On Integer Programming, Discrepancy, and Convolution*

Klaus Jansen

Lars Rohwedder

July 27, 2022

Abstract

Integer programs with m constraints are solvable in pseudo-polynomial time in Δ , the largest coefficient in a constraint, when m is a fixed constant. We give a new algorithm with a running time of $O(\sqrt{m}\Delta)^{2m} + O(nm)$, which improves on the state-of-the-art. Moreover, we show that improving on our algorithm for any m is equivalent to improving over the quadratic time algorithm for (min, +)-convolution. This is a strong evidence that our algorithm's running time is the best possible. We also present a specialized algorithm with running time $O(\sqrt{m}\Delta)^{(1+o(1))m} + O(nm)$ for testing feasibility of an integer program and also give a tight lower bound, which is based on the SETH in this case.

1 Introduction

Vectors $v^{(1)}, \ldots, v^{(n)} \in \mathbb{R}^m$ that sum up to 0 can be seen as a circle in \mathbb{R}^m that walks from 0 to $v^{(1)}$ to $v^{(1)} + v^{(2)}$, etc. until it reaches $v^{(1)} + \ldots + v^{(n)} = 0$ again. The Steinitz Lemma [42] says that if each of the vectors is small with respect to some norm, we can reorder them in a way that each point in the circle is not far away from 0 with respect to the same norm.

Recently Eisenbrand and Weismantel found a beautiful application of this lemma in the area of integer programming [22]. They looked at ILPs in standard form

$$\max\{c^T x : Ax = b, x \in \mathbb{Z}_{\geq 0}^n\},\tag{1}$$

where $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$ and obtained a pseudo-polynomial algorithm in Δ , the biggest absolute value of an entry in A, when m is treated as a constant. The running time they achieve is $n \cdot O(m\Delta)^{2m} \cdot ||b||_1^2$ for finding the optimal solution and $n \cdot O(m\Delta)^m \cdot ||b||_1$ for finding only a feasible solution. This improves on a classic algorithm by Papadimitriou [38], which has a running time of

$$O(n^{2m+2} \cdot (m \cdot \max\{\Delta, ||b||_{\infty}\})^{(m+1)(2m+1)}).$$

The central idea in [22] is that a solution x^* for the ILP above can be viewed as a walk in \mathbb{Z}^m starting at 0 and ending at b. Every step is a column of the matrix A: For every $i \in \{1, \ldots, n\}$ we step x_i^* times in the direction of A_i (see left picture in Figure 1). By applying the Steinitz Lemma they show that there is an ordering of these steps such that the walk never strays off far from the direct line between 0 and b (see right picture in Figure 1). They construct a directed graph with one vertex for every integer point near the line between 0 and b and create an edge from b0 to b1. The weight of the edge is the same as the b1. The column. An optimal solution to the ILP can now be obtained by finding a longest path from 0 to b2. This can be done in the mentioned time, if one is careful with cycles.

In this work we present a different algorithm for the same problem. In our approach we do not reduce to a longest path problem, but rather solve the ILP in a divide and conquer fashion. We use the (weaker) assumption that a walk from 0 to b visits a vector b' near b/2 at some point. The distance of this point to b/2 is closely related to the discrepancy of the matrix A, see Lemma 4

^{*}This research was supported by German Research Foundation (DFG) projects JA 612/20-1 and JA 612/16-1.

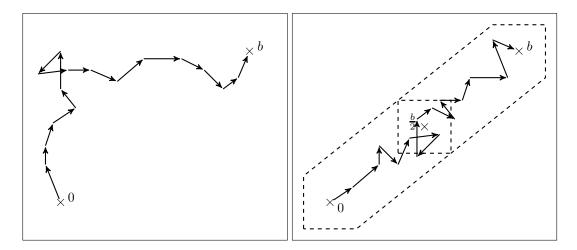


Figure 1: Steinitz Lemma in Integer Programming

for more details. A natural approach is to guess the vector b' and solve the problem with Ax = b' and Ax = b - b' independently. Both results can be merged to a solution for Ax = b. In the subproblems the norm of b and the norm of the solution are roughly divided in half. We use this idea in a dynamic program and speed up the process of merging solutions using algorithms for convolution. This approach leads to better running times for both the problem of finding optimal solutions and for finding any feasible solution. We complement our study by giving (almost) tight conditional lower bounds on the running time in which such ILPs can be solved. Finally, we discuss some applications to Knapsack, Change Making, and Scheduling problems.

We proceed by giving a detailed outline of the results.

Optimal solutions for ILPs. We show that a solution to (1) can be found in time

$$O(\sqrt{m}\Delta)^{2m} + O(nm). \tag{2}$$

We note that throughout the article we work with the assumption that arithmetics on the input numbers require constant time. Comparing to the state-of-the-art, we remove the dependence on b from the running time and save a factor of n without increasing the dependence on Δ and even mildly improving the dependence on m. The running time can be improved if there exists a truly sub-quadratic algorithm for (min, +)-convolution (see Section 3.2 for details on the problem). However, it has been conjectured that no such algorithm exists and this conjecture is the base of several lower bounds in fine-grained complexity [17, 31, 32, 4]. We show that for every m the running time above is essentially the best possible unless the $(\min, +)$ -convolution conjecture is false. More formally, for every m there exists no algorithm that solves ILP in time $f(m) \cdot (n^{2-\delta} + (\Delta + ||b||_{\infty})^{2m-\delta})$, where $\delta > 0$ and f is an arbitrary computable, unless there exists a truly sub-quadratic algorithm for (min, +)-convolution. Indeed, this means there is an equivalence between improving algorithms for (min, +)-convolution and for ILPs with fixed number of constraints. It may be surprising that the lower bound has a dependence on $\Delta + ||b||_{\infty}$ and the upper bound only on Δ . This implies that hardness cannot come from only letting b grow and, in particular, it rules out improvements by adding a dependence on $||b||_{\infty}$. Our lower bound does leave open some other trade-offs between n and $O(\sqrt{m}\Delta)^m$ such as $n\cdot O(\sqrt{m}\Delta)^m$, which would be an interesting improvement for sparse instances, i.e., when $n \ll (2\Delta + 1)^m$. Such an improvement has recently been made for Unbounded Knapsack [12], a notable special case of m=1, see also Definition 11. A running time of $n^{f(m)} \cdot (\Delta + ||b||_{\infty})^{m-\delta}$, however, is not possible (see feasibility below).

Feasibility of ILPs. Finding only a feasible solution of an ILP is easier than finding an optimal solution. It can be done in time

$$O(\sqrt{m}\Delta)^{(1+o(1))m} + O(nm) \tag{3}$$

by solving a Boolean convolution problem that has a more efficient algorithm than the (min, +)-convolution problem that arises in the optimization version. Under the Strong Exponential Time Hypothesis (SETH) this running time is tight except for sub-polynomial factors. The SETH and the Exponential Time Hypothesis (ETH) are conjectures commonly used to prove conditional lower bounds. The SETH asserts that the satisfiability problem (SAT) cannot be solved in time $O(2^{\delta n})$ for any $\delta < 1$, while the (weaker) ETH asserts that this holds for some $\delta > 0$. If the SETH holds, then there is no $n^{f(m)} \cdot (\Delta + ||b||_{\infty})^{m-\delta}$ time algorithm for testing feasibility of ILPs for any $\delta > 0$ and any computable function f.

Comparison to previous version. A preliminary version of this article has appeared in the proceedings of ITCS 2019 [25]. The analysis in that version has relied completely on the Steinitz Lemma, whereas the present article uses bounds on hereditary discrepancy, which is a cleaner fit given the requirements in the proof. Furthermore, this change leads to a slightly improved base $O(\sqrt{m}\Delta)$ in the running times instead of the previous base $O(m\Delta)$. This can be improved further in case of constraint matrices with small hereditary discrepancy. Moreover, by utilizing specialized algorithms for linear programming in fixed dimension we avoid the logarithmic dependency on $\|b\|_{\infty}$ as in the previous version. This also allows us to simplify the proof by removing a lemma that bounds the norm of the solution, which was required earlier. To the applications, we added the Coin Change problem.

Other related work

It is notable that the case where the number of variables n is fixed and not m as here behaves differently. There is a $2^{O(n\log(n))} \cdot |I|^{O(1)}$ time algorithm (|I| being the encoding length of the input), whereas an algorithm of the kind $f(m) \cdot |I|^{O(1)}$ (or even $|I|^{f(m)}$) is impossible for any computable function f, unless P = NP. This can be seen with a trivial reduction from Unbounded Knapsack (where m = 1). The $2^{O(n\log(n))} \cdot |I|^{O(1)}$ time algorithm is due to Kannan [27] improving over a $2^{O(n^2)} \cdot |I|^{O(1)}$ time algorithm by Lenstra [33]. It is a long open question whether $2^{O(n)} \cdot |I|^{O(1)}$ is possible instead; see also [18, 19] for progress towards this question.

