Recovering Best Statistical Guarantees via the Empirical Divergence-based Distributionally Robust Optimization

Henry Lam

Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, MI 48109, khlam@umich.edu

We investigate the use of distributionally robust optimization (DRO) as a tractable tool to recover the asymptotic statistical guarantees provided by the Central Limit Theorem, for maintaining the feasibility of an expected value constraint under ambiguous probability distributions. We show that using empirically defined Burg-entropy divergence balls to construct the DRO can attain such guarantees. These balls, however, are not reasoned from the standard data-driven DRO framework since by themselves they can have low or even zero probability of covering the true distribution. Rather, their superior statistical performances are endowed by linking the resulting DRO with empirical likelihood and empirical processes. We show that the sizes of these balls can be optimally calibrated using χ^2 -process excursion. We conduct numerical experiments to support our theoretical findings.

Key words: distributionally robust optimization, empirical likelihood, empirical process, chi-square process, central limit theorem

1. Statistical Motivation of Distributionally Robust Optimization

We consider an expected value constraint in the form

$$Z_0(x) := E_0[h(x;\xi)] \le 0 \tag{1}$$

where $\xi \in \Xi$ is a random object under the probability measure P_0 , $E_0[\cdot]$ denotes the corresponding expectation, $x \in \Theta \subset \mathbb{R}^m$ is the decision variable, and h is a known function. The generic constraint (1) has appeared in various applications such as resource allocation (Atlason et al. (2004)), risk management (Krokhmal et al. (2002), Fábián (2008)), among others.

In practice, the probability measure P_0 is often unknown, but rather is observed via a finite collection of data. Such uncertainty has been considered in the stochastic and the robust optimization literature. Our main goal in this paper is to investigate, in a statistical sense, the *best* data-driven reformulation of (1) in terms of feasibility guarantees.

1.1. Initial Attempt: Sample Average Approximation

To define what "best" means, we start by discussing arguably the most natural attempt for handling (1), namely the sample average approximation (SAA) (Shapiro et al. (2014), Wang and Ahmed

(2008), Kleywegt et al. (2002)). Suppose we have i.i.d. data ξ_1, \ldots, ξ_n . SAA entails replacing the unknown expectation $Z_0(x)$ with the sample average $(1/n) \sum_{i=1}^n h(x;\xi_i)$, leading to

$$\hat{h}(x) := \frac{1}{n} \sum_{i=1}^{n} h(x; \xi_i) \le 0$$
(2)

The issue with naively using SAA in this setting is that a solution feasible according to (2) may be mistakeably infeasible for (1). Since for any x the true mean $Z_0(x)$ can lie above or below its sample average, both with substantial probabilities, the x's close to the boundary of the feasible region according to (2) could, with overwhelming probabilities, be infeasible for the original constraint (1). Consequently, the probability

$$P\left(\hat{h}(x) \le 0 \ \Rightarrow \ Z_0(x) \le 0\right)$$

where P is with respect to the generation of data, can be much lower than an acceptable level.

One way to boost the confidence of SAA is to insert a margin, namely by using the constraint

$$\hat{h}(x) + \epsilon_n \le 0 \tag{3}$$

This idea has appeared in various contexts (e.g., Wang and Ahmed (2008), Nagaraj and Pasupathy (2014)). Choosing $\epsilon_n > 0$ suitably can guarantee that

$$P\left(\hat{h}(x) + \epsilon_n \le 0 \implies Z_0(x) \le 0\right) \ge 1 - \alpha \tag{4}$$

where $1 - \alpha$ is a prescribed confidence level chosen by the modeler (a typical choice is $\alpha = 0.05$). This is achieved by finding ϵ_n such that

$$P\left(Z_0(x) \le \hat{h}(x) + \epsilon_n \text{ for all } x \in \Theta\right) \ge 1 - \alpha$$
(5)

Such a choice of ϵ_n can be obtained in terms of the maximal variance of $h(x;\xi)$ over all $x \in \Theta$, and other information such as the diameter of the space Θ (e.g., Wang and Ahmed (2008) provides one such choice).

1.2. The Statistician's Approach: Confidence Bounds from the Central Limit Theorem

Though (5) could provide a good feasibility guarantee, the use of one single number ϵ_n as the margin adjustment may unnecessarily penalize x whose $h(x;\xi)$ bears only a small variation. From a "classical statistician"'s viewpoint, we adopt a margin adjustment that takes into account the variability of $h(x;\xi)$ at each point of x, and at the same time provides a $1 - \alpha$ confidence guarantee, by formulating the constraint as

$$\hat{h}(x) + z \frac{\hat{\sigma}(x)}{\sqrt{n}} \le 0 \tag{6}$$

where z is the critical value of a suitable sampling distribution, and $\hat{\sigma}(x)$ is an estimate of $\sqrt{Var_0(h(x;\xi))}$ ($Var_0(\cdot)$ denotes the variance under P_0), i.e., $\hat{\sigma}(x)/\sqrt{n}$ is the standard error. A judicious choice of z can lead to the *asymptotically exact* guarantee

$$\lim_{n \to \infty} P\left(Z_0(x) \le \hat{h}(x) + z \frac{\hat{\sigma}(x)}{\sqrt{n}} \text{ for all } x \in \Theta\right) = 1 - \alpha$$
(7)

Without making further assumption on the optimization objective, we set the reformulation (6) and the guarantee (7) as our benchmark in this paper, since they stem from the central limit theorem (CLT) widely used in statistics.

The problem with directly using (6) is that (sample) standard deviation is not a tractabilitypreserving operation, e.g., $\hat{\sigma}(x)$ may not be convex in x even though the function $h(x;\xi)$ is. Thus the constraint (6) can be intractable despite that (1) is tractable. This motivates the investigation of a distributionally robust optimization (DRO) approach, namely, by using

$$\max_{P \in \mathcal{U}} E_P[h(x;\xi)] \le 0 \tag{8}$$

where $E_P[\cdot]$ denotes the expectation under P, and $\mathcal{U} := \mathcal{U}(\xi_1, \ldots, \xi_n)$ is an uncertainty set (also known as ambiguity set), calibrated from data, that contains a collection of distributions. As documented in many previous work (e.g., Delage and Ye (2010), Ben-Tal et al. (2013)), (8) can be made tractable by suitably choosing \mathcal{U} . One central question in this paper is to ask:

Is there a tractable choice of \mathcal{U} that can recover the statistician's asymptotically exact guarantee, namely

$$\lim_{n \to \infty} P\left(Z_0(x) \le \max_{P \in \mathcal{U}} E_P[h(x;\xi)] \text{ for all } x \in \Theta\right) = 1 - \alpha$$
(9)

and that

$$\max_{P \in \mathcal{U}} E_P[h(x;\xi)] \approx \hat{h}(x) + z \frac{\hat{\sigma}(x)}{\sqrt{n}}$$
(10)

1.3. Data-driven Distributionally Robust Optimization and Statistically "Good" Uncertainty Sets

To answer the above question, let us first revisit the common argument in the literature of datadriven DRO. To facilitate discussion, we call an uncertainty set \mathcal{U} statistically "good" if it allows

$$\liminf_{n \to \infty} P\left(Z_0(x) \le \max_{P \in \mathcal{U}} E_P[h(x;\xi)] \text{ for all } x \in \Theta\right) \ge 1 - \alpha \tag{11}$$

In contrast, a statistically "best" uncertainty set in the sense of (9) sharpens the inequality in (11) to equality.

The data-driven DRO framework provides a general methodology in guaranteeing (11). First, one calibrates an uncertainty set \mathcal{U} from data so that it contains the true distribution with probability $1 - \alpha$, namely $P(P_0 \in \mathcal{U}) \ge 1 - \alpha$. Note that since $P_0 \in \mathcal{U}$ implies that $Z_0(x) = E_0[h(x;\xi)] \le \max_{P \in \mathcal{U}} E_P[h(x;\xi)]$ for all x, we have

$$P\left(Z_0(x) \le \max_{P \in \mathcal{U}} E_P[h(x;\xi)] \text{ for all } x \in \Theta\right) \ge P(P_0 \in \mathcal{U}) \ge 1 - \alpha$$
(12)

Similarly, a set \mathcal{U} constructed with the asymptotic property $\liminf_{n\to\infty} P(P_0 \in \mathcal{U}) \ge 1 - \alpha$ guarantees that (11) holds and, in fact, so is the stronger guarantee

$$\liminf_{n \to \infty} P\left(\min_{P \in \mathcal{U}} E_P[h(x;\xi)] \le Z_0(x) \le \max_{P \in \mathcal{U}} E_P[h(x;\xi)] \text{ for all } x \in \Theta\right) \ge 1 - \alpha$$

Thus, good uncertainty sets can be readily created as confidence regions for P_0 . Constructing these confidence regions and their tractability have been substantially investigated. A non-exhaustive list includes moment and deviation-type constraints (Delage and Ye (2010), Goh and Sim (2010), Wiesemann et al. (2014)), Wasserstein balls (Esfahani and Kuhn (2015), Gao and Kleywegt (2016)), ϕ -divergence balls (Ben-Tal et al. (2013)), likelihood-based (Wang et al. (2015)) and goodness-of-fit-based regions (Bertsimas et al. (2014)). Recently, Gupta (2015) further investigates the smallest of such confidence regions as a baseline to measure the degree of conservativeness of a given uncertainty set.

1.4. Our Contributions

Despite the availability of all the good uncertainty sets, finding the statistically best one in the sense of (9) has not been addressed in the literature. In this paper, we construct an uncertainty set that is close to the best (the meaning of "close to" will be apparent in our later exposition) by leveraging one of the good sets, namely the Burg-entropy divergence ball.

Intriguingly, the way we construct these balls, and the associated statistical explanation, is completely orthogonal to the standard data-driven DRO framework discussed above. These balls are empirically defined (as we will explain in detail) and do not have any interpretation as confidence regions by themselves. In fact, they have low, or even zero, probability of covering the true distribution. Yet the resulting DRO has the best statistical performances among all DRO formulations. This disentanglement between set coverage and ultimate performance can be explained by a duality relation between our resulting DRO and the empirical likelihood theory, a connection that has been briefly discussed in a few previous work (e.g., Wang et al. (2015), Lam and Zhou (2015)) but not been fully exploited as far as we know.

Importantly, through setting up such a connection, we study optimal calibration of the sizes of these sets by using a generalization of χ^2 -quantiles that involves the excursion of so-called χ^2 -processes. As a by-product, our proposed method also resolves some technical challenges reported in the previous literature in calibrating divergence balls (e.g., Jiang and Guan (2012), Esfahani and Kuhn (2015)). More precisely, since divergence is only properly defined between absolutely continuous distributions, it has been suggested, in the case of continuous distributions, that one needs to construct the ball using kernel estimation of density and the divergence, which is statistically challenging, or resorting to a parametric framework. The approach we take here, on the other hand, bypasses these issues.

To summarize, our main contributions of this paper are:

1. We systematically build an uncertainty set that, in a precise sense, is close to recovering the guarantees (9) and (10) provided by the CLT.

2. In doing so, we expand the view on the meaning of uncertainty sets beyond the notion of confidence regions, by showing that our empirical Burg-entropy divergence ball recovers the best guarantees despite being a low or zero-coverage set. This is achieved through connecting the dual of the resulting DRO with the empirical likelihood theory.

3. To achieve our claimed guarantees, we study an approach to optimally calibrate the sizes of these balls using quantiles of χ^2 -process excursion.

4. As a by-product, our approach resolves the technical difficulties in enforcing absolute continuity when calibrating divergence balls that are raised in previous works in data-driven DRO.

Finally, while the viewpoint taken by this paper is primarily statistical, we mention that there are other valuable perspectives in the DRO literature motivated from risk or tractability considerations (see, e.g., the survey Gabrel et al. (2014)); these are, however, beyond the scope of this work.

The rest of this paper is organized as follows. Section 2 motivates our proposed uncertainty sets. Section 3 presents methods to calibrate their sizes and the theoretical explanation of their statistical performances. Section 4 shows results of our numerical experiments. Section 5 concludes and discusses future directions. Section A provides all the proofs. Appendices B and C list some auxiliary concepts and theorems.

2. Towards the Empirical DRO

We first review some background in divergence-based inference and how to use it to create confidence regions for probability distributions in Section 2.1. Through a preliminary numerical investigation in Section 2.2, we motivate and present, in Section 2.3, the *empirical divergence ball* as our main tool.

2.1. Divergence-based Inference and Confidence Regions

A ϕ -divergence ball is in the form

$$\mathcal{U} = \{ P \in \mathcal{P}_Q : D_\phi(P, Q) \le \eta \}$$
(13)

where

$$D_{\phi}(P,Q) = \int \phi\left(\frac{dP}{dQ}\right) dQ$$

for some baseline distribution Q and suitable function $\phi(\cdot)$, and dP/dQ is the likelihood ratio given by the Radon-Nikodym derivative between P and Q. The latter is well-defined only for P within \mathcal{P}_Q , the set of all distributions absolutely continuous with respect to Q. The function $\phi: \mathbb{R}^+ \to \mathbb{R}$ is convex and satisfies $\phi(1) = 0$.

Suppose the random variable ξ lies on a finite discrete support $\{s_1, \ldots, s_k\}$. One way to construct a statistically good divergence ball is as follows (Ben-Tal et al. (2013)). Set the baseline distribution as the histogram of the i.i.d. data given by $\hat{\mathbf{p}} = (\hat{p}_i)_{i=1,\ldots,k}$, where $\hat{p}_i = n_i/n$, n_i is the counts on support s_i , and n is the total sample size. The divergence ball (13) can be written as

$$\mathcal{U} = \{ \mathbf{p} \in \mathcal{P}_{\hat{\mathbf{p}}} : D_{\phi}(\mathbf{p}, \hat{\mathbf{p}}) \le \eta \}$$
$$= \left\{ (p_1, \dots, p_k) : \sum_{i=1}^k \hat{p}_i \phi\left(\frac{p_i}{\hat{p}_i}\right) \le \eta, \sum_{i=1}^k p_i = 1, \ p_i \ge 0 \text{ for all } i = 1, \dots, k \right\}$$
(14)

Under twice continuous differentiability condition on ϕ , the theory of divergence-based inference (Pardo (2005)) stipulates that

$$\frac{2n}{\phi''(1)} D_{\phi}(\mathbf{p}, \hat{\mathbf{p}}) \Rightarrow \chi^2_{k-1} \text{ as } n \to \infty$$

where χ^2_{k-1} is the χ^2 -distribution with degree of freedom k-1, and " \Rightarrow " denotes convergence in distribution. This implies that taking $\eta = \frac{\phi''(1)}{2n}\chi^2_{k-1,1-\alpha}$ in (14), where $\chi^2_{k-1,1-\alpha}$ is the $1-\alpha$ quantile of χ^2_{k-1} , forms an uncertainty set \mathcal{U} that contains the true distribution with probability asymptotically $1-\alpha$. This in turn implies that \mathcal{U} is a good uncertainty set satisfying (11).

For instance, $\phi(x) = (x - 1)^2$ yields the χ^2 -distance, and setting η at $\chi^2_{k-1,1-\alpha}/n$ results in the confidence region associated with the standard χ^2 goodness-of-fit test for categorical data (Agresti and Kateri (2011)). On the other hand, $\phi(x) = -\log x + x - 1$ yields the Burg-entropy (or the Kullback-Leibler) divergence (Kullback and Leibler (1951)), and η in this case should be set at $\chi^2_{k-1,1-\alpha}/(2n)$. Since the Burg-entropy divergence is important in our subsequent discussion, for convenience, we denote its divergence ball as

$$\mathcal{U}_{Burg} = \left\{ (p_1, \dots, p_k) : -\sum_{i=1}^k \hat{p}_i \log \frac{p_i}{\hat{p}_i} \le \frac{\chi_{k-1,1-\alpha}^2}{2n}, \sum_{i=1}^k p_i = 1, \ p_i \ge 0 \text{ for all } i = 1, \dots, k \right\}$$
(15)

From the discussion above, \mathcal{U}_{Burg} is a good uncertainty set and moreover satisfies

$$\lim_{n \to \infty} P\left(P_0 \in \mathcal{U}_{Burg}\right) = 1 - \alpha \tag{16}$$

for a finite discrete true distribution P_0 .

