

On Cournot Equilibria in Electricity Transmission Networks

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Abstract

We consider electricity pool markets in radial electricity transmission networks in which the lines have no transmission losses, but have transmission capacities. At each node there is a strategic generator submitting generation quantities to the pool. Prices are determined by a linear competitive fringe at each node. We derive necessary and sufficient conditions on the line capacities that ensure that the unconstrained one-shot Cournot equilibrium remains an equilibrium in the constrained network. These conditions are characterized by a convex polyhedral set.

1 Introduction

There has been much interest in recent years in extending the classical Nash-Cournot equilibrium for oligopolistic competition to the setting of nodal electricity pool markets. In the Cournot setting, strategic generators inject fixed quantities of power at nodes of a transmission network, and receive nodal prices at these nodes that are determined by elastic demand functions at the nodes. In most wholesale electricity markets, demand is inelastic in the short term, at least for residential consumers, and so demand elasticity is interpreted liberally in this setting. It can correspond to industrial load shedding or the generation of a competitive fringe, who offer fixed increasing supply curves at some nodes.

As outlined by Yao *et al* [5], the different classes of equilibrium models one can derive depend on the assumptions made about the rationality of the players. The most comprehensive and arguably realistic is the *full-rationality* assumption. In the transmission congestion setting this assumes that generators anticipate the effect of their generation decisions on the congestion in the network, and therefore on the payments that must be paid as rentals to the system operator. In a full-rationality model, each generator acts as a leader choosing their generation quantity simultaneously, assuming that other generation quantities are fixed, but anticipating a follower stage in which the system operator computes clearing prices that maximize the welfare of the system. These prices are the nodal prices from an appropriate dispatch model, and the gross revenue (ignoring contracts and costs) earned by a generator at a node is the product of its injection quantity and the nodal price.

The development of algorithms for computing Nash-Cournot equilibria in congested electricity transmission networks is an active area of research. A fundamental contribution in the understanding of this field was provided by the paper of Borenstein *et al* [1], which showed, in a two-node symmetric setting, that a Nash-Cournot equilibrium in pure strategies might not exist if the transmission capacity of the line joining the nodes was not sufficiently large. Borenstein *et al* also show that transmission losses affect the nature of the equilibrium in this model. In equilibrium, symmetric generators will offer less generation than they would under the unconstrained Cournot solution that pertains if the line had no losses. In asymmetric situations with congestion, full-rationality models may result in either no equilibrium, a unique equilibrium, or more than one equilibrium.

The essential difficulty in modelling this situation is the assumption that generators will act strategically to exploit congestion in the line. If this is the case then we seek an equilibrium in a *generalized Nash game*, (see e.g. [4]) in which agents can affect the feasible strategies of other agents by exploiting their ability to contribute to congestion. It is known that solutions to generalized Nash games are not guaranteed to exist, or be unique (see e.g. [5] for a discussion of this issue in an electricity market setting). This has been illustrated in models of the European market by Neuhoff *et al* [3], and in a model of the New Zealand electricity market by Downward [2].

Since the lack of existence or non-uniqueness of Nash equilibria is a serious impediment to economic analysis of electricity markets with transmission systems, most authors seeking to quantify the effects of market power in transmission networks have chosen to relax the full-rationality assumption to so-called *bounded rationality*. Here it is assumed that the system operator is itself a player in the game who makes decisions simultaneously with the generators choosing their levels of generation. Observe that this is in contrast to the more realistic full-rationality model, in which the system operator determines the optimal transmission flows after the generators have chosen their generation levels in a Stackelberg-type game where the system operator's response is taken account of in the generators' choices of strategies.

One may interpret bounded rationality as an assumption that generators will act as price-takers with respect to transmission, but price-setters at their own node. Within this framework, there are several common variations, which are explored in [5] for example. In one variation, generators assume that competitors injections and all transmission flows are fixed, and then optimize their injection. Simultaneous optimization of injections gives an equilibrium. Unfortunately, in some circumstances, this approach can give solutions that are not valid. For example, when applied to the Borenstein *et al* example of symmetric generators at opposite ends of an uncongested line, the bounded rationality approach would yield two local monopolies as opposed to the symmetric Cournot duopoly solution. Alternative approaches (in which generators assume nodal price premiums are fixed rather than transmission quantities) will overcome this problem, but have other limitations, such as yielding the symmetric Cournot duopoly solution for the example in [1] when the line is congested.

