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An Achievable-Region-Based Approach for Kidney Allocation Policy Design with Endogenous Patient Choice

(Authors' names blinded for peer review)

Problem definition. The deceased-donor kidney transplant candidates in the US are ranked according to characteristics of both the donor and the recipient. We seek the ranking policy that optimizes the efficiency-equity tradeoff among all such policies, taking into account patients' strategic choices.

Relevance. Our approach considers a broad class of ranking policies, which provides approximations to the previously and currently used policies in practice. It also subsumes other policies proposed in the literature previously. As such it facilitates a unified way of characterizing good policies.

Methodology. We use a fluid model to approximate the transplant waitlist. Modeling patients as rational decision makers, we compute the resulting equilibria under a broad class of ranking policies, namely the achievable region. We then develop an algorithm that optimizes the system performance over the achievable region.

Results. We show analytically that it suffices to restrict attention to priority scores that are affine in the patient's waiting time. We also show through a numerical study that the total QALYs can be increased substantially by allowing patient rankings to depend on the kidney quality. Lastly, we observe that there is almost no improvement if only the healthier patients are prioritized for certain kidney types.

Managerial Implications. Our results verify that ranking patients differently for kidneys of different quality can reduce the survival mismatch and the kidney wastage significantly. Consequently, the policy change in 2014, that implemented prioritizing the healthiest patients when allocating the highest 20% quality organs, is a step in the right direction. For further improvement, one may consider revising the new policy by also prioritizing the least healthy patients on the waitlist for the lowest-quality organs.

Key words: kidney allocation, fluid model, multiclass queue, Nash equilibrium, achievable region

1. Introduction

Although patients with end-stage renal disease (ESRD) can sustain their renal function on dialysis for several years, the desired therapy for ESRD is transplantation. Unfortunately, the supply of kidneys for transplantation is far fewer than the demand for them. Thus, patients with ESRD join

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a waitlist managed by the United Network of Organ Sharing (UNOS). In 2017, there were 35,587 new patients who registered to the kidney waitlist, but only 14,038 kidneys from deceased donors were transplanted during the same year. As of June 5, 2018, the number of candidates awaiting transplantation is $95,102$[1]. The waiting time to receive a transplant can vary from several months to a decade, or even longer. Therefore, the ranking of patients on the transplant waitlist is a key factor determining who lives and who dies.

Given the limited supply of kidneys for transplantation, any allocation mechanism would inevitably favor certain patients over others. Although it is hard to pick a single performance metric for choosing an allocation scheme among a set of alternatives, a desirable allocation policy should strike a balance between efficiency and equity (OIG, 1991; Zenios et al., 2000; Akan et al., 2012). An allocation scheme is efficient if it maximizes the total quality-adjusted-life-years (QALY) of all patients, and is equitable if patients in different categories have equal access to kidneys (Zenios et al., 2000). Unfortunately, efficiency and equity cannot be optimized at the same time. For example, offering kidneys to younger patients may contribute a larger margin in QALYs, but can increase inequity across different age groups. Therefore, the policy maker should strive to balance efficiency and equity.

As alluded to above, the UNOS is responsible for formulating a nation-wide ranking policy that governs the allocation of deceased-donor kidneys in the US. The new policy has been implemented since December, 2014. The previous policy had been in place for more than twenty years without major changes. Under the previous policy, each patient on the kidney waitlist is assigned a score, which is a sum of the patient's cumulative waiting time since starting dialysis (in years), and a certain number of bonus points for being pediatric (Age < 14), highly sensitized (CPRA[2] $> 80\%$), or well matched with the donor's Human Leukocyte Antigen (HLA) type. The score, however, does not depend on the quality of the kidney, which is measured by kidney donor profile index (KDPI); see (Rao et al., 2009). Kidneys with lower KDPI are associated with better post-transplant survival.

Kenneth Andreoni, the former chair of the UNOS Kidney Transplantation Committee, commented that the old allocation policy "was based on good science, but it was at a different time" (Faherty, 2009). Critics objected to the previous allocation policy because it is responsible for two unintended consequences. First, the post-transplant survival time between a recipient and his donor are not well matched. Under the previous policy, it often happens that a high-quality kidney is transplanted to a senior recipient, who may then die with a well-functioning kidney, or a low-quality kidney, after being transplanted to a young patient, stops functioning and the patient needs another transplant (Committee, 2011). Both cases lead to welfare loss. Second, kidneys are frequently turned down by patients. If a kidney cannot find a recipient within its cold ischemia

time, which is typically 24-48 hours, it loses its function and has to be discarded. Sung et al. (2008) reported that about 11% of medically qualified kidneys were eventually discarded in year 2006.

These unintended consequences were due in part to the fact that the previous allocation policy was "donor-blind", i.e., the score of a patient for a given organ did not depend on the KDPI of the organ (OPTN/UNOS, 2008). Under the new policy that was implemented in 2014, kidneys of the highest quality (KDPI $< 20\%$) are first offered to candidates with estimated post-transplant survival time in the top 20^{th} percentile, and then to the rest of the patients; whereas the rest kidneys (KDPI $\geq 20\%$) will be offered to the entire patient population by each patient's score from high to low[3](Israni et al., 2014). Therefore, the new policy is not donor-blind, and has attempted to increase survival matching by offering high-quality kidneys first to healthier patients. An interesting research question is whether the new policy is effective in increasing survival matching. If not, can it be further improved by moving further toward this direction, e.g., by offering the low-quality kidneys first to less healthy patients?

Both the new and the previous policies take a patient's waiting time into account by adding it to other terms in computing the patient's ranking score for a given kidney offer. One may wonder if the allocation outcome can be further improved by incorporating the waiting time in a more sophisticated way, for example, by allowing the score to increase non-linearly with the waiting time. If yes, then what functional form of the score should be used?

Transplant researchers often resort to simulation experiments to evaluate a kidney allocation policy and glean insights, see for example, (Israni et al., 2014; Schold and Reese, 2014). Indeed, the Scientific Registration of Transplant Research (SRTR) has developed a simulation software called the kidney-pancreas simulated allocation model (KPSAM) for researcher use. The KPSAM is advantageous in capturing complex features of the kidney allocation policies in practice, such as geographical factors, previous transplants, etc. Nevertheless, it suffers from three limitations. First, the KPSAM can only be used to evaluate a specific policy, rather than to derive general insights for a set of policies which share common features, such as the three policy classes defined in this paper. Second, it is computationally costly to simulate a single policy, and the result is usually noisy. Lastly, under both the previous and the new policies, patients can turn down kidney offers without a penalty. In fact, some low-quality organs can be turned down hundreds of times (Wolfe et al., 2007) and the final allocation very much depends on the patients' accept/reject decisions (Su and Zenios, 2006). The KPSAM assumes that the patients' decision patterns follow the historical data and "cannot account changes in organ acceptance behavior. Therefore, if the new policy results in dramatic changes in organ acceptance behavior, the estimates of number of transplants from the simulations will differ from reality" (Israni et al., 2014).

Researchers have also investigated the use of a nonlinear score for kidney allocation. For example, with a data-driven optimization method, Bertsimas et al. (2013) searched for a scoring function which maximizes the total life years from transplant over a six-month period subject to equity constraints. The optimal score function they derived is a piecewise linear function of a patient's cumulative waiting time. Their optimal score has incorporated the KDPI, which also justifies the advantage of allowing the ranking mechanism to depend on kidney quality. In a similar fluid queueing system without patient choice, Ding et al. (2016) showed that the optimal score has to increase nonlinearly with a patient's waiting time. These studies, however, did not consider the patients' strategic behavior in response to the policy revision; while our paper is the first to analytically compare linear and nonlinear allocation scores in the presence of endogenous patient choice.

To systematically evaluate and compare different allocation policies, below we define three policy classes, all of which use health score and waiting time as the two main criteria to rank the patients on the waitlist. The *head-of-line-matching* policies rank patients differently according for organs of different quality (measured by the KDPI). This policy class covers a wide range of ranking mechanisms that could potentially be implemented in the future. The *head-of-line-healthier-first* policies, depending on the kidney quality, assign healthier patients with equal or higher priority over less healthy patients. The healthier-first policies are a subclass of the head-of-line-matching policies, as the latter can prioritize less healthy patients for certain types of kidneys. The head-of-line-healthier-first policies indeed cover the new policy implemented in 2014, in which kidneys of the highest quality are first offered to the healthiest candidates. The *head-of-line-donor-blind* policies rank all patients in the same order irrespective of the kidney quality, which is also a subclass of the head-of-line-matching policies. The previous policy uses the same ranking for all kidneys except for the expanded criterion donors[4], and therefore can be regarded as a head-of-line-donor-blind policy. For simplicity, we refer to the above three policies as the matching, healthier-first, and donor-blind policies, respectively.

We will mathematically define these policies in Section 2.2 after the necessary notations are introduced. We then take a modeling-based approach to investigate the waitlist system under these policies. Our model takes into account the impact of both kidney quality and waiting time on a patient's acceptance/rejection decision and attempts to model the patient's strategic behavior in response to the policy changes. For analytical tractability, we use a fluid model to approximate the dynamics of the transplant waitlist, in which patients and kidneys are regarded as continuous fluid that arrive at the system according to a deterministic process. We show that under the fluid approximation, the kidney waitlist system admits a unique equilibrium, in which all patients decide on whether to accept a kidney offer rationally based on their belief regarding the waiting times

for the various kidney offers they may receive, and the resulting waiting times are consistent with their beliefs. The equilibrium can be characterized as a solution to a nonlinear complementarity problem (NCP).

We assume that the social planner seeks a ranking policy to optimize a given (steady-state) performance metric. We refer to the set of (performance metrics associated with) achievable equilibria as the achievable region for a certain policy class. In queueing systems without strategic customer behavior, the achievable region is usually characterized as a polytope with conservation-law-type constraints; whereas in our setting, each equilibrium is the unique solution to a NCP, therefore the achievable region is non-convex. Consequently, optimization over the achievable region is more challenging in our setting. However, by exploiting the special structure of our policies, we express the achievable region as a union of subregions and derive a closed-form expression for each subregion that facilitates computation.

To summarize, below are the main contributions of this paper.

• We develop a novel fluid model for the kidney transplant waitlist. This model captures a patient's health state change as well as his acceptance/rejection strategy in response to a given allocation policy. We prove that there exists a unique equilibrium and characterize it as a solution to an NCP. To the best of our knowledge, such types of fluid models have not been previously studied, and our equilibrium analysis of this complex fluid model makes a methodological contribution to the related literature.

• The matching policy studied in this paper covers a broad set of policies, including those proposed in the extant literature, as well as the previous and the new policies used by the UNOS. The achievable-region method proposed in this paper allows the policy makers to compute the efficiency-equity Pareto frontier for all matching (or healthier-first, donor-blind) policies. As such, our numerical results, which use kidney transplant data, yield useful policy implications. Specifically, when the policy maker aims to maximize the total QALYs as well as to minimize disparity in transplantation likelihood across different patient classes, the Pareto frontier of the matching policies improves over the donor-blind policies by 10% to 26% in total QALYs, suggesting a great potential of incorporating kidney quality into the ranking mechanism. Therefore, the new policy, which allocates the top 20% and the rest kidneys using different ranking mechanisms, moves the needle in the right direction, i.e., from donor-blind to matching. Nevertheless, a possibly counter-intuitive result is that the healthier-first policies have comparable performance with the donor-blind policies. That means, the new policy, as a healthier-first policy, has not explored the full capability of the matching policy. In order to do that, future policy revisions might consider allocating low-quality kidneys to less healthy patients first.

• We prove that for matching or healthier-first policies, it suffices to use a linear waiting time score (as opposed to a more general function of the waiting time) in order to optimize the steady-state performance. However, for donor-blind policies, we show that a linear score may not recover the entire achievable region, and more sophisticated functional forms are needed in order to ensure optimality.

We conclude this section with a roadmap for the paper. In Section 1.1, we review the related literature and a set of representative policies that have been studied in the literature. In Section 2, we formally introduce our model setup, including definitions of the matching, healthier-first, and donor-blind policies, the calculation of the patient's post-transplant life time, etc. We also present the equilibrium analysis of the fluid model in Section 2, which provides the theoretical basis for the achievable region approach. Section 3 provides the mathematical formulation of the achievable region, and proves that the achievable region of matching policies can be achieved by a score function that is linear in the patient's waiting time. Section 4 discusses how to use the achievable-region approach to optimize certain efficiency metrics subject to equity constraints. Section 5 provides a numerical example of applying the achievable-region approach. Section 6 concludes the paper and summarizes the main policy implications.

Readers primarily interested in the practical/policy implications of our work can ship ahead to Section 5 before reading the earlier sections. Moreover, the derivations in Section 2 can be omitted by such readers on a first reading. However, Sections 3 and 4 are essential to understanding the crux of our approach, the achievable-region method. Similarly, Sections 2 and 3 are essential to understanding the mathematical foundations of our work; and the readers interested in the underlying mathematical development are encouraged to study those in detail as we expect that our approach to carry over to other settings.

1.1. Literature Review

One of the earliest models of organ allocation is the sequential stochastic assignment model studied by Derman et al. (1972). In this model, n resources are assigned to n candidates, with a payoff $c\phi$ if the resource has value ϕ and the candidate has value c. Because of the product form of the reward function, the well-known Hardy's theorem implies that the optimal situation would be an exact high-high match, i.e., the i^{th} highest resource is assigned to the candidate of the i^{th} highest value. This principle, when applied to kidney allocation, suggests that it maximizes total life years when there is a high degree of survival matching. The sequential stochastic assignment problem was later studied by Albright (1974), David and Yechiali (1995), and Righter (1989).

Zenios et al. (2000) considers the resource allocation problem in the context of the deceased-donor kidney allocation, where an important performance metric is equity across different patient groups.

Zenios et al. (2000) proposes a multi-objective function which includes three criteria: quality-adjusted life years, inequity in waiting time, and inequity in the likelihood of transplantation. A dynamic allocation policy is proposed to approximately optimize this multi-objective function. In reality, however, it is not always practical to include queue length as one of the ranking criteria, as in this case a patient's rank may depend on future arrivals of his queue.

To address this issue, Su and Zenios (2005, 2006); Su et al. (2004) propose two versions of partition-based policies which will not dynamically allocate kidneys depending on queue length information. In their first version (hereafter referred to as *Par1*), candidates and kidneys are partitioned into exclusive groups according to their survival expectancy, and high-survival kidneys are allocated to high-survival candidates (Su and Zenios, 2005). Because the allocation depends on both the kidney and the patient types, *Par1* is a special form of an matching policy. Because *Par1* matches the kidneys to recipients based on their health scores rather than waiting times, a patient cannot get a better kidney by waiting longer. Consequently, patients have no incentive to turn down kidneys. Thus, *Par1* addresses an important shortcoming of the previous policy. Moreover, *Par 1* requires that different patient classes have equitable probabilities of getting a kidney offer. *Par1* achieves equitable allocation probabilities, in part, by restricting the access of high-survival patients to medium- or low-quality organs. This is not the case for both the previous and the new policies.

In Su and Zenios' second partition-based policy (*Par2*), kidneys are also partitioned by their survival expectancy, but patients can specify a range of KDPIs of interest when they join the waitlist. The waitlist is therefore partitioned into multiple queues, each of which is waiting for kidneys whose KDPIs fall into a specific range. Although our paper takes a similar approach by modeling the recipient choice, the ranking policies considered in our model, i.e., matching (or healthier-first, donor-blind), are more general than the priority rule of *Par2*, which requires all patients to be served on a first-come-first-served basis regardless of their health scores. Moreover, while Su and Zenios (2006) has focused on the impact of patient choice on allocation outcomes, the focus of our paper is on searching for the optimal priority rule in the presence of patient choice.

Many researchers have used simulation or computational methods to compare different policies, e.g., (Abellán et al., 2004; Wujciak, 1997; Wujciak T, 1993; Gaston et al., 1993; Opelz and Wujciak, 1995; Bertsimas et al., 2013). A recent working paper Ata et al. (2019b) explores empirically the impact of policy changes under endogenous patient choice; also see Ata et al. (2019a).

A few recent papers have discussed some emerging issues related to organ allocation. For example, Segev et al. (2007); Dai et al. (2017) studied the impact of allocation rules on the likelihood of live donation; Kong et al. (2010); Gentry et al. (2015) studied the region design issue for the sake of minimizing geographic disparity and maximizing efficiency; Ata et al. (2016) considered the use

of jet services to overcome geographical disparities; whereas Arıkan et al. (2017) studied how to increase the decreased-donor procurement rate, hence, the supply of organs, through geographic sharing. Sandıkçı et al. (2013); Proon et al. (2017) studied the tradeoffs in releasing partial information to patients on the liver waitlist; whereas Akan et al. (2012) considered optimizing the liver allocation policy. Arora and Subramanian (2015) studied the incentive alignment problem in the organ donation value chain, that is, between the social planner, organ-procurement-organization, and the trauma center. We refer the reader to Friedewald et al. (2014) and Ata et al. (2018) for further overviews of the deceased-donor transplant system.

2. Fluid Model for the Transplant Waitlist

2.1. Model Setup

This section advances a fluid model to study the deceased-donor transplant waitlist of a moderate-to-large donor service area for a certain blood type (A, B, O, or AB). This modeling choice not only simplifies the analysis, which is otherwise intractable, but also is based on two empirical facts: First, a great majority of kidneys are transplanted within the same donor service area where it was procured[5]. Second, as of 1991, 93% of kidney transplantations in the U.S. were between identical blood types (Port et al., 1991), and this fraction has increased to 94.05% during 2005-2010 according to our data. In addition, we exclude highly sensitized patients (i.e., CPRA $\geq 80\%$) because their waiting times exhibit high variation due to HLA incompatibility, which has been suppressed in our fluid model for tractability. Those excluded patients correspond to about fifteen percent of the waitlist population as of 03/31/2010 (Cecka et al., 2011).