The computational tractability of divergence balls has been studied in depth in Ben-Tal et al. (2013), who reformulate $\max_{P \in \mathcal{U}} E_P[h(x;\xi)]$ in terms of the conjugate function of ϕ and propose efficient optimization algorithms. Because of this we will not drill further on tractability and instead refer interested readers therein.

2.2. An Initial Numerical Investigation on Coverage Accuracy

To get a sense of the coverage performance provided by \mathcal{U}_{Burg} , we run an experiment on estimating $Z_0(x) = E_0[h(x;\xi)]$, where we set h as

$$h(x;\xi) = -v\min(x,\xi) - s(x-\xi)^{+} + l(\xi-x)^{+} + cx + \rho$$
(17)

with v = 10, s = 5, l = 4, c = 3, and $\rho = 40$. This function h is adapted from the example in Section 6.3 in Ben-Tal et al. (2013). As an application, (17) can represent the loss amount in excess of the threshold ρ for a newsvendor. In this case, v is the selling price per unit, s the salvage value per unit, l the shortage cost per unit, c the cost per unit, ξ a random demand, and x the quantity to order.

For now, let us fix the solution at x = 30 (so it is purely about estimating $Z_0(30)$). We set the random variable ξ as an exponential random variable with mean 20 that is discretized uniformly over a k-grid on the interval [0, 50], or more precisely,

$$P\left(\xi = \frac{50j}{k}\right) = P\left(\frac{50(j-1)}{k} < Exp\left(\frac{1}{20}\right) < \frac{50j}{k}\right) \text{ for } j = 1, \dots, k-1$$

$$P\left(\xi = 50\right) = P\left(Exp\left(\frac{1}{20}\right) > \frac{50(k-1)}{k}\right)$$
(18)

We repeat 1,000 times:

1. Simulate n i.i.d. data ξ_1, \ldots, ξ_n from the k-discretized Exp(1/20).

2. Construct \mathcal{U}_{Burg} , and compute $\min_{\mathbf{p}\in\mathcal{U}_{Burg}} E_{\mathbf{p}}[h(x;\xi)]$ and $\max_{\mathbf{p}\in\mathcal{U}_{Burg}} E_{\mathbf{p}}[h(x;\xi)]$ with $\alpha = 0.05$.

3. Output $I\left(\min_{\mathbf{p}\in\mathcal{U}_{Burg}} E_{\mathbf{p}}[h(x;\xi)] \leq Z_0(x) \leq \max_{\mathbf{p}\in\mathcal{U}_{Burg}} E_{\mathbf{p}}[h(x;\xi)]\right)$, where $Z_0(x)$ is the true quantity calculable in closed-form, and $I(\cdot)$ is the indicator function.

We then output the point estimate and the 95% confidence interval (CI) of the coverage probability from the 1,000 replications.

Step 2 above is carried out by using duality and numerically solving

$$\min_{\mathbf{p}\in\mathcal{U}_{Burg}} E_{\mathbf{p}}[h(x;\xi)] = \max_{\lambda\geq0,\gamma} \sum_{i=1}^{n} \frac{\lambda}{n} \log\left(1 - \frac{-h(\xi_i) + \gamma}{\lambda}\right) - \lambda\eta + \gamma$$
$$\max_{\mathbf{p}\in\mathcal{U}_{Burg}} E_{\mathbf{p}}[h(x;\xi)] = \min_{\lambda\geq0,\gamma} - \sum_{i=1}^{n} \frac{\lambda}{n} \log\left(1 - \frac{h(\xi_i) + \gamma}{\lambda}\right) + \lambda\eta - \gamma$$

where $-0\log(1-t/0) := 0$ for $t \le 0$ and $-0\log(1-t/0) := \infty$ for t > 0 (see Ben-Tal et al. (2013)).

Table 1a shows the estimates of coverage probabilities for different support size k. The sample size for ξ is n = 30. The coverage probabilities are all greater than 95%, showing correct statistical guarantees. However, more noticeable is that they are all higher than 99%, and are consistently close to 100% for k = 10 or above, thus leading to severe over-coverage. Note that this phenomenon occurs despite that \mathcal{U}_{Burg} has asymptotically exactly $1 - \alpha$ probability of covering the true distribution as guaranteed in (16).





As a comparison, we repeat the experiment, but this time checking the coverage of the standard 95% CI generated from the CLT

$$\left[\hat{h}(30) - z_{1-\alpha/2} \frac{\hat{\sigma}(30)}{\sqrt{n}}, \hat{h}(30) + z_{1-\alpha/2} \frac{\hat{\sigma}(30)}{\sqrt{n}}\right]$$

where $\hat{h}(x) = \frac{1}{n} \sum_{i=1}^{n} h(x;\xi_i), \ \hat{\sigma}^2(x) = \frac{1}{n-1} \sum_{i=1}^{n} (h(x;\xi_i) - \bar{h})^2$, and $z_{1-\alpha/2}$ is the $(1-\alpha/2)$ -quantile of standard normal distribution. Table 1b shows that, unlike data-driven DRO, the coverage probabilities are now very close to 95%, regardless of the values of k. This result is, of course, as predicted by the CLT.

To investigate the source of inferiority in the data-driven DRO approach, we shall interpret the degree of freedom in the χ^2 -distribution from another angle: In maximum likelihood theory (Cox and Hinkley (1979)), the degree of freedom in the limiting χ^2 -distribution of the so-called loglikelihood ratio is equal to the number of effective parameters to be estimated. In our experiment, this number is one, because we are only interested in estimating a single quantity $Z_0(30)$. Indeed, Table 1c shows that the coverage probabilities of DRO, using the quantile of χ^2_1 instead of χ^2_{k-1} , are equally competitive as the CLT approach. This motivates us to propose our key definition of uncertainty set next.

2.3. The Empirical Divergence Ball

Given i.i.d. data ξ_1, \ldots, ξ_n , we define the *empirical Burg-entropy divergence ball* as

$$\mathcal{U}_{n}(\eta) = \left\{ \mathbf{w} = (w_{1}, \dots, w_{n}) : -\frac{1}{n} \sum_{i=1}^{n} \log(nw_{i}) \le \eta, \sum_{i=1}^{n} w_{i} = 1, w_{i} \ge 0 \text{ for all } i = 1, \dots, n \right\}$$
(19)

where **w** is the probability weight vector on the *n* support points from data (some possibly with the same values). The set (19) is well-defined whether the distribution of ξ is discrete or continuous. It is a Burg-entropy divergence ball centered at the empirical distribution, with radius $\eta > 0$, pretending that the support of the distribution is solely on the data. For convenience, we call the corresponding DRO over the empirical divergence ball as the *empirical DRO*.

The discussion in Section 2.2 suggests to put $\eta = \chi_{1,1-\alpha}^2/(2n)$. One intriguing observation is that $\mathcal{U}_n(\chi_{1,1-\alpha}^2/(2n))$ under-covers the true probability distribution. This can be seen by noting that, in the discrete case, $\mathcal{U}_n(\chi_{1,1-\alpha}^2/(2n))$ is equivalent to \mathcal{U}_{Burg} except that $\chi_{k-1,1-\alpha}^2$ in its definition is replaced by $\chi_{1,1-\alpha}^2$ (see Proposition 1 later). Since \mathcal{U}_{Burg} is asymptotically exact in providing $1 - \alpha$ coverage for the true distribution, and that $\chi_{1,1-\alpha}^2 < \chi_{k-1,1-\alpha}^2$, $\mathcal{U}_n(\chi_{1,1-\alpha}^2/(2n))$ must be asymptotically under-covering. What is more, in the continuous case, the empirical distribution is singular with respect to the true distribution. Thus $\mathcal{U}_n(\chi_{1,1-\alpha}^2/(2n))$ has as low as zero coverage. Clearly, the performance of the empirical uncertainty set cannot be reasoned using the standard data-driven DRO framework discussed in Section 1.3.

We close this section with Table 2, which shows additional experimental results for the same example as above, this time the data being generated from the *continuous* distribution Exp(1/20). As we can see in Table 2a, the coverages using $\mathcal{U}_n(\chi^2_{1,1-\alpha}/(2n))$ are maintained at close to 95% when n = 40 or above. As a comparison, Table 2b shows that the standard CLT performs similarly as the empirical DRO (except that it tends to over-cover instead of under-cover when n is small). Note that, unlike the discrete case, there is no well-defined choice of k in this setting.

n	Cover.	95% C.I. of	n	Cover.	95% C.I. of
	Prob.	Cover. Prob.		Prob.	Cover. Prob.
20	91.9%	(90.5%, 93.3%)	20	96.1%	(95.1%, 97.1%)
30	92.8%	(91.5%, 94.1%)	30	96.4%	(95.4%, 97.4%)
50	94.5%	(93.3%, 95.7%)	50	94.3%	(93.1%, 95.5%)
80	94.4%	(93.2%, 95.6%)	80	96.4%	(95.4%, 97.4%)

(a) Empirical DRO with ball size $\chi^2_{1,0.95}/(2n)$

(b) Standard CLT

 Table 2
 Coverage probabilities for different methods and sample sizes for continuous distributions

3. Statistical Guarantees

We present our theoretical justification in two subsections. Section 3.1 first connects the dual of the empirical DRO with the empirical likelihood (EL) method. Sections 3.2, 3.3 and 3.4 elaborate this connection to develop the calibration method for the radius η in the empirical divergence ball, via estimating the excursion of χ^2 -processes. We defer all proofs to Appendix A.

Throughout our exposition, " \Rightarrow " denotes weak convergence (or convergence in distribution), "a.s." abbreviates "almost surely", and "ev." abbreviates "eventually".

3.1. The Empirical Likelihood Method

The EL method, first proposed by Owens (Owen (1988, 2001)), can be viewed as a nonparametric counterpart of maximum likelihood theory. Given a set of i.i.d. data ξ_1, \ldots, ξ_n , one can view the empirical distribution, formed by putting probability weight 1/n on each data point, as a nonparametric maximum likelihood in the following sense. We define the nonparametric likelihood of any distributions supported on the data as

$$\prod_{i=1}^{n} w_i \tag{20}$$

where $\mathbf{w} = (w_1, \ldots, w_n) \in \mathcal{P}_n$ is any probability vector on $\{\xi_1, \ldots, \xi_n\}$. Then the likelihood of the empirical distribution, given by

$$\prod_{i=1}^{n} \frac{1}{n} \tag{21}$$

maximizes (20). This observation can be easily verified by a simple convexity argument. Moreover, (21) still maximizes even if one considers other distributions that are not only supported on the data, since these distributions would have $\sum_{i=1}^{n} w_i < 1$, making (20) even smaller.

The key of EL is a nonparametric analog of the celebrated Wilks' Theorem (Cox and Hinkley (1979)), stating the convergence of the so-called logarithmic likelihood ratio to χ^2 -distribution. In the EL framework, the nonparametric likelihood ratio is defined as the ratio between any nonparametric likelihood and the maximum likelihood, given by

$$\prod_{i=1}^{n} \frac{w_i}{1/n} = \prod_{i=1}^{n} (nw_i)$$

To carry out inference we need to specify a quantity of interest to be estimated. Suppose we are interested in estimating $\mu_0 = E_0[g(\xi)]$ for some function $g(\cdot)$, where $E_0[\cdot]$ is the expectation with respect to the true distribution generating the data (and similarly, $Var_0(\cdot)$ denotes its variance). The EL method utilizes the profile nonparametric likelihood ratio

$$R(\mu) = \max\left\{\prod_{i=1}^{n} (nw_i) : \sum_{i=1}^{n} g(\xi_i)w_i = \mu, \sum_{i=1}^{n} w_i = 1, w_i \ge 0 \text{ for all } i = 1, \dots, n\right\}$$
(22)

where the likelihood ratios are "profiled" according to the value of $\sum_{i=1}^{n} g(\xi_i) w_i$. With this definition, we have:

THEOREM 1 (The Empirical Likelihood Theorem; Owen (1988)). Let $\xi_1, \ldots, \xi_n \in \Xi$ be i.i.d. data under P_0 . Let $\mu_0 = E_0[g(\xi)] < \infty$, and assume that $0 < Var_0(g(\xi)) < \infty$. Then

$$-2\log R(\mu_0) \Rightarrow \chi_1^2 \quad as \ n \to \infty \tag{23}$$

where $-2\log R(\mu_0)$ is defined as ∞ if there is no feasible solution in defining $R(\mu_0)$ in (22).

The degree of freedom 1 in the limiting χ^2 -distribution in (23) counts the number of effective parameters, which is only μ_0 in this case.

Phrasing in terms of our problem setup, we define

$$R(x;Z) = \max\left\{\prod_{i=1}^{n} (nw_i) : \sum_{i=1}^{n} h(x;\xi_i)w_i = Z(x), \sum_{i=1}^{n} w_i = 1, w_i \ge 0 \text{ for } i = 1,\dots,n\right\}$$
(24)

and hence

$$-2\log R(x;Z) = \min\left\{-2\sum_{i=1}^{n}\log(nw_i): \sum_{i=1}^{n}h(x;\xi_i)w_i = Z(x), \sum_{i=1}^{n}w_i = 1, w_i \ge 0 \text{ for all } i = 1,\dots,n\right\}$$

From Theorem 1, we conclude $P(-2\log R(x;Z_0) \le \chi^2_{1,1-\alpha}) \to 1-\alpha$ as $n \to \infty$ for a fixed x. The important implication of Theorem 1 arises from a duality relation between $-2\log R(x;Z_0)$ and the optimal values of the empirical DRO, in the sense that $-2\log R(x;Z_0) \le \kappa$ if and only if

$$\min_{\mathbf{w}\in\mathcal{U}_{n}(\kappa/(2n))}\sum_{i=1}^{n}h(x;\xi_{i})w_{i} \leq Z_{0}(x) \leq \max_{\mathbf{w}\in\mathcal{U}_{n}(\kappa/(2n))}\sum_{i=1}^{n}h(x;\xi_{i})w_{i}$$

where $\mathcal{U}_n(\eta)$ is the empirical divergence ball defined in (19). This implies:

THEOREM 2. Fix $x \in \Theta$, and let $\xi_1, \ldots, \xi_n \in \Xi$ be i.i.d. data under P_0 . Assume that $0 < Var_0(h(x;\xi)) < \infty$, and $Z_0(x) = E_0[h(x;\xi)] < \infty$. We have

$$\lim_{n \to \infty} P\left(\underline{Z}_n(x) \le \overline{Z}_n(x)\right) = 1 - \alpha \tag{25}$$

where

$$\underline{Z}_n(x) = \min_{\mathbf{w} \in \mathcal{U}_n\left(\chi_{1,1-\alpha}^2/(2n)\right)} \sum_{i=1}^n h(x;\xi_i) w_i \tag{26}$$

$$\overline{Z}_n(x) = \max_{\mathbf{w} \in \mathcal{U}_n\left(\chi_{1,1-\alpha}^2/(2n)\right)} \sum_{i=1}^n h(x;\xi_i) w_i$$
(27)

Next, we argue that, in the discrete case, the empirical DRO given by $\underline{Z}_n(x)$ and $\overline{Z}_n(x)$ reduces to the standard divergence-based DRO given by max / min_{$\mathbf{p} \in \mathcal{U}_{Burg}} E_{\mathbf{p}}[h(x;\xi)]$, except that the degree of freedom in the χ^2 -quantile is replaced by 1. This explains the experimental results in Section 2.2.}

PROPOSITION 1. Fix $x \in \Theta$. When ξ is discrete on the support set $\{s_1, \ldots, s_k\}$, $\underline{Z}_n(x)$ and $\overline{Z}_n(x)$ defined in (26) and (27) are equal to $\min_{\mathbf{p} \in \mathcal{U}'_{Burg}} E_{\mathbf{p}}[h(x;\xi)]$ and $\max_{\mathbf{p} \in \mathcal{U}'_{Burg}} E_{\mathbf{p}}[h(x;\xi)]$ respectively, where

$$\mathcal{U}_{Burg}' = \left\{ (p_1, \dots, p_k) : -\sum_{i=1}^k \hat{p}_i \log \frac{p_i}{\hat{p}_i} \le \frac{\chi_{1,1-\alpha}^2}{2n}, \sum_{i=1}^k p_i = 1, \ p_i \ge 0 \ \text{for all } i = 1, \dots, k \right\}$$
(28)

and $\hat{p}_i = n_i/n$, the proportion of data falling onto s_i .