In this paper we take a slightly different approach to previous work. We retain the full rationality assumption, but seek to derive conditions on the capacities of the transmission lines (assuming no losses) that will ensure that the unconstrained Nash-Cournot equilibrium is also a generalized Nash equilibrium for these capacities. For linear demand curves and constant marginal generation costs, we show in the case that the network has a radial structure that the set of arc capacities supporting an unconstrained Nash-Cournot equilibrium is a convex polyhedral set, which we call the *competitive capacity set*.

This result is significant for several reasons. Primarily, electricity transmission grid owners are concerned with providing grid capacity to enable competition between geographically separate agents. As demonstrated by [1], these transmission lines can have a beneficial effect even if they contain no flow in equilibrium. The results in this paper provide explicit bounds on the capacities of the lines so as to yield the maximum benefit in this respect. In other words, if the line capacities lie in the competitive capacity set then expanding any of them will yield no further competitive benefits. Moreover, as electricity demand and generation grow, the competitive capacity set will change. Since there are many capacity vectors that lie in the competitive capacity set, it makes sense to seek a plan of capacity expansion comprising a sequence of such vectors (ideally one that maximizes the welfare of the plan). Finally, an understanding of the boundary of the competitive capacity set enables us to shed some light on the circumstances in which bounded rationality models yield the correct answers when applied to full rationality settings.

The paper is laid out as follows. In the next section, we formulate the version of the economic dispatch problem we work with and give its optimality conditions. We then use these conditions in section 3 to establish a key property of the residual demand curve faced by a generator at a given node. In section 4, this property is used to derive conditions on the line capacities that guarantee an unconstrained Nash-Cournot equilibrium.

2 The Economic dispatch problem

We shall consider a radial (tree) network of nodes $i \in \mathcal{N}$, and directed lossless links denoted $ij \in \mathcal{A}$, where i is the tail node and j is the head node. The flow on link ij is denoted f_{ij} and the capacity of link ij is denoted K_{ij} . At each node i there is a known demand d_i and a competitive fringe defined by a linear supply function $S_i(p) = a_i p$. Observe that this is equivalent to assuming a linear demand function of the form

$$D_i(p) = d_i - a_i p.$$

We assume at each node, i , that there is a single generator with marginal cost c_i which injects power q_i . We denote by x_i the dispatch of the competitive fringe. The economic dispatch problem is then

$$\begin{aligned}
P(q) := \min \quad & \sum_{i \in \mathcal{N}} \frac{1}{2a_i} x_i^2 \\
\text{s.t.} \quad & -x_i + \sum_{j, ij \in \mathcal{A}} f_{ij} - \sum_{j, ji \in \mathcal{A}} f_{ji} = q_i - d_i \quad [\pi_i] \quad \forall i \in \mathcal{N} \\
& -K_{ij} \leq f_{ij} \leq K_{ij} \quad [\eta_{ij}^1, \eta_{ij}^2] \quad \forall ij \in \mathcal{A}
\end{aligned}$$

The optimality conditions of the above dispatch is given by the following:

$$\begin{aligned}
& -x_i + \sum_{j, ij \in \mathcal{A}} f_{ij} - \sum_{j, ji \in \mathcal{A}} f_{ji} = q_i - d_i \quad \forall i \in \mathcal{N} \\
& \frac{1}{a_i} x_i - \pi_i = 0 \quad \forall i \in \mathcal{N} \\
& \pi_i - \pi_j + \eta_{ij}^1 - \eta_{ij}^2 = 0 \quad \forall ij \in \mathcal{A} \\
& \eta_{ij}^1 (f_{ij} - K_{ij}) = 0 \quad \forall ij \in \mathcal{A} \\
& \eta_{ij}^2 (f_{ij} + K_{ij}) = 0 \quad \forall ij \in \mathcal{A} \\
& \eta_{ij}^1, \eta_{ij}^2 \geq 0 \quad \forall ij \in \mathcal{A} \\
& -K_{ij} \leq f_{ij} \leq K_{ij} \quad \forall ij \in \mathcal{A}
\end{aligned} \tag{1}$$

The following lemmas are simple consequences of these optimality conditions.