Another intriguing question is whether a similar running time as in this work, e.g., $(\sqrt{m}\Delta)^{O(m)}$. $n^{O(1)}$, is possible when upper bounds on variables are added to the ILP and they are not counted in m. In [22] an algorithm for this extension is given, but the exponent of Δ is $O(m^2)$.

As for other lower bounds on pseudo-polynomial algorithms for integer programming, Fomin et al. [23] prove that the running time cannot be $n^{o(m/\log(m))} \cdot \|b\|_{\infty}^{o(m)}$ unless the ETH (a weaker conjecture than the SETH) fails. Their reduction implies that there is no algorithm with running time $n^{o(m/\log(m))} \cdot (\Delta + \|b\|_{\infty})^{o(m)}$, since in their construction the matrix A is non-negative and therefore columns with entries larger than $\|b\|_{\infty}$ can be discarded; thus leading to $\Delta \leq \|b\|_{\infty}$. Very recently, Knop et al. [30] show that under the ETH there is also no $2^{o(m\log(m))} \cdot (\Delta + \|b\|_{\infty})^{o(m)}$ time algorithm. An interesting aspect of this function is that it matches the dependency in m achieved here and in [22] up to a constant in the exponent. Our lower bound differs substantially from the two above. We concentrate on the dependency on Δ and give a precise value of the constant in its exponent.

Linear programming in fixed dimension, that is, solving (1) where $x \in \mathbb{R}^n_{\geq 0}$ instead of $x \in \mathbb{Z}^n_{\geq 0}$, has also been studied extensively. In a seminal work [37], Megiddo gave the first linear time algorithm for m = O(1). Since then there have been numerous improvements [36, 15, 20, 21, 16, 39, 26, 2, 14, 10, 11]. The currently best randomized algorithm has a running time of $m^2n + 2^{O(\sqrt{m\log(m)})}$ (a combination of [16, 26, 36]) and the best deterministic algorithm [11] has a running time of $O(m)^{m/2} \cdot \log^{3m}(m) \cdot n$. These works typically solve the dual of this problem,

which is equivalent by standard complementary slackness arguments. Our algorithm for ILP uses these results as a subroutine.

2 Preliminaries

In the remainder of the article we will assume that A has no duplicate columns. Note that we can completely ignore a column i, if there is another identical column i' with $c_{i'} \geq c_i$. This implies that in time $O(nm) + O(\Delta)^m$ we can reduce to an instance without duplicate columns and, in particular, with $n \leq (2\Delta + 1)^m$. The running time can be achieved as follows. We create a new matrix for the ILP with all $(2\Delta + 1)^m$ possible columns (in lexicographic order) and objective value $c_i = -\infty$ for all columns i. Now we iterate over all n old columns and compute in time O(m) the index of the new column corresponding to the same entries. We then replace its objective value with the current one if this is bigger. In the upcoming running times we will omit the additive term O(nm) and assume the duplicates are already eliminated $(O(\Delta)^m)$ is always dominated by actual algorithms running time).

Eisenbrand and Weismantel observed that using the Steinitz Lemma (with ℓ_{∞} norm) one can solve integer programs efficiently, if all entries of the matrix are small integers and the number of constraints is fixed.

Theorem 1 (Steinitz Lemma). Let $\|\cdot\|$ be a norm in \mathbb{R}^m and $v^{(1)}, \ldots, v^{(t)} \in \mathbb{R}^m$ such that $\|v^{(i)}\| \leq \Delta$ for all i and $v^{(1)} + \cdots + v^{(t)} = 0$. Then there exists a permutation $\pi \in S_t$ such that for all $j \in \{1, \ldots, t\}$

$$\|\sum_{i=1}^{j} v^{(\pi(i))}\| \le m\Delta.$$

The proof of the bound $m\Delta$ is due to Sevastyanov [40] (see also [22] for a good overview). Our algorithmic results rely on a similar, but weaker property. Roughly speaking, we only need that there is some $j \approx t/2$ with $\|\sum_{i=1}^{j} v^{(\pi(i))}\|$ being small. All other partial sums are insignificant. As it is a weaker property, we can hope for better bounds than $m\Delta$, which is indeed true. The bounds we need come from discrepancy theory, for which we now state relevant definitions and results.

Definition 2. For a matrix $A \in \mathbb{R}^{m \times n}$ its discrepancy is

$$\operatorname{disc}(A) = \min_{z \in \{0,1\}^n} \left\| A \left(z - \left(\frac{1}{2}, \dots, \frac{1}{2}\right)^T \right) \right\|_{\infty}.$$

Discrepancy theory originates in the problem of coloring the elements of a ground set with two colors such that a given family of subsets are all colored evenly, i.e., the number of elements of each color is approximately the same. When A is the incidence matrix of this family of sets, z in the definition above gives a coloring and the ℓ_{∞} norm its discrepancy. Discrepancy, however, is also studied for arbitrary matrices. If A is the matrix of a linear program as in our case, this definition corresponds to finding an integral solution that approximates $x = (1/2, \ldots, 1/2)^T$. Our algorithm is based on dividing a solution into two similar parts. Therefore, discrepancy is a natural measure. However, we need a definition that is stable when restricting to a subset of the columns.

Definition 3. The hereditary discrepancy of a matrix $A \in \mathbb{R}^{m \times n}$ is

$$herdisc(A) = \max_{I \subseteq \{1, \dots, n\}} disc(A_I),$$

where A_I denotes the matrix A restricted to the columns I.

Hereditary discrepancy is often used in the context of rounding non-integral solutions, see for example [34]. For our algorithm we need to split a solution x into two similar parts, which can be seen as rounding x/2. The following lemma shows that by paying a factor of 2 in the discrepancy we can also get a balanced split of the ℓ_1 norm of the solutions.

Lemma 4. Let $x \in \mathbb{Z}_{\geq 0}^n$. Then there is a vector $z \in \mathbb{Z}_{\geq 0}^n$ with $z_i \leq x_i$ for all i and

$$\left\| A\left(z - \frac{x}{2}\right) \right\|_{\infty} \le \operatorname{herdisc}(A).$$

Furthermore, if $||x||_1 > 1$, then there is a vector $z' \in \mathbb{Z}_{\geq 0}^n$ with $z_i' \leq x_i$ for all i, $1/6 \cdot ||x||_1 \leq ||z'||_1 \leq 5/6 \cdot ||x||_1$, and

 $\left\| A\left(z' - \frac{x}{2}\right) \right\|_{\infty} \le 2 \cdot \operatorname{herdisc}(A).$

We emphasize that the lemma is symmetric in the sense that the same properties hold when substituting z for x-z (z' for x-z'). For completeness a proof of the lemma is given in the appendix. Our algorithm's running time will depend on $\operatorname{herdisc}(A)$, so we will state some standard bounds on it.

Theorem 5 (Spencer's Six Standard Deviations Suffice [41]). For every matrix $A \in \mathbb{R}^{m \times n}$ with biggest absolute value of an entry Δ ,

$$herdisc(A) \leq 6\sqrt{m} \cdot \Delta.$$

This slightly differs from the original statement. The original paper considers square matrices (n=m) with biggest absolute value 1 and gives a bound of $6\sqrt{n}=6\sqrt{m}$. However, the proof easily holds also for $6\sqrt{m}$ in non-square matrices, as mentioned for example in [34]. By scaling both sides we obtain $6\sqrt{m} \cdot \Delta$ for matrices with biggest absolute value Δ .

Spencer's proof is not constructive, that is, it is unclear how to compute the z from the defintion of discrepancy. There has been significant work towards making it constructive [5, 35]. For our algorithm, however, we will not need a constructive variant.

There are matrices for which Spencer's bound is tight up to a constant factor. For specific matrices it might be lower. The linear dependency on Δ , however, is required for any matrix A.

Lemma 6. For every matrix $A \in \mathbb{R}^{m \times n}$ with absolute value of an entry $\leq \Delta$,

$$herdisc(A) \ge \frac{\Delta}{2}$$
.

This can be seen by taking $I = \{i\}$ in the definition of herdisc(A) with A_i being a column with an entry of absolute value Δ . An example where the dependency on m is lower than in Spencer's theorem are matrices with a small ℓ_1 norm in every column.

Theorem 7 (Beck, Fiala [6]). For every matrix $A \in \mathbb{R}^{m \times n}$, where the ℓ_1 norm of each column is at most t it holds that $\operatorname{herdisc}(A) < t$.

3 Algorithm

First, we will show how to compute the best solution x^* to (1) with the additional constraint $||x^*||_1 \leq K$. Here the running time has a logarithmic dependence on K. Then, we will remove this dependence while allowing arbitrarily large solutions. Further, we will elaborate an improvement for finding any feasible solution and show how to cope with unbounded problems. Finally, we give a more fine-grained study of the problem when the maximum entries of the rows differ.