We complement Theorem 2 with a consistency result:

THEOREM 3. Under the same conditions in Theorem 2, for any fixed $x \in \Theta$, both $\underline{Z}_n(x) \xrightarrow{a.s.} Z_0(x)$ and $\overline{Z}_n(x) \xrightarrow{a.s.} Z_0(x)$ as $n \to \infty$.

We note that, in the data-driven DRO framework, if ξ is continuous, the absolute continuity condition requires a divergence ball to center at a continuous distribution to have any chance of containing the true distribution. This observation has been pointed out by several authors (e.g., Jiang and Guan (2012), Esfahani and Kuhn (2015)) and forces the use of kernel density estimators to set the baseline. Unless one assumes a parametric framework, calibrating the ball radius requires nonparametric divergence estimation, which involves challenging statistical analyses on bandwidth tuning and loss of estimation efficiency (e.g., Moon and Hero (2014), Nguyen et al. (2007), Pál et al. (2010)). The empirical DRO based on the EL framework cleanly bypasses these issues.

Our discussion in this subsection is also related to likelihood robust optimization studied in Wang et al. (2015), which also discusses EL as well as other connections such as Bayesian statistics. Wang et al. (2015) focuses on finite discrete distributions. The work Lam and Zhou (2015) also investigates EL, among other techniques like the bootstrap, in constructing confidence bounds for the optimal values of stochastic programs. However, none of these formalizes the connection, or more precisely, the *disconnection* between set coverage and the statistical performance of DRO. As our next subsection shows, this formalization is important in capturing a statistical price to attain our best guarantee in (9). This will be our focus next.

3.2. Asymptotically Exact Coverage via χ^2 -Process Excursion

The discussion so far presumes a fixed $x \in \Theta$. Recall in Section 1.3 that, in data-driven DRO, a confidence region given by \mathcal{U} guarantees $Z_0(x) \leq \max_{P \in \mathcal{U}} E_P[h(x;\xi)]$ with at least the same confidence level thanks to (12). This guarantee holds regardless of a fixed x or uniformly over all $x \in \Theta$. This is because the construction of such confidence regions is completely segregated from the expected value constraint of interest. In contrast, the statistical performance of our empirical divergence ball is highly coupled with h, since $E_0[h(x;\xi)]$ can be viewed as the parameter we want to estimate in the EL method. Consequently, the reasoning for Theorem 2 only applies to situations where x is fixed, and the empirical divergence ball constructed there is not big enough to guarantee (9), which requires a bound simultaneous for all $x \in \Theta$.

The main result in this section is to explain and to show how, depending on the "complexity" of h, one can suitably inflate the size of the ball to match a statistical performance close to (9).

We begin our discussion by imposing the following assumptions:

ASSUMPTION 1 (Finite mean). $Z_0(x) = E_0[h(x;\xi)] < \infty$ for all $x \in \Theta$. ASSUMPTION 2 (Non-degeneracy). $\inf_{x\in\Theta} E_0[h(x;\xi) - Z_0(x)] > 0$. ASSUMPTION 3 (L_2 -boundedness). $E_0 \sup_{x\in\Theta} |h(x;\xi) - Z_0(x)|^2 < \infty$

ASSUMPTION 4 (Function complexity). The collection of functions

$$\mathcal{H}_{\Theta} = \{h(x; \cdot) : \Xi \to \mathbb{R} | x \in \Theta\}$$
⁽²⁹⁾

is a P_0 -Donsker class.

The first three assumptions are mild moment conditions on the quantity $h(x;\xi)$. The last assumption, the so-called Donsker condition, means that the function class \mathcal{H}_{Θ} is "simple" enough to allow the associated empirical process indexed by \mathcal{H}_{Θ} to converge weakly to a Brownian bridge (see Definition 2 in Appendix B).

The following theorem precisely describes the radius of the empirical divergence ball needed to attain the best guarantee in (9):

THEOREM 4 (Optimal Calibration of Empirical Divergence Ball). Let $\xi_1, \ldots, \xi_n \in \Xi$ be i.i.d. data under P_0 . Suppose Assumptions 1, 2, 3 and 4 hold. Let q_n be the $(1 - \alpha)$ -quantile of $\sup_{x \in \Theta} J_n(x)$, i.e.

$$P_{\boldsymbol{\xi}}\left(\sup_{x\in\Theta}J_n(x)\geq q_n\right)=\alpha\tag{30}$$

where $J_n(x) = G_n(x)^2$ and $G_n(\cdot)$ is a Gaussian process indexed by Θ that is centered, i.e. mean zero, with covariance

$$Cov(G_n(x_1), G_n(x_2)) = \frac{\sum_{i=1}^n (h(x_1; \xi_i) - \hat{h}(x_1))(h(x_2; \xi_i) - \hat{h}(x_2))}{\sqrt{\sum_{i=1}^n (h(x_1; \xi_i) - \hat{h}(x_1))^2 \sum_{i=1}^n (h(x_2; \xi_i) - \hat{h}(x_2))^2}}$$
(31)

for any $x_1, x_2 \in \Theta$, and $\hat{h}(x) = (1/n) \sum_{i=1}^n h(x;\xi_i)$ is the sample mean of $h(x;\xi_i)$'s. P_{ξ} denotes the probability conditional on the data ξ_1, \ldots, ξ_n .

We have

$$\lim_{n \to \infty} P(\underline{Z}_n^*(x) \le Z_0(x) \le \overline{Z}_n^*(x) \quad \text{for all} \quad x \in \Theta) = 1 - \alpha$$
(32)

where

$$\underline{Z}_n^*(x) = \min_{\mathbf{w} \in \mathcal{U}_n(q_n/(2n))} \sum_{i=1}^n h(x;\xi_i) w_i$$
$$\overline{Z}_n^*(x) = \max_{\mathbf{w} \in \mathcal{U}_n(q_n/(2n))} \sum_{i=1}^n h(x;\xi_i) w_i$$

Note that, other than being a two-sided bound instead of one-sided, the guarantee (32) is precisely (9).

The process $J_n(\cdot)$, as the square of a Gaussian process, is known as a χ^2 -process (or χ^2 random field; e.g., Adler and Taylor (2009)). Its covariance structure can be expressed explicitly in terms of the function h and the data. The quantity $P_{\boldsymbol{\xi}}(\sup_{x\in\Theta} J_n(x) \ge u)$ is the excursion probability of $J_n(\cdot)$ above u. Note that we have ignored some subtle measurability issues in stating our result. To avoid unnecessary diversion, we will stay silent on measurability throughout the paper and refer the reader to Van Der Vaart and Wellner (1996) for detailed treatments.

We observe some immediate connection of $\sup_x J_n(x)$ to the χ_1^2 -distribution used in Theorem 2. In addition to the fact that the marginal distribution of $J_n(x)$ at any x is a χ_1^2 -distribution, we also have, by the Borell-TIS inequality (Adler (1990)), that the asymptotic tail probability of $\sup_x J_n(x)$ has the same exponential decay rate as that of χ_1^2 , i.e.

$$\frac{\log P\left(\sup_{x\in\Theta}G^2(x)\geq\nu\right)}{\log P(Y\geq\nu)}\to 1$$

as $\nu \to \infty$, where Y is a χ_1^2 random variable. This suggests a relatively small overhead in using q_n instead of $\chi_{1,1-\alpha}^2$ in calibrating the empirical ball when α is small.

Nevertheless, Theorem 4 offers some insights beyond Theorem 2. First, it requires the Donsker condition on the class \mathcal{H}_{Θ} . One sufficient condition of P_0 -Donsker is:

LEMMA 1. Suppose that $Z_0(x) = E_0[h(x;\xi)] < \infty$ and $Var_0(h(x;\xi)) < \infty$ for all $x \in \Theta$. Also assume that there exists a random variable M with $E_0M^2 < \infty$ such that

$$|h(x_1;\xi) - h(x_2;\xi)| \le M ||x_1 - x_2||_2$$

a.s. for all $x_1, x_2 \in \Theta$. Then \mathcal{H}_{Θ} as defined in (29) is P_0 -Donsker.

Lemma 1 is a consequence of the Jain-Marcus Theorem (e.g., Van Der Vaart and Wellner (1996), Example 2.11.13). It is worth noting that the condition in Lemma 1 is also a standard sufficient condition in guaranteeing the central limit convergence for SAA (Shapiro et al. (2014), Theorem 5.7). This is not a coincidence, as the machinery behind Theorem 4 involves an underpinning CLT, much like in the convergence analysis of SAA.

Secondly, even though $q_n \approx \chi_{1,1-\alpha}^2$ when $\alpha \approx 0$, q_n is strictly larger than $\chi_{1,1-\alpha}^2$ since $\sup_{x\in\Theta} J_n(x)$ stochastically dominates χ_1^2 (unless degeneracy occurs). Thus the ball constructed in Theorem 4 is always bigger than that in Theorem 2. One way to estimate this inflation is by approximating the excursion probability of χ^2 -process using the theory of random geometry. We delegate this discussion to Section 3.4. For now, we will delve into more details underlying Theorem 4 and other properties of the empirical DRO.

3.3. The Profile Nonparametric Likelihood Ratio Process and Other Properties of the Empirical DRO

We explain briefly the machinery leading to Theorem 4, leaving the details to Appendix A. Our starting point is to define the profile nonparametric likelihood ratio in (24) at the process level

$$\{R(x;Z): x \in \Theta\}\tag{33}$$

We call (33) the profile nonparametric likelihood ratio process indexed by $x \in \Theta$. Denote the space

$$\ell^{\infty}(\Theta) = \left\{ y : \Theta \to \mathbb{R} \middle| \|y\|_{\Theta} < \infty \right\}$$
(34)

where we define $||y||_{\Theta} = \sup_{x \in \Theta} |y(x)|$ for any function $y : \Theta \to \mathbb{R}$. We have a convergence theorem for R(x; Z) uniformly over $x \in \Theta$, in the following sense:

THEOREM 5 (Limit Theorem of the Profile Nonparametric Likelihood Ratio Process). Under Assumptions 1, 2, 3 and 4, the profile likelihood ratio process defined in (33) satisfies

$$-2\log R(\cdot; Z_0) \Rightarrow J(\cdot) \quad in \quad \ell^{\infty}(\Theta)$$

where $J(x) = G(x)^2$ and $G(\cdot)$ is a Gaussian process indexed by $x \in \Theta$ that has mean zero and covariance

$$Cov(G(x_1), G(x_2)) = \frac{Cov_0(h(x_1; \xi), h(x_2; \xi))}{\sqrt{Var_0(h(x_1; \xi))Var_0(h(x_2; \xi))}}$$

for any $x_1, x_2 \in \Theta$.

Theorem 5 is the empirical-process generalization of Theorem 2. It implies that $P(\sup_{x\in\Theta} \{-2\log R(x;Z_0)\} \le q^*) \to 1 - \alpha$ for q^* selected such that $P(\sup_{x\in\Theta} J(x) \le q^*) = 1 - \alpha$. By a duality-type argument similar to that in Section 3.1, we have $-2\log R(x;Z_0) \le q^*$ for all $x \in \Theta$, if and only if $\min_{\mathbf{w}\in\mathcal{U}_n(q^*/(2n))} \sum_{i=1}^n h(x;\xi_i) w_i \le Z_0(x) \le \max_{\mathbf{w}\in\mathcal{U}_n(q^*/(2n))} \sum_{i=1}^n h(x;\xi_i) w_i$ for all $x \in \Theta$, which implies $\lim_{n\to\infty} P\left(\min_{\mathbf{w}\in\mathcal{U}_n(q^*/(2n))} \sum_{i=1}^n h(x;\xi_i) w_i \le Z_0(x) \le \max_{\mathbf{w}\in\mathcal{U}_n(q^*/(2n))} \sum_{i=1}^n h(x;\xi_i) w_i \le Z_0(x) \le \max_{\mathbf{w}\in\mathcal{U}_n(q^*/(2n))} \sum_{i=1}^n h(x;\xi_i) w_i \le Z_0(x) \le \max_{\mathbf{w}\in\mathcal{U}_n(q^*/(2n))} \sum_{i=1}^n h(x;\xi_i) w_i$ for all $x \in \Theta$, which implies $\lim_{n\to\infty} P\left(\min_{\mathbf{w}\in\mathcal{U}_n(q^*/(2n))} \sum_{i=1}^n h(x;\xi_i) w_i \le Z_0(x) \le \max_{\mathbf{w}\in\mathcal{U}_n(q^*/(2n))} \sum_{i=1}^n h(x;\xi_i) w_i$ for all $x \in \Theta$.

LEMMA 2. Under Assumptions 1, 2, 3 and 4, conditional on almost every data realization $(P_n : n \ge 1)$,

$$G_n(\cdot) \Rightarrow G(\cdot) \quad in \quad \ell^{\infty}(\Theta)$$

where $G_n(\cdot)$ and $G(\cdot)$ are defined in Theorems 4 and 5 respectively.

Theorem 4 can then be proved by combining Theorem 5 and Lemma 2. Moreover, consistency of the empirical DRO also holds uniformly over $x \in \Theta$:

THEOREM 6 (Uniform Strong Consistency). Under Assumptions 1, 2, 3 and 4,

$$\sup_{x \in \Theta} \frac{|\underline{Z}_n^*(x) - Z_0(x)| \stackrel{a.s.}{\to} 0}{\sup_{x \in \Theta} |\overline{Z}_n^*(x) - Z_0(x)| \stackrel{a.s.}{\to} 0}$$

as $n \to \infty$.

Lastly, the following theorem highlights that the width of the confidence band $[\underline{Z}_n^*(x), \overline{Z}_n^*(x)]$ varies with the standard deviation at each x:

THEOREM 7 (Pertaining to the Variability at Each Decision Point). Suppose Assumptions 1, 2, 3 and 4 hold. Additionally, suppose that $h(\cdot; \cdot)$ is bounded. Then

$$\underline{Z}_n^*(x) = \hat{h}(x) - \sqrt{q_n} \frac{\hat{\sigma}(x)}{\sqrt{n}} + O\left(\frac{1}{n}\right)$$
$$\overline{Z}_n^*(x) = \hat{h}(x) + \sqrt{q_n} \frac{\hat{\sigma}(x)}{\sqrt{n}} + O\left(\frac{1}{n}\right)$$

uniformly over $x \in \Theta$ a.s.. Here $\hat{h}(x) = \frac{1}{n} \sum_{i=1}^{n} h(x;\xi_i)$ is the sample mean, $\hat{\sigma}^2(x) = \frac{1}{n} \sum_{i=1}^{n} (h(x;\xi_i) - \hat{h}(x))^2$ is the sample variance at each x, and q_n is defined in Theorem 4.

Theorem 7 gives rise to (10). In particular, $\sqrt{q_n}$ is analogous to the critical value in a confidence band. In summary, Theorems 4 and 7 show that our empirical divergence ball $\mathcal{U}_n(q_n/(2n))$, calibrated via the quantile of χ^2 -process excursion q_n , satisfies our benchmark guarantees (9) and (10), except that it provides a two-sided bound instead of one-sided. The difference of two- versus one-sided bound is the reason we have claimed "close to" the best in Section 1.3.