Lemma 2.1. *Suppose that for a set of vectors x , f and π , we have that:*

1. *the node balance equations of $P(q)$ are satisfied by x and f ,*
2. *$|f| \leq K$,*
3. *and $\pi_i = \frac{1}{a_i} x_i \quad \forall i \in \mathcal{N}$.*

Then (x, f, π) is a solution of the economic dispatch problem if f and π satisfy:

4. *if $|f_{ij}| < K_{ij}$, then $\pi_i = \pi_j$,*
5. *if $f_{ij} = K_{ij}$, then $\pi_i \leq \pi_j$,*
6. *and if $-f_{ij} = K_{ij}$, then $\pi_i \geq \pi_j$.*

Furthermore, if (x, f, π) satisfy

7. *$\pi_i < \pi_j$ then $f_{ij} = K_{ij}$,*
8. *$\pi_i > \pi_j$ then $f_{ij} = -K_{ij}$,*
9. *$\pi_i = \pi_j$ then $|f_{ij}| \leq K_{ij}$*

then (x, f, π) is a solution to $P(q)$.

Proof. From the optimality conditions, it can be shown that:

4. if $|f_{ij}| < K_{ij}$, then from (1) we have that the associated multipliers η_{ij}^1 and η_{ij}^2 must be zero. This yields $\pi_i = \pi_j$,
5. if $f_{ij} = K_{ij}$, then $\eta_{ij}^2 = 0$ and $\pi_i - \pi_j + \eta_{ij}^1 = 0$, where $\eta_{ij}^1 \geq 0$, hence $\pi_i \leq \pi_j$,
6. similarly, if $-f_{ij} = K_{ij}$, then from (1) $\pi_i - \pi_j - \eta_{ij}^2 = 0$, where $\eta_{ij}^2 \geq 0$ yielding $\pi_i \geq \pi_j$.

Furthermore, suppose that f and π satisfy conditions (7-9) from the lemma statement. for each arc ij , let

$$\eta_{ij}^1 = \begin{cases} \pi_j - \pi_i & \text{if } f_{ij} = K_{ij} \\ 0 & \text{otherwise} \end{cases}$$

and let

$$\eta_{ij}^2 = \begin{cases} \pi_i - \pi_j & \text{if } f_{ij} = -K_{ij} \\ 0 & \text{otherwise.} \end{cases}$$

Now (x, f, π, η) (1) and we have a solution by construction. \square

Lemma 2.2. *Suppose that problem $P(q)$ is solved to optimality for injections q . Suppose $f_{ij} = K_{ij}$, $ij \in \mathcal{F}$, and $f_{ij} = -K_{ij}$, $ij \in \mathcal{G}$. Now let \hat{q} be a new set of injections, augmented by fixed flows f_{ij} , $ij \in \mathcal{F} \cup \mathcal{G}$. If the optimal solution to $P(\hat{q})$ for each connected component of $\mathcal{T} \setminus (\mathcal{F} \cup \mathcal{G})$ gives π with $\pi_i \leq \pi_j$, $ij \in \mathcal{F}$, $\pi_i \geq \pi_j$, $ij \in \mathcal{G}$, then these solutions together with f_{ij} , $ij \in \mathcal{F} \cup \mathcal{G}$ solve $P(q)$ for injections \hat{q} .*

Proof. By construction the solution is easily shown to satisfy the optimality conditions for each link in $\mathcal{T} \setminus (\mathcal{F} \cup \mathcal{G})$. The remaining optimality conditions pertain to links in $\mathcal{F} \cup \mathcal{G}$, which hold by assumption. \square

3 Residual demand curves in radial networks

In this section, we establish some properties of the residual demand curve faced by a generator at some arbitrary but fixed node, n . To do this, it is convenient to adopt the convention throughout this section that all arcs in the radial network are directed towards n . We first establish that the residual demand curve faced by each generator is piecewise linear. Furthermore, if q^U is the vector of injections at the unconstrained Nash-Cournot equilibrium (where the flows resulting from q^U are feasible for the network), then the residual demand curve faced by the generator at n is convex for all $q_n < q_n^U$ (and concave for all $q_n > q_n^U$). (See Figure 2.)

To establish this result, we consider a “decomposition scheme” for the network. A *decomposition* δ for node n is determined by choosing a subtree \mathcal{T}_δ of the network rooted at n . We denote the nodes within \mathcal{T}_δ by \mathcal{N}_δ . The network is therefore decomposed into \mathcal{T}_δ and several other subtrees, each rooted at the tail node of an arc with its head node in \mathcal{N}_δ . We denote by \mathcal{B}_δ the set of arcs that link these subtrees. (See Figure 1.)