3.1 Dynamic program

Let $H \ge \operatorname{herdisc}(A)$ be a given upper bound on the hereditary discrepancy of A. For every $i = 0, 1, \ldots, \ell = \lceil \log_{6/5}(K) \rceil$ and every b' with $||b' - 2^{i-\ell} \cdot b||_{\infty} \le 4H$ we solve

$$\max \left\{ c^T x : Ax = b', \|x\|_1 \le \left(\frac{6}{5}\right)^i, x \in \mathbb{Z}_{\ge 0}^n \right\}.$$
 (4)

We iteratively derive solutions for i using pairs of solutions for i-1. Ultimately, we will compute a solution for $i=\ell$ and b'=b.

If i=0 the solutions are trivial, since $||x||_1 \le 1$. This means they correspond exactly to the columns of A. Fix some i>0 and b' and let x^* be an optimal solution to (4). By Lemma 4 there exists a $0 \le z \le x^*$ with $||Az - b'/2||_{\infty} \le 2H$ and

$$||z||_1 \le \begin{cases} \frac{5}{6} ||x^*||_1 \le \frac{5}{6} \cdot \left(\frac{6}{5}\right)^i = \left(\frac{6}{5}\right)^{i-1} & \text{if } ||x^*||_1 > 1, \\ ||x^*||_1 \le 1 \le \left(\frac{6}{5}\right)^{i-1} & \text{otherwise.} \end{cases}$$

The same holds for $x^* - z$. Then z is an optimal solution to

$$\max \left\{ c^T x : Ax = b'', ||x||_1 \le \left(\frac{6}{5}\right)^{i-1}, x \in \mathbb{Z}_{\ge 0}^n \right\},\,$$

where b'' = Az. This is because if there was a solution z^* of higher value, then $z^* + x^* - z$ would be feasible for (4) and have a higher value than x^* , contradicting its optimality. Likewise, $x^* - z$ is an optimal solution to

$$\max \left\{ c^T x : Ax = b' - b'', ||x||_1 \le \left(\frac{6}{5}\right)^{i-1}, x \in \mathbb{Z}_{\ge 0}^n \right\}.$$

We will prove that $\|b'' - 2^{(i-1)-\ell} \cdot b\|_{\infty} \le 4H$ and $\|(b' - b'') - 2^{(i-1)-\ell} \cdot b\|_{\infty} \le 4H$. This implies that we can look up solutions for b'' and b' - b'' in the dynamic table and their sum is a solution for b'. Clearly it is also optimal. We do not know b'', but we can guess it: There are only $(8H+1)^m$ candidates. To compute an entry, we therefore enumerate all possible b'' and take the two partial solutions (for b'' and b' - b''), where the sum of both values is maximized. To verify that the inequalities above holds, we calculate

$$\begin{aligned} \left\| b'' - 2^{(i-1)-\ell} b \right\|_{\infty} &= \left\| Az - \frac{1}{2}b' + \frac{1}{2}b' - 2^{(i-1)-\ell} b \right\|_{\infty} \\ &\leq \left\| Az - \frac{1}{2}b' \right\|_{\infty} + \left\| \frac{1}{2}b' - 2^{(i-1)-\ell} b \right\|_{\infty} \\ &\leq 2 \cdot \operatorname{herdisc}(A) + \frac{1}{2} \left\| b' - 2^{i-\ell} b \right\|_{\infty} \leq 4H. \end{aligned}$$

The same holds for b'-b'', since z and x^*-z are interchangeable. The dynamic table has $O(H)^m \cdot \log(K)$ entries. To compute an entry, $O(n \cdot m) \leq O(\Delta)^m \leq O(H)^m$ operations are necessary during initialization and $O(H)^m$ in the iterative calculations. This gives a total running time of

$$O(H)^{2m} \cdot \log(K). \tag{5}$$

3.2 Convolution

The careful reader may wonder, whether the computation of entries in the dynamic table can be improved. Let D_i be the set of vectors b' with $\|b'-2^{i-\ell}\cdot b\|_{\infty} \leq 4H$. Recall, the dynamic programs computes values for each element in D_0, D_1, \ldots, D_ℓ . More precisely, for the value of $b' \in D_i$ we consider vectors b'' such that $b'', b' - b'' \in D_{i-1}$ and take the maximum sum of the values for b'', b' - b'' among all. For illustration consider the case of m = 1. Here we have that $b' \in D_i$ is equivalent to $-4H \leq b' - 2^{i-\ell} \cdot b \leq 4H$. It is not hard to see that then the problem can be formulated as the following well-studied problem.

Definition 8 ((min, +)-convolution). Given input variables $r_1, \ldots, r_n \in \mathbb{R}$ and $s_1, \ldots, s_n \in \mathbb{R}$, compute $t_1, \ldots, t_n \in \mathbb{R}$, where $t_k = \min_{i+j=k} r_i + s_j$.

We can also define (max, +)-convolution as the counterpart where the maximum is taken instead of the minimum. The two problems are equivalent as each of them can be transformed to the other by negating the elements. There is a trivial $O(n^2)$ time algorithm for (min, +)-convolution and it has been conjectured that there exists no truly sub-quadratic algorithm [17]. There does, however, exist an $O(n^2/\log(n))$ time algorithm [8], which we are going to use. In fact, there is an even faster algorithm that runs in time $O(n^2/2^{\Omega(\sqrt{\log(n)})})$ [13].

Also when m>1 the task of deriving D_i from D_{i-1} can be reformulated as a (min, +)-convolution instance. For this, the m dimensions of each $b'\in D_i$ are embedded in a single dimension with appropriate zero padding between them. The precise construction and its proof of correctness require some tedious calculations, which are deferred to the appendix. Using an algorithm for (min, +)-convolution with running time T(n) we get an algorithm for ILP with running time $T(O(H)^m) \cdot \log(K)$. Inserting $T(n) = n^2/\log(n)$ and using $H \geq \Delta/2$, we slightly improve on (5) and obtain a running time of

$$O(H)^{2m} \cdot \frac{\log(K)}{\log(\Delta)}.$$
 (6)

Even more interesting though, a sub-quadratic algorithm for (min, +)-convolution, where $T(n) = n^{2-\delta}$ for some $\delta > 0$, would directly improve the exponent. Next, we will consider the problem of only testing feasibility of an ILP. Since we only record whether or not there exists a solution for a particular right-hand side, the convolution problem reduces to the following.

Definition 9 (BOOLEAN CONVOLUTION). Given input variables $r_1, \ldots, r_n \in \{0, 1\}$ and $s_1, \ldots, s_n \in \{0, 1\}$ compute $t_1, \ldots, t_n \in \{0, 1\}$, where $t_k = \bigvee_{i+j=k} r_i \wedge s_j$.

This problem can be solved very efficiently via fast Fourier transform. We compute the $(+,\cdot)$ -convolution of the input. It is well known that this can be done using FFT in time $O(n \log(n))$. The $(+,\cdot)$ -convolution of r and s is the vector t, where $t_k = \sum_{i+j=k} r_i \cdot s_j$. To get the Boolean convolution instead, we simply replace each $t_k > 0$ by 1. Using $T(n) = O(n \log(n))$ for the convolution algorithm yields that a feasible solution can be found in time

$$O(H)^m \cdot \log(\Delta) \cdot \log(K).$$
 (7)

3.3 Proximity

Eisenbrand and Weismantel gave the following bound on the proximity between fractional and integral solutions.

Theorem 10 ([22]). Let $\max\{c^Tx: Ax = b, x \in \mathbb{Z}_{\geq 0}^n\}$ be feasible and bounded. Let x^* be an optimal vertex solution of the fractional relaxation. Then there exists an optimal solution z^* with

$$||z^* - x^*||_1 \le m(2m\Delta + 1)^m$$
.