3.4. Approximating the Quantile of χ^2 -process Excursion

We discuss how to estimate q_n in Theorem 4. One approach is to approximate the excursion probability of χ^2 -process by the mean Euler characteristic approximation (e.g., Adler and Taylor (2009), Theorem 13.4.1 and Section 15.10.2, and Adler and Taylor (2011), Theorem 4.8.1):

$$P\left(\sup_{x\in\Theta} J_n(x) \ge u\right) \approx \sum_{j=0}^m (2\pi)^{-j/2} \mathcal{L}_j(\Theta) \mathcal{M}_j(u)$$
(35)

Here *m* is the dimension of the decision space $\Theta \subset \mathbb{R}^m$. The coefficients $\mathcal{L}_j(\Theta)$ on the RHS of (35) are known as the Lipschitz-Killing curvatures of the domain Θ , which measure the "intrinsic volumes" of the domain Θ using the Riemannian metric induced by the Gaussian process G_n

(Adler and Taylor (2009), equation (12.2.2)). In particular, the highest-dimensional coefficient is given by

$$\mathcal{L}_m(\Theta) = \int_{\Theta} det(\Lambda(x))^{1/2} dx$$

(Adler and Taylor (2009), equation (12.2.22), and Adler and Taylor (2011), equation (5.4.1)) where $\Lambda(x) = (\Lambda_{ij}(x))_{i,j=1,\dots,m} \in \mathbb{R}^{m \times m}, \text{ and}$

$$\Lambda_{ij}(x) = Cov\left(\frac{\partial G_n(x)}{\partial x_i}, \frac{\partial G_n(x)}{\partial x_j}\right) = \frac{\partial^2}{\partial y_i \partial z_j} Cov(G_n(y), G_n(z))\Big|_{y=x, z=x}$$
(36)

for differentiable G_n (in the L^2 sense), with x_i and x_j the *i* and *j*-th components of *x*. Thus (36) can be evaluated by differentiating (31). Lower-dimensional coefficients can be evaluated by integration over lower-dimensional surfaces of Θ , and $\mathcal{L}_0(\Theta) = 1$.

On the other hand, the quantities $\mathcal{M}_j(u)$'s are the Gaussian Minkowski functionals for the excursion set, independent of Θ and h, and are given by

$$\mathcal{M}_{j}(u) = (-1)^{j} \frac{d^{j}}{dy^{j}} P(Y \ge y) \Big|_{y = \sqrt{u}}$$

where Y is the square root of a χ_1^2 random variable. Thus, for instance, $\mathcal{M}_0(u) = P(\chi_1^2 \ge u)$, and $\mathcal{M}_1(u) = 2\phi(\sqrt{u})$ where $\phi(\cdot)$ is the standard normal density.

(35) is a very accurate approximation for $P(\sup_{x\in\Theta} J_n(x) \ge u)$ in the sense

$$\left| P\left(\sup_{x \in \Theta} J_n(x) \ge u \right) - \sum_{j=0}^m (2\pi)^{-j/2} \mathcal{L}_j(\Theta) \mathcal{M}_j(u) \right| \le C e^{-\beta u/2}$$

where C > 0 and $\beta > 1$ (Adler et al. (in preparation), Section 5.3.2). In other words, the approximation error is exponentially smaller than all the terms in (35) as u increases. In practice, however, the formula for $\mathcal{L}_j(\Theta)$ could get increasingly complex as j decreases, in which case only the first and the second highest order coefficients of $\mathcal{L}_j(\Theta)$ are used.

In light of the above, an accurate approximation of q_n can be found by solving the root of

$$\sum_{j=0}^{m} (2\pi)^{-j/2} \mathcal{L}_j(\Theta) \mathcal{M}_j(u) = \alpha$$

As an explicit illustration, when m = 1, and h is twice differentiable almost everywhere, we have

$$P\left(\sup_{x\in\Theta}J_n(x)\geq u\right) = P(\chi_1^2\geq u) + \int_{\Theta}\sqrt{\frac{\partial^2}{\partial y\partial z}Cov(G_n(y),G_n(z))}\Big|_{y=x,z=x}dx \ \frac{e^{-u/2}}{\pi} + O(e^{-\beta u/2})$$

for some $\beta > 1$. An approximate q_n can then be found by solving the root of

$$P(\chi_1^2 \ge u) + \int_{\Theta} \sqrt{\frac{\partial^2}{\partial y \partial z}} Cov(G_n(y), G_n(z)) \Big|_{y=x, z=x} dx \ \frac{e^{-u/2}}{\pi} = \alpha$$
(37)

4. Numerical Illustrations for the Empirical DRO

This section shows some numerical results on the statistical performance of empirical DRO. We use the newsvendor loss function in (17) as our h. We repeat 1,000 times:

1. Simulate n i.i.d. data ξ_1, \ldots, ξ_n from the k-discretized Exp(1/20).

2. Estimate q_n using (37), and compute $\underline{Z}_n^*(x) = \min_{\mathbf{w} \in \mathcal{U}_n(q_n/(2n))} \sum_{i=1}^n h(x;\xi_i) w_i$ and $\overline{Z}_n^*(x) = \max_{\mathbf{w} \in \mathcal{U}_n(q_n/(2n))} \sum_{i=1}^n h(x;\xi_i) w_i$, with α set to be 0.05.

3. Output

$$I\left(\underline{Z}_n^*(x) \le Z_0(x) \le \overline{Z}_n^*(x) \text{ for } x = \frac{50j}{20}, \ j = 1, \dots, 20\right)$$

and

$$I\left(Z_0(x) \le \overline{Z}_n^*(x) \text{ for } x = \frac{50j}{20}, \ j = 1, \dots, 20\right)$$

where $Z_0(x)$ is the true function of interest that is calculable in closed-form.

We then output the point estimates and the 95% CIs of the two- and one-sided coverage probabilities, i.e. $P\left(\underline{Z}_n^*(x) \leq Z_0(x) \leq \overline{Z}_n^*(x) \text{ for } x = \frac{50j}{20}, j = 1, \dots, 20\right)$ and $P\left(Z_0(x) \leq \overline{Z}_n^*(x) \text{ for } x = \frac{50j}{20}, j = 1, \dots, 20\right)$, from the 1,000 replications. These probabilities serve as proxies for the probabilities $P\left(\underline{Z}_n^*(x) \leq Z_0(x) \leq \overline{Z}_n^*(x) \text{ for } x \in \Theta\right)$ and $P\left(Z_0(x) \leq \overline{Z}_n^*(x) \text{ for } x \in \Theta\right)$ respectively.

We first set ξ as a k-discretized Exp(1/20) as in (18). For comparison, we also repeat the above experiment but using $\chi^2_{k-1,0.95}$ and $\chi^2_{1,0.95}$ in place of q_n . Table 3 shows the results of two-sided coverage probabilities as we vary the sample size from n = 20 to 80. The coverage probabilities appear to be stable already starting at n = 20. As we can see, the coverages using the $\chi^2_{k-1,0.95}$ calibration (Table 3a) are around 99%, much higher than 95%, as k - 1 is over-determining the number of parameters we want to estimate from the EL perspective. The coverage probabilities using the $\chi^2_{1,0.95}$ calibration (Table 3b), on the other hand, are in the range 86% to 87%, significantly lower than 95%, since it does not account for simultaneous estimation errors. Lastly, the coverage probabilities using the χ^2 -process excursion (Table 3c) are very close to 95% in all cases, thus confirming the superiority of our approach.

Next, Table 4 shows the results for one-sided coverage instead of two-sided. These one-sided coverage probabilities are slightly higher than the two-sided counterparts as the coverage condition is now more relaxed. Nonetheless, the magnitudes of these changes are very small compared to the effects brought by the choice of calibration methods. In particular, using $\chi^2_{k-1,0.95}$ appears to be severely over-covering at about 99% to 100%, while using $\chi^2_{1,0.95}$ gives under-coverage at about 89% to 91%. Using the χ^2 -process excursion shows 95% to 96% coverage performances, thus significantly better than the other two methods. These show that, even though our statistical guarantees in Theorem 4 are two-sided, the loss of inaccuracy for one-sided coverage is very minor compared to

						n	2-sided	95% C.I. of	
n	2-sided	95% C.I. of	n	2-sided	95% C.I. of		Cover. Prob.	Cover. Prob.	
	Cover. Prob.	Cover. Prob.		Cover. Prob.	Cover. Prob.	20	94.4%	(93.8%, 95.0%)	
20	98.3%	(98.0%, 98.6%)	20	85.6%	(85.0%, 86.2%)	30	94.6%	(94.0%, 95.2%)	
30	98.8%	(98.7%, 100.0%)	30	85.9%	(85.4%, 86.5%)	40	94.7%	(94.2%, 95.3%)	
40	98.9%	(98.8%, 100.0%)	40	86.6%	(86.1%, 87.1%)	50	94.7%	(94.2%, 95.3%)	
50	98.8%	(98.7%, 100.0%)	50	86.6%	(86.1%, 87.1%)	60	94.4%	(93.8%, 94.9%)	
60	98.9%	(98.8%, 100.0%)	60	86.1%	(85.5%, 86.7%)	80	95.0%	(94.5%, 95.5%)	
80	98.8%	(98.6%, 98.9%)	80	86.8%	(86.3%, 87.3%)	()			
(a) $\chi^2_{k-1,0.95}/(2n)$				(b) $\chi^2_{1,0.95}/(2n)$			(c) Approximate 95%-qu		
							$J_n(x)$		

Table 3Two-sided coverage probabilities for different Burg-divergence ball sizes and sample sizes for a discrete
distribution with k = 5

the improvement in the calibration method used. This experimentally justifies our claim of "close to" the best at the end of Section 3.3.

n	1-sided Cover. Prob.	95% C.I. of Cover. Prob.	n	1-sided Cover. Prob.	95% C.I. of Cover. Prob.	$\frac{n}{20}$	1-sided Cover. Prob. 94.8%	95% C.I. of Cover. Prob.
20	98.6%	(98.1%, 99.2%)	20	88.5%	(87.2%, 89.8%)	30	95.1%	(94.0%, 96.3%)
30	99.6%	(99.4%, 100.0%)	30	89.2%	(88.0%, 90.4%)	40	95.4%	(94.3%, 96.5%)
40	99.7%	(99.6%, 100.0%)	40	90.5%	(89.4%, 91.6%)	50	95.4%	(94.3%, 96.5%)
50	99.6%	(99.4%, 100.0%)	50	90.5%	(89.4%, 91.6%)	60	94.7%	(93.5%, 95.8%)
60	99.7%	(99.6%, 100.0%)	60	89.5%	(88.3%, 90.7%)	80	96.0%	(95.0%, 97.0%)
80	99.5%	(99.3%, 99.8%)	80	90.9%	(89.9%, 91.9%)			
(-) - 2 - (-)				(1) = 2 $/(2)$		(c) 1	Approximate	95%-quantile of
(a) $\chi_{k-1,0.95}/(2\pi)$				(D) $\chi_{1,0.9}$	$_{95}/(2n)$	sup_	$J_n(x)$	

Table 4One-sided coverage probabilities for different Burg-divergence ball sizes and sample sizes for a discrete
distribution with k = 5

Finally, we repeat the experiments using the continuous distribution Exp(1/20). We compare the use of $\chi^2_{1,0.95}$ with the χ^2 -process excursion (there is no notion of k in this case). Table 5 shows that the two-sided coverages using $\chi^2_{1,0.95}$ are under-covering at between 82% and 85%. The χ^2 process excursion gives about 93% at n = 20 and converges to close to 95% at n = 80. Thus, similar to the discrete case, the calibration using χ^2 -process excursion gives significantly more accurate two-sided coverages than using $\chi^2_{1,0.95}$. Table 6 draws similar conclusion for one-sided coverages. For $\chi^2_{1,0.95}$, the coverage probability is about 84% at n = 20 and 90% at n = 80, therefore severely under-covering. On the other hand, χ^2 -process excursion gives 94% to 96% coverages among all the *n*'s. This once again shows the insignificance of one- versus two-sided coverage compared to the improvement in the choice of calibration method. In overall, our proposed scheme of using χ^2 -process excursion gives much more accurate coverages than using $\chi^2_{1,0.95}$.

n	2-sided	95% C.I. of	n	2-sided	95% C.I. of
	Cover. Prob.	Cover. Prob.		Cover. Prob.	Cover. Prob.
20	82.4%	(81.3%, 83.5%)	20	92.8%	(91.1%, 94.6%)
30	82.8%	(81.7%, 83.8%)	30	93.6%	(92.3%, 94.9%)
40	83.5%	(82.5%, 84.5%)	40	94.4%	(93.1%, 95.8%)
50	84.3%	(83.3%, 85.2%)	50	94.4%	(93.1%, 95.8%)
60	84.2%	(83.2%, 85.2%)	60	95.7%	(94.6%, 96.9%)
80	85.1%	(84.2%, 86.1%)	80	95.3%	(94.1%, 96.5%)
(a) $\chi^2_{1,0.95}/(2n)$			(b) Appro	oximate 95%-0	quantile of $\sup_x J_n(x)$

 Table 5
 Two-sided coverage probabilities for different Burg-divergence ball sizes and sample sizes for a

continuous distribution

n	1-sided	95% C.I. of	n	1-sided	95% C.I. of	
	Cover. Prob.	Cover. Prob.		Cover. Prob.	Cover. Prob.	
20	83.9%	(81.7%, 86.1%)	20	93.6%	(91.9%, 95.3%)	
30	84.6%	(82.4%, 86.8%)	30	94.4%	(93.1%, 95.7%)	
40	86.1%	(84.0%, 88.2%)	40	95.2%	(93.9%, 96.5%)	
50	87.7%	(85.7%, 89.7%)	50	95.2%	(93.9%, 96.5%)	
60	87.5%	(85.4%, 89.6%)	60	96.5%	(95.4%, 97.6%)	
80	89.5%	(87.6%, 91.4%)	80	96.1%	(94.9%, 97.3%)	
(a) $\chi^2_{1 0.95}/(2n)$			(b) Appro	oximate 95%-o	quantile of $\sup_x J$	$J_n(x)$

(a) $\chi^2_{1,0.95}/(2n)$ (b) Approximate 95%-quantile of $\sup_x J_n(x)$ **Table 6** One-sided coverage probabilities for different Burg-divergence ball sizes and sample sizes for a continuous distribution

5. Conclusion

We have motivated and investigated the construction of tractable uncertainty sets that can recover the feasibility guarantees on par with the implications of CLT. We have shown that the empirical Burg-entropy divergence balls are capable of achieving such guarantees. We have also shown, intriguingly, that these balls are invalid confidence regions in the standard framework of data-driven DRO, and can have low or zero coverages on the true underlying distributions. Rather, we have explained their statistical performances via linking the resulting DRO with empirical likelihood. This link allows us to derive the optimal sizes of these balls, using the quantiles of χ^2 -process excursion. Such a calibration approach also bypasses some documented difficulties in using divergence balls in the data-driven DRO literature. Future work includes further developments of the theory and calibration methods to incorporate optimization objectives and more general constraints.

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Appendix A: Technical Proofs

Theorem 2 is a simple consequence of the following proposition:

PROPOSITION 2. Under the same conditions as Theorem 2, $\underline{Z}_n(x) \leq \overline{Z}_n(x) \leq \overline{Z}_n(x)$ if and only if $-2\log R(Z_0(x)) \leq \chi^2_{1,1-\alpha}$.

Proof of Proposition 2. We first argue that the optimization defining (38) must have an optimal solution, if it is feasible. Since $-2\sum_{i=1}^{n} \log(nw_i) \to \infty$ as $w_i \to 0$ for any i, it suffices to consider only w_i such that $w_i \ge \epsilon$ for some small $\epsilon > 0$. Since the set $\{\sum_{i=1}^{n} h(x;\xi_i)w_i = Z_0(x), \sum_{i=1}^{n} w_i = 1, w_i \ge \epsilon \text{ for all } i = 1, ..., n\}$ is compact, by Weierstrass Theorem, there exists an optimal solution for (38).

