Targeting, Deployment and Loss-Tolerance in Lanchester Engagements

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August 1, 2019

Abstract

Existing Lanchester combat models focus on two force parameters: numbers (force size) and per-capita effectiveness (attrition rate). While these two parameters are central in projecting a battle's outcome, there are other important factors that affect the battlefield: (1) targeting capability, the capacity to identify live enemy units and not dissipate fire on non-targets; (2) tactical restrictions preventing full deployment of forces; and (3) morale and tolerance of losses, the capacity to endure casualties. In the spirit of Lanchester theory, we derive, for the first time, force-parity equations for various combinations of these effects, and obtain general implications and trade-offs. We show that more units and better weapons (higher attrition rate) are preferred over improved targeting capability and relaxed deployment restrictions unless these are poor. However, when facing aimed fire and unable to deploy more than half one's force it is better to be able to deploy more existing units than to have either additional reserve units or the same increase in attrition effectiveness. Likewise more relaxed deployment constraints are preferred over enhanced loss-tolerance when initial reserves are greater than the force level at which withdrawal occurs.

Keywords: Combat modelling, Lanchester equations

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1 Introduction

Military forces engaged in a battle of attrition have classically been described by Lanchester equations (Lanchester, 1916; Taylor, 1983; Bracken, 1995; Breton et al., 2006), and in particular by the Square Law (Taylor, 1983; Kress and Talmor, 1999). A Lanchester model – a pair of linear differential equations – determines, for each side in the battle, the balance between the effects of initial force size and attrition effectiveness. Lanchester's *aimed fire* model, in which forces cause attrition in proportion to their numbers, results in Lanchester's Square Law: that the effect of the initial force size is quadratic, while the effect of the attrition rates is linear.

Lanchester's model, while insightful and widely used in combat modeling, is overly simple. In particular, it implicitly assumes that no attrition effort is wasted on targets already destroyed, that each side can deploy all of its available forces at the outset, and that the loser is totally annihilated. In reality, these three assumptions do not hold. First, the identification of targets and then of their state – killed or alive – is a perennial military conundrum, which leads to wasted attrition efforts (Diehl and Sloan, 2005). Lanchester's *unaimed fire* model, one of his two models which result in Lanchester's Linear Law, addresses the case of total absence of such targeting capability. Second, due to tactical, operational or other constraints (e.g., of terrain), a force may only be able to deploy a fraction of its units, which, upon attrition, will be replenished from the remaining units held in reserve. Third, battles seldom continue until one force is annihilated. More typically, one force will opt to surrender or disengage if its attrition reaches its loss-tolerance threshold – an attrition level at which the competitor loses the will to fight.

A military example of some of these issues is the battle of Ein-A-Tinna during the 1982 Lebanon war (Gabriel, 1984). Southern Lebanon is a mountainous region, crisscrossed by narrow, steep and winding roads where mechanized units were forced to move in a single column. An Israeli tank battalion was approaching the village of Ein-A-Tinna assuming that there were no Syrian troops in that village. As the first tank in the column was turning around a horseshoe bend in the road, it encountered heavy fire from the village. It was hit and the second tank in the column passed it and took over the position in the front, resuming the fire duel. The second tank was also hit and a third tank in line, taking over and resuming the fight, was immediately hit too. At this point the battalion commander decided to retreat and regroup. A small Syrian force (about 6 tanks) successfully engaged a battalion (30 tanks) because it was able to exercise all of its firepower while that of the Israeli forces was reduced, due to the topography, to only a single tank.

In this paper we address the three aforementioned aspects: imperfect targeting, tactical restrictions on deployment, and limited tolerance of losses. The goal, in the spirit of classic advocacy of simple mathematical models (Richardson, 1960; Epstein, 2008), is to connect simple real constraints on Lanchester's aimed-fire square-law model with equally simple conclusions.

Section 2 presents a short review of Lanchester's Square Law. In section 3 we assume that all three disadvantages – imperfect targeting capability (TC), constrained deployment and limited loss-tolerance – apply to one side only. This enables a simpler initial exposition, and also allows us to clearly observe how the three effects combine. The first effect, of imperfect TC, would classically be thought of as leading to the Linear Law of Lanchester's unaimed fire model, but, in fact, its effect is more subtle: it leads to a Square Law with a penalty factor. The other two effects simply exacerbate this into a Square Law with an even greater penalty on the effective per-unit kill rate. In section 4 we apply the effects to both sides. Regarding loss-tolerance, our results extend the work of Taylor (1983), p.126. Regarding deployment constraints we extend the model in Kress and Talmor (1999). Section 5 presents the implications of our results as a series of operational and force-planning propositions.

