \mu < 1$, we have

$$\Pr\left\{Z(t) \ge z\right\} \le \mu, \forall t \in \{1, 2, \ldots\},$$

where

$$z = \theta + t_0 \delta_{\max} + t_0 \frac{4\delta_{\max}^2}{\zeta} \log\left[\frac{8\delta_{\max}^2}{\zeta^2}\right] + t_0 \frac{4\delta_{\max}^2}{\zeta} \log\left(\frac{1}{\mu}\right).$$

Lemma 2.8 Let $\{Z(t), t \ge 0\}$ be a supermartingale adapted to a filtration $\{\mathcal{F}(t), t \ge 0\}$ with Z(0) = 0 and $\mathcal{F}(0) = \{\emptyset, \Omega\}$, i.e. $\mathbb{E}[Z(t+1) | \mathcal{F}(t)] \le Z(t), \forall t \ge 0$. Suppose there exists a constant c > 0 such that $\{|Z(t+1) - Z(t)| > c\} \subseteq \{Y(t) > 0\}, \forall t \ge 0$, where Y(t)is process with Y(t) adapted to $\mathcal{F}(t)$ for all $t \ge 0$. Then, for all z > 0, we have

$$\Pr[Z(t) \ge z] \le e^{-z^2/(2tc^2)} + \sum_{j=0}^{t-1} \Pr[Y(j) > 0], \forall t \ge 1$$

3 Regret analysis of PMMSopt

In order to use Lemma 2.7 and Lemma 2.6 to analyze regrets of PMMSopt for Problem (1.1), we introduce the following notations. For $\theta = \vartheta(\sigma, \alpha, s)$, $\delta_{\max} = \sigma \nu_g$ and $\zeta = \frac{\sigma}{2} \epsilon_0$, and $t_0 = s$, define

$$\psi(\sigma, \alpha, s) = \theta + t_0 \delta_{\max} + t_0 \frac{4\delta_{\max}^2}{\zeta} \log\left[\frac{8\delta_{\max}^2}{\zeta^2}\right]$$

and

$$\phi(\sigma, \alpha, s, \mu) = \psi(\sigma, \alpha, s) + 8 \frac{\nu_g^2}{\epsilon_0} \log\left(\frac{1}{\mu}\right) \sigma s.$$

Then $\psi(\sigma, \alpha, s)$ is expressed as

$$\psi(\sigma, \alpha, s) = \vartheta(\sigma, \alpha, s) + \left[\nu_g + \frac{8\nu_g^2}{\epsilon_0}\log\frac{32\nu_g^2}{\epsilon_0^2}\right]\sigma s$$
$$= \kappa_0 + \kappa_1\frac{\alpha}{s} + \kappa_2 s + \kappa_3 \sigma + \kappa_4 \sigma s$$

and $\phi(\sigma, \alpha, s, \mu)$ is expressed as

$$\phi(\sigma, \alpha, s, \mu) = \kappa_0 + \kappa_1 \frac{\alpha}{s} + \kappa_2 s + \kappa_3 \sigma + \kappa_4 \sigma s + 8 \frac{\nu_g^2}{\epsilon_0} \log\left(\frac{1}{\mu}\right) \sigma s$$

where

$$\kappa_{0} = \frac{2\nu_{f}}{\epsilon_{0}}, \ \kappa_{1} = \frac{D_{0}^{2}}{\epsilon_{0}}, \ \kappa_{2} = 0,$$

$$\kappa_{3} = \frac{\nu_{g}^{2}}{\epsilon_{0}} - \nu_{g}, \ \kappa_{4} = \left[2\nu_{g} + \frac{\epsilon_{0}}{2} + \frac{8\nu_{g}^{2}}{\epsilon_{0}}\log\frac{32\nu_{g}^{2}}{\epsilon_{0}^{2}}\right].$$
(3.1)

Lemma 2.6 allows us to apply Lemma 2.7 to random process $Z(t) = \|\lambda^t\|$ and obtain $\mathbb{E}[\|\lambda^t\|] = O(1), \forall t$ by taking $s = \lceil \sqrt{T} \rceil, \alpha = \sqrt{T}$, where $\lceil \sqrt{T} \rceil$ represents the smallest integer no less than \sqrt{T} . By Corollary 2.1, this further implies the expected constraint violation bound

$$\mathbb{E}\left[\sum_{t=0}^{T-1} G_i(x^t, \xi_t)\right] \le \mathcal{O}(\sqrt{T}), i = 1, \dots, p$$

as summarized in the next theorem.