Various patient- and donor-related factors can influence a patient's post-transplant survival expectancy (PTSE). The SRTR developed a proportional hazard model to predict a patient's PTSE, which leads to the following expression (Su and Zenios, 2006; SRTR, 2007)

$$\text{PTSE} = c\,\phi, \tag{1}$$

where c and ϕ represent aggregate contributions of patient- and donor-related covariates to the PTSE, respectively; see, for example, (Su and Zenios, 2006) for a derivation of (1). In what follows, we refer to c as a patient's health score and refer to ϕ as kidney quality. The PTSE function, due to its product form, is supermodular in c and ϕ, implying that the benefit is larger when higher-quality kidneys are matched to younger and healthier patients than when they are matched to the older and unhealthier ones. This is consistent with the prevailing belief that survival matching improves total life years.

We let $c(t;\xi)$ denote the patient's health score at time t conditional on his health score being ξ when he joins the waitlist, i.e., $c(0;\xi) = \xi$. We assume that $c(t;\xi)$ is continuously differentiable

and strictly decreasing. We further assume that the future evolution of the patient's health score depends only on his current health score, i.e., it is (deterministically) Markovian. Using this property, one can show that

$$c(t;\xi) = H^{-1}(t + H(\xi)), \tag{2}$$

where H is a strictly decreasing, continuously differentiable function and H^{-1} is its inverse. In our numerical study, we estimate the function $c(\cdot;\bar{\xi})$ using the formula provided by Israni et al. (2014) (also see (Clayton et al., 2014) and (Rao et al., 2009)), and plot $c(\cdot;\bar{\xi})$ in Figure 1.

Patients may die as they wait for a transplant. We assume that the pre-transplant survival time has a hazard rate $h(c)$ which continuously decreases with the patient's current health score c. This generalizes the common assumption in the literature that pre-transplant survival time has constant hazard rate (e.g., Su and Zenios (2006), Zenios et al. (2000), Akan et al. (2012), and Ata et al. (2016)). Since a patient's health score $c(t;\xi)$ changes with his cumulative waiting time, we can compute the cumulative distribution function (cdf) of the pre-transplant survival time for a patient with initial health score ξ as $F_\xi(t) := 1 - \exp(-\int_0^t h(c(s;\xi))\,ds)$. We let $\bar{F}_\xi(t) := 1 - F_\xi(t) = \exp(-\int_0^t h(c(s;\xi))\,ds)$ denote the complementary cdf of the pre-transplant survival time. We assume that the patient dies with probability one when his health score drops to $\underline{\xi}$. That is, $h(c) \to \infty$ as $c \to \underline{\xi}$. Consequently, the pre-transplant survival time of a patient with initial health score ξ has an upper limit $\bar{\tau}_\xi := c^{-1}(\underline{\xi};\xi)$, where $c^{-1}(\cdot;\xi)$ denotes the inverse function of $c(\cdot;\xi)$.

We assume that there are J types of kidneys. Kidneys of type j have quality ϕ_j $(j = 1,\ldots,J)$, where $\phi_1 < \phi_2 < \ldots < \phi_J$. That is, lower-indexed kidneys correspond to lower quality. The kidney transplant waitlist is mathematically equivalent to a multi-server queueing system with abandonments (i.e. deaths). The J different types of kidneys are modeled as J heterogeneous servers, with the service times corresponding to the inter-arrival times of kidneys. Also, we assume

$$\frac{1}{h(c)} < c\,\phi_1 \text{ for all } c > 0, \tag{3}$$

In Equation (3), $1/h(c)$ provides an upper bound for the survival expectancy of a patient of health score c on dialysis. It is an upper bound because in reality the patient's hazard rate increases with time. Indeed, the constant hazard rate would have resulted in a life expectancy of $1/h(c)$. On the other hand, $c\phi_1$ gives the PTSE for a patient of health score c who has been transplanted the lowest-quality kidney. The preceding inequality then implies that the patient could live longer by transplanting even the lowest-quality kidney compared to staying on dialysis. In what follows, this assumption will be used to show that patients prefer receiving a transplant to staying on dialysis and that they prefer to receive a transplant sooner than later; see Lemma 1.

In our fluid model, patients and kidneys are modeled as continuous fluid which arrive at the waitlist system according to a fixed rate. We assume that the total arrival rates of kidneys and

patients are given by μ and λ, respectively. We assume $\lambda > \mu$ to model the realistic situation that the total supply of kidneys cannot meet the total demand. To be more specific, the kidneys of type j arrive at a constant rate μ_j for $j = 1, \ldots, J$. Note that $\mu = \sum_{j=1}^{J} \mu_j$. We also assume that the initial health score of patients ξ admits a continuous distribution over the interval $[\underline{\xi}, \bar{\xi}]$ with pdf $\rho(\cdot)$. In particular, the arrival rate of patients with initial health score $\xi \in A$ is given by $\lambda \int_{\xi \in A} \rho(\xi)\, d\xi$.

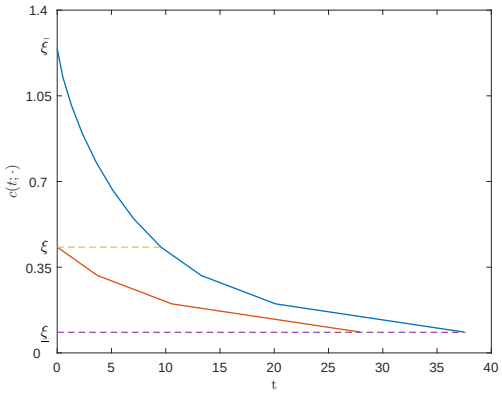

Figure 1 Health Score Curve (estimated from data)

The upper curve represents the evolution of a patient's health score with an initial value of $\bar{\xi}$. Once his health score degrades to ξ, the rest of the curve coincides with that of the patient whose initial health score is ξ. That is, shifting the lower curve to the right yields precisely the relevant part of the upper curve. A patient dies with a state-dependent hazard rate $h(c)$, with $\lim_{c \downarrow \underline{\xi}} h(c) = +\infty$.

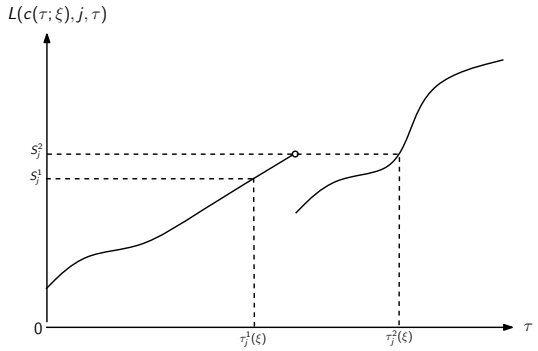

Figure 2 The Evolution of a Patient's Score

The score $L(c(\tau;\xi), j, \tau)$ is continuous and strictly increasing in τ except at finitely many points in $\mathcal{K}$. This figure (and the labels on the X- and Y-axis) will be used in Appendix E to illustrate the discontinuity of $\tau_j(\xi)$ in S_j.

When a deceased-donor kidney is procured, patients on the waitlist are ranked by a specific allocation policy. A more detailed discussion of the allocation policy will be provided in the next subsection. The kidney is then sequentially offered to the patients according to their rank from highest to lowest. When the kidney is offered to a patient, he has to decide whether to accept or reject it. By accepting the kidney offer, the patient departs the waitlist; otherwise, the kidney will be offered to the next highest ranked patient, the third, and so on. The patient seeks to maximize his expected QALYs. Without loss of generality, we normalize the quality-of-life coefficient after transplantation to one, but allow the quality-of-life coefficient before transplantation, denoted by η, to vary from patient to patient. According to the existing literature, e.g., Valderrábano et al. (2001) and Moreno et al. (1996), η depends on the patient's socioeconomic status as well as a few health-related attributes, mainly gender and comorbidities (particularly diabetes). Because the latter are correlated with the patient's initial health score ξ, we assume that the distribution of η varies according to ξ. Because the time that a patient stays on dialysis is usually much shorter than

the time that would cause a significant quality-of-life degradation for the patient by aging alone (all else being equal), we assume that η is time-stationary during a patient's wait and that it has a continuous distribution with cdf $G_\xi(\cdot)$ among patients with initial health score ξ. We assume that for all ξ, $G_\xi(\cdot)$ have a common support of $[0,1]$. The reader can find the estimates of the model quantities, such as $h(\cdot)$, $G_\xi(\cdot)$, $\rho(\cdot)$ in Section 5.

2.2. Allocation Policies

Given the above model setup, we next formally define the three policy classes introduced in Section 1. The head-of-line-matching policy uses the following formula to calculate a patient's kidney allocation score:

$$\text{Score under a matching policy} = L(c(\tau;\xi), j, \tau), \tag{4}$$

where τ denotes the patient's cumulative waiting time. A deceased-donor kidney is offered sequentially to the candidates on the waitlist in the order of their scores. We assume $L(c(\tau;\xi), j, \tau)$ satisfies the following properties: (1) its partial derivative $\partial L/\partial\tau$ exists and is continuous and positive everywhere; (2) Letting $\mathcal{K} = \{c_1, c_2, \ldots, c_D\}$ denote a finite set of cutoff values, we allow $L(\cdot, j, \tau)$ to be discontinuous at $c \in \mathcal{K}$ for all fixed j, τ. We also assume that for $c \notin \mathcal{K}$, the partial derivative $\partial L/\partial c$ exists, is continuous and satisfies the following head-of-line (HL) property:

$$\text{HL property:} \quad \frac{dL(c(\tau;\xi), j, \tau)}{d\tau} = \frac{\partial L}{\partial \tau} + \frac{\partial L}{\partial c} c'(\tau;\xi) > 0 \text{ for } \tau,\ \xi \text{ with } c(\tau;\xi) \notin \mathcal{K} \text{ and all } j. \tag{5}$$

The term "head-of-line" refers to that a patient's score increases as he waits except when his health score hits the cutoff values in $\mathcal{K}$. The term "matching" refers to that the score depends on both patient health score c and kidney type j. Note that a patient's score can have an upward or downward jump at the cutoff values in $\mathcal{K}$. We allow those jumps so that the class of matching policies subsumes the new policy, in which a patient can lose his priority for the top 20% kidneys once his health score c drops below the 20-percentile cutoff value. For technical convenience, we further assume that $L(c, j, \tau)$ is left-continuous with right limits (LCRL) in c, and thus $L(c(\tau;\xi), j, \tau)$ is right-continuous with left limits (RCLL) in τ. Figure 2 displays an illustrative example of how a patient's score changes over time.

The class of matching policies subsumes a wide range of policies, including the aforementioned partition-based policy *Par1*. Specifically, we define a partition function that maps each health score c to a specific patient class $j(c)\colon\ [0, +\infty) \to \{1, 2, \ldots, J\}$, and then let $L(c, j, \tau) = -M|j(c) - j| + \tau$ for some constant $M > 0$. Then a patient would suffer from a penalty when he attempts to transplant a kidney whose quality is not matched to his health score, which is similar to the mechanism of *Par1*.

In addition to the aforementioned assumptions made for the matching policies, one can further require the score $L(c, j, \tau)$ to be non-decreasing in c everywhere, i.e., healthier patients receive an equal or higher priority for all kidney types. Including this extra constraint leads to the *head-of-line healthier-first* policies mentioned in Section 1. Note that the new policy is similar to a healthier-first policy except for the geographical and tissue type matching constraints.

Another important subclass of matching policies is the class of *head-of-line donor-blind* policies, in which the patient's score does not depend on the kidney quality. So the score can be expressed as $L(c, \tau)$. As discussed above, the previous policy largely reassembles a donor-blind policy except for the expanded-criteria donors. However, the new allocation policy is not donor-blind, because it offers kidneys with KDPI $< 20\%$ first to patients with health score among the top 20%.

The three classes of policies, matching, healthier-first, and donor-blind, all require the patient's score to monotonically increase with patient's waiting time possibly except at $c \in \mathcal{K}$. A more general policy would relax the HL assumption by allowing the allocation score to either decrease with waiting time, e.g., last-come-first-serve (LCFS), or not to depend on the waiting time at all, e.g., a lottery (patients are ranked according to a randomly generated order). In section 4, we propose methods to compute the optimal allocation outcome under matching policies and the more general policies; and in Section 5, we numerically show that the more general policies achieve an improvement of up to 37% in total QALYs compared to the matching policies, a result that is consistent with the existing literature, e.g., Su and Zenios (2004). Despite this limitation, the main idea of the matching policies, i.e., using waiting time as a major criterion to rank the patients on the waitlist, is far more practical for implementation than other policies such as LCFS or a lottery. Therefore, we develop a modeling framework that facilitates the analysis of matching policies (thus also the healthier-first and donor-blind policies). To this end, we prove that there exists a unique equilibrium of the fluid model under matching policies, and characterize it as the unique solution to a nonlinear complementarity problem.

2.3. The Fluid Model Equilibrium

The state of the fluid model can be fully captured by the distribution of the waitlist population in terms of the parameters c, τ, and η. Because patients arrive continuously over time, the joint distributions of τ, $c = c(\tau; \xi)$, and η have a density function, denoted by $\pi(c, \tau, \eta)$. Note that $\pi(c, \tau, \eta)$ has a similar interpretation to the probability density function, except that $\iiint \pi(c, \tau, \eta)\, dc d\tau d\eta$ gives the total mass of patients on the waitlist rather than one. The equilibrium (or steady state) of the fluid model can be represented by a density function $\Pi = \pi(\cdot, \cdot, \cdot)$. In particular, starting from Π, the population distribution remains invariant at Π despite various dynamic events in the fluid model.

The dynamic events on the transplant waitlist include kidney and patient arrivals, patient deaths, and the matching process, which results in transplants. The kidney and patient arrival processes are exogenous and independent of Π. The reneging process can be solely determined by Π. The matching process depends on every patient' acceptance/rejection decision. Suppose a patient with parameters (ξ, η) is offered a kidney offer j after staying in the waitlist for t time units. According to the PTSE formula (1), by accepting that kidney, the patient receives expected QALYs of $U_A := c(t;\xi)\phi_j$; otherwise, the patient will stay on dialysis, and receive expected QALYs of U_R, which is a function of t, ξ, η, Π, and the other patients' strategies. We assume that the patient has complete information. In particular, he knows his parameters ξ and η, his cumulative waiting time t, and the population distribution Π at the equilibrium. So the patient's strategy profile at the equilibrium can be represented by a function $a(\cdot,\cdot|\xi,\eta)$, which maps a combination of offer time and offered kidney type (t, j) to a binary response, *Accept(A)* or *Reject(R)*. If the patient can compute U_R, then his optimal strategy at the equilibrium is

$$a(t, j|\xi, \eta) = \begin{cases} \text{A} & \text{if } U_A \geq U_R, \\ \text{R} & \text{if } U_A < U_R. \end{cases} \tag{6}$$

This assumes that a patient will accept the offered kidney whenever $U_A = U_R$, which only happens to patients with a particular η (because U_R includes a proportion of the pre-transplant life time and thus strictly increases in η). The measure of such patients is zero among the waitlist population, thus the way we break the tie will have no impact on the system dynamics.

However, in order to compute U_R, the patient needs to know his waiting time for kidneys of higher types than the current offer. Moreover, the patient needs to know who else will accept kidneys of those types. Thus, the patient's U_R and consequently his optimal strategy depends on other patients' strategies. We therefore need to study the Nash equilibrium at the steady state. To characterize the Nash equilibrium, we first prove that it is always better off for a patient to accept a kidney earlier than later, if the offered kidneys have the same quality. The proof for Lemma 1 is provided in Appendix B.

Lemma 1 *Fix $t \geq 0$ and let $QALY_1$, $QALY_2$, and $QALY_3$ denote the expected QALYs that a patient receives starting at time t in each of the following three scenarios, respectively:*

1. *The patient is transplanted a type j kidney at time t.*
2. *The patient is transplanted a type j kidney at time $s > t$.*
3. *The patient stays on dialysis forever.*

Then the following holds for all $t \geq 0$:

$$QALY_1 > QALY_2 > QALY_3. \tag{7}$$

According to Lemma 1, the best kidneys (i.e., type J) will always be accepted immediately upon being offered. Therefore, we can leverage the properties of the fluid model and show that for type-J kidneys, there is a constant threshold S_J such that a patient receives a kidney if and only if his score is equal to or larger than S_J. We can then compute each individual patient's expected waiting time for type-J kidneys. That will allow every patient to calculate his expected utility if he would turn down kidneys of the second best type, i.e., type $J-1$. So one can figure out every patient's acceptance/rejection decision whenever he receives a type $J-1$ kidney. Iteratively repeating this procedure leads to an Nash equilibrium at which kidneys of type j have a score threshold S_j (for $j=1,\ldots,J$). This important characterization of the Nash equilibrium is proved in the next proposition.

Proposition 1 *At every Nash equilibrium under the matching policy, a patient with initial health score ξ will first receive a type-j kidney offer after waiting for $\tau_j(\xi)$ time units, where*

$$\tau_j(\xi) := \inf\{\tau \geq 0 : L(c(\tau;\xi), j, \tau) \geq S_j\} \tag{8}$$

with the convention that $\inf \emptyset = +\infty$, and S_j is a constant threshold ($j=1,\ldots,J$). Furthermore, if $\tau_j(\xi) < +\infty$, then

$$L(c(\tau_j(\xi);\xi), j, \tau_j(\xi)) \geq S_j, \tag{9}$$

and the inequality is strict only if $c(\tau_j(\xi);\xi) \in \mathcal{K}$.

We refer to $(\tau_j(\xi))$ defined in Proposition 1 as the *allocation times*. An important property of the allocation time $\tau_j(\xi)$ is that all patients of initial health score ξ will be first offered a type-j kidney at the allocation time $\tau_j(\xi)$. This characterization, however, crucially uses the HL property of the matching policies; see Appendix A for an example that the characterization of allocation times fails when the HL property was violated. The proof for Proposition 1 is also provided in Appendix A.

2.4. Characterization of the Equilibrium Allocation Times and Patient Choice

The properties of the allocation times $\tau_j(\xi)$ as a function of S_j is summarized in the next lemma.

Lemma 2 *Viewing $\tau_j(\xi)$ as a function of S_j for given ξ and j in light of Equation* (8)*, it satisfies the following:*

(a) $\tau_j(\xi)$ is a non-decreasing, right-continuous function of S_j with left limits.