We use the theorem to bound the value of K at the expense of computing the optimum of the fractional relaxation. This follows a similar approach as used in [22]. Note that $z_i^* \geq \ell_i := \max\{0, \lceil x_i^* \rceil - m(2m\Delta + 1)^m\}$. By setting $y = x - \ell$ we obtain the equivalent ILP $\max\{c^T y : Ay = b - A\ell, y \in \mathbb{Z}_{\geq 0}^n\}$. It suffices to find an optimal solution to it. Notice that $z^* - \ell$ is optimal for this ILP and we can bound

$$||z^* - \ell||_1 \le ||z^* - x^*||_1 + ||\ell - x^*||_1 \le m(2m\Delta + 1)^m + m^2(2m\Delta + 1)^m = O(m\Delta)^m.$$

Here, we use that x^* and ℓ can only differ in the m many non-zero components of x^* and in those by at most $m(2m\Delta + 1)^m$. Also, note that the O-notation hides polynomial terms in m. Using $K = O(m\Delta)^m$, $H \leq O(\sqrt{m}\Delta)$ and the $O(m)^{m/2} \log^{3m}(m) \cdot n$ time algorithm [11] for solving the

relaxation, we derive a running time of

$$O(m)^{m/2} \log^{3m}(m) \cdot n + O(H)^{2m} \cdot \frac{\log(K)}{\log(\Delta)}$$

$$\leq O(\sqrt{m})^m \log^{3m}(m) \cdot O(\Delta)^m + O(\sqrt{m}\Delta)^{2m} \cdot \frac{m \log(m\Delta)}{\log(\Delta)} \leq O(\sqrt{m}\Delta)^{2m}. \quad (8)$$

Similarly, we can improve the running time for finding any feasible solution to

$$O(m)^{m/2} \log^{3m}(m) \cdot n + O(H)^m \cdot \log(\Delta) \log(K)$$

$$\leq O(\sqrt{m})^m \log^{3m}(m) \cdot O(\Delta)^m + O(\sqrt{m}\Delta)^m \log(\Delta) \cdot m \log(m\Delta) \leq O(\sqrt{m}\Delta)^{(1+o(1))m}. \tag{9}$$

This proves the running times (2) and (3) in the case that the ILP is bounded. Testing whether an ILP is unbounded can be done without increasing the asymptotic running time, as we will lay out next.

3.4 Unbounded solutions

The ILP $\max\{c^Tx: Ax = b, x \in \mathbb{Z}_{\geq 0}^n\}$ is unbounded, if and only if it is feasible and $\max\{c^Tx: Ax = 0, x \in \mathbb{Z}_{\geq 0}^n\}$ has a solution with positive value. The former can be checked with our algorithm, hence it remains to check if the latter condition holds. We can simply solve the LP relaxation for this. If there is a fractional solution with positive value, there is also an integral one. This is because by Cramer's rule there exists a fractional solution with denominators $\det(A)$, hence multiplying by $\det(A)$ yields an integral solution.

3.5 Heterogeneous rows

Let $\Delta_1, \ldots, \Delta_m \leq \Delta$ denote the largest absolute values of each row in A. When some of these values are much smaller than Δ , the maximum among all, we can do better than $O(\sqrt{m}\Delta)^{2m}$. Define $A' = \operatorname{diag}(\Delta_1^{-1}, \ldots, \Delta_m^{-1}) \cdot A$, where

$$\operatorname{diag}(\Delta_1^{-1}, \dots, \Delta_m^{-1}) = \begin{pmatrix} \Delta_1^{-1} & 0 \\ & \ddots & \\ 0 & & \Delta_m^{-1} \end{pmatrix}.$$

We claim that in the dynamic program a table of size $\prod_{k=1}^m O(H'\Delta_k)$ suffices, where $H' \ge \operatorname{herdisc}(A')$. Clearly, the ILP $\max\{c^Tx, Ax = b, x \in \mathbb{Z}_{>0}^n\}$ is equivalent to

$$\max\{c^T x, A' x = b', x \in \mathbb{Z}_{>0}^n\},\$$

where $b'=\operatorname{diag}(\Delta_1^{-1},\ldots,\Delta_m^{-1})\cdot b$. At first glance, our algorithm cannot be applied to this problem, since the entries are not integral. However, in the algorithm we only use the fact that the number of points Ax with $x\in\mathbb{Z}_{\geq 0}^n$ close to some point b'', that is, with $\|Ax-b''\|_\infty\leq 4H$, is small and can be enumerated. The points A'x with $x\in\mathbb{Z}_{\geq 0}^n$ and $\|A'x-b''\|_\infty\leq 4H'$ are exactly those with $|(Ax)_k-b''_k\cdot\Delta_k|\leq 4H'\cdot\Delta_k$ for all k. These are $\prod_{k=1}^m O(H'\Delta_k)$ many and they can be enumerated. This way, we get a running time of $\prod_{k=1}^m O(H'\Delta_k)^2$, which using the bound from Theorem 5 yields

$$\prod_{k=1}^{m} O(m\Delta_k^2). \tag{10}$$

If one is only interested in a feasible solution, then this improves to

$$O(\sqrt{m})^{(1+o(1))m} \cdot \prod_{k=1}^{m} [m\Delta_k] \cdot \log^2(\Delta). \tag{11}$$

4 Lower bounds

In this section we give conditional lower bounds that match the running time of our algorithm both for finding an optimal solution and for finding a feasible solution.

4.1 Optimization problem

We use an equivalence between the problems Unbounded Knapsack and $(\min, +)$ -convolution regarding sub-quadratic algorithms.

Definition 11 (Unbounded Knapsack). Given $C \in \mathbb{N}$, $w_1, \ldots, w_n \in \mathbb{N}$, and $p_1, \ldots, p_n \in \mathbb{N}$ find integer multiplicities x_1, \ldots, x_n , such that $\sum_{i=1}^n x_i \cdot w_i \leq C$ and $\sum_{i=1}^n x_i \cdot p_i$ is maximized.

Note that when we instead require $\sum_{i=1}^{n} x_i \cdot w_i = C$ in the problem above, we can transform it to this form by adding an item of profit zero and weight 1.

Theorem 12 ([17, 31]). For any $\delta > 0$ there exists no $O((n+C)^{2-\delta})$ time algorithm for Unbounded Knapsack unless there is a truly sub-quadratic algorithm for (min, +)-convolution.

When using this theorem, we assume that the input already consists of the at most C relevant items only, $n \leq C$, and $w_i \leq C$ for all i. This preprocessing can be done in time O(n+C).

Theorem 13. Let $m \in \mathbb{N}$. For any $\delta > 0$ and any computable function f there does not exist an algorithm that solves ILPs with m constraints in time $f(m) \cdot (n^{2-\delta} + (\Delta + ||b||_{\infty})^{2m-\delta})$, unless there exists a truly sub-quadratic algorithm for (min, +)-convolution.

Proof. Proof. Let $\delta > 0$ and $m \in \mathbb{N}$. Assume that there exists an algorithm that solves ILPs of the form $\max\{c^Tx: Ax = b, x \in \mathbb{Z}_{\geq 0}^n\}$ where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $c \in \mathbb{Z}^n$ in time $f(m) \cdot (n^{2-\delta} + (\Delta + ||b||_{\infty})^{2m-\delta})$, where Δ is the greatest absolute value in A. We will show that this implies an $O((n+C)^{2-\delta'})$ time algorithm for Unbounded Knapsack for some $\delta' > 0$. Let $(C, (w_i)_{i=1}^n, (p_i)_{i=1}^n)$ be an instance of this problem. Let us first observe that the claim holds for m = 1. Clearly Unbounded Knapsack (with equality) can be written as the following ILP.

$$\max \sum_{i=1}^{n} p_i \cdot x_i$$

$$\sum_{i=1}^{n} w_i \cdot x_i = C$$

$$x \in \mathbb{Z}_{\geq 0}^n$$
(UKS1)

Since $w_i \leq C$ for all i (otherwise the item can be discarded), we can solve this ILP by assumption in time $f(1) \cdot (n^{2-\delta} + (2C)^{2-\delta}) \leq O((n+C)^{2-\delta})$. Now consider the case where m > 1. We want to reduce Δ by exploiting the additional rows. Let $\Delta = \lfloor C^{1/m} \rfloor + 1 > C^{1/m}$. We write C in base- Δ notation, that is,

$$C = C^{(0)} + \Delta C^{(1)} + \dots + \Delta^{m-1} C^{(m-1)}$$
.

where $0 \leq C^{(k)} < \Delta$ for all k. Likewise, write $w_i = w_i^{(0)} + \Delta w_i^{(1)} + \dots + \Delta^{m-1} w_i^{(m-1)}$ with

 $0 \le w_i^{(k)} < \Delta$ for all k. We claim that (UKS1) is equivalent to the following ILP.

$$\max \sum_{i=1}^{n} p_i \cdot x_i$$

$$\sum_{i=1}^{n} [w_i^{(0)} \cdot x_i] - \Delta \cdot y_1 = C^{(0)}$$
(12)

$$\sum_{i=1}^{n} [w_i^{(1)} \cdot x_i] + y_1 - \Delta \cdot y_2 = C^{(1)}$$
(13)

$$\sum_{i=1}^{n} [w_i^{(m-2)} \cdot x_i] + y_{m-2} - \Delta \cdot y_{m-1} = C^{(m-2)}$$
(14)

$$\sum_{i=1}^{n} [w_i^{(m-1)} \cdot x_i] + y_{m-1} = C^{(m-1)}$$

$$x \in \mathbb{Z}_{\geq 0}^n$$

$$y \in \mathbb{Z}_{\geq 0}^{m-1}$$
(15)