Theorem 3.1 (Expected Constraint Violation Bound). If $\alpha = \sqrt{T}$ and $\sigma = T^{-1/2}$ in *PMMSopt*, then for all $T \ge 1$, we have

$$\mathbb{E}\left[\sum_{t=0}^{T-1} G_i(x^t, \xi_t)\right] \le \kappa_c \sqrt{T}, i = 1, \dots, p,$$
(3.2)

where

$$\kappa_c = \kappa_* + 4\kappa_g \kappa_f + 2\sqrt{p}(\kappa_* + \nu_g)\kappa_g^2,$$

with $\kappa_* = \kappa_0 + \kappa_1 + \kappa_3 + \kappa_4$.

Proof. Define random process Z(t) with Z(0) = 0 and $Z(t) = ||\lambda^t||, t \ge 1$ and filtration $\mathcal{F}(t)$ with $\mathcal{F}(0) = \{\emptyset, \Omega\}$ and $\mathcal{F}(t) = \xi_{[t-1]}, t \ge 1$. Then we have Z(t) is adapted to $\mathcal{F}(t)$. It follows from Lemma 2.6 that $Z(t) = ||\lambda^t||$ satisfies Lemma 2.7 with $\delta_{\max} = \sigma \nu_g$, $t_0 = s$ and $\zeta = \sigma \epsilon_0/2$. We have from Lemma 2.7 that the following inequality holds for every t:

$$\mathbb{E}\|\lambda^t\| \le \psi(\sigma, \alpha, s) = \kappa_0 + \kappa_1 \frac{\alpha}{s} + \kappa_3 \sigma + \kappa_4 \sigma s.$$

Taking $s = \lceil \sqrt{T} \rceil, \alpha = \sqrt{T}, 2\sqrt{T} - p\kappa_g^2 \sigma > \sqrt{T}/2$ and $\sigma = T^{-1/2}$, we have $\mathbb{E} \|\lambda^t\| \le \kappa_*$ for $t \in \{1, 2, \dots, T\}$, where $\kappa_* = \kappa_0 + \kappa_1 + \kappa_3 + \kappa_4$.

From Corollary 2.1, we have

$$\mathbb{E}\left[\sum_{t=0}^{T-1} G_i(x^t, \xi_t)\right] \leq \frac{1}{\sigma} \mathbb{E}\lambda_i^T + \frac{2\kappa_g \kappa_f}{2\alpha - p\kappa_g^2 \sigma} T + \frac{\sqrt{p}\kappa_g^2}{2\alpha - p\kappa_g^2 \sigma} \sum_{t=0}^{T-1} \mathbb{E}\|\lambda^t\| + \frac{\sqrt{p}\nu_g \kappa_g^2 \sigma}{2\alpha - p\kappa_g^2 \sigma} T \\ \leq \kappa_* \sqrt{T} + 4\kappa_g \kappa_f \sqrt{T} + 2\sqrt{p}\kappa_* \kappa_g^2 \sqrt{T} + 2\sqrt{p}\nu_g \kappa_g^2 \leq \kappa_c \sqrt{T},$$

which implies (3.2). The proof is completed.

Lemma 3.1 Let (x^t, λ^t) be generated by PMMSopt and Assumption (A1)–Assumption (A4) be satisfied. Then for $x \in X_0$,

$$\sum_{t=0}^{T-1} F(x^{t},\xi_{t}) \leq \sum_{t=0}^{T-1} F(x,\xi_{t}) + \frac{\kappa_{f}^{2}}{2\alpha}T + \frac{\alpha}{2}D_{0}^{2} + \frac{\sigma}{2}\nu_{g}^{2}T + \sum_{t=0}^{T-1} \left[\langle \lambda^{t}, G(x,\xi_{t}) \rangle \right].$$
(3.3)