(b) In a neighborhood of a given $S_j > 0$, $\tau_j(\xi)$ changes continuously with S_j for a.e. ξ.

Lemma 2 can be proved by leveraging the properties of $L(c(\tau;\xi),j,\tau)$ and $c(\tau;\xi)$; see Appendix E. Although $\tau_j(\xi)$ is right-continuous as a function of S_j, it may not be left-continuous in S_j due to the non-monotonicity of $L(c(\tau;\xi),j,\tau)$ with respect to τ at points in $\mathcal{K}$; see Figure 2 for such an example of $L(c(\tau;\xi),j,\tau)$.

The allocation times facilitate a simple characterization of a patient's equilibrium strategy. Consider a patient with parameters (ξ,η), who faces an allocation time $\tau_j(\xi)$ for the type-j kidneys. Kidney type j' is said to dominate kidney class j if $j'>j$ and $\tau_{j'}(\xi)\le\tau_j(\xi)$, that is, a higher-quality kidney requires a shorter waiting time. Clearly, a patient will never accept a kidney type that is dominated by other kidney types. By eliminating those dominated kidney types, suppose there are m kidney types left, i.e., $j_1<j_2<\ldots<j_m$ with the corresponding allocation times $\tau_{j_1}(\xi)\le\tau_{j_2}(\xi)\le\ldots\le\tau_{j_m}(\xi)$. Let $\{a_k|k=1,2,\ldots,m\}$ denote a sequence of binary decisions such that $a_k=$A or R if the patient accepts or rejects the kidney offer of type j_k at time $\tau_{j_k}(\xi)$, respectively, conditional on not accepting any previous offers. The patient's optimal strategy is characterized by the decision sequence $\{a_k|k=1,2,\ldots,m\}$, because we will show in Proposition 2 that the patient would never change his acceptance/rejection decision for the same kidney type since it was first offered. Thus, after $\tau_j(\xi)$, the patient can be considered oblivious to all future type-j kidney offers, though the patient may keep receiving them as long as his score stays above S_j.

We next derive the patient's optimal decision sequence $\{a_k|k=1,2,\ldots,m\}$ by solving a sequential decision problem. Let $U_{A,k}$ and $U_{R,k}$ denote the expected QALYs after the patient accepts or rejects kidney type j_k, respectively, and let V_k denote the optimal expected future QALYs that the patient could gain starting at time τ_{j_k}. We first note that $U_{A,k}=\xi g(\tau_{j_k}(\xi))\phi_{j_k}$ for all k. We also note that $a_m=$A, that is, if the patient is still on the waitlist at time $\tau_{j_m}(\xi)$, then he must accept the kidney offer of type j_m immediately. Because otherwise, he has to wait further without the possibility of getting a better kidney, which is suboptimal by Lemma 1. The patient's earlier decisions can be characterized with the help of the following recursive formulas:

$$V_m=U_{A,m}, \tag{10}$$

$$\begin{aligned}U_{R,k}=\eta\left(\int_{\tau_{j_k}(\xi)}^{\tau_{j_{k+1}}(\xi)}(t-\tau_{j_k}(\xi))\frac{f_\xi(t)}{\bar F_\xi(\tau_{j_k}(\xi))}\,dt+\frac{\bar F_\xi(\tau_{j_{k+1}}(\xi))}{\bar F_\xi(\tau_{j_k}(\xi))}(\tau_{j_{k+1}}(\xi)-\tau_{j_k}(\xi))\right)\\ +\frac{\bar F_\xi(\tau_{j_{k+1}}(\xi))}{\bar F_\xi(\tau_{j_k}(\xi))}V_{k+1},\quad k=1,\ldots,m-1,\end{aligned} \tag{11}$$

$$V_k=\max\{U_{A,k},\,U_{R,k}\},\quad k=1,\ldots,m-1. \tag{12}$$

The first term on the right-hand side of Equation (11) computes the expected QALYs that the patient could gain from time $\tau_{j_k}(\xi)$ until time $\tau_{j_{k+1}}(\xi)$, taking into account that the patient may die while he waits for the offer of type j_{k+1} kidney which will be offered at time $\tau_{j_{k+1}}(\xi)$. The last

term on the right-hand side of Equation (11) is the expected future QALYs that the patient can gain starting at time $\tau_{j_{k+1}}(\xi)$, which is the maximum QALYs of either accepting the offer of type $j_{k+1}(\xi)$ at time $\tau_{j_{k+1}}(\xi)$ or waiting further for a future offer as reflected in Equation (12).

By recursively solving $U_{R,k}$ and V_k for $k=m,m-1,\ldots,1$, we obtain the optimal decision sequence $\{a_k\}$ by Equation (6). Let $j^*(\eta,\xi)$ denote the kidney type that is matched to a patient with parameters (η,ξ) ex ante who anticipates kidney offers of type $j_1,\ldots,j_m$ at times $\tau_{j_1},\ldots,\tau_{j_m}$. The following proposition characterizes $j^*(\eta,\xi)$ and shows that it enjoys a monotonicity property with respect to η; see Appendix C for its proof.

Proposition 2 *Once a patient turns down a kidney offer, he would never accept a kidney of the same (or lower) type at a later time. Therefore, his choice throughout his wait is consistent with that upon joining the transplant waitlist, which is given by*

$$j^*(\eta,\xi) \ := \min\{j_k \,|\, k=1,2,\ldots,m,\ a_k=A\}. \tag{13}$$

Moreover, $j^*(\eta,\xi)$ is non-decreasing in η.

Once we computed $j^*(\eta,\xi)$, we can use $(\tau_j(\xi))$ to recover the steady-state distribution $\Pi := (\pi_{c,\tau,\eta})$; see Appendix C. Thus, the equilibrium can be described using $(\tau_j(\xi))$ instead of Π.

It follows from Proposition 2 that there exist thresholds $\Gamma_j(\xi)$ for $j=1,\ldots,J-1$ such that

$$\Gamma_0(\xi)\le\Gamma_1(\xi)\le\Gamma_2(\xi)\le\cdots\le\Gamma_{J-1}(\xi)\le\Gamma_J(\xi), \tag{14}$$

where $\Gamma_0(\xi):=0$ and $\Gamma_J(\xi):=1$ and that

$$j^*(\eta,\xi)=j \quad \text{if } \eta\in(\Gamma_{j-1}(\xi),\Gamma_j(\xi)],\ j=1,\ldots,J. \tag{15}$$

In particular, if $\Gamma_{j-1}(\xi)=\Gamma_j(\xi)$, no patient with initial health score ξ chooses kidney type j. The proportion of patients with initial health score ξ who choose kidney type j is given by

$$Q_j(\xi) := G_\xi(\Gamma_j(\xi)) - G_\xi(\Gamma_{j-1}(\xi)). \tag{16}$$

In what follows, we say that $j^*(\eta,\xi)$, $(\Gamma_j(\xi))$, and $(Q_j(\xi))$ are associated with $(\tau_j(\xi))$ if they are computed from $(\tau_j(\xi))$ using the above procedure, cf. Equations (6), (10)-(12),(13), (15) and (16).

According to Proposition 2, a patient with parameters (ξ,η) will eventually transplant a kidney type $j^*(\xi,\eta)$ if he survives until then. Thus, the kidney waitlist can be partitioned into J exclusive *virtual queues* according to the values of $j^*(\xi,\eta)$. Patients in the j^{th} virtual queue, despite having different parameters (ξ,η), will all accept and transplant kidneys of type j, conditional on being alive until their allocation time. Since a proportion $Q_j(\xi)$ of patients with initial health score

ξ will have $j^*(\xi,\eta)=j$, the aggregate patient arrival rate to the j^{th} virtual queue is given by $\lambda\int_{\xi} Q_j(\xi)\rho(\xi)\,d\xi$, and the service rate is given by μ_j.

The above characterization allows us to develop a compact characterization of the equilibrium that is based on $(\tau_j(\xi))$ as done in the next proposition; see Appendix D for its proof.

Proposition 3 *Under a matching policy, $(\tau_j(\xi))$ forms the allocation times in equilibrium if and only if $(\tau_j(\xi))$ are given by* (8) *for some function $L(c,j,\tau)$, and $(\tau_j(\xi))$ and the associated $(Q_j(\xi))$ and $(\Gamma_j(\xi))$ solve the following (NCP): For $j=1,\ldots,J$,*

$$\begin{aligned} y_j &:= \mu_j - \lambda\textstyle\int_{\underline{\xi}}^{\bar{\xi}} Q_j(\xi)\bar{F}_\xi(\tau_j(\xi))\rho(\xi)\,d\xi \geq 0,\\ z_j &:= \sup_{\xi\in[\underline{\xi},\bar{\xi}]}\tau_j(\xi) \geq 0,\\ y_j\,z_j &= 0. \end{aligned} \tag{17}$$

Intuitively, in equilibrium, either a virtual queue is empty, which is the case of $z_j=0$; or a virtual queue has its supply rate μ_j balanced with the reneging rate and patient arrival rate, which is the case of $y_j=0$. This leads to the NCP characterization. It is proved in Appendix F.

The next result establishes the existence and uniqueness of the equilibrium.

Theorem 1 *There exists a unique equilibrium under every matching policy, which can be described by the allocation times $(\tau_j(\xi))$.*

To prove the existence of the equilibrium, we need to construct a mapping $\Psi(\cdot)$, which maps a score-threshold vector $S:=(S_j)$ to the domain of S itself. This mapping must satisfy the following property: $S=\Psi(S)$ if and only if the $(Q_j(\xi),\tau_j(\xi))$ associated with S solves the NCP (17). The existence of such a fixed point thus leads to the existence of the equilibrium. To prove that the mapping $\Psi(\cdot)$ is well-defined and continuous, we exploit the special structure of the model, such as continuous density of ξ and properties of $\tau_j(\xi)$ as a function of S_j (Lemma 2). The uniqueness of the equilibrium builds on the intuition that when a subset of kidney types have a smaller score threshold (so a shorter waiting time for each patient) in one equilibrium than the other, then those queues will attract more customers, which, however, contradicts that those queues end up with a shorter waiting time.

An important property of the equilibrium allocation times $(\tau_j(\xi))$ is that if patients with initial health score ξ have accepted kidney types $j_1<j_2<\ldots<j_m$, then $\tau_{j_1}(\xi)\leq\tau_{j_2}(\xi)\leq\cdots\leq\tau_{j_m}(\xi)$. Because otherwise, the lower kidney type will not be accepted by any of those patients as a result of Proposition 2.

Clearly, different scoring formulas lead to different equilibria. Next, we focus on searching for the scoring rules to optimize certain efficiency and equity metrics. To do that, one needs to characterize what equilibria are achievable by a certain matching policy (or healthier-first, donor-blind), i.e., a characterization of the achievable set of equilibria.

3. Achievable Region of Equilibria

In the literature of queueing theory, the achievable region is defined as the space of performance vectors such as average queue lengths that can be achieved under certain priority rules (Coffman and Mitrani, 1980; Federgruen and Groenevelt, 1988; Bertsimas, 1995; Chen and Yao, 2001). In a typical queueing system, the achievable region can be represented as a polytope, whose facets are derived from the conservation law dictated by the capacity constraints. In our setting, the ex-post allocation of kidneys to patients depends not only on the allocation mechanism but also on the patients' choices. Therefore, our definition of the achievable region differs significantly from its usual definition. For brevity, we will use the term "achievable region", but the reader should interpret it as the "achievable region of equilibria".

As we previously argued, the equilibrium can be described by the collection of allocation times $\boldsymbol{\tau} := (\tau_j(\xi))$. For technical convenience, we only consider allocation times that are piecewise continuously differentiable (p.c.d.), that is, $\tau_j(\xi)$ has continuous partial derivatives with respect to ξ except at finitely many points. Therefore, we denote the achievable region of matching policies as

$$\mathcal{A}_M := \{(\tau_j(\xi)) \,|\, (\tau_j(\xi)) \text{ are p.c.d. equilibrium allocation times under a matching policy}\}. \quad (18)$$

The achievable regions for the healthier-first policies and the donor-blind policies, i.e., $\mathcal{A}_{HF}$, and $\mathcal{A}_{DB}$, are defined similarly. We next derive a closed-form representation of $\mathcal{A}_M$. For notational brevity, we let $Q_j(\xi;\boldsymbol{\tau})$ denote the choice probabilities associated with the allocation times $\boldsymbol{\tau}$; and let $p_j(\xi)$ be a density function such that for all $\underline{\xi} \le \xi_1 < \xi_2 \le \overline{\xi}$, $\int_{\xi_1}^{\xi_2} p_j(\xi)\, d\xi$ denotes the fraction of type-j kidneys that are transplanted to patients of initial health scores on $[\xi_1, \xi_2]$. Note that type-j kidneys are fully utilized if $\int_{\underline{\xi}}^{\bar{\xi}} p_j(\xi)\, d\xi = 1$.

To facilitate the statement of the following theorem, for $k = 1, \ldots, J$, define the following sets:

$$\mathcal{P}_k := \left\{ \text{p.c.d. functions } \boldsymbol{\tau} \;\middle|\; \begin{array}{ll} \mu_j p_j(\xi) = \lambda \rho(\xi) Q_j(\xi;\boldsymbol{\tau}) \bar{F}_\xi(\tau_j(\xi)) \text{ for all } j, \xi & \text{(C.1)} \\ \int_\xi p_j(\xi)\, d\xi = 1 \text{ for } j > k & \text{(C.2)} \\ \int_\xi p_k(\xi)\, d\xi \le 1 & \text{(C.3)} \\ Q_j(\xi;\boldsymbol{\tau}) = 0 \text{ for } j < k,\ \xi \in [\underline{\xi}, \bar{\xi}] & \text{(C.4)} \\ \tau_j(\xi) = 0,\ \text{for } j \le k,\ \xi \in [\underline{\xi}, \bar{\xi}] & \text{(C.5)} \\ 0 \le \tau_j(\xi) \le \bar{\tau}_\xi \text{ for } j > k,\ \xi \in [\underline{\xi}, \bar{\xi}] & \text{(C.6)} \end{array} \right\}. \quad (19)$$

Also, we define

$$\mathcal{P}^M := \left\{ \text{p.c.d. functions } \boldsymbol{\tau} \,|\, c(\tau_j(\xi);\xi) \ge c(\tau_j(\xi');\xi') \text{ for } \xi > \xi' \quad \text{(C.M)} \right\}.$$

Given an equilibrium allocation-time vector $(\tau_j(\xi))$, we also define the following function

$$\gamma_j(\xi) := c(\tau_j(\xi);\xi), \quad (20)$$

which corresponds to the health score of the patient (with initial health score ξ) at the time of receiving a type-j kidney transplant.

Let $\mathcal{C}_j$ denote the range of the function $\gamma_j(\cdot)$, i.e., $\mathcal{C}_j := \{\gamma_j(\xi)|\xi \in [\underline{\xi}, \bar{\xi}]\}$. The following lemma establishes useful properties of the function $\gamma_j(\cdot)$ and its inverse $\gamma_j^{-1}(\cdot)$; see Appendix G for its proof.

Lemma 3 *Suppose $\boldsymbol{\tau}$ are equilibrium allocation times under a matching policy. Then for $j = 1, \ldots, J$, the function $\gamma_j(\cdot)$ is non-decreasing, left continuous and with right limits (LCRL). Furthermore, for all $c \notin \mathcal{K}$, $\gamma_j^{-1}(c) := \{\xi|\gamma_j(\xi) = c\}$ is a singleton.*

We use Lemma 3 to prove the following theorem that characterizes the $\mathcal{A}_M$; see Appendix H for its proof.

Theorem 2 *We have that*

$$\mathcal{A}_M = \cup_{k=0}^{J} \mathcal{P}_k \cap \mathcal{P}_M. \tag{21}$$

Moreover, every equilibrium in $\mathcal{A}_M$ can be achieved by a score function $L(c, j, \tau)$ with the following parametric form

$$L(c, j, \tau) = M(c, j) + \tau, \tag{22}$$

where

$$M(c, j) = \begin{cases} -\inf\{\tau_j(\xi) \,|\, \xi \in \gamma_j^{-1}(c)\} & \text{if } c \in \mathcal{C}_j, \\ -\bar{\tau}_{\bar{\xi}} & \text{if } c \notin \mathcal{C}_j. \end{cases} \tag{23}$$

Furthermore, the score $L(c, j, \tau) = M(c, j) + \tau$ satisfies all assumptions of a matching policy.

We next provide some intuition towards the statements and the proof for the above Theorem. To derive the expression of the achievable region in Equation (19), we first partition the achievable region into sub-regions $\mathcal{P}_k$ for $k = 0, 1, \ldots, J$, where k represents the *minimum acceptable level* of kidney class. Formally,

$$k := \begin{cases} \max\{j = 1, \ldots, J \,|\, \sup_\xi \tau_j(\xi) = 0\} & \text{if the set is nonempty,} \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Intuitively, k denotes the highest type of kidneys that do not require any waiting. Given that, all patients prefer to accept a kidney of type k rather than those of type $j < k$ by Lemma 1. Thus, kidneys of type $1, 2, \ldots, k-1$ are completely wasted. Note that $k = 0$ represents the desired case when all kidneys are fully utilized, though in practice low-quality kidneys are usually wasted because of the externality introduced by the head-of-line discipline (see (Su and Zenios, 2004)).

The set $\mathcal{P}_k$ consists of equilibria at which the lowest type of kidney transplanted is k. In the expression of $\mathcal{P}_k$, condition (C.1) calculates the amount of kidneys allocated to patients with

parameter ξ; condition (C.2) requires all kidneys of types $j > k$ to be fully utilized; condition (C.3) states that kidneys of type k can be partially utilized or even fully utilized; condition (C.4) states that no patient would choose kidney types lower than k. This is because type-k kidneys require no waiting, and hence dominate kidneys of type $1, \ldots, k-1$. Condition (C.5) is implied by the definition of the minimum acceptable level; condition (C.6) gives the lower and upper bound for each $\tau_j(\xi)$ for $j > k$. Recall from Equation (8) that if a patient's health score can never reach the threshold S_j, his allocation time for kidney type j is $+\infty$. However, because a patient with initial health score ξ can live at most $\bar{\tau}_\xi$ time units, we impose the upper bound $\tau_j(\xi) \le \bar{\tau}_\xi$ in condition (C.6) without excluding any feasible equilibria. Finally, condition (C.M) considers patients receiving the same type of kidney offers, say type j, and requires that their health scores $c(\tau_j(\xi);\xi)$ at the time $\tau_j(\xi)$ of the offer are non-decreasing in their health scores. This constraint results from the HL property of the matching policy.