Implication $x \in (USK1) \Rightarrow x \in (USKm)$. Let x be a solution to (UKS1). Then for all $1 \le \ell \le m$,

$$\sum_{i=1}^n \sum_{k=0}^{\ell-1} \Delta^k w_i^{(k)} \cdot x_i \equiv \sum_{i=1}^n w_i \cdot x_i \equiv C \equiv \sum_{k=0}^{\ell-1} \Delta^k C^{(k)} \mod \Delta^\ell.$$

This is because all $\Delta^{\ell} w_i^{(\ell)}, \ldots, \Delta^{m-1} w_i^{(m-1)}$ and $\Delta^{\ell} C^{(\ell)}, \ldots, \Delta^{m-1} C^{(m-1)}$ are multiples of Δ^{ℓ} . It follows that there exists an $y_{\ell} \in \mathbb{Z}$ such that

$$\sum_{i=1}^{n} \sum_{k=0}^{\ell-1} [\Delta^k w_i^{(k)} \cdot x_i] - \Delta^\ell \cdot y_\ell = \sum_{k=0}^{\ell-1} \Delta^k C^{(k)}.$$

Furthermore, y_{ℓ} is non-negative, because otherwise

$$\begin{split} \sum_{k=0}^{\ell-1} \Delta^k C^{(k)} & \leq \sum_{k=0}^{\ell-1} \Delta^k (\Delta - 1) < \Delta^{\ell-1} (\Delta - 1) \sum_{k=0}^{\infty} \Delta^{-k} \\ & = \Delta^{\ell-1} \frac{\Delta - 1}{1 - \frac{1}{\Delta}} = \Delta^{\ell} \leq -\Delta^{\ell} y_{\ell} \leq \sum_{i=1}^{n} \sum_{k=0}^{\ell-1} [\Delta^k w_i^{(k)} \cdot x_i] - \Delta^{\ell} y_{\ell}. \end{split}$$

We choose y_1, \ldots, y_m exactly like this. The first constraint (12) follows directly. Now let $\ell \in \{2, \ldots, m\}$. By choice of $y_{\ell-1}$ and y_{ℓ} we have that

$$\sum_{i=1}^{n} \left[\underbrace{\left(\sum_{k=0}^{\ell-1} \Delta^k w_i^{(k)} - \sum_{k=0}^{\ell-2} \Delta^k w_i^{(k)} \right)}_{=\Delta^{\ell-1} w_i^{(\ell-1)}} \cdot x_i \right] + \Delta^{\ell-1} \cdot y_{\ell-1} - \Delta^{\ell} \cdot y_{\ell} = \underbrace{\sum_{k=0}^{\ell-1} \Delta^k C^{(k)} - \sum_{k=0}^{\ell-2} \Delta^k C^{(k)}}_{=\Delta^{\ell-1} C^{(\ell-1)}} .$$
 (16)

Dividing both sides by $\Delta^{\ell-1}$ we get every constraint (13) - (14) for the correct choice of ℓ . Finally, consider the special case of the last constraint (15). By choice of y_m we have that

$$\sum_{i=1}^{n} \sum_{k=0}^{m-1} \Delta^{k} w_{i}^{(k)} \cdot x_{i} - \Delta^{m} \cdot y_{m} = \underbrace{\sum_{k=0}^{m-1} \Delta^{k} C^{(k)}}_{-C}.$$

Thus, $y_m = 0$ and (16) implies the last constraint (with $\ell = m$).

Implication $x \in (USKm) \Rightarrow x \in (USK1)$. Let $x_1, \ldots, x_n, y_1, \ldots, y_{m-1}$ be a solution to (UKSm) and set $y_m = 0$. We show by induction that for all $\ell \in \{1, \ldots, m\}$ it holds that

$$\sum_{i=1}^{n} \sum_{k=0}^{\ell-1} \Delta^k w_i^{(k)} \cdot x_i - \Delta^{\ell} y_{\ell} = \sum_{k=0}^{\ell-1} \Delta^k C^{(k)}.$$

With $\ell = m$ this implies the claim as $y_m = 0$ by definition. For $\ell = 1$ the equation is exactly the first constraint (12). Now let $\ell > 1$ and assume that the equation above holds. We will show that it also holds for $\ell + 1$. From (USKm) we have

$$\sum_{i=1}^{n} [w_i^{(\ell)} \cdot x_i] + y_{\ell} - \Delta \cdot y_{\ell+1} = C^{(\ell)}.$$

Multiplying each side by Δ^{ℓ} we get

$$\sum_{i=1}^{n} [\Delta^{\ell} w_i^{(\ell)} \cdot x_i] + \Delta^{\ell} y_{\ell} - \Delta^{\ell+1} \cdot y_{\ell+1} = \Delta^{\ell} C^{(\ell)}.$$

By adding and subtracting the same elements, it follows that

$$\sum_{i=1}^{n} \left[\left(\sum_{k=0}^{\ell} \Delta^k w_i^{(k)} - \sum_{k=0}^{\ell-1} \Delta^k w_i^{(k)} \right) \cdot x_i \right] + \Delta^{\ell} \cdot y_{\ell} - \Delta^{\ell+1} \cdot y_{\ell+1} = \sum_{k=0}^{\ell} \Delta^k C^{(k)} - \sum_{k=0}^{\ell-1} \Delta^k C^{(k)}.$$

By inserting the induction hypothesis we conclude

$$\sum_{i=1}^{n} \sum_{k=0}^{\ell} [\Delta^k w_i^{(k)} \cdot x_i] - \Delta^{\ell+1} y_{\ell+1} = \sum_{k=0}^{\ell} \Delta^k C^{(k)}.$$

Constructing and solving the ILP. The ILP (UKSm) can be constructed easily in $O(Cm+nm) \leq O((n+C)^{2-\delta/m})$ operations (recall that m is a constant). We obtain $\Delta = \lfloor C^{1/m} \rfloor + 1$ by guessing: More precisely, we iterate over all numbers $\Delta_0 \leq C$ and find the one where $(\Delta_0 - 1)^m < C \leq \Delta_0^m$. Although there are more efficient, non-trivial ways to compute the rounded m-th root, this is not required here. The base- Δ representation for w_1, \ldots, w_n and C can be computed with O(m) operations for each of these numbers.

All entries of the matrix in (UKSm) and the right-hand side are bounded by $\Delta = O(C^{1/m})$. Therefore, by assumption this ILP can be solved in time

$$f(m) \cdot (n^{2-\delta} + O(C^{1/m})^{2m-\delta}) \le f(m) \cdot O(1)^{2m-\delta} \cdot (n+C)^{2-\delta/m} = O((n+C)^{2-\delta/m}).$$

This yields a truly sub-quadratic algorithm for Unbounded Knapsack.

4.2 Feasibility problem

We will show that our algorithm for solving feasibility of ILPs is optimal (except for sub-polynomial improvements). We use a recently discovered lower bound for k-SUM based on the SETH.

Definition 14 (k-SUM). Given $T \in \mathbb{N}_0$ and $Z_1, \ldots, Z_k \subset \mathbb{N}_0$ where $|Z_1| + |Z_2| + \cdots + |Z_k| = n \in \mathbb{N}$ find $z_1 \in Z_1, z_2 \in Z_2, \ldots, z_k \in Z_k$ such that $z_1 + z_2 + \cdots + z_k = T$.

Theorem 15 ([1]). If the SETH holds, then for every $\delta > 0$ there exists a value $\gamma > 0$ such that k-SUM cannot be solved in time $O(T^{1-\delta} \cdot n^{\gamma k})$.

This implies that for every $p \in \mathbb{N}$ there is no $O(T^{1-\delta} \cdot n^p)$ time algorithm for k-SUM if $k \geq p/\gamma$.

Theorem 16. Let $m \in \mathbb{N}$. If the SETH holds, then for every $\delta > 0$ and every computable function f, there does not exist an algorithm that solves feasibility of ILPs with m constraints in time $n^{f(m)} \cdot (\Delta + ||b||_{\infty})^{m-\delta}$.

Proof. Proof. Like in the previous reduction we start with the case of m = 1. For higher values of m the result can be shown in the same way as before.

Suppose there exists an algorithm for solving feasibility of ILPs with one constraint in time $n^{f(1)} \cdot (\Delta + ||b||_{\infty})^{1-\delta}$ for some $\delta > 0$ and $f(1) \in \mathbb{N}$. Let γ be the constant given by Theorem 15 for this δ and set $k = \lceil f(1)/\gamma \rceil$. Now consider an instance (T, Z_1, \ldots, Z_k) of k-SUM. We will show that this can be solved in $O(T^{1-\delta} \cdot n^{f(1)})$, which contradicts the SETH. For every $i \leq k$ and every $z \in Z_i$ we use a binary variable $x_{i,z}$ that describes whether z is used. We can easily model k-SUM as the following ILP:

$$\sum_{i=1}^{k} \sum_{z \in Z_i} z \cdot x_{i,z} = T$$

$$\sum_{z \in Z_i} x_{i,z} = 1 \qquad \forall i \in \{1, \dots, k\}$$

$$x_{i,z} \in \mathbb{Z}_{\geq 0} \qquad \forall i \in \{1, \dots, k\}, z \in Z_i$$

However, since we want to reduce to an ILP with one constraint, we need a slightly more sophisticated construction. We will show that the cardinality constraints can be encoded into the k-SUM instance by increasing the numbers by a factor of $2^{O(k)}$, which is in O(1) since k is some constant depending on f(1) and γ only. We will use this to obtain an ILP with only one constraint and values of size at most O(T). A similar construction is also used in [1].