Suppose $-2\log R(Z_0(x)) \le \chi^2_{1,1-\alpha}$. Then the optimization in $-2\log R(Z_0(x))$ is feasible, and there must exist a probability vector $\mathbf{w} = (w_1, \dots, w_n)$ such that $-2\sum_{i=1}^n \log(nw_i) \le \chi^2_{1,1-\alpha}$ and $\sum_{i=1}^n h(x;\xi_i)w_i = Z_0(x)$. This implies $\underline{Z}_n(x) \le Z_0(x) \le \overline{Z}_n(x)$.

To show the reverse direction, note first that the set

$$\left\{\sum_{i=1}^{n} h(x;\xi_i)w_i: -2\sum_{i=1}^{n} \log(nw_i) \le \chi^2_{1,1-\alpha}, \sum_{i=1}^{n} w_i = 1, w_i \ge 0 \text{ for all } i = 1,\dots,n\right\}$$

is an interval, since $\sum_{i=1}^{n} h(x;\xi_i) w_i$ is a linear function of the convex set

$$\left\{ (w_1, \dots, w_n) : -2\sum_{i=1}^n \log(nw_i) \le \chi^2_{1,1-\alpha}, \sum_{i=1}^n w_i = 1, \ w_i \ge 0 \text{ for all } i = 1, \dots, n \right\}$$

Moreover, since the latter set is compact, by Weierstrass Theorem again, there must exist optimal solutions in the optimization pair

$$\max / \min\left\{\sum_{i=1}^{n} h(x;\xi_i)w_i : -2\sum_{i=1}^{n} \log(nw_i) \le \chi^2_{1,1-\alpha}, \sum_{i=1}^{n} w_i = 1, w_i \ge 0 \text{ for all } i = 1, \dots, n\right\}$$

Therefore, $\underline{Z}_n(x) \leq Z_0(x) \leq \overline{Z}_n(x)$ implies that there exists a probability vector \mathbf{w} such that $\sum_{i=1}^n h(x;\xi_i)w_i = Z_0(x)$ and $-2\sum_{i=1}^n \log(nw_i) \leq \chi^2_{1,1-\alpha}$, leading to $-2\log R(Z_0(x)) \leq \chi^2_{1,1-\alpha}$. \Box

Proof of Theorem 2. By Theorem 1, we have $\lim_{n\to\infty} P(-2\log R(Z_0(x)) \le \chi^2_{1,1-\alpha}) = 1 - \alpha$ for a fixed $x \in \Theta$, where

$$-2\log R(Z_0(x)) = \min\left\{-2\sum_{i=1}^n \log(nw_i) : \sum_{i=1}^n h(x;\xi_i)w_i = Z_0(x), \sum_{i=1}^n w_i = 1, w_i \ge 0 \text{ for all } i = 1,\dots,n\right\} (38)$$

Thus, to show (25), it suffices to prove that $\underline{Z}_n(x) \leq Z_0(x) \leq \overline{Z}_n(x)$ if and only if $-2 \log R(Z_0(x)) \leq \chi^2_{1,1-\alpha}$. Proposition 2 finishes the proof. \Box

Proof of Proposition 1. By relabeling the weights under membership of the support points, we rewrite

$$\underline{Z}_{n}(x) = \min\left\{\sum_{i=1}^{k} h(x;s_{i})\sum_{j=1}^{n_{i}} w_{ij} : -\frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n_{i}} \log(nw_{ij}) \le \frac{\chi_{1,1-\alpha}^{2}}{2n}, \\ \sum_{i=1}^{k}\sum_{j=1}^{n_{i}} w_{ij} = 1, \ w_{ij} \ge 0 \text{ for } i = 1,\dots,k, \ j = 1,\dots,n_{i}\right\}$$
(39)

and

$$\overline{Z}_{n}(x) = \max\left\{\sum_{i=1}^{k} h(x;s_{i})\sum_{j=1}^{n_{i}} w_{ij} : -\frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n_{i}} \log(nw_{ij}) \le \frac{\chi_{1,1-\alpha}^{2}}{2n}, \\ \sum_{i=1}^{k}\sum_{j=1}^{n_{i}} w_{ij} = 1, \ w_{ij} \ge 0 \text{ for } i = 1, \dots, k, \ j = 1, \dots, n_{i}\right\}$$
(40)

To avoid repetition, we focus on the maximization formulation. We show that, for any feasible **p** in $\max_{\mathbf{p}\in\mathcal{U}'_{Burg}} E_{\mathbf{p}}[h(x;\xi)]$, we can construct a feasible **w** for $\overline{Z}_n(x)$ that attains the same objective value, and vice versa.

To this end, for any $\mathbf{p} = (p_1, \ldots, p_k) \in \mathcal{U}'_{Burg}$, we define $w_{ij} = p_i/n_i$ for all $j = 1, \ldots, n_i$. Then

$$-\frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n_{i}}\log(nw_{ij}) = -\sum_{i=1}^{k}\frac{n_{i}}{n}\log\frac{np_{i}}{n_{i}} = -\sum_{i=1}^{k}\hat{p}_{i}\log\frac{p_{i}}{\hat{p}_{i}} \le \frac{\chi_{1,1-\alpha}^{2}}{2n}$$

as well as $\sum_{i=1}^{k} \sum_{j=1}^{n_i} w_{ij} = \sum_{i=1}^{k} p_i = 1$, and $w_{ij} \ge 0$ for all *i* and *j*. Hence w_{ij} is feasible for (40). Moreover, $\sum_{i=1}^{k} h(x;s_i) \sum_{j=1}^{n_i} w_{ij} = \sum_{i=1}^{k} h(x;s_i) p_i$, thus the same objective value is attained.

On the other hand, suppose $\mathbf{w} = (w_{ij})$ is a feasible solution for (40). We then define $p_i = \sum_{j=1}^{n_i} w_{ij}$. By Jensen's inequality we have $-\log(p_i/n_i) \leq -(1/n_i) \sum_{j=1}^{n_i} \log w_{ij}$, and so

$$-\sum_{i=1}^{k} \frac{n_i}{n} \log \frac{np_i}{n_i} \le -\frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \log(nw_{ij}) \le \frac{\chi_{1,1-\alpha}^2}{2n}$$

Together with the simple observation that $\sum_{i=1}^{k} p_i = \sum_{i=1}^{k} \sum_{j=1}^{n_i} w_{ij} = 1$ and $p_i \ge 0$ for all i, we get that $\mathbf{p} = (p_i)$ is feasible for $\max_{\mathbf{p} \in \mathcal{U}_{Burg}} E_{\mathbf{p}}[h(x;\xi)]$. Moreover, $\sum_{i=1}^{k} h(x;s_i)p_i = \sum_{i=1}^{k} h(x;s_i) \sum_{j=1}^{n_i} w_{ij}$, thus the same objective value is attained in this case as well.

Similar arguments apply to the minimization formulation, and we conclude the proof. \Box

Proof of Theorem 5. First, Assumption 1 allows us to define $\tilde{h}(x;\xi) = h(x;\xi) - Z_0(x)$. Also, we denote the classes of functions $\Xi \to \mathbb{R}$

$$\mathcal{H}_{\Theta}^{1} = \{ |\tilde{h}(x; \cdot)| : x \in \Theta \}$$
$$\mathcal{H}_{\Theta}^{2} = \{ \tilde{h}(x; \cdot)^{2} : x \in \Theta \}$$

$$\begin{split} \mathcal{H}_{\Theta}^{+} &= \{\tilde{h}(x;\cdot)^{+} : x \in \Theta\} \\ \mathcal{H}_{\Theta}^{-} &= \{\tilde{h}(x;\cdot)^{-} : x \in \Theta\} \end{split}$$

where

$$y^{+} = \begin{cases} y & \text{if } y \ge 0\\ 0 & \text{if } y < 0 \end{cases} \quad \text{and} \quad y^{-} = \begin{cases} 0 & \text{if } y > 0\\ -y & \text{if } y \le 0 \end{cases}$$

Since \mathcal{H}_{Θ} is a P_0 -Donsker class, it is P_0 -Glivenko-Cantelli (GC) (e.g., the discussion before Example 2.1.3 in Van Der Vaart and Wellner (1996)). By the preservation theorem (Theorem 8 in Appendix B), since $E_0 \|\tilde{h}(\cdot;\xi)^2\|_{\Theta} = E \|\tilde{h}(\cdot;\xi)\|_{\Theta}^2 < \infty$ by Assumption 3, \mathcal{H}_{Θ}^2 is also P_0 -GC. Moreover, since $E \|\tilde{h}(\cdot;\xi)^{\pm}\|_{\Theta} \leq E \|\tilde{h}(\cdot;\xi)\|_{\Theta} \leq \sqrt{E} \|\tilde{h}(\cdot;\xi)\|_{\Theta}^2 < \infty$, \mathcal{H}_{Θ}^+ , \mathcal{H}_{Θ}^- and \mathcal{H}_{Θ}^1 are all P_0 -GC as well. Letting P_n be the empirical measure generated from ξ_1, \ldots, ξ_n , the above imply

$$\|P_n - P_0\|_{\mathcal{H}^+_D} \stackrel{a.s.}{\to} 0 \tag{41}$$

$$\|P_n - P_0\|_{\mathcal{H}_{O}^-} \stackrel{a.s.}{\to} 0 \tag{42}$$

$$\|P_n - P_0\|_{\mathcal{H}^2_{\Theta}} \stackrel{a.s.}{\to} 0 \tag{43}$$

where $||P_n - P_0||_{\mathcal{F}} = \sup_{f \in \mathcal{F}} |P_n(f) - P_0(f)|$ for P_n indexed by \mathcal{F} , and similarly defined for P_0 (see Appendix B).

Note that (43) in particular implies the uniform convergence of the empirical variance

$$\left\|\frac{1}{n}\sum_{i=1}^{n}(h(\cdot;\xi_{i})-Z_{0}(\cdot))^{2}-\sigma_{0}(\cdot)\right\|_{\Theta} \stackrel{a.s.}{\to} 0$$

$$(44)$$

where $\sigma_0(x) = Var_0(h(x;\xi)).$

Now, for each x,

$$E_0 h(x;\xi)^+ + E_0 h(x;\xi)^- = E_0 |h(x;\xi)|$$
$$E_0 \tilde{h}(x;\xi)^+ - E_0 \tilde{h}(x;\xi)^- = E_0 \tilde{h}(x;\xi) = 0$$

which gives $E_0 \tilde{h}(x;\xi)^+ = E_0 \tilde{h}(x;\xi)^- = E_0 |\tilde{h}(x;\xi)|/2$. Hence

$$\inf_{x \in \Theta} E_0 \tilde{h}(x;\xi)^{\pm} = \inf_{x \in \Theta} \frac{E_0 |h(x;\xi)|}{2} \ge \frac{c}{2}$$
(45)

where we define c as a constant such that $\inf_{x\in\Theta} E_0|h(x;\xi) - Z_0(x)| \ge c$, which exists by Assumption 2. By Jensen's inequality,

$$\inf_{x\in\Theta} E_0 \tilde{h}(x;\xi)^2 \ge \left(\inf_{x\in\Theta} E_0 |\tilde{h}(x;\xi)|\right)^2 \ge c^2 \tag{46}$$

by using Assumption 2 again. From (41), (42) and (45), we have $\inf_{x\in\Theta}(1/n)\sum_{i=1}^{n}\tilde{h}(x;\xi_i)^+$ and $\inf_{x\in\Theta}(1/n)\sum_{i=1}^{n}\tilde{h}(x;\xi_i)^- > 0$ for large enough n a.s.. When this occurs, $\min_{1\leq i\leq n}h(x;\xi_i) < Z_0(x) < \max_{1\leq i\leq n}h(x;\xi_i)$ for every x, and the optimization defining $-\log R(x;Z_0)$, namely

$$\min\left\{-\sum_{i=1}^{n}\log(nw_i)\left|\sum_{i=1}^{n}h(x;\xi_i)w_i=Z_0(x), \sum_{i=1}^{n}w_i=1, w_i\geq 0 \text{ for } i=1,\dots,n\right\}\right\}$$
(47)

has a unique optimal solution $\mathbf{w}(x) = (w_1(x), \dots, w_n(x))$ with $w_i(x) > 0$ for all *i*, for any *x*. This is because setting any $w_i(x) = 0$ would render $-2\sum_{i=1}^n \log(nw_i) = \infty$ which is clearly suboptimal. Hence it suffices to replace $w_i \ge 0$ with $w_i \ge \epsilon$ for all *i* for some small enough $\epsilon > 0$. In this modified region, the optimum exists and is unique since $-\sum_{i=1}^n \log(nw_i)$ is strictly convex.

Now consider the optimization (47) when $\min_{1 \le i \le n} h(x; \xi_i) < Z_0(x) < \max_{1 \le i \le n} h(x; \xi_i)$. We adopt the proof technique in Section 11.2 of Owen (2001), but generalize at the empirical process level. For convenience, we write $\tilde{h}_i = \tilde{h}(x; \xi_i) = h(x; \xi_i) - Z_0(x)$, and also suppress the x in $w_i = w_i(x)$ and $Z_0 = Z_0(x)$. The Lagrangian is written as

$$-\sum_{i=1}^{n}\log(nw_i) + \lambda\left(\sum_{i=1}^{n}\tilde{h}_iw_i - Z_0\right) + \gamma\left(\sum_{i=1}^{n}w_i - 1\right)$$

where $\lambda = \lambda(x)$ and $\gamma = \gamma(x)$ are the Lagrange multipliers. Differentiating with respect to w_i and setting it to zero, we have

$$-\frac{1}{w_i} + \lambda \tilde{h}_i + \gamma = 0 \tag{48}$$

Setting $\sum_{i=1}^{n} \tilde{h}_i w_i = 0$ and $\sum_{i=1}^{n} w_i = 1$, multiplying both sides of (48) by w_i and summing up over i, we get $\gamma = n$. Using (48) again, we have

$$w_i = \frac{1}{n} \frac{1}{1 + \lambda \tilde{h}_i} \tag{49}$$

where the λ in (49) is rescaled by a factor of n. Note that we can find λ such that

$$\sum_{i=1}^{n} \frac{1}{n} \frac{\tilde{h}_i}{1+\lambda \tilde{h}_i} = 0 \tag{50}$$

and $\frac{1}{n}\frac{1}{1+\lambda\tilde{h}_i} > 0$ for all i, upon which the Karush-Kuhn-Tucker (KKT) condition can be seen to hold and conclude that w_i in (49) is the optimal solution. Indeed, let $\tilde{h}^* = \max_i \tilde{h}_i > 0$ and $\tilde{h}_* = \min_i \tilde{h}_i < 0$. Note that $\sum_{i=1}^n \frac{1}{n}\frac{\tilde{h}_i}{1+\lambda\tilde{h}_i} \to \infty$ as $\lambda \to -1/\tilde{h}^*$, and $\to -\infty$ as $\lambda \to -1/\tilde{h}_*$. Since $\sum_{i=1}^n \frac{1}{n}\frac{\tilde{h}_i}{1+\lambda\tilde{h}_i}$ is a continuous function in λ between $-1/\tilde{h}^*$ and $-1/\tilde{h}_*$, there must exist a λ that solves (50). Moreover, $\frac{1}{n}\frac{1}{1+\lambda\tilde{h}_i} > 0$ for all i for this λ .