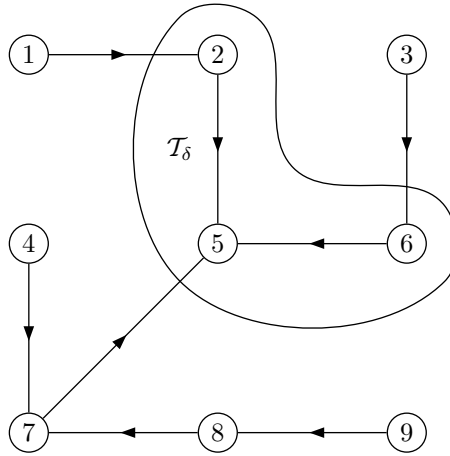


Figure 1: An example of a decomposition. Note that $\mathcal{N}_\delta = \{2, 5, 6\}$

For each node n , we denote by \mathcal{D}_n the set of all decompositions pertaining to n . Given a decomposition $\delta \in \mathcal{D}_n$ and a vector of injections q , we compute nodal prices π^δ by setting $f_{ij} = K_{ij}$ for arcs $ij \in \mathcal{B}_\delta$, and solving the dispatch problem for \mathcal{T}_δ , including these flows but ignoring arc capacity constraints for the arcs within \mathcal{T}_δ . (Note that the prices computed this way will not necessarily result in a corresponding set of feasible flows.)

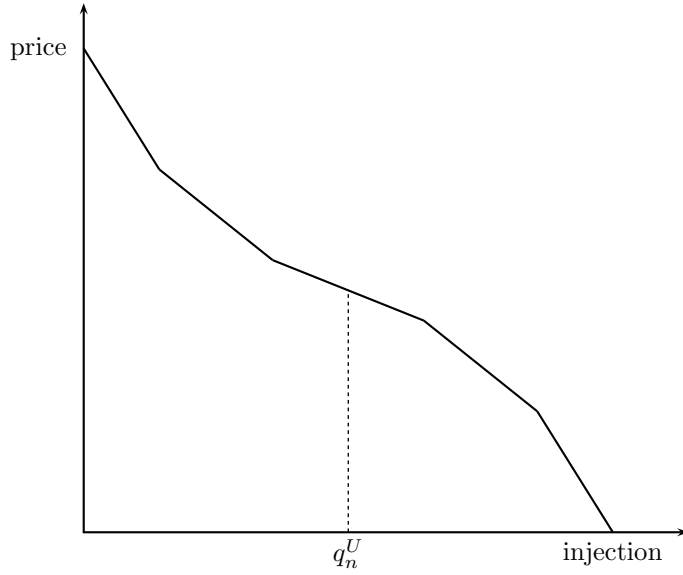


Figure 2: Residual Demand Curve for Player n

Lemma 3.1. *Given decomposition $\delta \in \mathcal{D}_n$, the nodal price at all nodes in \mathcal{N}_δ , is given by:*

$$\pi^\delta = \frac{\sum_{i \in \mathcal{N}_\delta} d_i - \sum_{i \in \mathcal{N}_\delta} q_i - \sum_{ij \in \mathcal{B}_\delta} K_{ij}}{\sum_{i \in \mathcal{N}_\delta} a_i}.$$

Proof. The dispatch problem for \mathcal{T}_δ defined by the decomposition $\delta \in \mathcal{D}_n$ is

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{N}_\delta} \frac{1}{2a_i} x_i^2 \\ \text{s.t.} \quad & - \sum_{i \in \mathcal{N}_\delta} x_i = \sum_{i \in \mathcal{N}_\delta} q_i - \sum_{i \in \mathcal{N}_\delta} d_i + \sum_{ij \in \mathcal{B}_\delta} K_{ij} \quad [\pi^\delta] \end{aligned}$$

Note that at optimality we must have,

$$\frac{1}{a_i} x_i - \pi^\delta = 0 \quad \forall i \in \mathcal{N}_\delta. \quad (2)$$

If we sum over the equation group (2) we obtain

$$\sum_{i \in \mathcal{N}_\delta} x_i - \pi^\delta \sum_{i \in \mathcal{N}_\delta} a_i = 0,$$

whereby

$$\pi^\delta = \frac{\sum_{i \in \mathcal{N}_\delta} x_i}{\sum_{i \in \mathcal{N}_\delta} a_i} = \frac{\sum_{i \in \mathcal{N}_\delta} d_i - \sum_{i \in \mathcal{N}_\delta} q_i - \sum_{ij \in \mathcal{B}_\delta} K_{ij}}{\sum_{i \in \mathcal{N}_\delta} a_i}$$

as required. \square

Corollary 3.2. *Suppose two distinct radial networks with node sets \mathcal{N}_1 and \mathcal{N}_2 have optimal dispatches with no congested lines, and node $i \in \mathcal{N}_1$ has demand d and price π_i while node $j \in \mathcal{N}_2$ has demand $-d$ and price $\pi_j < \pi_i$. Then connecting \mathcal{N}_1 and \mathcal{N}_2 with a line of infinite capacity gives a new price $\pi'_j > \pi_j$.*