Not only does Theorem 2 provide an explicit expression of the achievable region for the matching policy, but also it shows that every equilibrium can be achieved by a scoring rule of the form (22). In fact, both the previous and new kidney allocation policies have used Equation (22) to calculate a patients' score – a patient receives an extra point of score for each additional year he spends on the waitlist. Theorem 2 states that there is no need to use further sophisticated formulae for a patient's waiting score. The policy maker only needs to solve the allocation times and then set $M(c,j)$ accordingly by Equation (23). To prove this result, the main idea is to show that the nonlinear curvature of the function $\tau_j(\xi)$ can be fully captured by the function $M(c(\tau_j(\xi);\xi),j)$ with respect to ξ. The proof leverages two important facts. First, the policy is not donor-blind, so the curvature of $M(c(\tau_j(\xi);\xi),j)$ can vary for different kidney type j. In fact, we will later show that the same conclusion may not hold under the donor-blind policies. Second, constraint (C.M.) ensures monotonicity of $c(\xi;\tau_j(\xi))$ in ξ. This property allows us to design the $M(c,j)$ in a way that a patient with initial health score ξ cannot receive a kidney before his allocation time (see Appendix H for further details).

Using a similar logic, we characterize the achievable region $\mathcal{A}_{HF}$ for the class of healthier-first policies in the next proposition. To facilitate this characterization, let

$$\mathcal{P}_{HF} := \big\{\text{p.c.d. functions } \boldsymbol{\tau} \mid \tau_j(\xi) \le \tau_j(\xi') \text{ for all } \xi > \xi' \qquad \text{(C.HF).}\big\}$$

Proposition 4 *We have that*

$$\mathcal{A}_{HF} = \cup_{k=0}^{J} \mathcal{P}_k \cap \mathcal{P}_{HF}. \tag{25}$$

Moreover, every equilibrium in $\mathcal{A}_{HF}$ can be achieved by a score $L(c,j,\tau)$ of the form given in Equation (22)*, where $M(c,j)$ is defined in Equation* (23)*. Furthermore, $M(c,j)$ is non-decreasing in c and $L(c,j,\tau) = M(c,j) + \tau$ satisfies all assumptions of a healthier-first policy.*

The characterization of $\mathcal{A}_{HF}$ differs from that of $\mathcal{A}_M$. Namely, the constraint (C.M) used to characterize $\mathcal{A}_M$ is replaced with the stronger constraint (C.HF) in characterizing $\mathcal{A}_{HF}$. Constraint (C.HF) requires that a patient with a larger initial health score does not wait longer than a patient with a lower initial health score. This constraint is directly implied by the additional property of the healthier-first policy compared to the matching policy that a healthier patient always receives a higher priority and thus waits less. The p.c.d. function $M(c,j)$ is constructed in exactly the same way as the function $M(c,j)$ is constructed for the matching policy. The expression of $M(c,j)$, which is identical to that of $M(c,j)$ in Equation (23), implies that $M(c,j)$ is non-decreasing in c. The proof of Proposition 4 is similar to that of Theorem 2 and is omitted.

In the characterizations of both $\mathcal{P}_M$ and $\mathcal{P}_{HF}$, we have not imposed any constraints on the scoring function $L(c,\tau,j)$. Instead, we proved that as long as $(\tau_j(\xi))$ satisfies condition (C.M) or (C.HF), a score function $L(c,\tau,j)$ is guaranteed to exist such that $(\tau_j(\xi))$ gives the allocation time under $L(c,\tau,j)$. However, under the donor-blind policy, we need to ensure that the allocation times $(\tau_j(\xi))$ for different kidney types are associated with the same score function $L(c,\tau)$ that is independent of j. This can be achieved by imposing the following condition in place of (C.M) or (C.HF).

$$\mathcal{P}_{DB} := \left\{ \text{p.c.d. functions } \boldsymbol{\tau} \;\middle|\; \begin{array}{l} \tau_j(\xi) := \min\{\bar{\tau}_\xi, \min\{\tau \geq 0 \,|\, L(c(\tau;\xi),\tau) \geq S_j\}\} \\ \text{for some } L(c,\tau) \text{ such that for all } c \notin \mathcal{K},\ \tau, \\ dL(c,\tau)/d\tau > 0, \quad \partial L(c,\tau)/\partial\tau > 0 \end{array} \right\} \quad \text{(C.DB)}$$

If the score of a patient with initial health score ξ can never reach S_j, we let $\tau_j(\xi) = \bar{\tau}_\xi$ (rather than $+\infty$ to have a bounded achievable region). Otherwise, $\tau_j(\xi)$ will be the first time at which the score of the patient reaches S_j, and the minimum is always attained by Proposition 1. We require that $L(c,\tau)$ has positive derivatives so it satisfies the assumptions of the donor-blind policy, which are essential to guarantee the allocation-times to exist at the equilibrium. The next proposition provides an expression for $\mathcal{P}_{DB}$. We omit its proof as it is straightforward.

Proposition 5 *We have that*

$$\mathcal{A}_{DB} = \cup_{k=0}^{J} \mathcal{P}_k \cap \mathcal{P}_{DB}. \tag{26}$$

For matching policies, Theorem 2 states that it suffices to consider scores in the form of $M(c,j)+\tau$. One may wonder if the same result holds in the donor-blind case, that is, whether we can achieve all equilibria in $\mathcal{P}_{DB}$ using a score in the form of $\tilde{M}(c)+\tau$ for some univariate function $\tilde{M}(c)$. Unfortunately, we illustrate that this is not the case by providing an example in Appendix I. As a result, to design a donor-blind scoring policy, one has to explore more general functional forms.

4. Optimization over the Achievable Region

4.1. Performance Metrics

Having characterized the achievable region, we formulate the policy design problem as one of finding an allocation time $\boldsymbol{\tau} \in \mathcal{P}_M$ which optimizes a certain performance metric. A standard efficiency measure in the kidney allocation literature is the QALY (Zenios et al., 2000). To calculate the QALYs, we follow the medical literature (Axelrod et al., 2018; Wyld et al., 2012) and assume the quality-of-life coefficient to be 0.69 and 1 before and after kidney transplantation, respectively. Then if a patient with initial health score ξ chooses to wait for a type-j kidney which has an allocation time $\tau_j(\xi)$, then the patient's expected QALYs since starting dialysis are given by

$$\text{QALY}(\xi, j, \tau_j(\xi)) = 0.69 \int_0^{\tau_j(\xi)} f_\xi(t) t dt + 0.69 \bar{F}_\xi(\tau_j(\xi))\tau_j(\xi) + \bar{F}_\xi(\tau_j(\xi)) c(\tau_j(\xi); \xi)\phi_j, \tag{27}$$

where the two terms on the right-hand-side represent the patient's QALYs before and after transplantation, respectively. The efficiency of the system, which is measured by the average QALYs of all patients since arrival (starting dialysis), is expressed as

$$\text{Eff}(\boldsymbol{\tau}) := \int_{\underline{\xi}}^{\overline{\xi}} \sum_{j=1}^{J} Q_j(\xi) \text{QALY}(\xi, j, \tau_j(\xi)) \rho(\xi)\, d\xi. \tag{28}$$

We use the notation $\text{Eff}(\boldsymbol{\tau})$ and to signify that it is essentially a function of $\boldsymbol{\tau}$ (note that $Q_j(\xi)$ is a also function of $\boldsymbol{\tau}$).

Unlike efficiency, there has not been consensus on the measure of inequity. For example, as in max-min fairness, one can measure inequity by the difference between the average probability of transplantation for the entire patient population and the smallest probability of transplantation among all patient types, that is,

$$\text{Ineq1}(\boldsymbol{\tau}) := \frac{1}{\lambda} \int_{\underline{\xi}}^{\bar{\xi}} \sum_{j=1}^{J} \mu_j p_j(\xi)\, d\xi - \inf_{\xi} \frac{1}{\lambda \rho(\xi)} \sum_{j=1}^{J} \mu_j p_j(\xi), \tag{29}$$

Alternatively, Zenios et al. (2000) proposed to measure inequity by the variance of the likelihood of transplantation across different patient classes. We use the standard deviation instead of the variance to keep its scale consistent with *Ineq1*. We name this inequity metric as *Ineq2*. Formally,

$$\text{Ineq2}(\boldsymbol{\tau}) := \left(\int_{\underline{\xi}}^{\overline{\xi}} \left(\frac{\sum_{j=1}^{J} \mu_j p_j(\xi)}{\lambda \rho(\xi)}\right)^2 \rho(\xi)\, d\xi - \left(\frac{1}{\lambda} \int_{\underline{\xi}}^{\overline{\xi}} \sum_{j=1}^{J} \mu_j p_j(\xi)\, d\xi\right)^2 \right)^{1/2}. \tag{30}$$

In the rest of the paper, we will simply denote the inequity metric with $\text{Ineq}(\boldsymbol{\tau})$, which can refer to Ineq1, Ineq2, or other inequity metrics that can be represented as a function of $\boldsymbol{\tau}$.

With the efficiency and inequity measures defined in Equations (28) and (29) (or (30)), the policy maker's problem is formulated as a constrained optimization problem:

$$\begin{aligned} \nu_M(\varrho)\ (\nu_{HF}(\varrho), \nu_{DB}(\varrho)) := \max\ & \text{Eff}(\boldsymbol{\tau}) \\ \text{s.t.}\ & \text{Ineq}(\boldsymbol{\tau}) \le \varrho, \\ & \boldsymbol{\tau} \in \mathcal{A}_M(\mathcal{A}_{HF}, \mathcal{A}_{DB}), \end{aligned} \tag{31}$$

where the parameter ϱ represents the maximum inequity level with which the policy maker can tolerate. The larger the ϱ is, the more emphasis has been put on efficiency optimization (over equity). The optimal objective value $\nu_M(\varrho)$ ($\nu_{HF}(\varrho)$, $\nu_{DB}(\varrho)$) represents the maximal efficiency that can be achieved by a certain matching (healthier-first, donor-blind) policy subject to the inequality constraint[6]. Note that (31) is an infinite-dimensional optimization problem and the achievable region contains infinitely many constraints (indexed by ξ). To facilitate computation, we thus develop a finite-dimensional approximation of the optimization problem; see Appendix J.

4.2. General Policies that Allow a Lottery

As shown above, patients with identical health scores experience the same waiting times to receive an offer of a particular kidney type under a matching policy. In policies more general than the matching policies, patients can be ranked in any order irrespective of their waiting time. Therefore, the system manger can offer a certain kidney to a patient when his waiting time is exactly τ, but do not offer any kidneys to him later. Therefore, the patient has to accept whatever kidney is offered to him at τ as that is the only kidney he will receive. In this way, the system manager has full control on which patient to transplant which kidney at what time. Thus, the equilibrium allocation outcome is characterized by a density function $\tilde{p}_j(\xi,\tau)$ for $j=1,\dots,J$. Here $\tilde{p}_j(\xi,\tau)$ has a similar definition as $p_j(\xi)$: for all $\underline{\xi}\le\xi_1<\xi_2\le\overline{\xi}$ and all $0\le\tau_1<\tau_2$, $\int_{\tau_1}^{\tau_2}\int_{\xi_1}^{\xi_2}\tilde{p}_j(\xi,\tau)\,d\xi d\tau$ denotes the fraction of type-j kidneys that are transplanted to patients of initial health scores on $[\xi_1,\xi_2]$ with cumulative waiting times in $[\tau_1,\tau_2]$.

The efficiency and inequity metrics that we derived in the previous section can be expressed as a function of $\tilde{\boldsymbol{p}}:=(\tilde{p}_j(\cdot,\cdot))$. For example,

$$\mathrm{Eff}(\tilde{\boldsymbol{p}}):=\textstyle\sum_{j=1}^{J}\int_{\underline{\xi}}^{\bar{\xi}}\int_{0}^{\bar{\tau}_\xi}\frac{\tilde{p}_j(\xi,\tau)\mu_j}{\lambda\bar{F}_\xi(\tau)}\,\mathrm{QALY}(\xi,j,\tau)\,d\tau\,d\xi, \tag{32}$$

where the ratio $\tilde{p}_j(\xi,\tau)\mu_j/(\lambda\bar{F}_\xi(\tau))$ stands for the proportion of patients who have an initial health score ξ and will receive a kidney of type j at time τ among all patients, and the expression of $\mathrm{QALY}(\xi,j,\tau)$ is given in (27). The inequity metrics Ineq1 and Ineq2 can be calculated using the same formulae (i.e., (29) and (30)) using $p_j(\xi)=\int_0^{\bar{\tau}_\xi}\tilde{p}_j(\xi,\tau)\,d\tau$.

We then formulate the policy design problem as follows:

$$\nu_{\mathrm{G}}(\varrho):=\max\ \mathrm{Eff}(\tilde{\boldsymbol{p}}) \tag{33}$$

$$\text{s.t. }\ \mathrm{Ineq}(\tilde{\boldsymbol{p}})\le\varrho \tag{34}$$

$$\int_0^{\bar{\tau}_\xi}\sum_{j=1}^{J}\frac{\tilde{p}_j(\xi,\tau)\mu_j}{\lambda\rho(\xi)\bar{F}_\xi(\tau)}\le 1\ \text{ for all }\xi\in[\underline{\xi},\overline{\xi}] \tag{35}$$

$$\int_{\underline{\xi}}^{\bar{\xi}}\int_0^{\bar{\tau}_\xi}\tilde{p}_j(\xi,\tau)\,d\tau\,d\xi\le 1,\ j=1,\dots,J \tag{36}$$

To elaborate on the above formulation, the subscript G stands for general policies, which allows the use of a lottery. Constraint (35) requires that the likelihood of receiving a kidney transplant is not greater than one for all patients; Constraint (36) requires that the total fraction of kidneys transplanted to patients is not greater than one.

Although a lottery-based policy is controversial to be implemented in practice, the optimal value of the above problem provides an upper bound for the kidney allocation problem. In Section 5, we will use $\nu_{\mathrm{G}}(\varrho)$ as a benchmark for comparison with other policies in the head-of-line category.

5. Numerical Study

This section conducts a numerical experiment using recent transplant data to investigate how the four policy classes, i.e., the general, matching, healthier-first, and donor-blind, perform towards maximizing the system efficiency subject to certain equity constraints. Investigating the allocation outcomes under different policies provides important insights into potential policy revisions.

5.1. Parameter Estimation

Our data covers kidney transplantation related information in all the 58 donor service areas (DSA) in the study period from 07/01/2005 through 09/01/2010. For each kidney procured during that period, the data records the donor's age, blood type, cause of death, comorbidities, race, height, weight, serum creatinine, and other related attributes; for each patient that has registered on the waitlist, the data records the patient's age upon starting the dialysis, any prior organ transplants, and comordbidities. Using the above covariates, we calculate each kidney's quality ϕ and each patient's initial health score ξ using the formulas in Israni et al. (2014). As done in the new kidney allocation policy (Israni et al., 2014), we classify kidneys into $J = 4$ equal-sized types according to their estimated values of ϕ, and record the cutoff values for each type. We also derive $\rho(\xi)$ as the probability density function from the empirical distribution of ξ. To estimate the hazard rate function $h(c)$, we count how many patients were alive when their health score just hits c, and how many of them have died during the next Δt time units. When Δt is small, the reciprocal of the ratio between these two numbers approximates the hazard rate $h(c)$. To estimate the health score curve $c(t;\xi)$ for each ξ, it suffices to estimate the function $H(\cdot)$ in Equation (2). The function $H(c)$ represents the time duration it takes the health score to deteriorate from $\bar{\xi}$ to c. Thus, $1/H'(c)$ gives the change rate of a patient's health score at c, and can be estimated from the data as

$$\frac{1}{H'(c)} = E[c'(t;\xi)|c(t;\xi) = c]. \tag{37}$$

Building on these, we estimate the function values of $h(c)$, $\rho(\xi)$ and $H(c)$ at forty points that are evenly distributed over the continuous domain $[\underline{\xi}, \bar{\xi}]$, and recover function values on the rest of the

domain via a linear interpolation. We plot the estimated functions $c(t;\bar{\xi}) = H^{-1}(t)$ in Figure 1 in Section 2, and plot $h(c)$ and $\rho(\xi)$ in Figure 4.

We consider the following setting in our numerical experiment: a large donor service area (CAOP-OP1) with more than six thousand patients on the waitlist, only blood type-O adult patients and donors, and patients with CPRA $\leq 80\%$. Focusing on this setting allows us to suppress factors that have not been considered in our model, such as blood type matching, HLA type matching, and geographic factors. We classify the kidneys into $J = 4$ classes, respectively, using the cutoff values computed earlier from the nation-wide data. Then the following parameters can be estimated in a straightforward manner with the data in the study period: the total arrival rate of patients who satisfy our inclusion criteria λ, the kidney arrival rate μ_j and the mean quality ϕ_j for kidney type $(j = 1, 2, 3, 4)$. To facilitate the computation, we develop a finite-dimensional representation for the achievable region following the procedures in Appendix J, in which the domain of ξ has been discretized into grids with $N = 10$. In other words, patients are classified into ten classes depending on their initial health score ξ, while their health scores change with time continuously. Finally, we follow the literature (Axelrod et al., 2018; Wyld et al., 2012) and assume that the quality-of-life coefficients for dialysis, η, follows a Beta distribution with mean and standard deviation 0.690 and 0.037, respectively[7]. Table 1 summarizes the parameter estimates based on this DSA.

Table 1 Summary of Estimated Parameters

Patient Arrival Rate λ (people/year)			801.72		
Kidney Class j	1	2	3	4	Total/Average
Kidney Arrival Rate (kidney/year) μ_j	46.78	50.07	49.88	59.25	206.08
Mean Kidney Quality ϕ_j	15.54	21.37	28.09	35.56	25.75

5.2. Results

This section compares the performance of the four policy classes numerically. The comparisons is based on the predictions of the fluid model developed in the preceding analysis.