Given this characterization of the optimal solution, the rest of the proof is to derive the asymptotic behavior of $-2\log R(x; Z_0)$ as $n \to \infty$. First, we write

$$\frac{1}{1+\lambda \tilde{h}_i} = 1 - \frac{\lambda h_i}{1+\lambda \tilde{h}_i}$$

Multiplying both sides by h_i/n and summing up over *i*, we get

$$\sum_{i=1}^{n} \frac{1}{n} \frac{\tilde{h}_i}{1+\lambda \tilde{h}_i} = \bar{h} - \lambda \sum_{i=1}^{n} \frac{1}{n} \frac{\tilde{h}_i^2}{1+\lambda \tilde{h}_i}$$

where $\bar{h} := \bar{h}(x) = (1/n) \sum_{i=1}^{n} \tilde{h}(x;\xi_i)$, and hence

$$\bar{h} = \lambda \sum_{i=1}^{n} \frac{1}{n} \frac{\tilde{h}_i^2}{1 + \lambda \tilde{h}_i}$$
(51)

by (50). Now let

$$s := s(x) = \frac{1}{n} \sum_{i=1}^{n} \tilde{h}(x;\xi_i)^2$$

be the empirical variance. Note that, from (49), $w_i > 0$ implies $1 + \lambda \tilde{h}_i > 0$. Together with $s \ge 0$, we get

$$\begin{split} |\lambda|s &\leq \left|\lambda \sum_{i=1}^{n} \frac{1}{n} \frac{\tilde{h}_{i}^{2}}{1 + \lambda \tilde{h}_{i}}\right| \left(1 + |\lambda| \max_{1 \leq i \leq n} |\tilde{h}_{i}|\right) \\ &= |\bar{h}| \left(1 + |\lambda| \max_{1 \leq i \leq n} |\tilde{h}_{i}|\right) \end{split}$$

by using (51), and hence

$$|\lambda| \left(s - |\bar{h}| \max_{1 \le i \le n} |\tilde{h}_i| \right) \le |\bar{h}| \tag{52}$$

By Lemma 3 in Appendix C, $E\|\tilde{h}(x;\xi)\|_{\Theta}^2<\infty$ in Assumption 3 implies that

$$\max_{1 \le i \le n} \|\tilde{h}_i\|_{\Theta} = o(n^{1/2}) \text{ a.s.}$$
(53)

Moreover, since \mathcal{H}_{Θ} is P_0 -Donsker, we have $\sqrt{n}\bar{h} = \sqrt{n}(P_n(h(\cdot; \cdot)) - P_0(h(\cdot; \cdot))) \Rightarrow \tilde{G}$ in $\ell^{\infty}(\mathcal{H}_{\Theta})$, where $\tilde{G}(\cdot)$ is a tight Gaussian process indexed by $h(x; \cdot) \in \mathcal{H}_{\Theta}$ that is centered and has covariance $Cov(\tilde{G}(h(x_1; \cdot)), \tilde{G}(h(x_2; \cdot))) = Cov_0(h(x_1; \xi), h(x_2; \xi))$ for any $h(x_1; \cdot), h(x_2; \cdot) \in \mathcal{H}_{\Theta}$. Noting that the map $\ell^{\infty}(\mathcal{H}_{\Theta}) \to \ell^{\infty}(\Theta)$ defined by $y(\cdot) \mapsto y(h(\cdot; \cdot))$ is continuous, by the continuous mapping theorem (Theorem 9 in Appendix C), we have $\sqrt{n}\bar{h} \Rightarrow \tilde{G}$ in $\ell^{\infty}(\Theta)$ where \tilde{G} is now indexed by $x \in \Theta$. As the norm in $\ell^{\infty}(\Theta), \|\cdot\|_{\Theta}$ is a continuous map. By the continuous mapping theorem again, $\sqrt{n}\|\bar{h}\|_{\Theta} = \|\sqrt{n}\bar{h}\|_{\Theta} \Rightarrow \|\tilde{G}\|_{\Theta}$, so that $\|\bar{h}\|_{\Theta} = O_p(n^{-1/2})$. Moreover, $\||\bar{h}|\max_{1\leq i\leq n}|\tilde{h}_i|\|_{\Theta} \leq \|\bar{h}\|_{\Theta} \max_{1\leq i\leq n}\|\tilde{h}_i\|_{\Theta} = O_p(n^{-1/2})o(n^{1/2}) = o_p(1)$.

Next, from (44) and (46), we have $\inf_{x \in \Theta} s(x) \ge c_1$ for some $c_1 > 0$ ev.. Pick any constant $\varepsilon < c_1$. We have

$$P\left(\inf_{x\in\Theta}\left\{s(x)-|\bar{h}(x)|\max_{1\leq i\leq n}|\tilde{h}_i(x)|\right\}\geq c_1-\varepsilon\right)\geq P\left(\inf_{x\in\Theta}s(x)\geq c_1, \left\||\bar{h}(x)|\max_{1\leq i\leq n}|\bar{h}_i(x)|\right\|_{\Theta}\leq\varepsilon\right)\to 1$$
(54)

Over the set $\{\inf_{x\in\Theta}\{s(x) - |\bar{h}(x)| \max_{1\leq i\leq n} |\bar{h}_i(x)|\} > c_1 - \varepsilon\}, (52)$ implies

$$|\lambda(x)| \le \frac{|h(x)|}{s - |\bar{h}(x)| \max_{1 \le i \le n} |\bar{h}_i(x)|}$$

for all $x \in \Theta$, so that

$$\|\lambda\|_{\Theta} \le \frac{\|h\|_{\Theta}}{c_1 - \varepsilon} \tag{55}$$

We argue that $\|\lambda\|_{\Theta} = O_p(n^{-1/2})$. This is because, for any given $\delta > 0$, we can find a large enough B > 0 such that

by (54) and that $\|\bar{h}\|_{\Theta} = O_p(n^{-1/2})$ as shown above. This and (53) together gives

$$\max_{1 \le i \le n} \sup_{x \in \Theta} |\lambda(x)\tilde{h}_i(x)| \le \|\lambda\|_{\Theta} \max_{1 \le i \le n} \|\tilde{h}_i\|_{\Theta} = O_p(n^{-1/2})o(n^{1/2}) = o_p(1)$$
(56)

Now (50) can be rewritten as

$$0 = \sum_{i=1}^{n} \frac{1}{n} \tilde{h}_{i} \left(1 - \lambda \tilde{h}_{i} + \frac{\lambda^{2} \tilde{h}_{i}^{2}}{1 + \lambda \tilde{h}_{i}} \right)$$
$$= \bar{h} - \lambda s + \sum_{i=1}^{n} \frac{1}{n} \frac{\lambda^{2} \tilde{h}_{i}^{3}}{1 + \lambda \tilde{h}_{i}}$$
(57)

The last term in (57) satisfies

$$\left|\sum_{i=1}^{n} \frac{1}{n} \frac{\lambda^2 \tilde{h}_i^3}{1+\lambda \tilde{h}_i}\right| \leq \frac{1}{n} \sum_{i=1}^{n} \tilde{h}_i^2 \lambda^2 \max_{1 \leq i \leq n} |\tilde{h}_i| \max_{1 \leq i \leq n} (1+\lambda \tilde{h}_i)^{-1}$$

Taking $\sup_{\theta\in\Theta}$ on both sides, we get

$$\left\|\sum_{i=1}^{n} \frac{1}{n} \frac{\lambda^2 \tilde{h}_i^3}{1+\lambda \tilde{h}_i}\right\|_{\Theta} \le \|s\|_{\Theta} \|\lambda\|_{\Theta}^2 \max_{1 \le i \le n} \|\tilde{h}_i\|_{\Theta} \max_{1 \le i \le n} \sup_{x \in \Theta} (1+\lambda \tilde{h}(x,\xi_i))^{-1}$$
(58)

Now, $||s||_{\Theta} \to ||\sigma_0||_{\Theta}$ by (44). Moreover, for any small $\varepsilon > 0$,

$$\begin{split} P\left(\max_{1\leq i\leq n}\sup_{x\in\Theta}(1+\lambda\tilde{h}(x,\xi_i))^{-1} > \frac{1}{1-\varepsilon}\right) &= P\left(\frac{1}{1+\lambda\tilde{h}(x,\xi_i)} > \frac{1}{1-\varepsilon} \text{ for some } 1\leq i\leq n \text{ and } \theta\in\Theta\right)\\ &\leq P(\lambda\tilde{h}(x,\xi_i) < -\varepsilon \text{ for some } 1\leq i\leq n \text{ and } x\in\Theta)\\ &\leq P\left(\max_{1\leq i\leq n}\sup_{x\in\Theta}|\lambda\tilde{h}(x,\xi_i)| > \varepsilon\right) \to 0 \end{split}$$

by (56). Thus (58) is bounded from above by

$$O(1)O_p(n^{-1})o(n^{1/2})O_p(1) = o_p(n^{-1/2})$$
(59)

From (57) and (59), we have

$$0 = h - \lambda s + \epsilon$$

where $\|\epsilon\|_{\Theta} = o_p(n^{-1/2})$. Since (44) and (46) implies $\inf_{x \in \Theta} s(x) \ge c_1$ for some $c_1 > 0$ ev., we further get

$$\lambda = s^{-1}(\bar{h} + \epsilon) \tag{60}$$

Now consider

$$-2\log R(x; Z_0) = -2\sum_{i=1}^{n} \log(nw_i)$$

= $2\sum_{i=1}^{n} \log(1 + \lambda \tilde{h}_i)$
= $2\sum_{i=1}^{n} \left(\lambda \tilde{h}_i - \frac{1}{2}\lambda^2 \tilde{h}_i^2 + \nu_i\right)$ (61)

where

$$\nu_i = \frac{1}{3} \frac{1}{(1+\zeta_i)^3} (\lambda \tilde{h}_i)^3$$

with $\zeta_i = \zeta_i(x)$ between 0 and $\lambda \tilde{h}_i$ by Taylor's expansion. So

$$|\nu_i| \le \frac{1}{3} \frac{1}{|1+\zeta_i|^3} |\lambda \tilde{h}_i|^3 \tag{62}$$

For any large enough $B_1 > 0$, we have

$$P(|\nu_{i}(x)| > B_{1}|\lambda(x)\tilde{h}(x,\xi_{i})|^{3} \text{ for all } x \in \Theta \text{ and } 1 \leq i \leq n)$$

$$\leq P\left((3B_{1})^{1/3} \max_{1 \leq i \leq n} ||1+\zeta_{i}||_{\Theta} < 1\right) \text{ from (62)}$$

$$= P\left((3B_{1})^{1/3} \max_{1 \leq i \leq n} ||1+\zeta_{i}||_{\Theta} < 1, \max_{1 \leq i \leq n} ||\zeta_{i}||_{\Theta} < \varepsilon\right)$$

$$+ P\left((3B_{1})^{1/3} \max_{1 \leq i \leq n} ||1+\zeta_{i}||_{\Theta} < 1, \max_{1 \leq i \leq n} ||\zeta_{i}||_{\Theta} > \varepsilon\right) \text{ for some sufficiently large } 0 < \varepsilon < 1$$

$$\leq P\left((3B_{1})^{1/3} \max_{1 \leq i \leq n} (1-\varepsilon) < 1\right) + P\left(\max_{1 \leq i \leq n} ||\zeta_{i}||_{\Theta} > \varepsilon\right)$$

$$\to 0 \tag{63}$$

since $\max_{1 \le i \le n} \|\zeta_i\|_{\Theta} \le \max_{1 \le i \le n} \|\lambda \tilde{h}_i\|_{\Theta} = o_p(1)$ by (56). Now (61) gives

$$2n\lambda\bar{h} - \lambda^2 ns + 2\sum_{i=1}^{n} \nu_i$$

= $2ns^{-1}(\bar{h} + \epsilon)\bar{h} - nss^{-2}(\bar{h}^2 + 2\epsilon\bar{h} + \epsilon^2) + 2\sum_{i=1}^{n} \nu_i$ by (60)
= $ns^{-1}\bar{h}^2 - ns^{-1}\epsilon^2 + 2\sum_{i=1}^{n} \nu_i$ (64)

Note that

$$\|ns^{-1}\epsilon^2\|_{\Theta} \le nO(1)o_p(n^{-1}) = o_p(1)$$
(65)

since $\inf_{x\in\Theta} s(x) \ge c_1 > 0$ ev., and $\|\epsilon\|_{\Theta} = o_p(n^{-1/2})$. Moreover, over the set $\{|\nu_i(x)| \le B_1 |\lambda(x)\tilde{h}_i(x)|^3$ for all $x \in \Theta$ and $1 \le i \le n\}$, we have

$$\left|\sum_{i=1}^{n} \nu_i\right| \le B_1 |\lambda|^3 \sum_{i=1}^{n} |\tilde{h}_i|^2 \max_{1 \le i \le n} |\tilde{h}_i|$$

for all $x \in \Theta$. Note that

$$\left\| B_1 |\lambda|^3 \sum_{i=1}^n |\tilde{h}_i|^2 \max_{1 \le i \le n} |\tilde{h}_i| \right\|_{\Theta} \le B_1 O_p(n^{-3/2}) n O(1) o(n^{1/2}) = o_p(1)$$
(66)

since $\|\lambda\|_{\Theta} = O_p(n^{-1/2}), \|s\|_{\Theta} = O(1)$, and $\max_{1 \le i \le n} \|\tilde{h}_i\|_{\Theta} = o(n^{1/2})$. Now, for any $\varepsilon > 0$,

$$\begin{split} &P\left(\left\|\sum_{i=1}^{n}\nu_{i}\right\|_{\Theta} > \varepsilon\right) \\ &\leq P\left(\left\|\sum_{i=1}^{n}\nu_{i}\right\|_{\Theta} > \varepsilon, \ |\nu_{i}(x)| \leq B_{1}|\lambda(x)\tilde{h}_{i}(x)|^{3} \text{ for some } x \in \Theta \text{ and } 1 \leq i \leq n\right) \\ &+ P(|\nu_{i}(x)| > B_{1}|\lambda(x)\tilde{h}_{i}(x)|^{3} \text{ for all } x \in \Theta \text{ and } 1 \leq i \leq n) \\ &\leq P\left(\left\|B_{1}|\lambda|^{3}\sum_{i=1}^{n}|\tilde{h}_{i}|^{2}\max_{1\leq i\leq n}|\tilde{h}_{i}|\right\|_{\Theta} > \varepsilon\right) + P(|\nu_{i}| > B_{1}|\lambda\tilde{h}_{i}|^{3} \text{ for all } x \in \Theta \text{ and } 1 \leq i \leq n) \\ &\to 0 \end{split}$$

by (63) and (66). Hence we have

$$\sum_{i=1}^{n} \nu_i \bigg\|_{\Theta} = o_p(1) \tag{67}$$

Using (65) and (67), (64) implies that

$$-2\log R(x;Z_0) = ns^{-1}\bar{h}^2 + \epsilon_1$$
(68)

where $\|\epsilon_1\|_{\Theta} = o_p(1)$.

Note that $\sqrt{n}\bar{h} \Rightarrow \tilde{G}$ in $\ell^{\infty}(\Theta)$ where $\tilde{G}(\cdot)$ is defined previously as the centered Gaussian process with covariance $Cov(\tilde{G}(x_1), \tilde{G}(x_2)) = Cov(h(x_1;\xi), h(x_2;\xi))$ for any $x_1, x_2 \in \Theta$. By Slutsky's Theorem (Theorem 10 in Appendix C) and (44), $(\sqrt{n}\bar{h}, s) \Rightarrow (\tilde{G}, \sigma_0)$ in $(\ell^{\infty} \times \ell^{\infty})(\Theta)$ defined as

$$(\ell^{\infty} \times \ell^{\infty})(\Theta) = \left\{ (y_1, y_2) : \Theta \to \mathbb{R}^2 \middle| \|y_1\|_{\Theta} + \|y_2\|_{\Theta} < \infty \right\}$$

Note that pointwise division and $(\cdot)^2$ are continuous maps on $(\ell^{\infty} \times \ell^{\infty})(\Theta) \to \ell^{\infty}(\Theta)$ and $\ell^{\infty}(\Theta) \to \ell^{\infty}(\Theta)$ and $\ell^{\infty}(\Theta) \to \ell^{\infty}(\Theta)$ respectively. Also, $\inf_{x \in \Theta} \sigma_0(x) > 0$ by Assumption 2 and Jensen's inequality. By continuous mapping theorem, we have $ns^{-1}\bar{h}^2 \Rightarrow J$ in $\ell^{\infty}(\Theta)$ where $J(\cdot)$ is as defined in the theorem. Finally, from (68) and $\|\epsilon_1\|_{\Theta} = o_p(1)$, we get further that $-2\log R(\cdot; Z_0) \Rightarrow J(\cdot)$ in $\ell^{\infty}(\Theta)$. This concludes the proof. \Box

Proof of Lemma 2. First define $\tilde{G}(\cdot)$ as a centered Gaussian process indexed by $x \in \Theta$ with covariance $Cov(\tilde{G}(x_1), \tilde{G}(x_2)) = Cov_0(h(x_1; \xi), h(x_2; \xi))$. Conditional on almost every data realization $(P_n : n \ge 1)$, define $\tilde{G}_n(\cdot)$ as a centered Gaussian process indexed by $x \in \Theta$ with covariance $Cov(\tilde{G}_n(x_1), \tilde{G}_n(x_2)) = (1/n) \sum_{i=1}^n (h(x_1; \xi) - \hat{h}(x_1))(h(x_2; \xi) - \hat{h}(x_2))$, and $\hat{h}(x) = (1/n) \sum_{i=1}^n h(x; \xi_i)$.