Proof. In view of (2.5), we have for $x \in X_0$,

$$F(x^{t},\xi_{t}) \leq F(x,\xi_{t}) + [F(x^{t},\xi_{t}) - F(x^{t+1},\xi_{t})] - \frac{1}{2\sigma} \left[\|\lambda^{t+1}\|^{2} - \|\lambda^{t}\|^{2} \right] + \frac{1}{2\sigma} \left[\|\lambda^{t} + \sigma G(x,\xi_{t})]_{+}\|^{2} - \|\lambda^{t}\|^{2} \right] + \frac{\alpha}{2} \left[\|x - x^{t}\|^{2} - \|x - x^{t+1}\|^{2} - \|x^{t+1} - x^{t}\|^{2} \right] = F(x,\xi_{t}) + \left[F(x^{t},\xi_{t}) - F(x^{t+1},\xi_{t}) - \frac{\alpha}{2} \|x^{t+1} - x^{t}\|^{2} \right] - \frac{1}{2\sigma} \left[\|\lambda^{t+1}\|^{2} - \|\lambda^{t}\|^{2} \right] + \frac{1}{2\sigma} \left[\|[\lambda^{t} + \sigma G(x,\xi_{t})]_{+}\|^{2} - \|\lambda^{t}\|^{2} \right] + \frac{\alpha}{2} \left[\|x - x^{t}\|^{2} - \|x - x^{t+1}\|^{2} \right].$$

$$(3.4)$$

From the convexity of $F(\cdot, \xi_t)$, we obtain from Assumption (A4) that

$$\begin{bmatrix}
F(x^{t},\xi_{t}) - F(x^{t+1},\xi_{t}) - \frac{\alpha}{2} \|x^{t+1} - x^{t}\|^{2} \\
\leq \left[\langle v_{0}(x^{t},\xi_{t}), x^{t} - x^{t+1} \rangle - \frac{\alpha}{2} \|x^{t+1} - x^{t}\|^{2} \right] \\
= -\frac{\alpha}{2} \left\{ \|x^{t+1} - x^{t} + v_{0}(x^{t},\xi_{t})/\alpha\|^{2} - \|v_{0}(x^{t},\xi_{t})\|^{2}/\alpha^{2} \right\} \\
\leq \frac{\kappa_{f}^{2}}{2\alpha}.$$
(3.5)

Since $[a]_{+}^{2} \leq a^{2}$ for scalar $a \in \Re$, we obtain

$$\begin{aligned} \|[\lambda^{t} + \sigma G(x, \xi_{t})]_{+}\|^{2} &= \|\lambda^{t}\|^{2} \\ &\leq \|\lambda^{t} + \sigma G(x, \xi_{t})\|^{2} - \|\lambda^{t}\|^{2} \\ &= 2\sigma \langle \lambda^{t}, G(x, \xi_{t}) \rangle + \sigma^{2} \|G(x, \xi_{t})\|^{2}. \end{aligned}$$
(3.6)

Substituting (3.5) and (3.6) into (3.4), we get

$$F(x^{t},\xi_{t}) \leq F(x,\xi_{t}) + \frac{\kappa_{f}^{2}}{2\alpha} - \frac{1}{2\sigma} \left[\|\lambda^{t+1}\|^{2} - \|\lambda^{t}\|^{2} \right] \\ + \frac{1}{2} \left[2\langle\lambda^{t}, G(x,\xi_{t})\rangle + \sigma \|G(x,\xi_{t})\|^{2} \right] \\ + \frac{\alpha}{2} \left[\|x - x^{t}\|^{2} - \|x - x^{t+1}\|^{2} \right]$$
(3.7)

Making a summation, we obtain from Assumption (A3)

$$\sum_{t=0}^{T-1} F(x^{t},\xi_{t}) \leq \sum_{t=0}^{T-1} F(x,\xi_{t}) + \frac{\kappa_{f}^{2}}{2\alpha}T - \frac{1}{2\sigma} \left[\|\lambda^{T}\|^{2} - \|\lambda^{0}\|^{2} \right] \\
+ \frac{1}{2} \sum_{t=0}^{T-1} \left[2\langle\lambda^{t}, G(x,\xi_{t})\rangle + \sigma \|G(x,\xi_{t})\|^{2} \right] \\
+ \frac{\alpha}{2} \left[\|x - x^{0}\|^{2} - \|x - x^{T}\|^{2} \right] \\
\leq \sum_{t=0}^{T-1} F(x,\xi_{t}) + \frac{\kappa_{f}^{2}}{2\alpha}T + \frac{\alpha}{2}D_{0}^{2} + \frac{\sigma}{2}\nu_{g}^{2}T \\
+ \sum_{t=0}^{T-1} \left[\langle\lambda^{t}, G(x,\xi_{t})\rangle \right],$$
(3.8)

which follows from $\lambda^0 = 0$.