Our goal is to construct an instance (T', Z'_k, \dots, Z'_k) such that for every x^* it holds that x^* is a solution to the first ILP if and only if

$$x^* \in \{x : \sum_{i=1}^k \sum_{z \in Z_i'} z \cdot x_{i,z} = T', x \in \mathbb{Z}_{\geq 0}^n\}.$$
 (17)

We will use one element to represent each element in the original instance. Consider the binary representation of numbers in $Z'_1 \cup \cdots \cup Z'_k$ and of T'. The numbers in the new instance will consist of three parts and $\lceil \log(k) \rceil$ many 0s between them to prevent interference. For an illustration of the construction see Figure 2. The $\lceil \log(k) \rceil$ most significant bits ensure that exactly k elements are selected; the middle part are k bits that ensure of every set Z'_i exactly one element is selected; the least significant $\lceil \log(T) \rceil$ bits represent the original values of the elements. Set the values in the first part of the numbers to 1 for all elements $Z'_1 \cup \cdots \cup Z'_k$ and to k in T'. Clearly this ensures that at most k elements are chosen. The sum of at most k elements cannot be larger than $k \leq 2^{\lceil \log(k) \rceil}$ times the biggest element. This implies that the buffers of $\lceil \log(k) \rceil$ zeroes cannot overflow and we can consider each of the three parts independently. It follows that exactly k elements must be chosen by any feasible solution. The system $\{x: \sum_{i=1}^k 2^i x_i = 2^{k+1} - 1, \|x\|_1 = k, \mathbb{Z}_{\geq 0}^k\}$ has exactly one solution and this solution is $(1, 1, \ldots, 1)$: Consider summing up k powers of 2 and envision the binary representation of the partial sums. When we add some 2^i to the partial sum, the number of ones in the binary representation increases by one, if the i'th bit of the current sum is zero. Otherwise, it does not increase. However, since in the binary representation of the final sum there are k ones, it has to increase in each addition. This means no power of two can be added twice and therefore each has to be added exactly once.

It follows that the second part of the numbers enforces that of every Z_i' exactly one element is chosen. We conclude that (17) solves the initial k-SUM instance. By assumption this can be done in time $n^{f(1)} \cdot (\Delta + \|b\|_{\infty})^{1-\delta} = n^{f(1)} \cdot O(T')^{1-\delta} = O(n^{f(1)} \cdot T^{1-\delta})$. Here we use that $T' \leq 2^{3 \log(k) + k + \log(T) + 4} = O(k^3 2^k T) = O(T)$, since k is a constant.

For m > 1 we can use the same construction as in the reduction for the optimization problem: Suppose there is an algorithm that finds feasible solutions to ILPs with m constraints in time

$$Z_i'\ni z'=\underbrace{0\dots0001}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots010\dots0}_{k}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace{0\dots0}_{\lceil\log(k)\rceil}|\underbrace$$

Figure 2: Construction of Z'_i and T'

 $n^{f(m)} \cdot (\Delta + \|b\|_{\infty})^{m-\delta}$. Choose γ such that there is no algorithm for k-SUM with running time $O(T^{1-\delta/m} \cdot n^{\gamma k})$ (under SETH). We set $k = \lceil f(m)/\gamma \rceil$. By splitting the one constraint of (17) into m constraints we can reduce the upper bound on elements from O(T) to $O(T^{1/m})$. This means the assumed running time for solving ILPs can be used to solve k-SUM in time

$$n^{f(m)} \cdot O(T^{1/m})^{m-\delta} \le n^{\gamma k} O(1)^{m-\delta} T^{1-\delta/m} = O(n^{\gamma k} T^{1-\delta/m}).$$

5 Applications

In this section we apply our results to some well-known problems, which can be formulated using ILPs with few constraints and small entries. In particular, we give examples, where the reduction of the running time by a factor n improves on the state-of-the-art, the heterogeneity of rows plays a role, and one where the use of hereditary discrepancy and the removal of the dependence on $||b||_{\infty}$ are relevant.

5.1 Unbounded Knapsack and Unbounded Subset-Sum

Recall Definition 11, which introduces the problem Unbounded Knapsack. Traditionally, C is only an upper bound on $\sum_{i=1}^n w_i \cdot x_i$ in most of the literature, but that variant easily reduces to the problem above by adding a slack variable. Unbounded Subset-Sum is the same problem without an objective function, i.e., the problem of finding a multi-set of items whose weights w_i sum up to exactly C. We assume that no two items have the same weight. Otherwise in time $O(n + \Delta)$ we can remove all duplicates by keeping only the most valuable ones. This gives algorithms with running time $O(n + \Delta^2)$ and $O(n + \Delta \log^2(\Delta))$ for Unbounded Knapsack and Unbounded Subset-Sum, respectively, where Δ is the maximum weight among all items (using the results from Section 3.3). The previously best pseudo-polynomial algorithms for Unbounded Knapsack, have running times O(nC) (standard dynamic programming; see e.g. [28]), $O(n\Delta^2)$ [22], or very recently $O(\Delta^2 \log(C))$ [3]. We note that the last algorithm, which was discovered simultaneously and independently to ours, follows a very similar approach to ours when restricted to the Unbounded Knapsack case. After our work Chan and He gave an interesting improvement, which achieves a running time of $O(n\Delta \log^3(\Delta))$ [12]. Note that n is potentially much smaller than Δ , but not vice versa

For Unbounded Subset-Sum the state-of-the-art is a $O(C \log(C))$ time algorithm [9]. Hence, our algorithm is preferable when $\Delta \ll C$. Very recently Klein [29] studied this problem and showed the perhaps surprising fact that there is also a pseudo-polynomial algorithm in terms of the smallest weight (and not the largest), but then the dependence on it is quadratic and cannot be improved unless the (min, +)-convolution conjecture is false.

5.2 Change Making

In the Change Making problem we are given an infinite supply of coins with values $c_1 < c_2 < \cdots < c_n$ and a target t. The goal is to match t with as few coins as possible. In the decision variant, where we want to find a solution with at most k coins, this can be written as finding a solution to the ILP

$$\left\{ \sum_{i=1}^{n} c_i x_i = t, \sum_{i=1}^{n} x_i + s = k, x \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_{\geq 0} \right\}.$$

In other words, this is a feasibility ILP with two rows, where the first row has maximum coefficient c_n and the second row has maximum coefficient 1. Using (9) this can be solved in time

$$O(c_n \log^2(c_n)).$$

This matches exactly the running time in [12]. In fact, that algorithm behaves very similar to ours when restricted to this problem.

5.3 Scheduling Jobs on Identical Parallel Machines

The problem Scheduling Jobs on Identical Parallel Machines asks to distribute N jobs onto $M \leq N$ machines. Each job j has a processing time p_j and the objective is to minimize the makespan, that is, the maximum sum of processing times on a single machine. Since an exact solution cannot be computed unless P = NP, we are satisfied with a $(1 + \epsilon)$ -approximation, where $\epsilon > 0$ is part of the input. We will outline how this problem can be solved using our algorithm. This gives the best known running time, which is even a slight improvement over the sophisticated algorithm for this problem in [24].

We consider here the variant, in which a makespan τ is given and we have to find a schedule with makespan at most $(1+\epsilon)\tau$ or prove that there exists no schedule with makespan at most τ . This suffices by using a standard dual approximation framework. It is easy to see that one can discard all jobs of size at most $\epsilon \cdot \tau$ and add them greedily after a solution for the other jobs is found. The big jobs can each be rounded to the next value of the form $\epsilon \cdot \tau \cdot (1+\epsilon)^i$ for some i. This reduces the number of different processing times to $O(1/\epsilon \log(1/\epsilon))$ many and increases the makespan by at most a factor of $1+\epsilon$. We are now ready to write this problem as an ILP. A configuration is a way to use a machine. It describes how many jobs of each size are assigned to this machine. Since we aim for a makespan of $(1+\epsilon) \cdot \tau$, the sum of these sizes must not exceed this value. The configuration ILP has a variable for every valid configuration and it describes how many machines use this configuration. Let \mathcal{C} be the set of valid configurations and C_k the multiplicity of size k in a configuration $C \in \mathcal{C}$. The following ILP solves the rounded instance. We note that there is no objective function in it.

$$\sum_{C \in \mathcal{C}} x_C = M$$

$$\sum_{C \in \mathcal{C}} C_k \cdot x_C = N_k \qquad \forall k \in \mathcal{K}$$

$$x_C \in \mathbb{Z}_{\geq 0} \qquad \forall C \in \mathcal{C}$$

Here \mathcal{K} are the rounded sizes and N_k the number of jobs with rounded size $k \in \mathcal{K}$. The first constraint enforces that the correct number of machines is used, the next $|\mathcal{K}|$ many enforce that for each size the correct number of jobs is scheduled.