2 Lanchester's Square Law

Let $B(t)$ and $R(t)$ denote the force sizes at time t of two adversaries, called henceforth *Blue* and *Red* respectively. For notational simplicity we suppress the explicit time dependence and write $B(t) = B$ and $R(t) = R$. Let b and r denote their respective kill rates. The initial conditions are given, $B(0) = B_0$ and $R(0) = R_0$. The Lanchester equations are

$$\begin{aligned} \frac{dB}{dt} &= -rR, \\ \frac{dR}{dt} &= -bB. \end{aligned} \tag{1}$$

Essentially, the conditions for these to hold are that all units on both sides are in action, aim their fire, know when they have incapacitated their targets, and can quickly acquire new ones. For this reason (1) is often known as the *aimed-fire model*.

Dividing the first equation by the second one we obtain $\frac{dB}{dR} = \frac{rR}{bB}$ and thus

$$bB\,dB = rR\,dR. \tag{2}$$

Integrating, we obtain the *state equation* of the Lanchester Square Law:

$$b(B_0^2 - B^2) = r(R_0^2 - R^2). \tag{3}$$

Now suppose that Blue and Red fight to annihilation, with the battle ending at time t^*, where t^* is the earliest time such that $\min(B(t^*), R(t^*)) = 0$. We define *parity* as mutual annihilation: $B(t^*) = R(t^*) = 0$. From (3), mutual annihilation occurs if and only if

$$\frac{rR_0^2}{bB_0^2} = 1, \tag{4}$$

which is called the *parity equation*.

Hence the *Square Law*: the effect of the initial force size is *squared* compared to the attrition rates, R_0^2 against r and B_0^2 against b. The solution curves (3) are hyperbolae, which show an increasing deviation from parity as the battle progresses, due to increasing 'ganging up' by the winning side on the depleted losing force. That is, assuming (4) does not hold and one side wins the battle, then the ratio $\frac{rR^2}{bB^2}$ moves further away from 1 as the battle progresses.

Lanchester contrasted this with two models in which the more intuitive *Linear Law* holds. The simplest such model is the *ancient model*, in which both sides engage the same number of units in a series of one-on-one duels. More interesting is the *unaimed-fire model*, in which

$$\begin{aligned} \frac{dB}{dt} &= -rRB \\ \frac{dR}{dt} &= -bBR. \end{aligned} \tag{5}$$

Here losses are proportional not only to attacking but also to defending numbers. This could be due to density-dependence in the effect of indirect artillery fire, or because of the effects on direct fire of poor TC, causing fire to be wasted on decoy or inactive targets. Linearity follows because, when we divide one equation by the other, $\frac{dB}{dR}$ no longer depends on force sizes; the state equation is then

$$b(B_0 - B) = r(R_0 - R), \tag{6}$$

and the parity equation thereby becomes

$$\frac{rR_0}{bB_0} = 1. \tag{7}$$

However, note that b and r now mean something different: they are attacking units' kill rates per unit time and *per enemy unit*. We shall address this subtlety, and its connection with the Square Law, in the next section.

3 An asymmetric Lanchester engagement

Consider a generalized engagement in the spirit of Ein-A-Tinna, described in Section 1. B Blue tanks are forced to attack along a road or other defile, so that only a fixed number $B_{\max}$ can deploy. That is, $B - B_{\max}$ Blue tanks initially start in reserve and are neither effective nor vulnerable to Red's fire. Red's R defenders, in contrast, are all able to engage. We give Blue two further disadvantages: limited loss-tolerance, and poor targeting due to absent or limited battle damage assessment (BDA) regarding the status of Red's targets. We first incorporate absent BDA into the model and then introduce the other two factors.