Theorem 3.2 (Expected Regret Bound) Let $x^* \in \Phi$ be any fixed solution that satisfies

$$x^* = \operatorname{argmin}_{x \in \Phi} \sum_{t=0}^{T-1} f_t(x).$$

If $\alpha = \sqrt{T}, \sigma = T^{-1/2}$ in PMMSopt and Assumption (A1)-Assumption (A4) are satisfied. Then for all $T \ge 2$,

$$\mathbb{E}\left[\sum_{t=0}^{T-1} F(x^t, \xi_t)\right] \le \mathbb{E}\left[\sum_{t=0}^{T-1} F(x^*, \xi_t)\right] + \kappa_o \sqrt{T},\tag{3.9}$$

where

$$\kappa_o = \frac{\kappa_f^2}{2} + \frac{1}{2}D_0^2 + \frac{1}{2}\nu_g^2.$$

Proof. For fixed $T \ge 2$, taking $x = x^*$ in (3.3), we get

$$\sum_{t=0}^{T-1} \mathbb{E}F(x^{t},\xi_{t}) \leq \sum_{t=0}^{T-1} \mathbb{E}F(x^{*},\xi_{t}) + \frac{\kappa_{f}^{2}}{2\alpha}T + \frac{\alpha}{2}D_{0}^{2} + \frac{\sigma}{2}\nu_{g}^{2}T + \sum_{t=0}^{T-1} \mathbb{E}\left[\langle\lambda^{t},G(x^{*},\xi_{t})\rangle\right].$$
(3.10)

Since $G(x^*, \xi_t)$ is independent on λ^t , which is determined by $\xi_{[t-1]}$, we have

$$\mathbb{E}\left[\langle \lambda^t, G(x^*, \xi_t) \rangle\right] = \mathbb{E}\left[\langle \lambda^t, \mathbb{E}[G(x^*, \xi_t) | \xi_{[t-1]}] \rangle\right] \le 0,$$

which follows from the fact that $\lambda^t \ge 0$ and $\mathbb{E}[G(x^*, \xi_t)|\xi_{[t-1]}] = \mathbb{E}G(x^*, \xi_t) \le 0$. Thus we have from (3.10) and $\alpha = \sqrt{T}, \sigma = T^{-1/2}$ that

$$\sum_{t=0}^{T-1} \mathbb{E}F(x^t, \xi_t) \le \sum_{t=0}^{T-1} \mathbb{E}F(x^*, \xi_t) + \left[\frac{\kappa_f^2}{2} + \frac{1}{2}D_0^2 + \frac{1}{2}\nu_g^2\right]\sqrt{T}.$$

The proof is completed.

4 High probability performance analysis

First of all, we will use (2.11) and part 2 of Lemma 2.6 to establish a high probability constraint violation bound.

Theorem 4.1 Let $\eta \in (0,1)$. Let (x^t, λ^t) be generated by PMMSopt, and Assumption (A1)-Assumption (A4) be satisfied. If $\sigma = T^{-1/2}$, $\alpha = T^{1/2}$ in PMMSopt, then for all $i \in \{1, \ldots, p\}$,

$$\Pr\left[\sum_{t=0}^{T-1} G_i(x^t, \xi_t) \le \pi(T, \eta)\right] \ge 1 - \eta, \tag{4.1}$$

where

$$\pi(T,\eta) = \kappa_3 \left[1 + 2\sqrt{p}\kappa_g^2 \right] + 2\sqrt{p}\nu_g \kappa_g^2 + \left[\left(1 + 2\sqrt{p}\kappa_g^2 \right) \left(\kappa_0 + \kappa_1 + \kappa_4 \right) + 4\kappa_g \kappa_f \right] T^{1/2} + 8 \left(1 + 2\sqrt{p}\kappa_g^2 \right) \frac{\nu_g^2}{\epsilon_0} T^{1/2} \log\left(\frac{T+1}{\eta}\right).$$

$$(4.2)$$

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Proof. Define $Z(t) = \|\lambda^t\|$ for $\forall t \in \{0, 1, 2, ...\}$. From Lemma 2.6, Z(t) satisfies the conditions in Lemma 2.7 with $\delta_{\max} = \sigma \nu_g$, $t_0 = s$ and $\zeta = \frac{1}{2} \epsilon_0 \sigma$ and

$$\vartheta = \frac{\epsilon_0 \sigma s}{2} + \nu_g \sigma(s-1) + \frac{\alpha D_0^2}{\epsilon_0 s} + \frac{2\nu_f}{\epsilon_0} + \frac{\sigma \nu_g^2}{\epsilon_0}.$$