We first show that $\tilde{G}_n(\cdot) \Rightarrow \tilde{G}(\cdot)$ in $\ell^{\infty}(\Theta)$. Note that, by the property of Gaussian processes, any finite-dimensional vector $(\tilde{G}_n(x_1), \ldots, \tilde{G}_n(x_d))$ is distributed as $N(\mathbf{0}, \Sigma_n)$, where **0** is the zero vector and

$$\Sigma_n = \left(\frac{1}{n} \sum_{k=1}^n (h(x_i; \xi_k) - \hat{h}(x_i))(h(x_j; \xi_k) - \hat{h}(x_j))\right)_{i,j=1,\dots,d}$$

On the other hand, $(G(x_1), \ldots, G(x_d))$ is distributed as $N(\mathbf{0}, \Sigma)$, where $\Sigma = (Cov_0(h(x_i;\xi), h(x_j;\xi))_{i,j=1,\ldots,d})$. Note that $\Sigma_n \to \Sigma$ a.s. in each entry, and hence $(\tilde{G}_n(x_1), \ldots, \tilde{G}_n(x_d)) \Rightarrow (\tilde{G}(x_1), \ldots, \tilde{G}(x_d))$ (by using for example convergence of the characteristic function).

Next, note that by Assumption 4, \mathcal{H}_{Θ} is P_0 -Donsker and hence is totally bounded equipped with the semi-metric $\rho_0(h(x_1; \cdot), h(x_2; \cdot)) := (Var_0(h(x_1; \xi) - h(x_2; \xi)))^{1/2}$ (Section 2.1.2 in Van Der Vaart and Wellner (1996)). Equivalently, Θ is totally bounded under the semi-metric $\rho_0(x_1, x_2) := (Var_0((h(x_1; \xi) - h(x_2; \xi)))^{1/2}.$

We shall also show that $\tilde{G}_n(\cdot)$ is uniformly equicontinuous in probability under the same semimetric. To this end, we want to show that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P_{\boldsymbol{\xi}} \left(\sup_{\rho_0(x_1 x_2) < \delta} |\tilde{G}_n(x_1) - \tilde{G}_n(x_2)| > \epsilon \right) = 0$$
(69)

where $P_{\boldsymbol{\xi}}(\cdot)$ is the probability conditional on the data $\boldsymbol{\xi}$. First, by using the covariance structure of the Gaussian process $\tilde{G}_n(\cdot)$,

$$E(\tilde{G}_n(x_1) - \tilde{G}_n(x_2))^2 = \widehat{Var}_n(h(x_1;\xi) - h(x_2;\xi))$$

= $\frac{1}{n} \sum_{i=1}^n (h(x_1;\xi) - \hat{h}(x_1))^2 + \frac{1}{n} \sum_{i=1}^n (h(x_2;\xi) - \hat{h}(x_2))^2 - \frac{2}{n} \sum_{i=1}^n (h(x_1;\xi) - \hat{h}(x_1))(h(x_2;\xi) - \hat{h}(x_2))$

Now define $\tilde{h}(x;\xi) = h(x;\xi) - Z_0(x)$ under Assumption 1. Note that

$$E\left[\sup_{x,y\in\Theta} (\tilde{h}(x;\xi) - \tilde{h}(y;\xi))^2\right] \le E\left[\sup_{x\in\Theta} \tilde{h}(x;\xi)^2 + \sup_{y\in\Theta} \tilde{h}(y;\xi)^2 + 2\sup_{x\in\Theta} |\tilde{h}(x;\xi)| \sup_{y\in\Theta} |\tilde{h}(x;\xi)|\right]$$
$$= 4E\|\tilde{h}(\cdot;\xi)\|_{\Theta}^2 < \infty$$

by Assumption 3. Viewing $\tilde{h}(x; \cdot)$ and $\tilde{h}(y; \cdot)$ each as a function $(x, y) \in \Theta^2 \to \mathbb{R}$, we can apply the preservation theorem to conclude that the class of functions

$$\mathcal{H}^{\Pi}_{\Theta} = \{ (\tilde{h}(x;\cdot) - \tilde{h}(y;\cdot))^2 : (x,y) \in \Theta^2 \}$$

is a P_0 -GC class. Therefore,

$$\sup_{x,y\in\Theta} \left| \frac{1}{n} \sum_{i=1}^{n} (\tilde{h}(x;\xi_i) - \tilde{h}(y;\xi_i))^2 - E_0 (\tilde{h}(x;\xi_i) - \tilde{h}(y;\xi_i))^2 \right| \to 0 \quad \text{a.s.}$$
(70)

Now, note that

$$\widehat{Var}_{n}(h(x_{1};\xi) - h(x_{2};\xi)) = \frac{1}{n} \sum_{i=1}^{n} \left(\left(h(x_{1};\xi_{i}) - \hat{h}(x_{1}) \right) - \left(h(x_{2};\xi_{i}) - \hat{h}(x_{2}) \right) \right)^{2} \\
= \frac{1}{n} \sum_{i=1}^{n} \left(\left(\tilde{h}(x_{1};\xi_{i}) - \tilde{h}(x_{2};\xi_{i}) \right) - \left(\left(\hat{h}(x_{1}) - Z_{0}(x_{1}) \right) - \left(\hat{h}(x_{2}) - Z_{0}(x_{2}) \right) \right) \right)^{2} \\
= \frac{1}{n} \sum_{i=1}^{n} \left(\tilde{h}(x_{1};\xi_{i}) - \tilde{h}(x_{2};\xi_{i}) \right)^{2} - \left(\left(\hat{h}(x_{1}) - Z_{0}(x_{1}) \right) - \left(\hat{h}(x_{2}) - Z_{0}(x_{2}) \right) \right)^{2} \tag{71}$$

Since \mathcal{H}_{Θ} is P_0 -GC, $\|\hat{h}(\cdot) - Z_0(\cdot)\|_{\Theta} \to 0$ a.s.. Hence $\sup_{x_1, x_2 \in \Theta} \left(\left(\hat{h}(x_1) - Z_0(x_1) \right) - \left(\hat{h}(x_2) - Z_0(x_2) \right) \right)^2 \to 0$ a.s.. Combining with (70), we have, from (71),

$$\sup_{x_1, x_2 \in \Theta} |\widehat{Var}_n(h(x_1; \xi) - h(x_2; \xi)) - Var_0(h(x_1, \xi) - h(x_2, \xi))| \to 0 \quad \text{a.s.}$$
(72)

by noting that $Var_0(h(x_1,\xi) - h(x_2,\xi)) = E_0(\tilde{h}(x;\xi) - \tilde{h}(y;\xi))^2$.

Therefore, by (72), for any $\gamma > 1$, we have

$$E_{\xi}(\tilde{G}_{n}(x_{1}) - \tilde{G}_{n}(x_{2}))^{2} = \widehat{Var}_{n}(h(x_{1};\xi) - h(x_{2};\xi)) \leq \gamma Var_{0}(h(x_{1};\xi) - h(x_{2};\xi)) = E(\tilde{G}_{\gamma}(x_{1}) - \tilde{G}_{\gamma}(x_{2}))^{2}$$

a.s. for any $x_1, x_2 \in \Theta$, when *n* is sufficiently large, where $\tilde{G}_{\gamma}(\cdot) := \gamma G(\cdot)$ and $E_{\xi}[\cdot]$ denotes the expectation conditional on ξ . Thus, by the argument for the Sudakov-Fernique inequality (the first equation in the proof of Theorem 2.9 in Adler (1990)), we have

$$E_{\boldsymbol{\xi}}\left[\sup_{\boldsymbol{\rho}_0(x_1,x_2)<\delta} |\tilde{G}_n(x_1) - \tilde{G}_n(x_2)|\right] \le E\left[\sup_{\boldsymbol{\rho}_0(x_1,x_2)<\delta} |\tilde{G}_{\boldsymbol{\gamma}}(x_1) - \tilde{G}_{\boldsymbol{\gamma}}(x_2)|\right] = \boldsymbol{\gamma} E\left[\sup_{\boldsymbol{\rho}_0(x_1,x_2)<\delta} |\tilde{G}(x_1) - \tilde{G}(x_2)|\right]$$

when n is large. Note that

$$\lim_{\delta \to 0} E \left[\sup_{\rho_0(x_1, x_2) < \delta} \left| \tilde{G}(x_1) - \tilde{G}(x_2) \right| \right] = 0$$

since $\tilde{G}(\cdot)$ is tight by the P_0 -Donsker property of \mathcal{H}_{Θ} . Thus

$$\begin{split} & \limsup_{n \to \infty} P_{\boldsymbol{\xi}} \left(\sup_{\substack{\rho_0(x_1 x_2) < \delta} |\tilde{G}_n(x_1) - \tilde{G}_n(x_2)| > \epsilon \right)} \\ & \leq \limsup_{n \to \infty} \frac{E_{\boldsymbol{\xi}} \left[\sup_{\rho_0(x_1 x_2) < \delta} |\tilde{G}_n(x_1) - \tilde{G}_n(x_2)| \right]}{\epsilon} \quad \text{by Chebyshev's inequality} \\ & \leq \frac{\gamma E_{\boldsymbol{\xi}} \left[\sup_{\rho_0(x_1 x_2) < \delta} |\tilde{G}(x_1) - \tilde{G}(x_2)| \right]}{\epsilon} \\ & \to 0 \end{split}$$

as $\delta \to 0$. We have therefore proved (69). Together with total boundedness, we have $\hat{G}_n(\cdot) \Rightarrow \hat{G}(\cdot)$ in $\ell^{\infty}(\Theta)$ (Section 2.1.2 in Van Der Vaart and Wellner (1996)).

Finally, note that $\inf_{x\in\Theta} \sigma_0(x) > 0$ by Assumption 2 and Jensen's inequality. Using (44) and that pointwise division is a continuous map $(\ell^{\infty} \times \ell^{\infty})(\Theta) \to \ell^{\infty}(\Theta)$, Slutsky's Theorem and the continuous mapping theorem conclude that $G_n(\cdot) \Rightarrow G(\cdot)$ in $\ell^{\infty}(\Theta)$. \Box

Proof of Theorem 4. By Theorem 5 and Lemma 2, and using the fact that $(\cdot)^2$ and $\sup_{x\in\Theta} \cdot$ are continuous maps $\ell^{\infty}(\Theta) \to \ell^{\infty}(\Theta)$ and $\ell^{\infty}(\Theta) \to \mathbb{R}$ respectively, we have $\sup_{x\in\Theta} \{-2\log R(x;Z_0)\} \Rightarrow \sup_{x\in\Theta} J(x)$ and $\sup_{x\in\Theta} J_n(x) \Rightarrow \sup_{x\in\Theta} J(x)$, where $J_n(\cdot)$ and $J(\cdot)$ are defined in Theorems 4 and 5. Moreover, since $\sup_{x\in\Theta} J(x)$ has a continuous distribution function, pointwise convergence of distribution functions to that of $\sup_{x\in\Theta} J(x)$ implies uniform convergence. Hence we have

$$\sup_{q \in \mathbb{R}^+} \left| P_{\xi} \left(\sup_{x \in \Theta} J_n(x) \le q \right) - P \left(\sup_{x \in \Theta} J(x) \le q \right) \right| \to 0$$
(73)

and

$$\sup_{q \in \mathbb{R}^+} \left| P\left(\sup_{x \in \Theta} \{-2\log R(x; Z_0)\} \le q \right) - P\left(\sup_{x \in \Theta} J(x) \le q \right) \right| \to 0$$
(74)

Selecting q_n such that $P_{\boldsymbol{\xi}}(\sup_{x\in\Theta} J_n(x) \leq q_n) = 1 - \alpha$, (73) and (74) implies that

$$P\left(\sup_{x\in\Theta}\{-2\log R(x;Z_0)\}\leq q_n\right)\to 1-\alpha$$

By applying Proposition 2 to every point $x \in \Theta$ and with $\chi^2_{1,1-\alpha}$ with q_n , we have $-2\log R(x; Z_0) \le q_n$ if and only if $\underline{Z}_n^*(x) \le Z_0(x) \le \overline{Z}_n^*(x)$, for each $x \in \Theta$. Hence

$$P\left(\sup_{x\in\Theta}\{-2\log R(x;Z_0)\}\leq q_n\right) = P\left(-2\log R(x;Z_0)\leq q_n \text{ for all } x\in\Theta\right)$$
$$= P(\underline{Z}_n^*(x)\leq Z_0(x)\leq \overline{Z}_n^*(x) \text{ for all } x\in\Theta) \to 1-\alpha$$

Proof of Theorem 6. To avoid repetition, we focus on $\overline{Z}_n(x)$. Consider

$$\overline{Z}_n(x) - Z_0(x) = \max_{\mathbf{w} \in \mathcal{U}_n(q_n/(2n))} \sum_{i=1}^n \tilde{h}(x;\xi_i) w_i$$
(75)

where $\tilde{h}(x;\xi) = h(x;\xi) - Z_0(x)$. With Lagrangian relaxation, the program (75) can be written as

$$\min_{\lambda \ge 0, \gamma} \max_{\mathbf{w} \ge 0} \sum_{i=1}^{n} \tilde{h}(x; \xi_i) w_i - \lambda \left(-\frac{1}{n} \sum_{i=1}^{n} \log(nw_i) - \frac{q_n}{2n} \right) + \gamma \left(\sum_{i=1}^{n} w_i - 1 \right)$$

$$= \min_{\lambda \ge 0, \gamma} \sum_{i=1}^{n} \frac{\lambda}{n} \max_{w_i \ge 0} \left\{ \frac{\tilde{h}(x; \xi_i) + \gamma}{\lambda} nw_i + \log(nw_i) - nw_i + 1 \right\} + \lambda \frac{q_n}{2n} - \gamma$$

$$= \min_{\lambda \ge 0, \gamma} - \sum_{i=1}^{n} \frac{\lambda}{n} \log \left(1 - \frac{\tilde{h}(x; \xi_i) + \gamma}{\lambda} \right) + \lambda \frac{q_n}{2n} - \gamma$$
(76)

where $-0\log(1 - t/0) := 0$ for $t \le 0$ and $-0\log(1 - t/0) := \infty$ for t > 0, by using the conjugate function of $-\log r + r - 1$ as $\sup_{r \ge 0} \{tr + \log r - r + 1\} = -\log(1 - t)$ for t < 1, and ∞ for $t \ge 1$ (e.g., Ben-Tal et al. (2013)).