3.1 Targeting

When Blue's targeting capability (TC) is poor, its probability to accurately target a live Red unit is reduced, and therefore the total aimed fire rate bB is subject to some multiplier less than one. The simplest case is absent BDA, so that Blue targets live units randomly among the live and dead. This absent BDA case produces a multiplier $\frac{R}{R_0}$ and the asymmetric model

$$\begin{aligned} \frac{dB}{dt} &= -rR, \\ \frac{dR}{dt} &= -bB\frac{R}{R_0}. \end{aligned} \tag{8}$$

This asymmetric model, with aimed fire from Red and unaimed fire from Blue, is the *guerrilla model* of Deitchman (1962).

The effect of lack of BDA is seen by observing the state equation

$$\frac{1}{2}b(B_0^2 - B^2) = r(R_0^2 - RR_0), \tag{9}$$

which is obtained by steps similar to those leading to Eq. (3).

The parity condition is now

$$\frac{rR_0^2}{bB_0^2} = \frac{1}{2}. \tag{10}$$

Thus the Square Law, in initial numbers, still applies! Blue's penalty for its lack of BDA is seen rather in the additional factor $\frac{1}{2}$ on the right-hand side: Blue would need to double its kill rate, or increase its numbers by $\sqrt{2}$, to remedy this.

Note that poor BDA is just one cause of poor TC. We could easily make the engagement still less favorable for Blue by giving Red further decoying or cover, with even the location of its units unclear to Blue. In this case, Blue's incapacity to identify targets goes beyond mere lack of BDA and becomes a more wide-ranging lack of targeting capability – Blue is reduced to 'firing into the brown'. This would require the replacement of R_0 in the denominator in (8) by some fixed parameter R_+ greater than R_0. Then, setting $\sigma = R_0/R_+$ (so that $\sigma < 1$), the parity equation becomes

$$\frac{rR_0^2}{bB_0^2} = \frac{\sigma}{2}. \tag{11}$$

Thus any further decoying and cover beyond mere absence of BDA, any 'firing into the brown', is equivalent to a proportionate reduction in kill-rate.

A different variation is to give Blue imperfect but not totally absent BDA, parametrized by δ, with $0 \leq \delta \leq 1$: a proportion δ of Blue's fire is directed only at currently-live targets, while a fraction $1 - \delta$ of its fire is uniformly directed at any of the R_0 (live or dead) targets available initially. The equations are then

$$\begin{aligned} \frac{dB}{dt} &= -rR, \\ \frac{dR}{dt} &= -bB\frac{\delta R_0 + (1-\delta)R}{R_0}. \end{aligned} \tag{12}$$

This is the system previously put forward as a model for the effects of partial intelligence

(Kress and MacKay, 2014). The fraction in the $\frac{dR}{dt}$ part of equation (12) varies linearly between 1 and $R/R_0 \leq 1$, so that δ interpolates between aimed fire ($\delta = 1$) and Deitchman's model (8) ($\delta = 0$). The parity equation for (12) is

$$\frac{rR_0^2}{bB_0^2} = \frac{1-\delta}{2}\left(1 + \frac{\delta \log \delta}{1-\delta}\right)^{-1}. \tag{13}$$

Throughout this paper $\log$ refers to the natural logarithm. Equation (13) is a variant of equation (8) of Kress and MacKay (2014). When $\delta = 0$ (absent BDA) this is (10), while when $\delta \simeq 1$ (by Taylor expansion) it is

$$\frac{rR_0^2}{bB_0^2} = 1 - (1-\delta)/3 + ..., \tag{14}$$

reducing to (4) at $\delta = 1$ (perfect BDA).

3.2 Deployment

Now we add the effect of the defile. If $B_0 > B_{\max}$, then for as long as $B > B_{\max}$ Blue's deployment is constrained. We assume that each time Blue loses a combatant, another is able to take its place, so that bB is replaced by $bB_{\max}$, whether in the simple aimed-fire model (1) or in (8) above. Thus, for the latter, we now have a two-stage battle:

$$\begin{aligned} \frac{dB}{dt} &= -rR, \\ \frac{dR}{dt} &= -bB_{\max}\frac{R}{R_0}. \end{aligned} \tag{15}$$

while $B > B_{\max}$, and (8) thereafter.

For the first stage, the equation which results from separating variables and integrating is

$$bB_{\max}(B_0 - B) = rR_0(R_0 - R). \tag{16}$$

If it happens that Red is annihilated, $R = 0$, before Blue is reduced to $B_{\max}$ then Blue has won for the loss of $rR_0^2/bB_{\max}$ units. There is then no parity equation to consider, for parity includes $R = 0, B = 0$ and thus requires Blue attrition to continue beyond $B = B_{\max}$.