Let $T \ge 1$ and $\lambda \in (0, 1)$. Taking $\mu = \frac{\eta}{T+1}$ in part 2 of Lemma 2.7, we obtain

$$\Pr\left[\|\lambda^t\| \ge \gamma(\sigma, \alpha, s, \eta)\right] \le \frac{\eta}{T+1}, \,\forall t \in \{0, 1, 2, \dots, T\},\tag{4.3}$$

where

$$\gamma(\sigma, \alpha, s, \eta) = \phi\left(\sigma, \alpha, s, \frac{\eta}{T+1}\right)$$
$$= \kappa_0 + \kappa_1 \frac{\alpha}{s} + \kappa_3 \sigma + \kappa_4 \sigma s + 8 \frac{\nu_g^2}{\epsilon_0} \log\left(\frac{T+1}{\eta}\right) \sigma s.$$

This implies

$$\Pr\left[\|\lambda^t\| \ge \gamma(\sigma, \alpha, s, \eta) \text{ for some } t \in \{0, 1, 2, \dots, T\}\right] \le \eta$$

or

$$\Pr\left[\|\lambda^t\| \le \gamma(\sigma, \alpha, s, \eta) \text{ for } \forall t \in \{0, 1, 2, \dots, T\}\right] \ge 1 - \eta.$$

$$(4.4)$$

It follows from (2.11) and Assumption (A4) that

$$\sum_{t=0}^{T-1} G_i(x^t,\xi_t) \leq \frac{1}{\sigma} \lambda_i^T + \frac{2\kappa_g \kappa_f}{2\alpha - p\kappa_g^2 \sigma} T + \frac{\sqrt{p}\kappa_g^2}{2\alpha - p\kappa_g^2 \sigma} \sum_{t=0}^{T-1} \|\lambda^t\| + \frac{\sqrt{p}\nu_g \kappa_g^2 \sigma}{2\alpha - p\kappa_g^2 \sigma} T.$$
(4.5)

for i = 1, ..., p. Thus, for $\sigma = T^{-1/2}$, $\alpha = T^{1/2}$, $2\sqrt{T} - p\kappa_g^2 \sigma > \sqrt{T}/2$ when T is very large, we have from (4.5) that

$$\sum_{t=0}^{T-1} G_i(x^t, \xi_t) \leq T^{1/2} \|\lambda^T\| + 4\kappa_g \kappa_f T^{1/2} + 2\sqrt{p\nu_g} \kappa_g^2 + 2\sqrt{p\kappa_g^2} T^{-1/2} \sum_{t=0}^{T-1} \|\lambda^t\|.$$
(4.6)

Let $s = \lceil T^{1/2} \rceil$. Noting, from (4.6), that

$$\begin{split} \sum_{t=0}^{T-1} G_i(x^t, \xi_t) &\leq \left[1 + 2\sqrt{p}\kappa_g^2\right] T^{1/2} \gamma(T^{-1/2}, T^{1/2}, T^{1/2}, \eta) + 4\kappa_g \kappa_f T^{1/2} + 2\sqrt{p}\nu_g \kappa_g^2 \\ &= \kappa_3 \left[1 + 2\sqrt{p}\kappa_g^2\right] + 2\sqrt{p}\nu_g \kappa_g^2 + \left[\left(1 + 2\sqrt{p}\kappa_g^2\right) (\kappa_0 + \kappa_1 + \kappa_4) + 4\kappa_g \kappa_f\right] T^{1/2} \\ &\quad + 8 \left(1 + 2\sqrt{p}\kappa_g^2\right) \frac{\nu_g^2}{\epsilon_0} T^{1/2} \log\left(\frac{T+1}{\eta}\right) \\ &= \pi(T, \eta), \end{split}$$

when $\|\lambda^t\| \leq \gamma(T^{-1/2}, T^{1/2}, T^{1/2}, \eta)$ for $\forall t \in \{0, 1, 2, \dots, T\}$, we obtain the probability inequality (4.1) from (4.4).

Define

$$\omega_c(T) = \pi(T, e^{-T^{1/4}})/T,$$

then

$$\omega_c(T) = \left[8 \left(1 + 2\sqrt{p}\kappa_g^2 \right) \frac{\nu_g^2}{\epsilon_0} \right] T^{-1/4} + o(T^{-1/4}).$$

We can obtain the following result from Theorem 4.1 directly.