It is notable that this ILP has only few constraints (a constant for a fixed choice of ϵ) and also the ℓ_1 -norm of each column is small. More precisely, it is at most $1/\epsilon$, since every size is at least $\epsilon \cdot \tau$ and therefore no more than $1/\epsilon$ jobs fit in one configuration. By the Theorem 7 we know that $H = 1/\epsilon$ is an upper bound on the hereditary discrepancy, $\Delta \leq 1/\epsilon$, $m = O(1/\epsilon \log(1/\epsilon))$, $||b||_{\infty} \leq N$, and $n \leq (1/\epsilon)^{O(1/\epsilon \log(1/\epsilon))}$. Notice also that K = N is a trivial upper bound on the

 ℓ_1 -norm of any solution. Using (7) and rounding in time $O(N+1/\epsilon \log(1/\epsilon))$ yields a running time of

$$\begin{split} O(H)^m \log(\Delta) \log(K) + O(nm) + O\left(N + \frac{1}{\epsilon} \log\left(\frac{1}{\epsilon}\right)\right) \\ &\leq 2^{O(1/\epsilon \log^2(1/\epsilon))} \log(N) + O\left(N + \frac{1}{\epsilon} \log\left(\frac{1}{\epsilon}\right)\right) \leq 2^{O(1/\epsilon \log^2(1/\epsilon))} + O(N). \end{split}$$

The inequality above follows from distinguishing between $2^{O(1/\epsilon \log^2(1/\epsilon))} \leq \log(N)$ and $2^{O(1/\epsilon \log^2(1/\epsilon))} > \log(N)$. The same running time (except for a higher constant in the exponent) could be obtained with [22]. However, in order to avoid a multiplicative factor of N, one would have to solve the LP relaxation first and then use proximity. Our approach gives an easier, purely combinatorial algorithm. The advantage of our algorithm comes from removing the dependence on $||b||_{\infty}$. Recently, the authors together with Berndt and Deppert [7] introduced a more involved ILP for this problem, which reduces the ℓ_1 -norm of each column to $O(\log(1/\epsilon))$ while maintaining the other bounds. Still using the algorithm from this work, this leads to a mild improvement of the running time to

$$2^{O(1/\epsilon \log(1/\epsilon)\log\log(1/\epsilon))} + O(N).$$

This improvement relies on the low hereditary discrepancy and does not follow with the weaker bounds on the Steinitz Lemma as in [22].

References

- [1] Amir Abboud, Karl Bringmann, Danny Hermelin, and Dvir Shabtay. SETH-based lower bounds for subset sum and bicriteria path. In *Proceedings of SODA*, pages 41–57, 2019.
- [2] Pankaj K Agarwal, Micha Sharir, and Sivan Toledo. An efficient multi-dimensional searching technique and its applications. In *Technical Report CS-1993-20*, 1993.
- [3] Kyriakos Axiotis and Christos Tzamos. Capacitated dynamic programming: Faster knapsack and graph algorithms. In *Proceedings of ICALP*, pages 19:1–19:13, 2019.
- [4] Arturs Backurs, Piotr Indyk, and Ludwig Schmidt. Better approximations for tree sparsity in nearly-linear time. In *Proceedings of SODA*, pages 2215–2229, 2017.
- [5] Nikhil Bansal. Constructive algorithms for discrepancy minimization. In *Proceedings of FOCS*, pages 3–10, 2010.
- [6] József Beck and Tibor Fiala. "integer-making" theorems. Discrete Applied Mathematics, 3(1):1–8, 1981.
- [7] Sebastian Berndt, Max A Deppert, Klaus Jansen, and Lars Rohwedder. Load balancing: The long road from theory to practice. In *Proceedings of the ALENEX*, pages 104–116, 2022.
- [8] David Bremner, Timothy M. Chan, Erik D. Demaine, Jeff Erickson, Ferran Hurtado, John Iacono, Stefan Langerman, Mihai Patrascu, and Perouz Taslakian. Necklaces, convolutions, and X+Y. *Algorithmica*, 69(2):294–314, 2014.
- [9] Karl Bringmann. A near-linear pseudopolynomial time algorithm for subset sum. In *Proceedings of SODA*, pages 1073–1084, 2017.
- [10] Hervé Brönnimann, Bernard Chazelle, and Jirí Matousek. Product range spaces, sensitive sampling, and derandomization. SIAM Journal on Computing, 28(5):1552–1575, 1999.
- [11] Timothy M. Chan. Improved deterministic algorithms for linear programming in low dimensions. ACM Transactions on Algorithms, 14(3):30:1–30:10, 2018.

- [12] Timothy M. Chan and Qizheng He. More on change-making and related problems. *Journal of Computer and System Sciences*, 124:159–169, 2022.
- [13] Timothy M. Chan and Moshe Lewenstein. Clustered integer 3SUM via additive combinatorics. In *Proceedings of STOC*, pages 31–40, 2015.
- [14] Bernard Chazelle and Jirí Matousek. On linear-time deterministic algorithms for optimization problems in fixed dimension. *Journal of Algorithms*, 21(3):579–597, 1996.
- [15] Kenneth L. Clarkson. Linear programming in $O(n \times 3^{d^2})$ time. Information Processing Letters, 22(1):21-24, 1986.
- [16] Kenneth L. Clarkson. Las vegas algorithms for linear and integer programming when the dimension is small. *Journal of the ACM*, 42(2):488–499, 1995.
- [17] Marek Cygan, Marcin Mucha, Karol Wegrzycki, and Michal Włodarczyk. On problems equivalent to (min, +)-convolution. In *Proceedings of ICALP*, pages 22:1–22:15, 2017.
- [18] Daniel Dadush. A randomized sieving algorithm for approximate integer programming. *Al-qorithmica*, 70(2):208–244, 2014.
- [19] Daniel Nicolas Dadush. Integer programming, lattice algorithms, and deterministic volume estimation. Georgia Institute of Technology, 2012.
- [20] Martin E. Dyer. On a multidimensional search technique and its application to the euclidean one-centre problem. SIAM Journal on Computing, 15(3):725–738, 1986.
- [21] Martin E. Dyer and Alan M. Frieze. A randomized algorithm for fixed-dimensional linear programming. *Mathematical Programming*, 44(1-3):203–212, 1989.
- [22] Friedrich Eisenbrand and Robert Weismantel. Proximity results and faster algorithms for integer programming using the Steinitz lemma. *ACM Transactions on Algorithms*, 16(1):5:1–5:14, 2020.
- [23] Fedor V. Fomin, Fahad Panolan, M. S. Ramanujan, and Saket Saurabh. On the optimality of pseudo-polynomial algorithms for integer programming. In *Proceedings of ESA 2018*, pages 31:1–31:13, 2018.
- [24] Klaus Jansen, Kim-Manuel Klein, and José Verschae. Closing the gap for makespan scheduling via sparsification techniques. *Mathematics of Operations Research*, 45(4):1371–1392, 2020.
- [25] Klaus Jansen and Lars Rohwedder. On integer programming and convolution. In Proceedings of ITCS, pages 43:1–43:17, 2019.
- [26] Gil Kalai. A subexponential randomized simplex algorithm (extended abstract). In S. Rao Kosaraju, Mike Fellows, Avi Wigderson, and John A. Ellis, editors, *Proceedings of STOC*, pages 475–482, 1992.
- [27] Ravi Kannan. Minkowski's convex body theorem and integer programming. *Mathematics of Operations Research*, 12(3):415–440, 1987.
- [28] Hans Kellerer, Ulrich Pferschy, and David Pisinger. Knapsack problems. Springer, 2004.
- [29] Kim-Manuel Klein. On the fine-grained complexity of the unbounded subsetsum and the Frobenius problem. In *Proceedings of SODA*, pages 3567–3582, 2022.
- [30] Dusan Knop, Michal Pilipczuk, and Marcin Wrochna. Tight complexity lower bounds for integer linear programming with few constraints. In *Proceedings of STACS*, pages 44:1–44:15, 2019.