Proof. To simplify notation let

$$D_i = \sum_{k \in \mathcal{N}_1 \setminus \{i\}} d_k - \sum_{k \in \mathcal{N}_1} q_k,$$

and

$$D_j = \sum_{k \in \mathcal{N}_2 \setminus \{j\}} d_k - \sum_{k \in \mathcal{N}_2} q_k.$$

Since $\pi_i > \pi_j$ we have

$$\frac{D_i + d}{\sum_{k \in \mathcal{N}_1} a_k} > \frac{D_j - d}{\sum_{k \in \mathcal{N}_2} a_k}.$$

Thus

$$\begin{aligned} \left(\sum_{k \in \mathcal{N}_2} a_k \right) (D_i + D_j) - \left(\sum_{k \in \mathcal{N}_2} a_k \right) (D_j - d) &= \left(\sum_{k \in \mathcal{N}_2} a_k \right) (D_i + d) \\ &> \left(\sum_{k \in \mathcal{N}_1} a_k \right) (D_j - d), \end{aligned}$$

yielding

$$\left(\sum_{k \in \mathcal{N}_2} a_k \right) (D_i + D_j) > \left(\sum_{k \in \mathcal{N}_1} a_k \right) (D_j - d) + \left(\sum_{k \in \mathcal{N}_2} a_k \right) (D_j - d).$$

Dividing by $\sum_{k \in \mathcal{N}_1} a_k$ and $\sum_{k \in \mathcal{N}_1 \cup \mathcal{N}_2} a_k$ gives

$$\pi'_j = \frac{D_i + D_j}{\sum_{k \in \mathcal{N}_1 \cup \mathcal{N}_2} a_k} > \frac{D_j - d}{\sum_{k \in \mathcal{N}_1} a_k}$$

as required. \square

As the injection q_n decreases from q_n^U , we show that arcs only ever congest towards node n . Therefore one of the decompositions, above, will provide the optimal solution to the economic dispatch problem. We prove in Theorem 3.4 that this decomposition is one with price $\max_{\delta \in \mathcal{D}_n} \pi_n^\delta$ where π_n^δ is the price at node n with injection q_n under the decomposition scheme $\delta \in \mathcal{D}_n$. Therefore the convex part of the residual demand curve faced by generator n is the upper envelope of a finite number of linear curves.

Lemma 3.3. *Suppose for some vector q^U that no line is congested at the optimal solution to problem $P(q)$. Suppose now that for some node n , q_n is decreased. Then as q_n decreases the flow in every line is non-decreasing in the direction of node n .*

Proof. Note that initially the network contains no congested lines. The price at node n is a decreasing function of the injection at node n . Therefore, at least until any flows reach their bounds, as q_n decreases the nodal prices at every node increase uniformly. Since at optimality

$$\pi_i = \frac{1}{a_i} x_i$$

each fringe dispatch x_i increases as q_n decreases, by a total amount that equals the change in q_n . Because the network has a radial structure the change in line flows are unique and non-decreasing.

If any flow f_{ij} hits a bound K_{ij} then this defines a decomposition $\delta \in \mathcal{D}_n$ with $ij \in \mathcal{B}_\delta$. For $j \notin \mathcal{N}_\delta$, we fix π_j at its current value. As q_n decreases further, then π_j increases to π_j^δ , for $j \in \mathcal{N}_\delta$. It is easy to see from Lemma 2.2 that this process yields an optimal solution to the dispatch problem for every $q_n < q_n^U$ for which it is feasible, and that the flow in every line is non-increasing as q_n decreases. \square

The results above show that the price at node n is a non-increasing piecewise linear function of q_n . We now show that this is convex in the region $q_n < q_n^U$ and concave in the region $q_n > q_n^U$.