Corollary 4.1 Let (x^t, λ^t) be generated by PMMSopt, and Assumptions (A1)–(A4) be satisfied. If $\sigma = T^{-1/2}$, $\alpha = T^{1/2}$ in PMMSopt, then for all $i \in \{1, \ldots, p\}$,

$$\Pr\left[\frac{1}{T}\sum_{t=0}^{T-1}G_i(x^t,\xi_t) \le \omega_c(T)\right] \ge 1 - e^{-T^{1/4}}.$$
(4.7)

For $\hat{x} \in \Phi$, define $\hat{Z}(0) = 0$ and

$$\widehat{Z}(t) = \sum_{l=0}^{t-1} \langle \lambda^l, G(\widehat{x}, \xi_l) \rangle.$$

Recall $\xi_{[0]} = \{\emptyset, \Omega\}$ and $\xi_{[t]} = \sigma(\xi_1, \ldots, \xi_t)$. In the following lemma, we will show that for any c > 0, $\widehat{Z}(t)$ satisfies conditions in Lemma 9 with $\mathcal{F}(t) = \xi_{[t]}$ and $Y(t) = \|\lambda^{t+1}\| - c/\nu_g$.

Lemma 4.1 Let $\hat{x} \in \Phi$. Let c > 0 be arbitrary. Let (x^t, λ^t) be generated by PMMSopt and Assumption (A3) be satisfied. Define $\hat{Z}(0) = 0$ and $\hat{Z}(t) = \sum_{l=0}^{t-1} \langle \lambda^l, G(\hat{x}, \xi_l) \rangle, \forall t \ge 1$, then $\{\hat{Z}(t), t \ge 0\}$ is a supermartingale adapted to filtration $\{\xi_{[t]}, t \ge 0\}$ such that

$$\{|\hat{Z}(t+1) - \hat{Z}(t)| > c\} \subseteq \{Y(t) > 0\}, \forall t \ge 0$$

where $Y(t) = \|\lambda^{t+1}\| - c/\nu_g$ is a random variable adapted to $\xi_{[t]}$.

Proof. It is very easy to check $\{\widehat{Z}(t), t \ge 0\}$ is adapted to $\{\xi_{[t]}, t \ge 0\}$. Now we prove that $\{\widehat{Z}(t), t \ge 0\}$ is a supermartingale. Since

$$\widehat{Z}(t+1) = \widehat{Z}(t) + \langle \lambda^{t+1}, G(\widehat{x}, \xi_{t+1}) \rangle,$$

we have

$$\begin{split} \mathbb{E}[\widehat{Z}(t+1) \,|\, \xi_{[t]}] &= \mathbb{E}[\widehat{Z}(t) + \langle \lambda^{t+1}, G(\widehat{x}, \xi_{t+1}) \rangle \,|\, \xi_{[t]}] \\ &= \widehat{Z}(t) + \langle \lambda^{t+1}, \mathbb{E}[G(\widehat{x}, \xi_{t+1})] \rangle \\ &= \widehat{Z}(t) + \langle \lambda^{t+1}, g(\widehat{x}) \rangle \\ &\leq \widehat{Z}(t), \end{split}$$

which follows from $\widehat{Z}(t) \in \xi_{[t]}, \lambda^{t+1} \in \xi_{[t]}, G(\widehat{x}, \xi_{t+1})$ is independent of $\xi_{[t]}$ and $g(\widehat{x}) \leq 0$. Thus we obtain that $\{\widehat{Z}(t), t \geq 0\}$ is a supermartingale.

From Assumption (A3), we get

$$|\widehat{Z}(t+1) - \widehat{Z}(t)| = |\langle \lambda^{t+1}, G(\widehat{x}, \xi_{t+1})| \le \nu_g ||\lambda^{t+1}||.$$

This implies that $\|\lambda^{t+1}\| > c/\nu_g$ if $|\widehat{Z}(t+1) - \widehat{Z}(t)| > c$ and

$$\{|\widehat{Z}(t+1) - \widehat{Z}(t)| > c\} \subseteq \{\|\lambda^{t+1}\| > c/\nu_g\}.$$

Since λ^{t+1} is adapted to $\xi_{[t]}$, we have that $Y(t) = \|\lambda^{t+1}\| - c/\nu_g$ is a random variable adapted to $\xi_{[t]}$.

Next, we will use (3.3) and Lemma 2.8 to establish a high probability objective regret bound.