- [31] Marvin Künnemann, Ramamohan Paturi, and Stefan Schneider. On the fine-grained complexity of one-dimensional dynamic programming. In *Proceedings of ICALP*, pages 21:1–21:15, 2017.
- [32] Eduardo Sany Laber, Wilfredo Bardales Roncalla, and Ferdinando Cicalese. On lower bounds for the maximum consecutive subsums problem and the (min, +)-convolution. In *Proceedings* of ISIT, pages 1807–1811, 2014.
- [33] Hendrik W. Lenstra. Integer programming with a fixed number of variables. *Mathematics of Operations Research*, 8(4):538–548, 1983.
- [34] László Lovász, Joel Spencer, and Katalin Vesztergombi. Discrepancy of set-systems and matrices. European Journal of Combinatorics, 7(2):151–160, 1986.
- [35] Shachar Lovett and Raghu Meka. Constructive discrepancy minimization by walking on the edges. SIAM Journal on Computing, 44(5):1573–1582, 2015.
- [36] Jirí Matousek, Micha Sharir, and Emo Welzl. A subexponential bound for linear programming. *Algorithmica*, 16(4/5):498–516, 1996.
- [37] Nimrod Megiddo. Linear programming in linear time when the dimension is fixed. *Journal* of the ACM, 31(1):114–127, 1984.
- [38] Christos H. Papadimitriou. On the complexity of integer programming. *Journal of the ACM*, 28(4):765–768, 1981.
- [39] Raimund Seidel. Small-dimensional linear programming and convex hulls made easy. *Discrete & Computational Geometry*, 6:423–434, 1991.
- [40] Sergey V. Sevastyanov. Approximate solution of some problems in scheduling theory. Metody Diskretnogo Analiza, 32:66-75, 1978. in Russian.
- [41] Joel Spencer. Six standard deviations suffice. Transactions of the American mathematical society, 289(2):679–706, 1985.
- [42] Ernst Steinitz. Bedingt konvergente reihen und konvexe systeme. Journal für die reine und angewandte Mathematik, 143:128–176, 1913. in German.

Omitted proofs

Splitting a solution into two even parts. Recall that Lemma 4 says that a vector x can be split into two parts z and x-z such that Az, $A(x-z) \approx 1/2 \cdot Ax$ and the ℓ_1 -norm of each part is at least a constant fraction of that of x.

Proof. Proof of Lemma 4 Let $x_i' = \lfloor x_i/2 \rfloor$ and $x_i'' = \lceil x_i/2 \rceil - \lfloor x_i/2 \rfloor \in \{0,1\}$ for all i. Then $x_i = \lfloor x_i/2 \rfloor + \lceil x_i/2 \rceil = 2x_i' + x_i''$. Now apply the definition of $\operatorname{disc}(A_I)$ to x'', where $I = \operatorname{supp}(x'')$ are the indices i with $x_i'' = 1$. This way we obtain a vector $z'' \in \{0,1\}^n$ with $\|A(z'' - x''/2)\|_{\infty} \leq \operatorname{disc}(A_I) \leq \operatorname{herdisc}(A)$. We now use z = x' + z'' to show the first part of the lemma. Then

$$\left\|A\left(z-\frac{x}{2}\right)\right\|_{\infty} = \left\|A\left((x'+z'')-\frac{2x'+x''}{2}\right)\right\|_{\infty} = \left\|A\left(z''-\frac{x''}{2}\right)\right\|_{\infty} \le \operatorname{herdisc}(A).$$

Furthermore, for all i

$$0 \le \underbrace{x'_i + z''_i}_{=z_i} \le x'_i + x''_i \le 2x'_i + x''_i = x_i.$$

In order to control the ℓ_1 norm in the second part of the lemma, we first split x into two non-empty $y', y'' \in \mathbb{Z}^n_{\geq 0}$ with y' + y'' = x and $\lfloor \|x\|_1/2 \rfloor = \|y'\|_1 \leq \|y''\|_1 = \lceil \|x\|_1/2 \rceil$. Now apply

the first part of the lemma to obtain $z' \leq y'$ and $z'' \leq y''$ with $||A(z'-y'/2)||_{\infty} \leq \operatorname{herdisc}(A)$ and $||A(z''-y''/2)||_{\infty} \leq \operatorname{herdisc}(A)$. We can assume w.l.o.g. that $||z'||_1 \leq ||y'||_1/2 \leq ||x||_1/4$ and $||z''||_1 \geq ||y''||_1/2 \geq ||x||_1/4$, since otherwise we can swap them for y'-z' and y''-z'', respectively. We will use z=z'+z'' for the second part of the lemma. As for the lower bound,

$$||z||_1 \ge ||z''||_1 \ge \frac{||x||_1}{4}.$$

For the upper bound we first consider the case where $||x||_1 \le 5$ and note that $||z'||_1 \le ||y'||_1/2 = ||y'||_1 - ||y'||_1/2 < ||y'||_1$. Thus,

$$||z||_1 = ||z' + z''||_1 \le ||y' + y''||_1 - 1 \le ||x||_1 - \frac{1}{5}||x||_1 = \frac{4}{5}||x||_1.$$

If $||x||_1 \ge 6$,

$$\|z\|_1 = \|z' + z''\|_1 \le \frac{\|x\|_1}{4} + \|y''\|_1 = \frac{\|x\|_1}{4} + \left\lceil \frac{\|x\|_1}{2} \right\rceil \le \frac{\|x\|_1}{2} + \frac{\|x\|_1}{4} + \frac{1}{2} \le \frac{3}{4} \|x\|_1 + \frac{1}{12} \|x\|_1 \le \frac{5}{6} \|x\|_1.$$

Finally, $z_i = z'_i + z''_i \le y'_i + y''_i = x_i$ and

$$\begin{split} \left\| A \left(z - \frac{x}{2} \right) \right\|_{\infty} &= \left\| A \left((z' + z'') - \frac{y' + y''}{2} \right) \right\|_{\infty} \\ &\leq \left\| A \left(z' - \frac{y'}{2} \right) \right\|_{\infty} + \left\| A \left(z'' - \frac{y''}{2} \right) \right\|_{\infty} \leq 2 \cdot \operatorname{herdisc}(A). \quad \Box \end{split}$$

Computing the dynamic table using convolution. In the following we explain the details on how to reduce the computation of the entries of the dynamic table to a 1-dimensional convolution. We first need to handle that $2^{i-1-k}b$ might not be integral. Let $b^0 = \lfloor 2^{i-1-k}b \rfloor$ denote the vector rounded down in every component. Then D_{i-1} is completely covered by the points with ℓ_{∞} -distance 4H+2 from b^0 . Likewise, D_i is covered by the points with distance 4H+2 from $2b^0$. We project a vector $b' \in D_{i-1}$ to

$$f_{i-1}(b') = \sum_{j=1}^{m} (16H + 11)^{j-1} \underbrace{(4H + 3 + b'_j - b^0)}_{\in \{1, \dots, 8H + 5\}}.$$
 (18)

Notice that 16H + 11 is always bigger than the sum of two values of the form $4H + 3 + b'_j - b^0$. We define $f_i(b')$ for all $b' \in D_i$ in the same way, except we substitute b^0 for $2b^0$. For all $a, a' \in D_{i-1}, b' \in D_i$, it holds that $f_{i-1}(a) + f_{i-1}(a') = f_i(b')$, if and only if $a + a' = b' - (4H + 3, ..., 4H + 3)^T$:

Implication \Rightarrow . Let $f_{i-1}(a) + f_{i-1}(a') = f_i(b')$. Then, in particular,

$$f_{i-1}(a) + f_{i-1}(a') \equiv f_i(b') \mod 16H + 11$$

Since all but the first element of the sum (18) are multiples of 16H + 11, i.e., they are equal 0 modulo 16H + 11, we can omit them in the equation. Hence,

$$(4H+3+a_1-b_1^0)+(4H+3+a_1'-b_1^0)\equiv (4H+3+b_1'-2b_1^0)\mod 16H+11.$$

We even have equality (without modulo) here, because both sides are smaller than $16m\Delta + 11$. Simplifying the equation gives $a_1 + a'_1 = b'_1 - (4H + 3)$. Now consider again the equation $f_{i-1}(a) + f_{i-1}(a') = f_i(b')$. In the sums leave out the first element. The equation still holds, since by the elaboration above this changes the left and right hand-side by the same value. We can now repeat the same argument to obtain $a_2 + a'_2 = b'_2 - (4H + 3)$ and the same for all other dimensions.

Implication \Leftarrow . Let $a + a' = b' - (4H + 3, ..., 4H + 3)^T$. Then for every j,

$$(4H + 3 + a_j - b_j^0) + (4H + 3 + a_j' - b_j^0) = 4H + 3 + b_j' - 2b_j^0.$$

It directly follows that $f_{i-1}(a) + f_{i-1}(a') = f_i(b')$. This means when we write the value of each $b'' \in D_{i-1}$ to r_j and s_j , where $j = f_{i-1}(b'')$ and every entry not used is set to $-\infty$, the correct solutions will be in t. More precisely, we can read the result for some $b' \in D_i$ at t_j where $j = f_i(b' + (4H + 3, ..., 4H + 3)^T)$.