We first plot the efficiency-equity Pareto frontier for the inequity metric *Ineq1* defined in (29). We vary the inequity allowance ϱ, solve the optimization problem (31), and obtain the optimal efficiency level $\nu_M(\varrho)$, $\nu_{HF}(\varrho)$, and $\nu_{DB}(\varrho)$. We also solve the optimization policy (33) and obtain $\nu_{\mathrm{G}}(\varrho)$ for each ϱ. We plot the efficiency levels and the corresponding values of ϱ in the left panel of Figure 3. The plots thus characterize the efficiency-equity Pareto frontier of each policy class. For some policy classes, the Pareto frontier may not cover the entire domain of ϱ, which is $[0, \mu/\lambda] = [0, 0.257]$. That can happen when some ϱ have not been achieved by any policy in that class. For example, our numerical test shows no healthier-first policy can achieve an inequity level lower than 0.24.

As shown in the left panel of Figure 3, when ϱ takes the highest value 0.257 (the no inequity constraint case), the matching, healthier-first, and donor-blind policies achieve the same QALYs. Because without the inequity constraint, they all will simply prioritize healthier patients for kidney types and thus their optimal policies coincide.

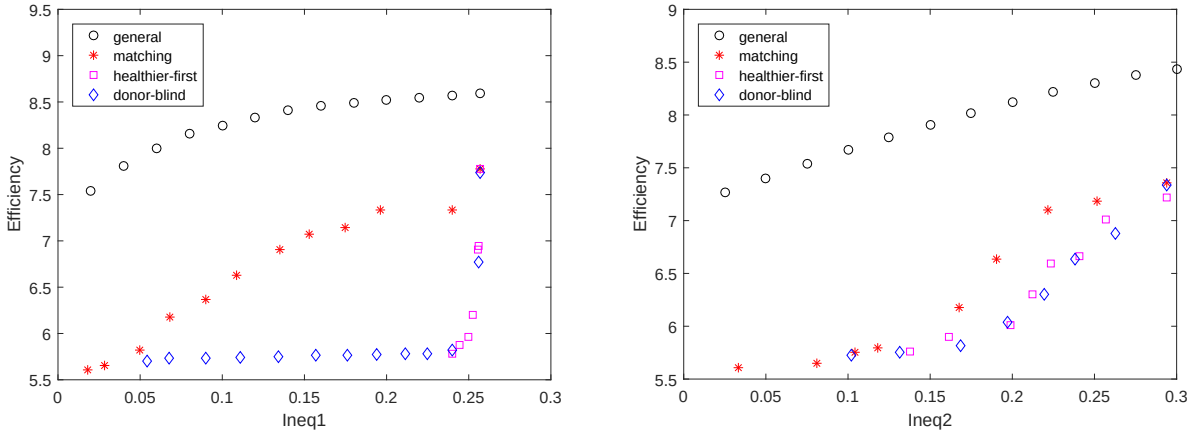

Figure 3 **The Efficiency-Equity Pareto Frontier under Ineq1 (Average Transplant Probability – Minimal Transplant Probability) and Ineq2(Standard Deviation of Transplant Probability)**

However, as ϱ decreases, the performances of the donor-blind and healthier-first policies degrade quickly. To shed light on this, recall that *Ineq1* measures the difference between the average transplantation likelihood (0.257) and the minimal transplantation likelihood among all patient classes. An inequity allowance lower than 0.257 would require all patient class to receive a positive amount of kidneys. This is difficult to achieve by a healthier-first policy, in which the low-class patients hardly receive any kidneys because they have the lowest priority but the highest death rate. As a result, no healthier-first policy can achieve an inequity level lower than 0.24. Compared to the healthier-first policies, the donor-blind policies can achieve a much lower inequity level by providing higher priority to the least healthy (thus most risky) patients. However, in order to do that, the donor-blind policies have to prioritize the least healthy patients for all kidney types. It thus increases the chance for less healthy patients to transplant a high-quality kidneys, which increases survival mismatch.

Table 2 supports that the ordering above stems from differences in the abilities of the four policy classes to achieve survival matching. It shows the allocation outcomes at $\varrho = 0.24$ under the optimal policy in each class. Under the optimal general policy, almost all kidneys are transplanted to patients in classes 5-10 (healthier patients). This can be achieved by assigning each patient with a particular kidney type according to the lottery outcome upon the patient's arrival. The patients, as

Table 2 Comparison of Allocation Outcomes at $\varrho = 0.24$

	General	Matching
Objective	8.566	7.335
Allocation	$\begin{pmatrix} 0.017 & 0 & 0 & 0 \\ 0.017 & 0 & 0 & 0 \\ 0.017 & 0 & 0 & 0 \\ 0.017 & 0 & 0 & 0 \\ 0.473 & 0.380 & 0 & 0 \\ 0 & 0.401 & 0.598 & 0 \\ 0 & 0.000 & 0.619 & 0.379 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.061 & 0 & 0 & 0 \\ 0.064 & 0 & 0 & 0 \\ 0.144 & 0 & 0 & 0 \\ 0.002 & 0.189 & 0 & 0 \\ 0 & 0.307 & 0 & 0 \\ 0 & 0 & 0.782 & 0 \\ 0 & 0 & 0.330 & 0.379 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
	Healthier-First	**Donor-Blind**
Objective	5.780	5.819
Allocation	$\begin{pmatrix} 0.017 & 0 & 0 & 0 \\ 0.173 & 0 & 0 & 0 \\ 0.018 & 0.216 & 0 & 0 \\ 0 & 0.021 & 0.052 & 0.145 \\ 0 & 0 & 0 & 0.327 \\ 0 & 0 & 0 & 0.379 \\ 0 & 0 & 0.541 & 0.013 \\ 0 & 0 & 0.591 & 0 \\ 0.165 & 0 & 0.576 & 0 \\ 0.698 & 0 & 0.218 & 0 \end{pmatrix}$	$\begin{pmatrix} 0.001 & 0.010 & 0.006 & 0 \\ 0.027 & 0 & 0 & 0 \\ 0.180 & 0.054 & 0 & 0 \\ 0 & 0.217 & 0.132 & 0 \\ 0 & 0.002 & 0.116 & 0.221 \\ 0 & 0 & 0.261 & 0.223 \\ 0 & 0 & 0.166 & 0.376 \\ 0 & 0.076 & 0 & 0.528 \\ 0 & 0.321 & 0.016 & 0.361 \\ 0 & 0.097 & 0.651 & 0.001 \end{pmatrix}$

The entry at the i^{th}-row and the j^{th}-column in the matrix represents the proportion of class-i patients that transplant a type-j kidney. A highly indexed row (column) represents a patient (kidney) class with better survival. The inequity constraint is binding at $\varrho = 0.24$ for all policies except for the matching policy.

a consequence of not being offered any other kidney types, have to accept whatever types of kidneys offered to them. If we use the waiting time instead of lottery to rank the patients as in a matching policy, a patient may turn down a low-type kidney in order to receive a high-type kidney later. For example, under the optimal matching policy at $\varrho = 0.24$, not enough class-7 patients are willing to accept type-3 kidneys, because the likelihood of transplanting a (better) type-4 kidney is as high as 37.8%. Consequently, only a small proportion of type-3 kidneys are allocated to class-7 patients and the rest have to be allocated to class-6 instead. Therefore, the optimal matching policy does less well in survival matching compared to the general policy due to the extra incentive compatibility constraint. Compared to the matching policy, the healthier-first and donor-blind policies are even less effective in promoting surviving matching for different reasons. To meet the equity constraint, healthier-first policies need to offer class 1 patients the required amount of kidneys. Note that class 1 patients have the lowest priority under healthier-first policies; and they have the highest death rates. To address this challenge, the optimal healthier-first policy has used type-4 kidneys to attract

the class-4 and -5 patients so those patients, which have a high density, will not compete with class-1 patients for type-1 kidneys in our numerical analysis. An unintended consequence, however, is that a majority of the class-4 kidneys are allocated to patients with intermediate health values, which results in survival mismatch. The reason for the performance loss of donor-blind policies (as ϱ decreases) is different. The donor-blind policies can give priority to the low-class patients, but it has to be for all kidney types. Consequently, some high-class kidneys are allocated to low-type patients, which results in survival mismatch. Despite that, the allocation outcome of donor-blind still exhibits a certain degree of survival matching, because a patient with longer survival time has a stronger incentive to wait for high-quality kidneys due to the supermodularity of the survival function; see (Su and Zenios, 2006) for a detailed explanation.

When ϱ decreases further, we observe that the performance gap between matching and donor-blind closes again. Intuitively, for small ϱ, both matching and donor-blind policies strive to meet the inequity constraint. Consequently, neither could achieve survival matching. The matching policies gradually loses its advantage over donor-blind policies in achieving survival matching, and the two policies exhibit comparable performance for small ϱ, though the donor-blind policies cannot achieve inequity allowance lower than 0.05.

We also compute the Pareto frontier for a different inequity metric *Ineq2* – the variance of transplantation probability of each patient class, which is plotted in the right panel of Figure 3. We observe that the slopes of the frontier for healthier-first and donor-blind policies are less steep, because they are less sensitive to *Ineq2*. The gap between these two policies and the matching policy is smaller, and healthier-first slightly outperforms the donor-blind policies for intermediate ϱs. However, the Pareto frontiers of the four policy classes largely follow the same order as that under *Ineq1*, suggesting that the qualitative insights we have derived for *Ineq1* are robust for similar inequity metrics.

For both inequity metrics, we note that the general policy outperforms the matching, healthier-first, and donor-blind policies by a large margin ($11\% - 36\%$). In fact, by looking into the allocation outcomes the Pareto frontier of the general policies, we find that a patient either receives a transplant upon arrival or never receives a kidney offer. This is more socially efficient compared to performing the transplantation after a patient's health has deteriorated. Despite its potential for increasing the QALYs, the general policy, which ranks patients regardless of their waiting time, cannot be justified as ethical or implemented in practice.

In addition to improving survival matching and maximizing QALYs, the matching policy can efficiently reduce kidney wastage compared to the donor-blind policy. However, in the above parametric setting, no kidney is ever wasted at the Pareto frontier[8]. This is because the patient arrival rate λ significantly outnumbers the kidney arrival rate μ in the DSA we consider. Nevertheless, if

we scale down λ by 60% percentage (note that 0.4λ is still larger than μ), then we observe that a significant number of class-1 kidneys are wasted under the donor-blind policies (on the Pareto frontier) when the inequity allowance ϱ for *Ineq1* is small; see Figure 5. Intuitively, when ϱ is large, the optimal donor-blind policy simply assigns the lowest priority to the lowest-class patients, who have to accept whatever kidneys that are available to them, so no kidneys are wasted. However, when ϱ is small, each patient class has an equitable likelihood of receiving kidneys of all types because the ranking under a donor-blind policy does not depend on kidney type. Once λ is scaled down, this likelihood is not small. Consequently, a patient has an incentive to turn down low-quality organs in order to get a better one. As a result, up to 44.0% of the class-1 kidneys are wasted in this example; see Figure 5. In contrast, under the matching policies, no class-1 kidneys are wasted, because the matching policy can provide the low-class patients only with class-1 kidneys without violating the equity constraint.

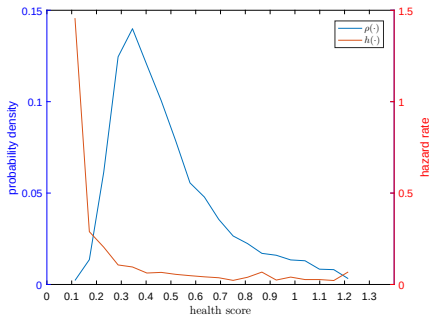

Figure 4 **The Estimated Values of $\rho(\xi)$ and $h(c)$**

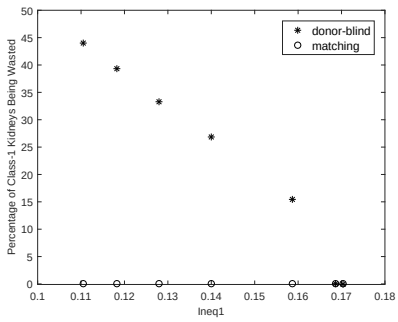

Figure 5 **Kidney Wastage at the Pareto Frontier**

Finally, we comment on the necessity of our methodological framework in obtaining the Pareto frontier. If one uses a simulation approach to obtain the above comparison, one needs to simulate the outcomes of all ranking policies in each policy class. That would be a formidable task as each policy class covers numerous functional forms of ranking scores. While using the fluid model and the achievable region approach, we can calculate the optimal policy in each class in real time. In fact, since the transplant waitlist is an overloaded queueing system, the fluid model provides an accurate approximation to the original stochastic system. We demonstrate in Appendix K that the predictions made by the fluid model are close to the simulated results in the stochastic setting. We also perform a sensitivity analysis in Appendix L, which shows the qualitative insights derived in this section are robust under a smaller demand rate.

6. Concluding Remarks

This paper develops a fluid model to predict the allocation outcome under a ranking policy. Unlike many fluid models in which the steady state depends solely on service disciplines, the steady state in our fluid model is the unique Nash equilibrium of a queue-joining game. The policy designer's problem is therefore to search for the ranking policy that leads to a Nash equilibrium with the most desirable characteristics. We derive an efficient characterization of the achievable region for each policy class. This characterization allows us to search for the best allocation in each policy class for given efficiency and inequity metrics. We apply the above method to the recent kidney transplant data and plot the Pareto frontier for each policy class.

The analytical and numerical results lead to important managerial/qualitative insights into the design of allocation policy. First, compared to the previous policy, the new policy implemented in 2014 prioritizes healthier patients for the access to high-quality kidneys. This major policy revision is a step in the right direction – from donor-blind (the previous policy) to matching (the new policy). The latter has at least two advantages: improves survival matching and reduces kidney wastage. Second, the new policy is a healthier-first policy, as it only prioritizes the healthiest patients for certain kidney types. In order to provide patients of different types with equitable opportunity for kidney transplantation, the future policy revision might consider prioritizing less healthy patients for low-quality kidneys. Such a revision can also improve the total QALYs if executed properly. Third, in both the previous and the new policies, a patient's score is an affine function of waiting time with the same slope. Our analysis shows that for matching policies, there is no need to consider scores with more complex functional forms; though this conclusion might not hold for donor-blind policies. Finally, our numerical study shows that a substantial improvement could be achieved by policies that do not use waiting time as a primary ranking criterion. That motivates the policy maker to seek alternative criteria to replace the waiting time.

Endnotes

1. `https://optn.transplant.hrsa.gov/data/`.
2. The Calculated Panel Reactive Antibodies (CPRA) calculates the probability that a recipient is HLA-incompatible with a randomly selected donor.
3. The allocation policy is further embedded in a geographically-tiered structure. We suppress how the policy depends on geography for simplicity; see (Israni et al., 2014) for those details. Also see (Ata et al., 2016) for a study of the issues stemming from the geographic structure.
4. In the previous policy, the kidney donors had been classified into standard criterion donors and expanded criterion donors. The expanded-criterion-donors kidneys had a small proportion and were only offered to patients who expressed interest in transplanting those kidneys.
5. http://surgery.ucsf.edu/conditions–procedures/kidney-transplantation.aspx.
6. If (31) is infeasible for a given ϱ, we let $\nu_M(\varrho) = -\infty$ by convention.

7. Our model allows the distribution of η to vary by ξ. However, our data does not exhibit correlation between ξ and η. As such, η has the same distribution across different values of ξ in our numerical study.

8. In this paper, we only count wastage as kidneys that are not of any patient's choice. In reality, a kidney can be wasted for other reasons.

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Appendix A: Proof of Proposition 1

We first prove that if $\tau_j(\xi) < +\infty$, then (9) holds and the inequality is strict only if $c(\tau_j(\xi);\xi) \in \mathcal{K}$. The infimum in Equation (8) is always attained by the assumption that $L(c(\tau;\xi), j, \tau)$ is right-continuous in τ. So when a patient's waiting time reaches $\tau_j(\xi)$, his score must be either equal to or greater than S_J. We next prove that the latter case could happen only when the patient's health score $c(\tau_j(\xi);\xi) \in \mathcal{K}$. We prove by contradiction. Suppose $L(c(\tau_j(\xi);\xi), j, \tau_j(\xi)) > S_j$. If $c(\tau_j(\xi);\xi) \notin \mathcal{K}$, then $L(c(\tau;\xi)), j, \tau)$ continuously and strictly increases with τ in a neighborhood of $\tau_j(\xi)$. Thus, we can always find a time t which is slightly smaller than $\tau_j(\xi)$, but with $L(c(t;\xi), j, t)$ still having a larger value than S_j. That, however, contradicts with $\tau_j(\xi)$ being the first time that the patient's score has reached or exceeded S_j.

We next construct the strategy $a(\cdot,\cdot|\xi,\eta)$ of a patient with parameters (ξ,η) at a Nash equilibrium, which also proves the existence of the Nash equilibrium. For $j=1,2,\ldots,J$, define

$$S_j = \inf\{s \mid \iiint \chi\left(L(c(\tau;\xi),j,\tau) > s,\, a(\tau,j|\xi,\eta) = \mathrm{A}\right)\, \pi(c(\tau;\xi),\tau,\eta)\, d\xi d\tau d\eta = 0\}, \tag{38}$$

where $\chi(\cdot)$ represents a characteristic function. Intuitively, S_j represents the minimum threshold such that no patients with score greater than S_j are willing to accept kidneys of type j. We prove the Proposition by induction on the type of kidney offer $j=J,J-1,\ldots,1$.

If $j=J$ (the best kidney type), then by Lemma 1, every patient's strictly dominating strategy is to accept a type-J kidney immediately when being offered. Therefore, the equilibrium strategy must have $a(\tau,J|\xi,\eta)=\mathrm{A}$ for all ξ,η,τ. Once we have determined $a(\cdot,J|\cdot,\cdot)$, S_J can be uniquely determined by (38) for any given population distribution $\Pi=(\pi(\cdot,\cdot,\cdot))$. We next prove that if $\tau_J(\xi)$ is calculated from S_J from Equation (38) and is finite, then $\tau_J(\xi)$ must be the first time that the score of patient ξ is equal to or larger than S_J.