Otherwise, the first stage ends when $B = B_{\max}$, at which $R = R_1$, say. At this point,

$$bB_{\max}(B_0 - B_{\max}) = rR_0(R_0 - R_1). \tag{17}$$

For the second stage, which is the Deitchman model (8) but beginning at $B = B_{\max}, R = R_1$, the parity equation (10) is replaced by

$$\frac{rR_0R_1}{bB_{\max}^2} = \frac{1}{2}. \tag{18}$$

Writing $\mu = B_{\max}/B_0$ (so that $0 < \mu < 1$) and substituting (17) into (18), we obtain

$$\frac{rR_0^2}{bB_0^2} = \frac{\mu(2-\mu)}{2}. \tag{19}$$

So we now see a Square Law further modified by the constraint on deployment: beyond the factor of $\frac{1}{2}$ already seen due to lack of BDA, we now have a further factor of $\mu(2-\mu)$. Note that since $0 < \mu < 1$ this factor is less than one: for example, if $\mu = 1/2$ so that Blue can deploy only half its force initially, then $\mu(2-\mu) = 3/4$. In order to compensate, its kill rate must improve by a factor of $4/3$, or its numbers be increased by the square root of this. In the extreme case of Ein-A-Tinna, where $\mu = 1/B_0$, the factor needed to compensate is approximately $B_0/2$.

3.3 Loss-tolerance

Finally, suppose Blue has limited loss-tolerance: it will disengage if its numbers are reduced from B_0 to βB_0, where β is Blue's *withdrawal proportion*. The lower the withdrawal proportion of a force, the higher its loss-tolerance. We assume that $\beta B_0 \leq B_{\max}$, or $\mu > \beta$

– that is, the battle continues into the second stage, beyond the point where the constraint on deployment ceases to apply. Adding this effect to those of the previous two subsections, with $\delta = 0$ (no BDA), and with the battle now finishing at $R = 0$, $B = \beta B_0$, the parity equation becomes

$$\frac{rR_0^2}{bB_0^2} = \frac{2\mu - \mu^2 - \beta^2}{2}, \tag{20}$$

which is positive as a consequence of $\mu > \beta$.

Equation (20) captures the compounded effect of the three disadvantages suffered by Blue: no BDA, constrained deployment and limited loss-tolerance. It describes the balance-of-forces which will lead to the outcome that Red is annihilated precisely when Blue is about to withdraw: on one side of this threshold, Blue annihilates Red just before Blue reaches its withdrawal level; on the other, Red forces Blue to withdraw just before Red is annihilated.

For example, suppose $\mu = 1/3$ (Blue can only deploy a third of its initial force) and $\beta = 1/4$ (Blue is willing to lose up to three quarters of its force before withdrawing). Then

$$\frac{2\mu - \mu^2 - \beta^2}{2} = \frac{1}{2}\left(\frac{2}{3} - \frac{1}{9} - \frac{1}{16}\right) = 0.247, \tag{21}$$

so that Blue needs to be roughly four times as effective or twice as numerous as Red to achieve parity.

These effects – perhaps combined with poor TC beyond poor BDA, realized as the further multiplier $\sigma < 1$ of 'firing into the brown' from (11) – provide a natural theoretical context for the classic empirical '3:1' rule of offense: that attackers need to be three times as numerous as defenders for parity (Mearsheimer, 1989; Epstein, 1989; Dupuy, 1989; Yigit, 2000).

To conclude, we examine the trade-off between deployment and loss-tolerance in Figure 1. We set $R_0 = B_0$ and define $\alpha \equiv \frac{r}{b}$ as the relative combat effectiveness. The y-axis is the withdrawal proportion β, which is the complement of loss-tolerance. For several values of α we plot the (μ, β) combination that produces parity, which corresponds to (20) if $\beta \le \mu$. When $\beta > \mu$ the parity condition can be derived by substituting $R = 0$ and $B = \beta B_0$ into equation (16). The y-axis in Figure 1 is flipped; we construct the figure this way so that maximum loss-tolerance ($\beta = 0$) corresponds to the top of the figure. The upper right-hand corner of the figure is the "Deitchman point" with $\alpha = \frac{1}{2}$ (see (10)). If Red's kill rate is within a factor of two of Blue's, then Blue cannot win. As Blue's loss-tolerance and/or deployment decrease, Red's combat effectiveness must be substantially less than Blue's to maintain parity.