Theorem 3.4. *Suppose that for some vector of injections q that all lines are uncongested at the optimal dispatch. Suppose now that q_n , the injection at node n , is decreased. Then the nodal price at node n , for any value of q_n is given by $\pi_n^* = \max_{\delta \in \mathcal{D}_n} \pi_n^\delta$, where π^δ is defined by Lemma 3.1.*

Proof. As there are only a finite number of decompositions of the type described in \mathcal{D}_n , we know that the maximum nodal price π_n^* is attained for some decomposition. Furthermore since the economic dispatch problem is a strictly convex quadratic program it has a unique optimal primal solution, and so there is a unique solution π satisfying (1). We will proceed by demonstrating that any decomposition $\delta \in \mathcal{D}$ with $\pi_n^\delta = \pi_n^*$ satisfies (1). We will therefore have that the (unique) solution to (1) will satisfy $\pi_n = \pi_n^*$.

Suppose that we have a decomposition $\delta \in \mathcal{D}_n$ with $\pi_n^\delta = \pi_n^*$. By construction, we know that within each subtree of \mathcal{D}_n , the conditions (1) are satisfied. Therefore if the solution defined by δ does not satisfy (1) then it must be the case that either

1. an arc capacity connecting 2 nodes in \mathcal{N}_δ is exceeded, or
2. there is a congested arc $ij \in \mathcal{B}_\delta$ where $\pi_i > \pi_j$.

For (1), suppose that $f_{lm} > K_{lm}$ where $lm \in \mathcal{T}_\delta$. Removing lm from \mathcal{T}_δ creates a new decomposition δ' with $\mathcal{N}_{\delta'}$ consisting of the node set containing n , and setting $f_{lm} = K_{lm}$. The net flow into $\mathcal{N}_{\delta'}$ is thus smaller than it was for decomposition δ , and so the price at n in the new decomposition is greater than π_n^δ which violates the assumption that π_n^δ is maximal over all decompositions.

Now consider case (2). Here again we will create a new decomposition where the price at node n will increase from π_n^δ . Define $\tilde{\mathcal{N}}_i$ by the set of nodes in $\mathcal{T} \setminus \mathcal{N}_\delta$ that are connected to node i via an uncongested path, and construct the decomposition γ with $\mathcal{N}_\gamma = \tilde{\mathcal{N}}_j \cup \mathcal{N}_\delta$. Since $\pi_i > \pi_j$, is easy to show from Corollary 3.2 that $\pi_n^\gamma > \pi_n^\delta$, which violates the assumption that π_n^δ is maximal over all decompositions. \square

Theorem 3.5. *Suppose that for some vector of injections q^U , the arcs are uncongested at the optimal dispatch for the economic dispatch problem. The residual demand curve faced by the generator at node i is a convex piecewise linear function.*

Proof. The proof is a direct consequence of Theorem 3.4, and the fact that the upper envelope of a set a linear functions is a convex function. \square

Note that so far we have established that the residual demand curve, faced by the generator located at node n , for any n in our radial network, is a convex, piecewise linear curve in all quantities $q_n \leq q_n^U$. The analogs of all of the above technical arguments will hold by symmetry and we can obtain that for all $q_n > q_n^U$, the residual demand curve, faced by the generator located at node n , is piecewise linear and concave (see Figure 2).

4 Competitive play

4.1 Uncongested Cournot Equilibrium

In absence of arc capacities in the network a unique Nash-Cournot equilibrium will exist. This is found when each player's optimality conditions hold simultaneously, with respect to their own injection. Recall that we assumed the constant marginal cost of c_n for each generator. The profit, ρ_n , of an arbitrary player, n , is given by

$$\rho_n = q_n (\pi - c_n).$$

Recall that $\pi = \frac{\sum_{i \in \mathcal{N}} d_i - \sum_{i \in \mathcal{N}} q_i}{\sum_{i \in \mathcal{N}} a_i}$. Then

$$\rho_n = q_n \left(\frac{\sum_{i \in \mathcal{N}} d_i - \sum_{i \in \mathcal{N}} q_i}{\sum_{i \in \mathcal{N}} a_i} - c_n \right) \quad (3)$$

Note that ρ_n , above, is a concave function of q_n , therefore q_n^* , the maximizer of the profit is found from solving $\frac{\partial \rho_n}{\partial q_n} = 0$;

$$\begin{aligned}\frac{\partial \rho_n}{\partial q_n}(q_n^*) &= \frac{\sum_{i \in \mathcal{N}} d_i - 2q_n^* - \sum_{i \in \mathcal{N}, i \neq n} q_i}{\sum_{i \in \mathcal{N}} a_i} - c_n = 0 \\ q_n^* &= \frac{1}{2} \left(\sum_{i \in \mathcal{N}} d_i - \sum_{i \in \mathcal{N}, i \neq n} q_i - c_n \sum_{i \in \mathcal{N}} a_i \right)\end{aligned}\quad (4)$$