When a patient has waited less than $\tau_J(\xi)$ time units, then his score s must be strictly smaller than S_J by the definition of $\tau_J(\xi)$. Then by the way that S_J is constructed, c.f., Equation (38), there are a positive mass of patients who are willing to accept type-J kidneys and also have their scores strictly larger than s. Since the population distribution at the steady state is stationary, the mass of those patients will stay invariant. Thus, all new arrived type-J kidneys will be consumed by those patients before being offered to the patient with a lower score s. Consequently, the patient would never receive any kidneys of type J before his waiting time has accumulated to $\tau_J(\xi)$.

When the patient has waited for exactly $\tau_J(\xi)$ time units, his score has to be equal or greater than S_J by the "furthermore" part of the proposition that we have proved at the beginning of this proof. Then by the HL property of the score, the patient's score must continue increasing during the time period $[\tau_J(\xi),\tau_J(\xi)+\delta]$ for sufficiently small $\delta>0$. Thus, immediately after $\tau_J(\xi)$, the patient's score will be strictly larger than S_j, and thus be larger than the scores of all other patients on the waitlist who are willing to accept a type-J kidney. Consequently, the patient will receive a type-J kidney offer either at or immediately after $\tau_J(\xi)$. It is complicated to differentiate these two scenarios. However, for all kidney types $j=1,\ldots,J$, these two scenarios only result in an infinitesimal difference in the values of $\tau_j(\xi)$ and U_R. Thus, the two scenarios make no difference for a patient's acceptance/rejection decision except when the patient has $U_A=U_R$. However, as we argued earlier, a tie can only happen to patients with a particular η, which has a measure of zero. Therefore, there is no need to differentiate these two scenarios, and we will simply assume in the remainder of our paper that a patient receives a kidney offer at the allocation time.

After we obtain S_J and $\tau_J(\xi)$, then whenever a patient is offered a type $J-1$ kidney, we can compute his expected QALYs by rejecting that kidney, U_R. For example, if a patient with parameters (ξ,η) is offered a type $J-1$ kidney at time t, then by knowing his waiting time for type-J kidney, $\tau_J(\xi)$, his U_R is exactly given by $V(\tau_J(\xi)-t)$ as defined in Equation (39) with $c=c(t;\xi)$ and $j=J$. Consequently, we can compute almost every patient's optimal strategy $a(\tau,J-1|\xi,\eta)$. We then iteratively repeat the above procedure for $j=J-1,\ldots,1$. In each iteration, by knowing almost every patient's strict dominating strategy $a(\tau,j|\xi,\eta)$, we can compute the score threshold S_j and the required waiting time $\tau_j(\xi)$ for that type of kidneys. This

allows us to calculate every patient's U_R and to determine $a(\tau, j-1|\xi,\eta)$, the strictly dominating strategy for the type-$(j-1)$ kidneys for almost all patients (except for those with $U_A = U_R$) given that almost everyone will play the strategy $a(\tau, j|\xi,\eta)$ for the type-j kidneys. Following the above procedure, which is well known as iterated elimination of dominated strategies, we can find the unique Nash equilibrium strategy. While we have proved that the $\tau_j(\xi)$ constructed in the above procedure satisfies the properties stated in the proposition.

Finally, we provide an example to illustrate that there might be not such a score threshold or allocation time when the ranking policy does not satisfy the HL property. Consider a waitlist with a single kidney type, in which the total kidney supply rate μ is lower than the aggregate patient arrival rate λ. Suppose all patients are ranked according to a score $L(c, j, \tau) = -\tau$. This is the donor-blind, last-come-first-serve policy, and does not satisfy the HL property. It is easy to check that patients with scores lower than zero will never be offered a kidney. Among patients whose scores are equal to zero, which only happens at the time of their arrival, only a proportion μ/λ are offered a kidney; while the other patients would never receive a kidney offer. So there is not such a score threshold or allocation time as described in Proposition 1. ■

Appendix B: Proof of Lemma 1

Consider a patient with health score ξ at time 0 (here 0 refers to the current time, not necessarily the time of arrival), and the quality-of-life coefficient η for that patient living on dialysis. We define the function

$$V(t) := \eta \int_0^t s f_\xi(s)\, ds + \eta \bar{F}_\xi(t) t + \bar{F}_\xi(t) c(t;\xi)\phi_j. \tag{39}$$

In the above equation, $\eta \int_0^t s f_\xi(s)\, ds + \eta \bar{F}_\xi(t) t$ and $\bar{F}_\xi(t) c(t;\xi)\phi_j$ gives the expected QALYs before and after the transplantation, respectively. Thus, $V(t)$ gives the optimal expected QALYs that the patient may receive conditional on accepting a type-j kidney offer at time t. We note that

$$V'(t) = \eta \bar{F}_\xi(t) - f_\xi(t) c(t;\xi)\phi_j + \bar{F}_\xi(t) c'(t;\xi)\phi_j \le \bar{F}_\xi(t) - f_\xi(t) c(t;\xi)\phi_j < 0, \tag{40}$$

where the first inequality follows from $c'(t;\xi) \le 0$, and the second inequality follows from $\eta < 1$ and that

$$1 - \frac{f_\xi(t)}{\bar{F}_\xi(t)} c(t;\xi)\phi_j = 1 - h(c(t;\xi)) c(t;\xi)\phi_j < 0$$

by (3). Thus, $V(t)$ strictly decreases with t. Consequently, whenever a type j kidney is offered, the patient's strictly dominating strategy is to accept it immediately rather than accepting a kidney of equal or even lower quality at a later time. If a patient chooses to stay on dialysis forever, then it is equivalent to accepting a kidney with $\phi = 0$ at an infinitely later time. So this case is also covered by this proof. ■

Appendix C: Proof of Proposition 2 and Remark 1

Proof. We proceed by contradiction. Suppose a patient with parameters with parameters (ξ, η), by using his optimal strategy, rejects a kidney of type j_k at time $\tau_{j_k}(\xi)$, but accepts a kidney of the same type j_k at a later time $s > \tau_{j_k}(\xi)$. After the patient rejects the kidney at time $\tau_{j_k}(\xi)$, let us assume that the first kidney type he will accept is $j_{k'}$. If $j_{k'} \neq j_k$, then the patient would either die before transplantation, or accept a kidney of type j' and leave the waitlist. In whichever case, the patient will not accept a kidney of type j_k, which contradicts our assumption. If $j_k = j_{k'}$, then it is clear from Lemma 1 that accepting the

type-j_k kidney earlier at time $\tau_{j_k}(\xi)$ strictly dominates the postulated alternative, and hence, leads to a contradiction. Therefore, we conclude that the patient's choices throughout their waits are consistent and it suffices to restrict attention to decision times $\tau_{j_1}(\xi), \ldots, \tau_{j_m}(\xi)$.

We next prove that $j^*(\eta, \xi)$ is non-decreasing in η by contradiction. Suppose there are two patients with $\eta_1 > \eta_2$, but $j_1(:= j^*(\xi, \eta_1)) < j_2(:= j^*(\xi, \eta_2))$. Then kidney type j_1 must have been accepted by the first patient, and rejected by the second one; note that j_2 is the first kidney type accepted by the second patient subsequent to the offer of type j_1. Given that the second patient rejects kidney type j_1 and accepts j_2, we can derive the following inequality by recursively applying formula (10),

$$c(\tau_{j_1}(\xi);\xi)\phi_{j_1} < \eta_2 \int_{\tau_{j_1}(\xi)}^{\tau_{j_2}(\xi)} \frac{f_\xi(t)}{\bar{F}_\xi(\tau_{j_1}(\xi))}(t - \tau_{j_1}(\xi))\, dt + \frac{\bar{F}_\xi(\tau_{j_2}(\xi))}{\bar{F}_\xi(\tau_{j_1}(\xi))}(\eta_2(\tau_{j_2}(\xi) - \tau_{j_1}(\xi)) + c(\tau_{j_2}(\xi);\xi)\phi_{j_2}). \quad (41)$$

Intuitively, the above inequality implies that for the second patient, choosing to wait till the offer of kidney type j_2 strictly dominates accepting kidney type j_1. Since $\eta_1 > \eta_2$, the above inequality still holds by replacing η_2 with η_1. However, that implies that the first patient has to reject kidney j_1, which contradicts that $j^*(\xi, \eta_1) = j_1$. This concludes the proof. ■

Remark 1 *Since $j^*(\eta, \xi)$ can be computed from $(\tau_j(\xi))$ using the recursive equations* (10)-(12) *and* (13)*, we can then describe the equilibrium state using only $(\tau_j(\xi))$ instead of the waitlist density Π. In fact, given $(\tau_j(\xi))$, we can compute each patient's matched kidney type $j^*(\eta, \xi)$. That allows us to determine if a certain patient has already accepted a kidney offer and left by comparing the patient's current waiting time τ and his allocation time $\tau_{j^*(\eta,\xi)}(\xi)$. Specifically, for a patient with health score c and waiting time τ, we can solve his initial health score ξ from the equality $c = c(\tau;\xi)$. We can then recover $\Pi := (\pi_{c,\tau,\eta})$ as follows*

$$\pi_{c,\tau,\eta} = \lambda\, \rho(\xi)\, \bar{F}_\xi(\tau)\, \chi(\tau < \tau_{j^*(\eta,\xi)}(\xi)). \quad (42)$$

Appendix D: Proof of Proposition 3

Proof. We will only prove the "if" part. The "only if" part can be proved with a similar argument.

Suppose $(\tau_j(\xi))$ are the allocation times at the equilibrium. We discuss two possible situations that could happen to each of the virtual queues. Recall that the j^{th} virtual queue consists of patients who will be matched to kidneys of type j.

In the first situation, queue j is non-empty and thus $z_j = \sup_\xi \tau_j(\xi) > 0$. Since it is at the steady state, the inflow and out-flow rates must be balanced for each virtue queue, otherwise the queue-length cannot stay invariant. Note that the departure rate includes those having reneged as well as those having transplanted kidneys of type j. This leads to the following equality,

$$\lambda \int_{\underline{\xi}}^{\bar{\xi}} Q_j(\xi)\rho(\xi)\, d\xi = \lambda \int_{\underline{\xi}}^{\bar{\xi}} F_\xi(\tau_j(\xi))\rho(\xi)\, d\xi + \mu_j, \quad (43)$$

where $\lambda \int_{\underline{\xi}}^{\bar{\xi}} F_\xi(\tau_j(\xi))\rho(\xi)\, d\xi$ gives the aggregate reneging rate in queue j. If we define y_j as in (17), then the above equality implies that $y_j = 0$, and thus the complementary slackness condition $y_j z_j = 0$ holds for j.

In the second situation, queue j is empty, then $\tau_j(\xi) = 0$ for all ξs. Thus, we have $z_j = \sup_\xi \tau_j(\xi) = 0$ for patients in queue j, and the complementary slackness condition holds. If a queue is empty, there can be a

surplus in the kidney and thus y_j can be any non-negative number. So the constraint $y_j \geq 0$ in the NCP holds for that j.

The discussion of the above two situations show that the NCP (17) is solved by $(\tau_j(\xi))$ and the associated variables.

Appendix E: Proof of Lemma 2

Proof. (a): Since $\tau_j(\xi)$ is the first time that the patient's score is equal to or larger than S_j, $\tau_j(\xi)$ must be non-decreasing in S_j. Since $L(c_\xi(\tau), j) + \tau$ is right-continuous in τ, $\tau_j(\xi)$ must be right-continuous. Its left-limit exists due to monotonicity.

(b): By the assumption of the matching policy, a patient with initial score ξ has his score $L(c(t;\xi), j, t)$ strictly increasing in t almost everywhere. At all those points, S_j strictly increases in t and thus $\tau_j(\xi)$ continuously increases in S_j. However, when the patient's health score hits one of the cutoff values in $\mathcal{K}$, say, $\tilde{c}$, the function value $L(\cdot, j, t)$ can possibly take a downward jump at $\tilde{c}$, at which time $\tau_j(\xi)$ changes discontinuously with S_j. To see that, we refer the readers to Figure 2 –when the threshold S_j^1 approaches to $S_j^2 := \lim_{c\uparrow\tilde{c}} L(c, j, c^{-1}(\tilde{c};\xi))$ from left, the allocation time corresponding to S_j^1 for patient ξ, i.e., $\tau_j^1(\xi)$, does not to approach to $c^{-1}(\tilde{c};\xi)$ from left. This is because the $L(c(t;\xi), j, t)$ has a downward jump at $\tilde{c}$. Consequently, it takes much longer for the score of patient ξ to reach S_j^2. That implies that the allocation time $\tau_j(\xi)$ changes discontinuously at S_j^2. However, for any given S_j^2, since both $c^{-1}(c;\xi)$ is strictly increasing in ξ and $L(c, j, t)$ is strictly increasing in t, the left-hand-side of the following equation is strictly increasing in ξ and the equation must have at most one solution ξ for any given S_j^2 and $\tilde{c} \in \mathcal{K}$,

$$\lim_{c\uparrow\tilde{c}} L(c, j, c^{-1}(c;\xi)) = S_j^2. \tag{44}$$

Because the above equation is a necessary condition for $\tau_j(\xi)$ to be discontinuous at S_j^2, there is at most one ξ such that $\tau_j(\xi)$ is discontinuous at S_j^2. Since the set $\mathcal{K}$ contains finitely many points, there are finitely many ξs at which $\tau_j(\xi)$ may change discontinuously in S_j, and those ξs have a measure of zero. ■

Appendix F: Proof of Theorem 1

Proof. Let $S := (S_j)$ denote a vector of score thresholds for all kidney types. We next construct a mapping $\Psi(\cdot)$ to serve the following purpose: if S is a fixed point of this mapping (that means, $S = \Psi(S)$), then its associated allocation times $(\tau_j(\xi))$ must solve the NCP, c.f., Equation (17), and thus be the equilibrium allocation times. Thus, to prove the theorem, it suffices to prove that such a fixed point exists and is unique.

The mapping $\Psi(\cdot)$ is constructed according to the following steps. Given (S_j), we calculate the unique allocation times $\tau_j(\xi)$ for each patient ξ and j using Equation (8), and then find functions $\Gamma_j(\cdot)$ and $Q_j(\cdot)$ that are associated with $(\tau_j(\xi))$. Finally, with $Q_j(\cdot)$, we search for another vector of score thresholds $\tilde{S} := (\tilde{S}_j)$ and the allocation times $(\tilde{\tau}_\cdot(\cdot))$ associated with $\tilde{S}$, such that $\tilde{\tau}_\cdot(\cdot)$ and its associated $Q_j(\cdot)$, $\Gamma_j(\cdot)$ solve the NCP (17). If this requirement can be satisfied by multiple $\tilde{S}$s, then we let $\Psi(S)$ be the infimum of them.

We next show that the mapping $\Psi(\cdot)$ constructed above satisfy the desired properties, which allows us to invoke the Brouwer's fixed point theorem and prove the existence of the fixed point. First, the image of $\Psi(\cdot)$

must be contained in the compact set $\prod_{j=1}^{J}[0,\bar{S}_j]$, where $\bar{S}_j$ denote the upper limit of a patient's score for kidney type j,

$$\bar{S}_j := \sup\{L(c(t;\xi),j,t) | \underline{\xi} \le \xi \le \bar{\xi},\, 0 \le t \le \bar{\tau}_\xi\}. \tag{45}$$

Thus, the mapping from S to $\tilde{S}$ is from the compact domain $\prod_{j=1}^{J}[0,\bar{S}_j]$ to itself (not necessarily onto).

Second, $\Psi(\cdot)$ is well-defined. That means, we can always find a $\tilde{S}$ such that its associated allocation times $(\tilde{\tau}_j(\xi))$ solve the NCP, or equivalently, for all $j = 1, ldots, J$, $(\tilde{\tau}_j(\xi))$ either solves the following identity,

$$\lambda \int_{\xi\in\Omega} Q(\xi,j)\rho(\xi)\bar{F}_\xi(\tilde{\tau}_j(\xi))\,d\xi = \mu_j, \tag{46}$$

or $\tilde{\tau}_j(\xi) = 0$ for all ξ and the following inequality holds,

$$\lambda \int_{\xi\in\Omega} Q(\xi,j)\rho(\xi)\,d\xi \le \mu_j. \tag{47}$$

The detailed proof follows next. For each j, by Property (b) of Lemma 2, $\tilde{\tau}_j(\xi)$ increases continuously in $\tilde{S}_j$ for almost all ξs. We can deduce that the integral on the left-hand-side of Equation (46) has to decrease continuously with $\tilde{S}_j$. Note that if the threshold $\tilde{S}_j$ is set as low as $\min_\xi L(\xi,j,0)$, it does not require any patient to wait for kidneys of type j so the associated allocation time $\tilde{\tau}_j(\xi) = 0$ for all ξ; if the threshold $\tilde{S}_j$ is set as its upper limit $\bar{S}_j$, then all patients must have reneged before their allocation times by the definition of $\bar{S}_j$. Therefore, when $\tilde{S}_j$ increases from $\min_\xi L(\xi,j,0)$ to $\bar{S}_j$, the left-hand-side of Equation (46) continuously decreases from $\lambda\int_{\xi\in\Omega} Q_j(\xi)\rho(\xi)\,d\xi$ (which is the function value corresponding to $\tau_j(\xi) = 0$ for all ξ) to 0. Thus, either $\mu_j \in [0, \lambda\int_\xi Q(\xi,j)\rho(\xi)\,d\xi]$, in which case the intermediate value theorem implies the existence of a solution to equality (46), or μ_j is outside this interval, in which case inequality (47) must hold. Note that in the first case, it is possible that Equation (46) has multiple solutions. This could happen when the curve $L(c(\tau;\xi),j,\tau)$ takes an upward jump at $\tilde{\tau}_j(\xi)$. Then all $\tilde{S}_j \in [\lim_{\tau\uparrow\tilde{\tau}_j(\xi)} L(c(\tau;\xi),j,\tau), L(c(\tilde{\tau}_j(\xi);\xi),j,\tilde{\tau}_j(\xi))]$ are associated with the same allocation time $\tilde{\tau}_j(\xi)$. We then let $(\Psi(S))_j = \lim_{\tau\uparrow\tilde{\tau}_j(\xi)} L(c(\tau;\xi),j,\tau)$, which is associated with $\tilde{\tau}_j(\xi)$ by right continuity of the function $L(c(\tau;\xi),j,\tau)$ (i.e., Property (a) of Lemma 2).