Now, to get an upper bound for (76), pick $\gamma = 0$, and λ as $\lambda_n = \Theta(n^{\varepsilon})$ where $1/2 < \varepsilon < 1$. Then, by using (53), we have

$$\max_{1 \le i \le n} \|\tilde{h}(\cdot;\xi_i)\|_{\Theta} \le \frac{\lambda_n}{2} \quad \text{ev}$$

Using the fact that $-\log(1-t) \le t + 2t^2$ for any $|t| \le 1/2$, we have (76) bounded from above by

$$\sum_{i=1}^{n} \frac{\lambda_n}{n} \left(\frac{\tilde{h}(x;\xi_i)}{\lambda_n} + 2 \frac{\tilde{h}(x;\xi_i)^2}{\lambda_n^2} \right) + \lambda_n \frac{q_n}{2n}$$
$$= \frac{1}{n} \sum_{i=1}^{n} \tilde{h}(x;\xi_i) + 2 \frac{(1/n) \sum_{i=1}^{n} \tilde{h}(x;\xi_i)^2}{\lambda_n} + \lambda_n \frac{q_n}{2n}$$
(77)

where $\frac{1}{n} \sum_{i=1}^{n} \tilde{h}(x;\xi_i) \to 0$ and $\frac{1}{n} \sum_{i=1}^{n} \tilde{h}(x;\xi_i)^2$ satisfy

$$\begin{split} \left\| \frac{1}{n} \sum_{i=1}^{n} \tilde{h}(\cdot;\xi_i) \right\|_{\Theta} &\to 0 \quad \text{a.s.} \\ \left\| \frac{1}{n} \sum_{i=1}^{n} \tilde{h}(\cdot;\xi_i)^2 - \sigma_0^2(\cdot) \right\|_{\Theta} &\to 0 \quad \text{a.s.} \end{split}$$

by the P_0 -GC property of \mathcal{H}_{Θ} and (44). Moreover, by (73) in the proof of Theorem 4, we have $P(\sup_{x\in\Theta} J(x) \leq q_n) \to 1 - \alpha$ a.s.. By the continuity of $\sup_{x\in\Theta} J(x)$, we get $q_n \to q^*$ a.s. where q^* satisfies $P(\sup_{x\in\Theta} J(x) \leq q^*) = 1 - \alpha$. Hence $q_n/(2n) = \Theta(1/n)$. These imply that (77) converges to 0 uniformly over Θ .

On the other hand, plugging in $\mathbf{w} = (1/n)_{1 \le i \le n}$, $\overline{Z}_n^*(x) - Z_0(x)$ in (75) is bounded from below by $(1/n) \sum_{i=1}^n \tilde{h}(x;\xi_i)$, which converges to 0 uniformly over Θ . Combining with above, we get

$$\|\overline{Z}_{n}^{*}(\cdot) - Z_{0}(\cdot)\|_{\Theta} \to 0 \quad \text{a.s.}$$

$$(78)$$

Proof of Theorem 3. For any fixed $x, \underline{Z}_n(x) \to Z_0(x)$ and $\overline{Z}_n(x) \to Z_0(x)$ follows as a special case of (78). \Box

Proof of Theorem 7. To avoid redundancy, we focus only on the upper bound $\overline{Z}_n^*(x)$. Consider $\overline{Z}_n^*(x) - \hat{h}(x)$, which can be written as

$$\max\left\{\sum_{i=1}^{n} w_i \hat{h}(x;\xi_i) : -\frac{1}{n} \sum_{i=1}^{n} \log(nw_i) \le \frac{q_n}{2n}, \sum_{i=1}^{n} w_i = 1, w_i \ge 0 \text{ for all } i = 1, \dots, n\right\}$$
(79)

where $\hat{h}(x;\xi_i) = h(x;\xi_i) - \hat{h}_n(x)$. Similar to the proof of Theorem 6, a Lagrangian relaxation of (79) gives

$$\min_{\lambda \ge 0, \gamma} \max_{\mathbf{w} \ge 0} \sum_{i=1}^{n} w_i \hat{h}(x; \xi_i) - \lambda \left(-\frac{1}{n} \sum_{i=1}^{n} \log(nw_i) - \frac{q_n}{2n} \right) + \gamma \left(\sum_{i=1}^{n} w_i - 1 \right)$$

$$\le \min_{\lambda > 0, \gamma} \sum_{i=1}^{n} \frac{\lambda}{n} \max_{w_i \ge 0} \left\{ nw_i \frac{\hat{h}(x; \xi_i) + \gamma}{\lambda} + \log(nw_i) - nw_i + 1 \right\} + \frac{\lambda q_n}{2n} - \gamma$$

$$\le \min_{\lambda > 0, \gamma, \frac{\hat{h}(x; \xi_i) + \gamma}{\lambda} < 1 \text{ for all } i=1, \dots, n} - \sum_{i=1}^{n} \frac{\lambda}{n} \log \left(1 - \frac{\hat{h}(x; \xi_i) + \gamma}{\lambda} \right) + \frac{\lambda q_n}{2n} - \gamma$$
(80)

by using the fact that $-\log(1-t) = \sup_{r \ge 0} \{tr + \log r - r + 1\}$ is the conjugate function of $-\log r + r - 1$, defined for t < 1.

Now, given x, we choose $\gamma = 0$, and $\lambda = \frac{\sqrt{n}\hat{\sigma}(x)}{\sqrt{q_n}}$. Similar to the proof of Theorem 6, we have

$$\left\|\frac{1}{n}\sum_{i=1}^n \hat{h}(\cdot;\xi_i)^2 - \sigma_0^2(\cdot)\right\|_{\Theta} \to 0 \quad \text{a.s.}$$

by the P_0 -GC property of \mathcal{H}_{Θ} and (44), and $q_n \to q^*$ a.s. where q^* satisfies $P(\sup_{x \in \Theta} J(x) \le q^*) = 1 - \alpha$. Moreover,

$$\hat{\sigma}^2(x) = \frac{1}{n} \sum_{i=1}^n (h(x;\xi_i) - Z_0(x))^2 - (\hat{h}(x) - Z_0(x))^2$$

By Assumptions 1, 3 and 4, $\{(h(x; \cdot) - Z_0(x))^2 : x \in \Theta\}$ is P_0 -Donsker, and we have

$$\sup_{x\in\Theta} |\hat{\sigma}^2(x) - \sigma_0^2(x)| \to 0 \ \text{ as } n \to \infty \text{ a.s.}$$

Together with the assumption that h is bounded, we have

$$\frac{\hat{h}(x;\xi_i) + \gamma}{\lambda} = \frac{\hat{h}(x;\xi_i)\sqrt{q_n}}{\sqrt{n}\hat{\sigma}(x)} \to 0$$
(81)

a.s. uniformly for all i = 1, ..., n as $n \to \infty$.

Therefore, (80) is bounded from above ev. by

$$\begin{split} &-\sum_{i=1}^{n} \frac{\hat{\sigma}(x)}{\sqrt{nq_n}} \log \left(1 - \frac{\hat{h}(x;\xi_i)\sqrt{q}_n}{\sqrt{n}\hat{\sigma}(x)}\right) + \frac{\sqrt{q}_n \hat{\sigma}(x)}{2\sqrt{n}} \\ &= \sum_{i=1}^{n} \frac{\hat{\sigma}(x)}{\sqrt{nq_n}} \left(\frac{\tilde{h}(x;\xi_i)\sqrt{q}_n}{\sqrt{n}\hat{\sigma}(x)} + \frac{1}{2} \left(\frac{\tilde{h}(x;\xi_i)\sqrt{q}_n}{\sqrt{n}\hat{\sigma}(x)}\right)^2 + O\left(\left(\frac{\hat{h}(x;\xi_i)\sqrt{q}_n}{\sqrt{n}\hat{\sigma}(x)}\right)^3\right)\right) + \frac{\sqrt{q}_n \hat{\sigma}(x)}{2\sqrt{n}} \\ & \text{where } O(\cdot) \text{ is uniform over } x \in \Theta \\ &= \frac{\sqrt{q_n}\hat{\sigma}(x)}{2\sqrt{n}} + O\left(\frac{\hat{\mu}_3(x)q_n}{n\hat{\sigma}(x)^2}\right) + \frac{\sqrt{q}_n \hat{\sigma}(x)}{2\sqrt{n}} \\ & \text{ since } \sum_{i=1}^{n} \hat{h}(x;\xi_i) = 0 \text{ and } \frac{1}{n} \sum_{i=1}^{n} \hat{h}(x;\xi_i)^2 = \hat{\sigma}^2(x), \end{split}$$

where $\hat{\mu}_3(x) = \frac{1}{n} \sum_{i=1}^n |\hat{h}(x;\xi_i)|^3$, which is uniformly bounded over $x \in \Theta$ a.s. since h is bounded $= \frac{\sqrt{q_n}\hat{\sigma}(x)}{\sqrt{n}} + O\left(\frac{1}{n}\right)$ (82)

On the other hand, choose

$$w_i = \frac{1}{n} \left(1 + \frac{\hat{h}(x;\xi_i)\sqrt{q_n}}{\sqrt{n}\hat{\sigma}(x)} \left(1 - \frac{C}{\sqrt{n}} \right) \right)$$

for some large enough C > 0. When n is large enough, we have $w_i > 0$ a.s. for all i = 1, ..., nand $x \in \Theta$ since $\frac{\hat{h}(x;\xi_i)\sqrt{q_n}}{\sqrt{n}\hat{\sigma}(x)} < 1$ ev. by the same argument in (81). Note that $\sum_{i=1}^n w_i = 1$ since $\sum_{i=1}^n \hat{h}(x;\xi_i) = 0$ by definition. Moreover,

$$\begin{split} &-\frac{1}{n}\sum_{i=1}^{n}\log(nw_{i})\\ &=-\frac{1}{n}\sum_{i=1}^{n}\log\left(1+\frac{\hat{h}(x;\xi_{i})\sqrt{q_{n}}}{\sqrt{n}\hat{\sigma}(x)}\left(1-\frac{C}{\sqrt{n}}\right)\right)\\ &=-\frac{1}{n}\sum_{i=1}^{n}\frac{\hat{h}(x;\xi_{i})\sqrt{q_{n}}}{\sqrt{n}\hat{\sigma}(x)}\left(1-\frac{C}{\sqrt{n}}\right)+\frac{1}{n}\frac{1}{2}\sum_{i=1}^{n}\frac{\hat{h}(x;\xi_{i})^{2}q_{n}}{n\hat{\sigma}(x)^{2}}\left(1-\frac{C}{\sqrt{n}}\right)^{2}+O\left(\frac{\hat{\mu}_{3}(x)q_{n}^{3/2}}{n^{3/2}\hat{\sigma}(x)^{3}}\left(1-\frac{C}{\sqrt{n}}\right)^{3}\right)\\ &\text{where }O(\cdot) \text{ is uniform over } x\in\Theta\\ &=\frac{q_{n}}{2n}\left(1-\frac{2C}{\sqrt{n}}\right)+O\left(\frac{1}{n^{2}}\right)+O\left(\frac{\hat{\mu}_{3}(x)q_{n}^{3/2}}{n^{3/2}\hat{\sigma}(x)^{3}}\right)\\ &\text{where the last }O(\cdot) \text{ has leading term that is independent of }C \end{split}$$

$$\leq \frac{q_n}{2n}$$

when n is large enough, by choosing a large enough C. Therefore, the chosen w_i 's form a feasible solution in $\mathcal{U}_n(q_n/(2n))$. We have

$$\sum_{i=1}^{n} w_i \hat{h}(x;\xi_i) = \sum_{i=1}^{n} \hat{h}(x;\xi_i) \frac{1}{n} \left(1 + \frac{\hat{h}(x;\xi_i)\sqrt{q_n}}{\sqrt{n}\hat{\sigma}(x)} \left(1 - \frac{C}{\sqrt{n}} \right) \right)$$
$$= \sqrt{q_n} \frac{\hat{\sigma}(x)}{\sqrt{n}} \left(1 - \frac{C}{\sqrt{n}} \right)$$
$$= \sqrt{q_n} \frac{\hat{\sigma}(x)}{\sqrt{n}} + O\left(\frac{1}{n}\right)$$
(83)

Combining the bound for the dual and the primal bounds (82) and (83), we conclude that $\overline{Z}_n^*(x) = \sqrt{q_n} \frac{\hat{\sigma}(x)}{\sqrt{n}} + O\left(\frac{1}{n}\right)$ uniformly over $x \in \Theta$. The proof for $\underline{Z}_n^*(x)$ follows by merely replacing h with -h. This concludes the theorem. \Box

Appendix B: Review of Empirical Processes

We review some terminologies and results in the empirical process theory that are related to our developments. Given a class of functions $\mathcal{F} = \{f : \Xi \to \mathbb{R}\}$, we define the empirical measure \mathbb{P}_n , generated from i.i.d. ξ_1, \ldots, ξ_n each under P, as a map from \mathcal{F} to \mathbb{R} such that

$$\mathbb{P}_n(f) = \frac{1}{n} \sum_{i=1}^n f(\xi_i)$$

We also define $P(f) = \int f(\xi) dP(\xi) = E_P[f(\xi)]$ where $E_P[\cdot]$ is the expectation under P. The empirical process indexed by $f \in \mathcal{F}$ is defined as

$$\sqrt{n}(\mathbb{P}_n - P)$$

For any functions $y : \mathcal{F} \to \mathbb{R}$, we define $||y||_{\mathcal{F}} = \sup_{f \in \mathcal{F}} |y(f)|$. We also define the envelope of \mathcal{F} as a function that maps from Ξ to \mathbb{R} given by

$$\sup_{f\in\mathcal{F}}|f(\xi)|$$

DEFINITION 1. We call \mathcal{F} a *P*-Glivenko-Cantelli (GC) class if the empirical measure under *P* satisfies

$$\|\mathbb{P}_n - P\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |\mathbb{P}_n(f) - P(f)| \stackrel{a.s.}{\to} 0 \text{ as } n \to \infty$$

DEFINITION 2. We call \mathcal{F} a *P*-Donsker class if the empirical process under *P* satisfies

$$\sqrt{n}(\mathbb{P}_n - P) \Rightarrow \mathbb{G} \text{ in } \ell^{\infty}(\mathcal{F})$$
(84)

where \mathbb{G} is a Gaussian process indexed by \mathcal{F} , centered, with covariance function

$$Cov(\mathbb{G}(f_1),\mathbb{G}(f_2)) = Cov_P(f_1(\xi), f_2(\xi)) = P(f_1f_2) - P(f_1)P(f_2)$$

where $Cov_P(\cdot, \cdot)$ denotes the covariance under P, and

$$\ell^{\infty}(\mathcal{F}) = \left\{ y : \mathcal{F} \to \mathbb{R} \middle| \|y\|_{\mathcal{F}} < \infty \right\}$$

Moreover, the process \mathbb{G} has uniformly continuous sample paths with respect to the canonical semi-metric $\rho_P(f_1, f_2) = Var_P(f_1(\xi) - f_2(\xi))$, where $Var_P(\cdot)$ denotes the variance under P.

We have ignored the measurability issues, in particular the use of outer and inner probability measures, in the definitions (see Van Der Vaart and Wellner (1996)).

THEOREM 8 (Preservation of GC classes; Van Der Vaart and Wellner (2000), Theorem 3). Suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_k$ are P-GC classes of functions, and that $\varphi : \mathbb{R}^k \to \mathbb{R}$ is continuous. Then $\mathcal{H} = \varphi(\mathcal{F}_1, \ldots, \mathcal{F}_k)$ is P-GC provided that it has an integrable envelope function.

Appendix C: Other Useful Theorems

LEMMA 3 (Owen (2001), Lemma 11.2). Let Y_i be i.i.d. random variables on \mathbb{R} with $EY_i^2 < \infty$. Then $\max_{1 \le i \le n} |Y_i| = o(n^{1/2})$ a.s..

THEOREM 9 (Continuous Mapping Theorem; Van Der Vaart and Wellner (1996), Theorem 1.3.6). Let $g: \mathbb{D} \to \mathbb{E}$ be continuous at every point $\mathbb{D}_0 \subset \mathbb{D}$. If $X_n \Rightarrow X$ and X takes its values in \mathbb{D}_0 , then $g(X_n) \Rightarrow g(X)$.

THEOREM 10 (Slutsky's Theorem; Van Der Vaart and Wellner (1996), Example 1.4.7). If $X_n \Rightarrow X$ and $Y_n \Rightarrow c$ where X is separable and c is a constant, then $(X_n, Y_n) \Rightarrow (X, c)$ under the product topology.