Theorem 4.2 Let $\eta \in (0,1)$ and $\hat{x} \in \Phi$. Let (x^t, λ^t) be generated by PMMSopt, and Assumption (A1)-Assumption (A4) be satisfied. If $\sigma = T^{-1/2}$, $\alpha = T^{1/2}$ in PMMSopt, then

$$\Pr\left[\sum_{t=0}^{T-1} F(x^t, \xi_t) \le \sum_{t=0}^{T-1} F(\widehat{x}, \xi_t) + \beta(T, \eta)\right] \ge 1 - \eta,$$
(4.8)

where

$$\beta(T,\eta) = \frac{\kappa_f^2 + \nu_g^2}{2} T^{1/2} + \frac{D_0^2}{2} T^{1/2} + \sqrt{2}\nu_g \log^{1/2} \left(\frac{2}{\eta}\right) \left[(\kappa_0 + \kappa_1 + \kappa_4) T^{1/2} + \kappa_3 + \frac{8\nu_g^2}{\epsilon_0} T^{1/2} \log\left(\frac{2T}{\eta}\right) \right].$$
(4.9)

Proof. By Lemma 4.1, we know that $\widehat{Z}(t)$ satisfies conditions in Lemma 2.8. Fix T > 0, we obtain from Lemma 2.8 for any c > 0 that

$$\Pr\left[\sum_{t=0}^{T-1} \langle \lambda^t, G(\hat{x}, \xi_t) \rangle \ge \gamma\right] \le e^{-\gamma^2/(2Tc^2)} + \sum_{t=0}^{T-1} \Pr\left[\|\lambda^{t+1}\| > c/\nu_g\right].$$
(4.10)

For given $\eta \in (0, 1)$, we shall show how to choose γ and c such that each term in (4.10) is not larger than $\eta/2$.

Noting that by Lemma 2.6, random process $Z(t) = \|\lambda^t\|$ satisfies conditions in Lemma 2.7 with $\delta_{\max} = \nu_g \sigma$, $t_0 = s$, $\zeta = \frac{\epsilon_0}{2} \sigma$ and

$$\vartheta = \frac{\epsilon_0 \sigma s}{2} + \nu_g \sigma(s-1) + \frac{\alpha D_0^2}{\epsilon_0 s} + \frac{2\nu_f}{\epsilon_0} + \frac{\sigma \nu_g^2}{\epsilon_0}.$$

For the second term being not larger than $\eta/2$, it suffices to choose c such that

$$\Pr\left[\|\lambda^t\| > c/\nu_g\right] \le \frac{\eta}{2T}, \,\forall t \in \{1, 2, \dots, T\}$$

The above inequality holds from part 2 of Lemma 2.7 when we choose

$$c = c(\sigma, \alpha, s) = \left[\kappa_0 + \kappa_1 \frac{\alpha}{s} + \kappa_3 \sigma + \kappa_4 \sigma s + \frac{8\nu_g^2}{\epsilon_0} \log\left(\frac{2T}{\eta}\right) \sigma s\right] \nu_g, \tag{4.11}$$

where s is an arbitrary integer. Define

$$\gamma(\sigma, \alpha, s, \eta) = \sqrt{2T} \log^{1/2} \left(\frac{2}{\eta}\right) c(\sigma, \alpha, s).$$
(4.12)

Then, for $\gamma = \gamma(\sigma, \alpha, s, \eta)$ in (4.10), the first term in this equation is equal to $\eta/2$. Thus we have, for $c = c(\sigma, \alpha, s)$ and $\gamma = \gamma(\sigma, \alpha, s, \eta)$ defined by (4.11) and (4.12), respectively, that

$$\Pr\left[\sum_{t=0}^{T-1} \langle \lambda^t, G(\hat{x}, \xi_t) \rangle \ge \gamma(\sigma, \alpha, s, \eta)\right] \le \eta$$

or equivalently

$$\Pr\left[\sum_{t=0}^{T-1} \langle \lambda^t, G(\hat{x}, \xi_t) \rangle \le \gamma(\sigma, \alpha, s, \eta)\right] \ge 1 - \eta$$
(4.13)