Finally, we show that $\Psi(\cdot)$ is continuous, that is, $\|\Psi(S) - \Psi(S')\| \to 0$ if $\|S - S'\| \to 0$, where $\|\cdot\|$ denote the supremum norm. We will use the superscript $'$ to denote the variables associated with S'. For example, $\tau_j(\xi)$ and $\tau'_j(\xi)$ denote the allocation times associated with S and S', respectively. By Property (b) of Lemma 2, we have $\|\tau_j(\xi) - \tau_{j'}(\xi)\| \to 0$ for almost all ξ. Then the recursive equations (10) imply that $|V_k - V'_k| \to 0$ for for almost all ξ and all k, where V_k denotes the optimal expected QALYs that a patient can get from the time of receiving kidney offers of type j_k. Since V_k and V'_k stay very close for almost all ξ, by changing V to V', only the patients with η near the cutoff values in $\{\Gamma_j(\xi)\,|\,j = 1,\ldots,J-1\}$ may change their matched kidney type $j^*(\xi,\eta)$. Therefore, when $|V_k - V'_k| \to 0$, for almost all ξ, $|Q_j(\xi) - Q'_j(\xi)| \to 0$. Consequently, the left-hand-side of (46), i.e., $\lambda\int_{\xi\in\Omega} Q_j(\xi)\rho(\xi)\bar{F}_\xi(\tau_j(\xi))\,d\xi$ continuously decreases with S. According to the previous discussion, $\tilde{S}$ is either the constant $\min_\xi L(\xi,j)$, at which $\tau_j(\xi) = 0$ for all ξs, or the unique intermediate solution to (46). In the first case, $\tilde{S}$ is a constant and thus changes continuously with S; in the second case, since the derivative of the left-hand-side of (46) with respect to $\tilde{S}$ is strictly negative, a small perturbation to the left-hand-side led by replacing S with S', can result in only small perturbation of its solution $\tilde{S}$. That implies $|\tilde{S} - \tilde{S}'| \to 0$.

We thus proved the existence of a fixed point $S = \Psi(S)$. We next prove its uniqueness by contradiction. Suppose there are two different allocation-time vectors, $\tau_j(\xi)$ and $\tau'_j(\xi)$, which are associated with two different score-thresholds vectors, S and S'. Define the index set

$$\mathcal{J}^+ := \{j \,|\, S_j < S'_j\}. \tag{48}$$

Intuitively, $\mathcal{J}^+$ contains kidney types for which the allocation time at the first equilibrium is not longer than that at the second equilibrium. Since $S \neq S'$, we can assume $\mathcal{J}^+ \neq \emptyset$ without loss of generality. We note that if a patient has $j^*(\xi, \eta) \in \mathcal{J}^+$ under S, then she must still have $j^*(\xi, \eta) \in \mathcal{J}^+$ if the score thresholds S' was replaced by S, because the latter requires an even shorter waiting time for those kidney types. Thus, the total mass of patients will choose kidney types in $\mathcal{J}^+$ under S is no less than that under S'. By property (a) of Lemma 2, we have $\tau_j(\xi) \leq \tau'_j(\xi)$ for all ξ and all $j \in \mathcal{J}^+$. Since $(\tau_j(\xi))$ and $(\tau'_j(\xi))$ are different allocation-time vectors, we must have $\tau_j(\xi) < \tau'_j(\xi)$ for some j. Without loss of generality, we assume $j \in \mathcal{J}^+$ (otherwise we can define $\mathcal{J}^+$ by swapping S and S' to have $j \in \mathcal{J}^+$). That implies the total mass of patients who die during their waiting for kidney types in $\mathcal{J}^+$ is larger under S' compared to that under S, despite the fact that the patients who choose to accept kidney types in $\mathcal{J}^+$ under S' is a subset of that under S. Consequently, the total mass of patients who will transplant kidneys of types in $\mathcal{J}^+$ under S is strictly larger than that under S'. However, since $S'_j > S_j \geq 0$ for all $j \in \mathcal{J}^+$, no queue in $\mathcal{J}^+$ is empty under S', which implies that the utilization of kidneys in classes $\mathcal{J}^+$ have already been 100% under S', which contradicts that more kidneys in classes $\mathcal{J}^+$ are accepted under S. ■

Appendix G: Proof of Lemma 3

Because $\boldsymbol{\tau}$ satisfies constraint (C.M), $\gamma_j(\xi)$ must be non-decreasing in ξ. Consequently, $\gamma_j(\xi)$ must have both left and right limits. We next prove that $\gamma_j(\xi)$ is left-continuous. Suppose a sequence of values $\{\xi^k\} \uparrow \xi^*$. Since $\gamma_j(\xi^k)$ is non-decreasing, we have $\{\gamma_j(\xi^k)\} \uparrow c^*$ for some c^*, which further implies that $\{\tau_j(\xi^k)\} \to c^{-1}(c^*; \xi^*)$. Because $L(c, j, \tau)$ is continuous in τ and left-continuous in c for all j, we have

$$L(\gamma_j(\xi^k), j, \tau_j(\xi^k)) \to L(c^*, j, c^{-1}(c^*; \xi^*)) \text{ when } k \to \infty. \tag{49}$$

Since for each k, $\tau_j(\xi^k)$ is the allocation time of patient ξ^k, Proposition 1 implies that $L(\gamma_j(\xi^k), j, \tau_j(\xi^k)) \geq S_j$. Consequently, $L(c^*, j, c^{-1}(c^*; \xi^*)) \geq S_j$. That implies $c^{-1}(c^*; \xi^*) \leq \tau_j(\xi^*)$, so

$$c^* \geq c(\tau_j(\xi^*); \xi^*) = \gamma_j(\xi^*), \tag{50}$$

On the other hand, monotonicity of $\gamma_j(\cdot)$ implies that $c^* = \lim_{k\to\infty} \gamma_j(\xi^k) \leq \gamma_j(\xi^*)$. We thus deduce that $c^* = \lim_{k\to\infty} \gamma_j(\xi^k) = \gamma_j(\xi^*)$. Thus, $\gamma_j(\cdot)$ is left-continuous.

Finally, we prove that $\gamma^{-1}(c)$ is a singleton for $c \notin \mathcal{K}$. Suppose $\xi, \xi' \in \gamma^{-1}(c)$. Then since $c \notin \mathcal{K}$, we have $L(c, j, \tau_j(\xi)) = L(c, j, \tau_j(\xi')) = S_j$ by Proposition 1. Since $L(c, j, \tau)$ strictly increases in τ for all c and j, we deduce that $\tau_j(\xi) = \tau_j(\xi')$. That implies that $\xi = \xi'$ as $c(\tau_j(\xi); \xi) = c(\tau_j(\xi'); \xi')$. ■

Appendix H: Proof of Proposition 2

Proof. To prove the "$\subseteq$" direction in Equation (21), it suffices to prove that if $\boldsymbol{\tau}$ has a minimum acceptance level k, then $\boldsymbol{\tau} \in \mathcal{P}_k$. We have argued in Section 3 that $\boldsymbol{\tau}$ must satisfy constraint (C.1)-(C.6) by the properties of the minimum acceptance level. We next prove that $\boldsymbol{\tau}$ satisfies constraint (C.M) by contradiction. Suppose two patients have their initial scores $\xi > \xi'$, but $c(\tau_j(\xi);\xi) < c(\tau_j(\xi');\xi')$. Since $c(\cdot;\xi)$ is continuously decreasing, there exists a time $t < \tau_j(\xi)$, such that $c(t;\xi) = c(\tau_j(\xi');\xi')$. Then by $\xi > \xi'$, we deduce that $t > \tau_j(\xi')$. Since $\partial L/\partial \tau > 0$, we have

$$L(c(t;\xi),j,t) = L(c(\tau_j(\xi');\xi'),j,t) > L(c(\tau_j(\xi');\xi'),j,\tau_j(\xi')) \geq S_j \tag{51}$$

where the last inequality follows from Proposition 1. The above equality implies that at time $t < \tau_j(\xi)$, the score of patient ξ for kidney type j is already strictly greater than S_j, which contradicts with that the patient's score firstly reaches or exceeds S_j at time $\tau_j(\xi)$.

To prove the "$\supseteq$" direction, it suffices to prove that for $k = 0, 1, \ldots, J$, any $\boldsymbol{\tau} \in \mathcal{P}_k$ is a solution to the NCP, c.f., (17), and thus is an equilibrium allocation-time vector by Proposition 3. Let $y_j = \mu_j - \lambda \int_{\underline{\xi}}^{\bar{\xi}} Q_j(\xi)\bar{F}_\xi(\tau_j(\xi))\rho(\xi)\,d\xi$ and $z_j = \sup\{\tau_j(\xi) \,|\, \xi \in [\underline{\xi}, \overline{\xi}]\}$ following their definitions in the NCP. The inequality constraints $z_j \geq 0$ follows from the non-negative constraints for $\tau_j(\xi)$ in (C.6). The inequality constraint $y_j \geq 0$ follows from (C.2) for $j > k$, and (C.3) for $j = k$. For $j < k$, (C.4) implies $Q_j = 0$. Thus, $y_j = \mu_j \geq 0$. It remains to prove the complementary slackness condition $y_j z_j = 0$ for all j. For $j \leq k$, constraint (C.5) implies $z_j = 0$, which leads to the complementary slackness condition. For $j > k$, constraint (C.6) implies $y_j = 0$ and thus the complementary slackness condition. Thus, $(\tau_j(\xi))$ is an equilibrium allocation time.

We next prove that any $\boldsymbol{\tau} \in \mathcal{P}_k$ can be achieved by a score $L(c,j,\tau) = M(c,j) + \tau$ for score threshold $S_j = 0$ $(j = 1, \ldots, J)$, where $M(c,j)$ was constructed as in Equation (23) the the theorem. To that end, it suffices to show that $(\tau_j(\xi))$ solves Equation (8) for all j and ξ,. Because $S_j = 0$, Equation (8) reduces to the following equality

$$\tau_j(\xi) := \begin{cases} \bar{\tau}_\xi, & \text{if } \{\tau \geq 0 : M(c(\tau;\xi),j) + \tau \geq 0\} = \emptyset, \\ \min\{\tau \geq 0 : M(c(\tau;\xi),j) + \tau \geq 0\}, & \text{otherwise.} \end{cases} \tag{52}$$

Note that when the set is empty, i.e., a patient could never reach the threshold 0, we let his allocation time to be $\bar{\tau}_\xi$ instead of $+\infty$ to make the allocation time bounded. That does not make a difference because a patient with initial health score ξ cannot live no longer than $\bar{\tau}_\xi$.

We next prove Equation (52). It suffices to prove that $L(c(\tau_j(\xi);\xi),j,\tau_j(\xi)) \geq 0$, and $L(c(t;\xi),j,t) < 0$ for all $t < \tau_j(\xi)$. To prove the first inequality, we note that

$$L(\gamma_j(\xi),j,\tau_j(\xi)) = M(\gamma_j(\xi),j) + \tau_j(\xi) = -\inf\{\tau_j(z) \,|\, z \in \gamma_j^{-1}(\gamma_j(\xi))\} + \tau_j(\xi) \geq 0, \tag{53}$$

where the second equality follows the definition of $M(c,j)$ in Equation (23).

To prove the second inequality, we discuss the following two cases. If $c(t;\xi) \notin \mathcal{C}$, then by the definition of $M(c,j)$, we have

$$L(c(t;\xi),j,t) = M(c(t;\xi),j) + t = -\bar{\tau}_{\bar{\xi}} + t < 0, \tag{54}$$

as no patient can live longer than $\bar{\tau}_{\bar{\xi}}$. If $c(t;\xi) \in \mathcal{C}_j$, then by definition of $\mathcal{C}_j$, the set $\gamma_j^{-1}(c(t;\xi)) := \{z | \gamma_j(z) = c(t;\xi)\}$ is non-empty. For any $z \in \gamma_j^{-1}(\xi)$, since $t < \tau_j(\xi)$, we have $\gamma_j(\xi) = c(\tau_j(\xi);\xi) < c(t;\xi) = \gamma_j(z)$, that

implies $\xi < z$ by monotonicity of $\gamma_j(\cdot)$. Because $\xi < z$ but $c(t;\xi) = c(\tau_j(z);z)$, we deduce that $t < \tau_j(z)$. Therefore,

$$L(c(t;\xi),j,t) = M(c(t;\xi),j) + t < M(c(\tau_j(z);z),j) + \tau_j(z) = L(c(\tau_j(z);z),j,\tau_j(z)). \tag{55}$$

If $c(t;\xi) \notin \mathcal{K}$, then Proposition 1 implies that $L(c(\tau_j(z);z),j,\tau_j(z)) = S_j = 0$; so the above inequality implies $L(c(t;\xi),j,t) < 0$; if $c(t;\xi) = \gamma_j(z) \in \mathcal{K}$, then since the above inequality holds for all $z \in \gamma_j^{-1}(c(t;\xi))$, we have

$$L(c(t;\xi),j,t) \le \inf\{L(\gamma_j(z),j,\tau_j(z)) \,|\, z \in \gamma_j^{-1}(c(t;\xi))\} = M(\gamma_j(z),j) + \inf\{\tau_j(z) \,|\, z \in \gamma_j^{-1}(c(t;\xi))\} = 0, \tag{56}$$

where the last equality follows from $\inf\{\tau_j(z) \,|\, z \in \gamma_j^{-1}(c(t;\xi))\} = -M(c(t;\xi),j) = -M(\gamma_j(z),j)$. Therefore, regardless of $c(t;\xi) \in \mathcal{K}$ or $c(t;\xi) \notin \mathcal{K}$, we have proved that $L(c(t;\xi),j,t) < 0$ for all $t < \tau_j(\xi)$. We have thus proved that $\tau_j(\xi)$ is the first time that a patient's score reaches 0 and verified Equation (52).

It remains to prove that the score function $L(c,j,\tau) = M(c,j) + \tau$ satisfies the assumptions of a matching policy. It suffices to show that $M(c,j)$ is p.c.d. for all j, and

$$\frac{dM(c(\tau;\xi),j)}{d\tau} > -1 \text{ for all } j,\xi \text{ when } c(\tau;\xi) \notin \mathcal{K}. \tag{57}$$

To prove that $M(c,j)$ is p.c.d., for all $c \notin \mathcal{K}$, we note that $\gamma_j^{-1}(c)$ is a singleton by Lemma 3. Furthermore, since $\gamma_j(\xi)$ is left-continuous and non-decreasing in ξ, $\mathcal{C}_j$ must have the following form,

$$\mathcal{C}_j = [\underline{c}_{j1}, \overline{c}_{j1}] \cup (\underline{c}_{j2}, \overline{c}_{j2}] \cup \ldots \cup (\underline{c}_{jm}, \overline{c}_{jm}], \tag{58}$$

where m is a positive integer and the cutoff points satisfy $\bar{c}_{i-1,j} < \underline{c}_{ij} \le \bar{c}_{ij}$ for $i = 2,3,\ldots,m$. Therefore, over each interval $(\underline{c}_{j\ell}, \overline{c}_{j\ell}]$ with $\overline{c}_{j\ell} > \underline{c}_{j\ell}$ (let $(\underline{c}_{ij}, \underline{c}_{ij}] = \{\underline{c}_{ij}\}$ by abuse of notation). $d\gamma_j^{-1}(c)/dc = 1/(\gamma_j'(\gamma_j^{-1}(c)))$ exists due to the assumption that $\tau_j(\xi)$ (and therefore $\gamma_j(\xi)$) is p.c.d. in ξ. Since $\mathcal{K}$ contains finite points, $M(c,j) = \tau_j(\gamma^{-1}(c))$ is p.c.d.