To find the Nash-Cournot equilibrium, all players must be simultaneously maximizing their own profit, i.e. $q_n = q_n^* \quad \forall n \in \mathcal{N}$. We now sum over the above equation for all n , to give

$$\sum_{i \in \mathcal{N}} q_i^* = \frac{1}{2} \left(|\mathcal{N}| \sum_{i \in \mathcal{N}} d_i - (|\mathcal{N}| - 1) \sum_{i \in \mathcal{N}} q_i^* - \sum_{i \in \mathcal{N}} c_i \sum_{i \in \mathcal{N}} a_i \right).$$

Thus

$$(|\mathcal{N}| + 1) \sum_{i \in \mathcal{N}} q_i^* = |\mathcal{N}| \sum_{i \in \mathcal{N}} d_i - \sum_{i \in \mathcal{N}} c_i \sum_{i \in \mathcal{N}} a_i,$$

whereby

$$\sum_{i \in \mathcal{N}} q_i^* = \frac{|\mathcal{N}| \sum_{i \in \mathcal{N}} d_i - \sum_{i \in \mathcal{N}} c_i \sum_{i \in \mathcal{N}} a_i}{|\mathcal{N}| + 1}.\quad (5)$$

Solving for a specific player n 's injection, we obtain the (uncongested) Nash-Cournot equilibrium injection q_n^U . Thus

$$\frac{\partial \rho_n}{\partial q_n}(q_n^*) = \frac{\sum_{i \in \mathcal{N}} d_i - q_n^* - \sum_{i \in \mathcal{N}} q_i^*}{\sum_{i \in \mathcal{N}} a_i} - c_n = 0,$$

whereby

$$\begin{aligned}q_n^U = q_n^* &= \sum_{i \in \mathcal{N}} d_i - \frac{|\mathcal{N}| \sum_{i \in \mathcal{N}} d_i - \sum_{i \in \mathcal{N}} c_i \sum_{i \in \mathcal{N}} a_i}{|\mathcal{N}| + 1} - \sum_{i \in \mathcal{N}} a_i c_n \\ &= \frac{\sum_{i \in \mathcal{N}} d_i + \sum_{i \in \mathcal{N}} c_i \sum_{i \in \mathcal{N}} a_i}{|\mathcal{N}| + 1} - c_n \sum_{i \in \mathcal{N}} a_i.\end{aligned}\quad (6)$$

The uncongested Nash-Cournot equilibrium (6) is the most competitive equilibrium for this type of game.

4.2 Existence of Uncongested Nash Cournot

In a game with strategic generators choosing quantities, and competing over a network, a set of conditions, ensuring the existence of the uncongested Nash-Cournot equilibrium can be found. Recall that we denote the price associated with the uncongested Nash-Cournot equilibrium by π^U and we denote player n 's equilibrium injection quantity by q_n^U .

There is incentive to deviate from the Nash-Cournot equilibrium if there exists any player, n , which can increase its profit by changing only its own injection. First, note that because of the shape of the residual demand curve faced by the player (see Figure 2,) there is no incentive to deviate to a quantity higher than q_n^U . Therefore we only need to consider $q_n < q_n^U$.

To this end, we impose constraints (on the line capacities,) that ensure that the profit obtained from the uncongested Cournot quantity is greater than (or equal to,) profits obtained from any other quantity by considering all decompositions $\delta \in \mathcal{D}_n$. Each decomposition δ , has an associated residual demand curve. We consider the residual demand curve associated to δ and find the point $(q_n^{\delta*}, \pi_n^{\delta*})$ that maximizes the profit

ρ_n^δ , for the generator at node n . If we now impose conditions that guarantee $\rho_n^\delta \leq \rho_n^U$, then there will be no incentive for generator n to deviate from the Cournot equilibrium. We impose these conditions considering all decompositions and all generators.

Deviation to decomposition δ gives the the following profit function.

$$\begin{aligned}\rho_n^\delta &= q_n (\pi^\delta - c_n) \\ &= q_n \left(\frac{\sum_{i \in \mathcal{N}_\delta} d_i - \sum_{i \in \mathcal{N}_\delta} q_i^U - \sum_{ij \in \mathcal{B}_\delta} K_{ij}}{\sum_{i \in \mathcal{N}_\delta} a_i} - c_n \right).\end{aligned}$$