It follows from (3.3) that

$$\sum_{t=0}^{T-1} F(x^{t},\xi_{t}) \leq \sum_{t=0}^{T-1} F(\widehat{x},\xi_{t}) + \frac{1}{2\alpha} \kappa_{f}^{2} T + \frac{\sigma}{2} \nu_{g}^{2} T + \sum_{t=0}^{T-1} \langle \lambda^{t}, G(\widehat{x},\xi_{t}) \rangle + \frac{\alpha}{2} D_{0}^{2} \\
= \sum_{t=0}^{T-1} F(\widehat{x},\xi_{t}) + \frac{1}{2} (\kappa_{f}^{2} + \nu_{g}^{2}) T^{\frac{1}{2}} + \sum_{t=0}^{T-1} \langle \lambda^{t}, G(\widehat{x},\xi_{t}) \rangle + \frac{D_{0}^{2}}{2} T^{\frac{1}{2}}.$$
(4.14)

Taking $s = \lceil T^{1/2} \rceil$, we obtain from (4.14) that if

$$\sum_{t=0}^{T-1} \langle \lambda^t, G(\hat{x}, \xi_t) \rangle \le \gamma(T^{-1/2}, T^{1/2}, T^{1/2}, \eta),$$

then

$$\begin{split} \sum_{t=0}^{T-1} F(x^t, \xi_t) &\leq \sum_{t=0}^{T-1} F(\widehat{x}, \xi_t) + \frac{1}{2} (\kappa_f^2 + \nu_g^2) T^{1/2} \\ &+ \gamma (T^{-1/2}, T^{1/2}, T^{1/2}, \eta) + \frac{D_0^2}{2} T^{1/2} \\ &\leq \sum_{t=0}^{T-1} F(\widehat{x}, \xi_t) + \frac{1}{2} (\kappa_f^2 + \nu_g^2) T^{1/2} + \frac{D_0^2}{2} T^{1/2} \\ &+ \sqrt{2} \nu_g \log^{1/2} \left(\frac{2}{\eta}\right) \left[(\kappa_0 + \kappa_1 + \kappa_4) T^{1/2} + \kappa_3 \right] \\ &+ \frac{8 \nu_g^2}{\epsilon_0} T^{1/2} \log \left(\frac{2T}{\eta}\right) \right] \\ &= \sum_{t=0}^{T-1} F(\widehat{x}, \xi_t) + \beta(T, \eta). \end{split}$$

We obtain the probability bound (4.8) from (4.13). Define

$$\omega_o(T) = \beta(T, e^{-T^{1/4}})/T, \qquad (4.15)$$

then

$$\omega_o(T) = \frac{8\sqrt{2}\nu_g^3}{\epsilon_0} T^{-1/8} + o(T^{-1/8}).$$

We can obtain the following result from Theorem 4.2 directly.

Corollary 4.2 Let $\eta \in (0,1)$ and $\hat{x} \in \Phi$. Let (x^t, λ^t) be generated by PMMSopt, and Assumptions (A1)-(A4) be satisfied. If $\sigma = T^{-1/2}$, $\alpha = T^{1/2}$ in PMMSopt, then

$$\Pr\left[\frac{1}{T}\sum_{t=0}^{T-1}F(x^t,\xi_t) \le \frac{1}{T}\sum_{t=0}^{T-1}F(\widehat{x},\xi_t) + \omega_o(T)\right] \ge 1 - e^{-T^{1/4}}.$$
(4.16)

5 Conclusion

In this paper, for the first time we present a stochastic approximation proximal method of multipliers (PMMSopt) for solving convex stochastic programming with expectation constraints. We show that, when the objective and constraint functions are generally convex, this algorithm exhibits $O(T^{-1/2})$ objective regret and $O(T^{-1/2})$ constraint violation regret if parameters in the algorithm are properly chosen, where T denotes the number of iterations. Moreover, we show that, with at least $1 - e^{-T^{1/4}}$ probability, the algorithm has no more than $O(T^{-1/4})$ objective regret and no more than $O(T^{-1/8})$ constraint violation regret.

The research presented in this paper has provided a proximal point-type method for solving convex stochastic programming with expectation constraints, which has the complexity bound comparable to the currently best existing one in [9, 20]. However, there are several questions to answer for the proposed stochastic approximation proximal method of multipliers. One question that has not been answered here is how to obtain the same complexity bound when X_0 in the problem is unbounded. The complexity bound in this paper requires that x^{t+1} in Step 1 of PMMSopt is an exact solution to the subproblem, so the the second question is whether the same complexity bound can be derived when x^{t+1} is inexactly solved. The algorithms in [9, 20] only require solving simple optimization problems involving the stochastic gradients of f and g_i 's, the third question is whether we can construct a proximal point-type method in which subproblems for x^{t+1} are simple optimization problems, or even have explicit solutions.

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