We next prove condition (57). Suppose $c = c(\tau;\xi) \notin \mathcal{K}$. For each $j = 1,\ldots,J$, if $c \notin \mathcal{C}_j$, then $\partial M(c(\tau;\xi),j)/\partial c = 0$ implies $dM(c(\tau;\xi),j)/d\tau = 0$, and condition (57) is proved; if $c \in \mathcal{C}_j$, then by Lemma 3, $\gamma_j^{-1}(c)$ is a singleton. Thus, the following equation holds for all $c \in \mathcal{C}_j$,

$$c = c(\tau_j(\gamma_j^{-1}(c)) \,;\, \gamma_j^{-1}(c)) = H^{-1}\left(\tau_j(\gamma_j^{-1}(c)) + H(\gamma_j^{-1}(c))\right). \tag{59}$$

where the first equality follows from the definition of the inverse function $\gamma_j^{-1}(\cdot)$, and the second equality follows from Equation (2). We have argued earlier that the derivative $\gamma_j^{-1}(c)$ exists over each sub-interval of $\mathcal{C}_j$. Thus, by taking derivative at both sides of Equation (59), we have

$$\begin{aligned} 1 &= \tfrac{dH^{-1}(t)}{dt}\big|_{t=\tau_j(\gamma_j^{-1}(c))+c^{-1}(\gamma_j^{-1}(c))} \left(\tfrac{d\tau_j(\gamma_j^{-1}(c))}{dc} + \tfrac{dH(\gamma_j^{-1}(c))}{dc}\right) \\ &= \tfrac{1}{H'(c)}\left(\tfrac{d\tau_j(\gamma_j^{-1}(c))}{dc} + \tfrac{dH(\gamma_j^{-1}(c))}{dc}\right) \\ &> \tfrac{1}{H'(c)}\tfrac{d\tau_j(\gamma_j^{-1}(c))}{dc}. \end{aligned} \tag{60}$$

The second equality follows from that for all t and ξ, the derivative $\frac{dH^{-1}(t)}{dt}$ only depends on the patient's up-to-date health score $c = H^{-1}\left(\tau_j(\gamma_j^{-1}(c)) + H(\gamma_j^{-1}(c))\right)$. For the last inequality, we note that $\gamma_j(\cdot)$ is strictly increasing, so the inverse function $\gamma_j^{-1}(c)$ is strictly increasing in c. Since $H(\cdot)$ is strictly decreasing, we

deduce that $dH(\gamma_j^{-1}(c))/dc < 0$, which, together with $H'(c) < 0$, lead to the last inequality in (60). As a result, for all $c \in \mathcal{C}_j \backslash \mathcal{K}$, we have

$$\frac{dM(c,j)}{d\tau} = \frac{\partial M(c,j)}{\partial c} c'(\tau\,;\gamma^{-1}(c)) = -\frac{d\tau_j(\gamma_j^{-1}(c))}{dc}\frac{1}{H'(c)} > -1, \tag{61}$$

where the second equality follows from the definition of $M(c,j)$ on $c \in \mathcal{C}_j$, and the inequality follows from (60). Thus, for each $j = 1,\ldots,J$, we have proved condition (57) when $c \in \mathcal{C}_j$. This completes the proof. ■

Appendix I: An Example Showing that $\mathcal{P}_{DB}$ Cannot be Recovered by Affine Scores

Here we present an example to illustrate that the achievable region of donor-blind policies cannot be recovered by scores in the form of $\tilde{M}(C) + \tau$. Suppose two patients with initial health scores ξ^1 and ξ^2 satisfy $c(\tau_1(\xi^1);\xi^1) = c(\tau_2(\xi^2);\xi^2) \notin \mathcal{K}$, that is, the first patient, when being offered a kidney of type 1, has exactly the same health score as that of the second patient when being offered a kidney of type 2. We further assume that there are two other patients with initial health scores ξ^3 and ξ^4 such that $c(\tau_1(\xi^3);\xi^3) = c(\tau_2(\xi^4);\xi^4) \notin \mathcal{K}$. Then if the donor-blind score has a form $\tilde{M}(c) + \tau$, then by Proposition 1 we have

$$\begin{array}{rcl} S_1 &=& \tilde{M}(c(\tau_1(\xi^1);\xi^1)) + \tau_1(\xi^1) = \tilde{M}(c(\tau_1(\xi^3);\xi^3)) + \tau_1(\xi^3) \\ S_2 &=& \tilde{M}(c(\tau_2(\xi^2);\xi^2)) + \tau_1(\xi^2) = \tilde{M}(c(\tau_2(\xi^4);\xi^4)) + \tau_2(\xi^4), \end{array} \tag{62}$$

As $c(\tau_1(\xi^1);\xi^1) = c(\tau_2(\xi^2);\xi^2)$, $c(\tau_1(\xi^3);\xi^3) = c(\tau_2(\xi^4);\xi^4)$, we have the following equality

$$\tau_1(\xi^1) - \tau_1(\xi^3) = \tau_1(\xi^2) - \tau_1(\xi^4). \tag{63}$$

The above equality is not implied by any constraints in the expression of $\mathcal{P}_{DB}$. That means, if we assume the score of a donor-blind policy to take the special form of $\tilde{M}(c) + \tau$, then we have implicitly imposed an extra equality constraint (63). Thus, any allocation time that violates equality (63) cannot be achieved by the score $\tilde{M}(c) + \tau$, though it could always be achieved by a score in its general form. In fact, the above example also suggests that other special forms such as $L(c,\tau) = \tilde{M}(c) + \hat{M}(\tau)$ or $L(c,\tau) = \tilde{M}(c)\hat{M}(\tau)$ cannot recover the entire achievable region for similar reasons.

Appendix J: Numerical Procedure for Solving the Policy Design Problem

To facilitate real-time computation, we propose a finite-dimensional approximation of (31) by discretizing the domain of ξ into N grid points: $\{\ell(\overline{\xi} - \underline{\xi})/N \,|\, \ell = 0,1,\ldots,N\}$. The finite-dimensional optimization problem then searches for an $(N+1)$-by-J matrix of the allocation times: $\boldsymbol{\tau}^f := \{\tau_j(\xi_\ell) \,|\, j = 1,\ldots,J,\ \ell = 0,\ldots,N\}$, where $\xi_\ell := \ell(\overline{\xi} - \underline{\xi})/N$ and the superscript f stands for "finite-dimensional". After obtaining $\boldsymbol{\tau}^f$, we recover the continuous allocation time using linear interpolation.

We provide a finite-dimensional representation for the achievable region of the matching policies below.

$$\mathcal{A}_M^f := \cup_{k=0}^J \mathcal{P}_k^f \cap \mathcal{P}_M^f,$$

where

$$\mathcal{P}_k^f := \left\{ \boldsymbol{\tau}^f \in \mathbb{R}_+^{N+1,J} \;\middle|\; \begin{array}{ll} p_j(\xi_\ell) = \frac{1}{\mu_j}\lambda Q_j(\xi_\ell;\boldsymbol{\tau}^f)\rho(\xi_\ell)\bar{F}_\xi(\tau_j(\xi_\ell)) \text{ for all } j,\ell & \text{(Cf.1)} \\ \sum_{\ell=1}^N p_j(\xi_\ell) = 1 \text{ for } j > k & \text{(Cf.2)} \\ \sum_{\ell=1}^N p_k(\xi_\ell) \le 1 & \text{(Cf.3)} \\ Q_j(\xi_\ell;\boldsymbol{\tau}^f) = 0 \text{ for } j < k, \text{ all } \ell & \text{(Cf.4)} \\ \tau_j^f(\xi_\ell) = 0 \text{ for } j \le k, \text{ all } \ell & \text{(Cf.5)} \\ 0 \le \tau_j^f(\xi_\ell) \le \bar{\tau}_{\xi_\ell} \text{ for } j > k, \text{ all } \ell & \text{(Cf.6)} \end{array} \right\}.$$

$$\mathcal{P}_M^f := \left\{\text{p.c.d. functions } \boldsymbol{\tau} \mid c(\tau_j(\xi_\ell);\xi_\ell) \geq c(\tau_j(\xi_{\ell'});\xi_{\ell'}) \text{ for all } \ell > \ell' \qquad \text{(Cf.M)}\right\}.$$

Similarly, to derive a finite-dimensional representations for $\mathcal{A}_{HF}^f$, we just need to replace (Cf.M) with a stronger constraint as follows,

$$\mathcal{P}_{HF}^f := \left\{\text{p.c.d. functions } \boldsymbol{\tau}^f \mid \tau_j^f(\xi_\ell) \leq \tau_j^f(\xi_{\ell'}) \text{ for all } \ell > \ell' \qquad \text{(Cf.HF)}\right\}.$$

For the donor-blind policies, we need derive a finite-dimensional representation of constraint (C.DB). In particular, this calls for a finite-dimensional representation for the function $L(c(\tau;\xi),\tau)$ which has a continuous domain $\{(\xi,\tau)|\xi \in [\underline{\xi},\bar{\xi}],\ \tau \in [0,\bar{\tau}_\xi]\}$. For that purpose, we construct the grid $\{\xi_\ell|\ell = 1,\ldots,N\} \otimes \{\tau_r := r\bar{\tau}_\xi/R|r = 0,1,\ldots,R\}$ (let $\tau_0 = 0$) on the continuous domain, and represent the function using its values at the grids $\{L(c(\tau_r;\xi_\ell),\tau_r)\,|\,\ell = 1,\ldots,N,\ r = 0,1,\ldots,R\}$. We then recover its values on the continuous domain by linear interpolation. This leads to the finite-dimensional representation of (C.DB) as follows

$$\mathcal{P}_{DB}^f := \left\{ \boldsymbol{\tau}^f \in \mathbb{R}_+^{N+1,J} \;\middle|\; \begin{array}{c} \tau_j^f(\xi_\ell) := \min\{\bar{\tau}_{\xi_\ell}, \min\{\tau \geq 0 : L(c(\tau;\xi_\ell),\tau) \geq S_j\}\} \text{ for some } L(c,\tau) \\ \text{such that } L(c(\tau_{r+1}^f;\xi_\ell),\tau_{r+1}^f) - L(c(\tau_r^f;\xi_\ell),\tau_r^f) \geq \epsilon \text{ for all } \ell, r \\ L(c(\tau_r^f;\xi_\ell),\tau_{r+1}^f) - L(c(\tau_r^f;\xi_\ell),\tau_r^f) \geq \epsilon \text{ for all } \ell, r \end{array} \qquad \text{(Cf.DB)} \right\}$$

Given $\{L(c(\tau_r;\xi_\ell),\tau_r)\,|\,\ell = 1,\ldots,N,\ r = 0,1,\ldots,R\}$, we solve $\tau_j^f(\xi_\ell)$ from the first equality in (Cf.DB) as follows. First, we can find the smallest index r such that $L(c(\tau_r;\xi_\ell),\tau_r) \geq S_j$, if such an r exists; otherwise, assign $\tau_j^f(\xi_\ell) = \bar{\tau}_{\xi_\ell}$. Second, because the values of $L(c(\tau;\xi_\ell),\tau)$ on the continuous domain are assigned using linear interpolation, we can locate $\tau_j^f(\xi_\ell)$ as

$$\tau_j^f(\xi_\ell) = \tau_{r-1}^f + \frac{S_j - L(c(\tau_{r-1}^f;\xi_\ell),\tau_{r-1}^f)}{L(c(\tau_r^f;\xi_\ell),\tau_r^f) - L(c(\tau_{r-1}^f;\xi_\ell),\tau_{r-1}^f)}(\tau_r^f - \tau_{r-1}^f). \tag{64}$$

The second and third equality in (Cf.DB) provides a discrete approximation of the constraints $dL(c,\tau)/d\tau > 0$ and $\partial L(c,\tau)/\partial\tau > 0$, respectively. The parameter ϵ is set to be a small positive number, e.g., 10^{-10}, to ensure the derivatives to stay strictly positive.

Finally, we discuss how to formulate the constraint $Q_j(\xi_\ell;\boldsymbol{\tau}^f) = 0$ to facilitate the computation. If kidney type j is dominated by other kidney types for patient ξ_ℓ, then $Q_j(\xi_\ell;\boldsymbol{\tau}^f) = 0$; otherwise, $Q_j(\xi_\ell;\boldsymbol{\tau}^f)$ can be expressed according to Equation (16) using the cutoff values $(\Gamma_j(\xi_\ell;\boldsymbol{\tau}^f))$ associated with $\boldsymbol{\tau}^f$. The cutoff values $(\Gamma_j(\xi_\ell;\boldsymbol{\tau}^f))_{j=1,\ldots,J-1}$ can be computed according to the following procedure for given $\boldsymbol{\tau}^f$. First, for all $1 \leq j < j' \leq J$, we compute variables $\eta_{j,j'}$ as the unique solution to the following equation

$$c(\tau_j^f;\xi_\ell)\phi_j = \eta \int_{\tau_j^f(\xi_\ell)}^{\tau_{j'}^f(\xi_\ell)} \frac{f_{\xi_\ell}(t)}{\bar{F}_{\xi_\ell}(\tau_j^f(\xi_\ell))}(t - \tau_j^f(\xi_\ell))\,dt + \frac{\bar{F}_{\xi_\ell}(\tau_{j'}^f(\xi_\ell))}{\bar{F}_{\xi_\ell}(\tau_j^f(\xi_\ell))}(\eta(\tau_{j'}^f(\xi_\ell) - \tau_j^f(\xi_\ell)) + c(\tau_{j'}^f(\xi_\ell);\xi_\ell)\phi_{j'}). \tag{65}$$

In particular, $\eta_{j,j'}$ stands for the cutoff values at which the patient with initial health score ξ_ℓ is indifferent between accepting a kidney j or turning it down and wait for a kidney of type j'. Note that it could happen that $\eta_{j,j'} < 0$, which suggests that all patients prefer kidney type j' to j; or $\eta_{j,j} > 1$, which suggests that all patients prefer kidney type j to j'. Then starting from $j = 1$, we know for a fixed j, $\min\{\eta_{j,j'}\,|\,j' > j\}$ provides the exact cutoff such that all patients with η smaller that cutoff point prefers kidney type j then any kidney types larger than j'; and all other patients prefer some kidney type greater than j' rather than j. To ensure that Γ_j is non-decreasing in j and within the interval $[0,1]$, we let

$$\Gamma_j(\xi_\ell;\boldsymbol{\tau}^f) = \min\{1, \max\{\min\{\eta_{j,j'}\,|\,j' > j\}, \Gamma_{j-1}(\xi_\ell;\boldsymbol{\tau}^f)\}\}, \quad \text{for } j = 1,\ldots,J, \tag{66}$$

with $\Gamma_0(\xi_\ell;\boldsymbol{\tau}^f)=0$ by abuse of notation. Thus, $\Gamma_j(\xi_\ell;\boldsymbol{\tau}^f)$ and therefore $Q_j(\xi_\ell;\boldsymbol{\tau}^f)$ both have an analytical representation, which allow us to compute their sub-gradient with respect to $\boldsymbol{\tau}$ and use first-order methods to solve the policy design problem, e.g., (31) and (33).

Appendix K: Comparison to the Stochastic Setting

We simulate the stochastic waitlist system in which patients and kidneys arrive according to a homogeneous Poisson process, and patients use historical information (e.g., the average score thresholds for each kidney type in the past year) to predict their allocation times and decide whether to accept or reject an offered kidney. All the parameters in stochastic system, including $\rho(\cdot)$, $h(\cdot)$, $c(\cdot;\cdot)$, and $G_\xi(\cdot)$, take the same values as those in the fluid model. We simulate the stochastic waitlist under a matching policy and a healthier-first policy for illustration. Table 3 reports their allocation outcomes in terms of the percentage of patients in each class that transplant each type of kidneys. The reported percentages are averaged over a ten-year period after the waitlist population stops further growing, so these percentages characterize the steady-state allocation outcome. As shown in Table 3, the simulated percentages are all within 5% of those predicted by the fluid model, suggesting that the fluid model has provided accurate predictions. This justifies our model choice.

Table 3 Comparison of the Allocation Outcomes for Fluid and Stochastic Models

	Fluid Model	Simulation
Matching	$\begin{pmatrix} 0.070 & 0 & 0 & 0 \\ 0.070 & 0 & 0 & 0 \\ 0.163 & 0 & 0 & 0 \\ 0.003 & 0.181 & 0 & 0 \\ 0 & 0.296 & 0 & 0 \\ 0 & 0 & 0.811 & 0 \\ 0 & 0 & 0.344 & 0.394 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.069 & 0 & 0 & 0 \\ 0.067 & 0 & 0 & 0 \\ 0.160 & 0.001 & 0 & 0 \\ 0.012 & 0.179 & 0 & 0 \\ 0 & 0.292 & 0 & 0.016 \\ 0 & 0 & 0.815 & 0.002 \\ 0 & 0.011 & 0.337 & 0.417 \\ 0 & 0 & 0 & 0.960 \\ 0 & 0 & 0 & 0.982 \\ 0 & 0 & 0 & 0.969 \end{pmatrix}$
Healthier-First	$\begin{pmatrix} 0.002 & 0 & 0 & 0 \\ 0.028 & 0 & 0 & 0 \\ 0.184 & 0 & 0 & 0 \\ 0.050 & 0.180 & 0 & 0 \\ 0 & 0.300 & 0.115 & 0 \\ 0 & 0 & 0.597 & 0 \\ 0 & 0 & 0.406 & 0.495 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.003 & 0 & 0 & 0 \\ 0.028 & 0 & 0 & 0 \\ 0.185 & 0 & 0 & 0 \\ 0.046 & 0.169 & 0.004 & 0 \\ 0 & 0.317 & 0.129 & 0 \\ 0 & 0 & 0.629 & 0 \\ 0 & 0 & 0.400 & 0.475 \\ 0 & 0 & 0 & 0.987 \\ 0 & 0 & 0 & 0.951 \\ 0 & 0 & 0 & 0.971 \end{pmatrix}$

Appendix L: Sensitivity Analysis

We present a sensitivity analysis to validate the robustness of our results. We change the total patient arrival rate to 641.38, which is 80% of the value we used in Section 5. Keeping all other parameters the same, we use the achievable region to compute the Pareto frontier of the four policies, with the results plot in Figure 6. The plots show similar pattern as in Figure 3, which supports the robustness of our conclusions.

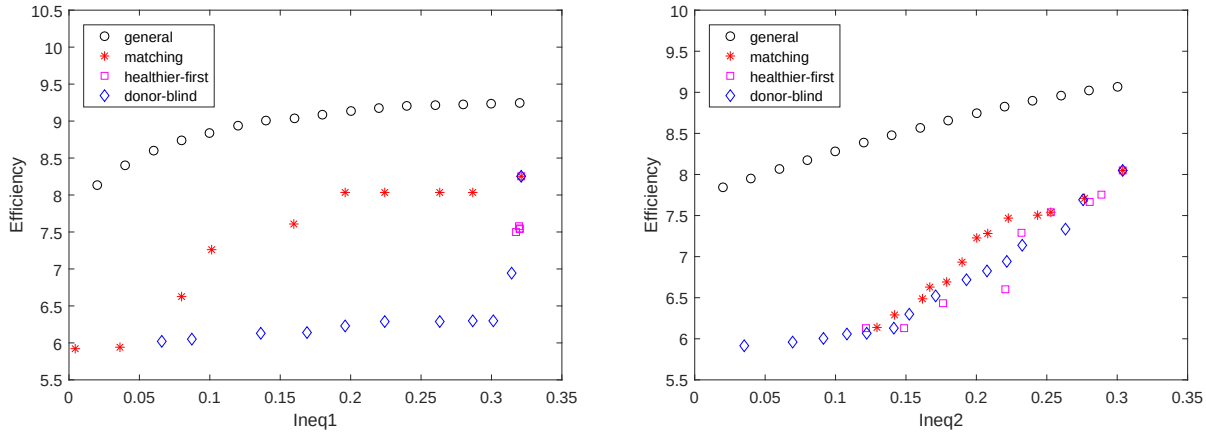

Figure 6 **The Efficiency-Equity Pareto Frontier for** $\lambda = 641.38$