Maximizing this profit with respect to q_n gives

$$\frac{\partial \rho_n^\delta}{\partial q_n} (q_n^{\delta*}) = \frac{\sum_{i \in \mathcal{N}_\delta} d_i - 2q_n^{\delta*} - \sum_{i \in \mathcal{N}_\delta, i \neq n} q_i^U - \sum_{ij \in \mathcal{B}_\delta} K_{ij}}{\sum_{i \in \mathcal{N}_\delta} a_i} - c_n = 0,$$

implying that

$$q_n^{\delta*} = \frac{1}{2} \left(\sum_{i \in \mathcal{N}_\delta} d_i - \sum_{i \in \mathcal{N}_\delta, i \neq n} q_i^U - \sum_{ij \in \mathcal{B}_\delta} K_{ij} - c_n \sum_{i \in \mathcal{N}_\delta} a_i \right). \quad (7)$$

Solving for $\pi_n^{\delta*}$ gives

$$\begin{aligned}\pi_n^{\delta*} &= \frac{\sum_{i \in \mathcal{N}_\delta} d_i - q_n^{\delta*} - \sum_{i \in \mathcal{N}_\delta, i \neq n} q_i^U - \sum_{ij \in \mathcal{B}_\delta} K_{ij}}{\sum_{i \in \mathcal{N}_\delta} a_i} \\ &= \frac{q_n^{\delta*}}{\sum_{i \in \mathcal{N}_\delta} a_i} + c_n.\end{aligned} \quad (8)$$

Solving for the deviation profit gives

$$\rho_n^{\delta*} = \frac{q_n^{\delta*2}}{\sum_{i \in \mathcal{N}_\delta} a_i}. \quad (9)$$

There is no incentive to deviate if

$$\rho_n^U \geq \rho_n^{\delta*}$$

whereby by (9)

$$\rho_n^U \geq \frac{q_n^{\delta*2}}{\sum_{i \in \mathcal{N}_\delta} a_i}.$$

With (7) this gives

$$\sqrt{\rho^U \sum_{i \in \mathcal{N}_\delta} a_i} \geq \frac{1}{2} \left(\sum_{i \in \mathcal{N}_\delta} d_i - \sum_{i \in \mathcal{N}_\delta, i \neq n} q_i^U - \sum_{ij \in \mathcal{B}_\delta} K_{ij} - c_n \sum_{i \in \mathcal{N}_\delta} a_i \right).$$

This yields the following inequality on the line capacities K_{ij} ,

$$\sum_{ij \in \mathcal{B}_\delta} K_{ij} \geq \sum_{i \in \mathcal{N}_\delta} d_i - \sum_{i \in \mathcal{N}_\delta, i \neq n} q_i^U - c_n \sum_{i \in \mathcal{N}_\delta} a_i - 2 \sqrt{\rho^U \sum_{i \in \mathcal{N}_\delta} a_i}. \quad (10)$$

If we impose inequalities analogous to (10) for all nodes n and for all possible decompositions $\delta \in \mathcal{D}_n$, we have a set of sufficient conditions that guarantee the existence of the unconstrained Nash-Cournot equilibrium. We term the set of arc capacities that satisfy these inequalities the competitive capacity set.

This set of constraints is also necessary. That is, for any combination of arc capacities $\{K_{ij}\}_{ij \in \mathcal{A}}$ that lie outside the competitive capacity set, there exists a generator n who has an incentive to deviate from the uncongested Cournot equilibrium. To see this, observe that if any inequality in (10) is violated, then there is a generator n and a decomposition $\delta \in \mathcal{D}_n$ giving

$$\rho_n^\delta = q_n^{\delta*} (\pi_n^\delta - c_n) > \rho_n^U.$$

Now by Theorem 3.4 an injection of $q_n^\delta < q_n^U$ yields a price

$$\pi_n^* = \max_{\delta \in \mathcal{D}_n} \pi_n^\delta,$$

and so the profit actually made by generator n injecting q_n^δ (while others inject q_i^U) is

$$\rho_n = q_n^{\delta*} (\pi_n^* - c_n) \geq \rho_n^\delta > \rho_n^U,$$

showing that generator n can deviate profitably from q_n^U .

Hence (10) gives an explicit description of the set of line capacities that support an uncongested Cournot equilibrium. Some open questions remain to be resolved, for example when generators are subject to capacity constraints and more general cost functions, or the competitive fringe is nonlinear. It is worth noting in the current case that since it is described by a system of linear inequalities in the arc capacities our competitive capacity set is a convex set. This makes it attractive to use in optimization models. For example, one might seek an optimal transmission capacity expansion plan to cater for supporting competitive equilibria when faced with growth in demand.

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