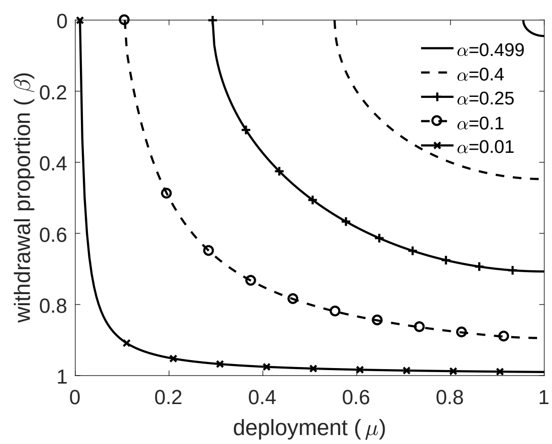


Figure 1: Parity contours for $R_0 = B_0$ and various values of $\alpha = \frac{r}{b}$.

4 Symmetric engagements

This section generalizes the previous section, applying the three effects to both sides.

4.1 Deployment

Suppose we give both sides perfect TC but constrain deployment, so that Blue can only deploy $B_{\max} < B_0$ units and Red $R_{\max} < R_0$. Again we write $\mu = B_{\max}/B_0$, and also set $\nu = R_{\max}/R_0$.

The engagement thus begins as Lanchester's ancient warfare model,

$$
\begin{aligned}
\frac{dB}{dt} &= -rR_{\max}, \\
\frac{dR}{dt} &= -bB_{\max}.
\end{aligned}
\tag{22}
$$

This holds from the initial values R_0, B_0 until (without loss of generality) $B = B_{\max}$ and $R = R_1 > R_{\max}$. The state equation at this stage of the battle is

$$
bB_{\max}(B_0 - B_{\max}) = rR_{\max}(R_0 - R_1). \tag{23}
$$

In the next stage we have

$$
\begin{aligned}
\frac{dB}{dt} &= -rR_{\max}, \\
\frac{dR}{dt} &= -bB
\end{aligned}
\tag{24}
$$

until $R = R_{\max}$ and $B = B_1$, with state equation

$$
rR_{\max}(R_1 - R_{\max}) = \frac{1}{2}b(B_{\max}^2 - B_1^2). \tag{25}
$$

The final stage, for which $B < B_{\max}$ and $R < R_{\max}$, is simple aimed fire, and obeys the Square Law. Combining the state equations to eliminate R_1 and B_1, we find

$$
\frac{rR_0^2}{bB_0^2} = \frac{2\mu - \mu^2}{2\nu - \nu^2}. \tag{26}
$$

The additional function of μ and ν on the right captures the effect of the deployment constraint on what remains, in terms of the relationship among R_0, B_0, r and b, a modified Square Law.

4.2 Targeting

In this symmetrical situation, there are various ways in which a force's deployment constraints can interact with its opponent's poor TC and BDA.

At its simplest, suppose for example that the constraint on deployment is a limited number of available foxholes. Each side observes the locations of the opponent's foxholes, but does not know whether a foxhole contains a live combatant. As long as force levels are sufficiently large, all foxholes are occupied and the battle is an exchange of aimed fire. Once the total attrition of Blue (respectively Red) exceeds $B_0 - B_{\max}$ (resp. $R_0 - R_{\max}$) some foxholes become "empty," and the fire becomes exceedingly unaimed. The battle begins with (22). The next stage, when $B < B_{\max}$ but $R > R_{\max}$, is (24), but with a multiplier of $B/B_{\max}$ in the first equation — which reduces the state equation back to precisely that of (22). Similarly for the final stage, so that the parity equation is simply the Linear Law $rR_0R_{\max} = bB_0B_{\max}$ of Lanchester's ancient model, or

$$\frac{rR_0^2}{bB_0^2} = \frac{\mu}{\nu}. \tag{27}$$

If there is poor TC beyond mere absence of BDA, the effect is to impose further penalty factors as in (11).

Alternatively, the most extreme case of absent BDA is to suppose a situation in which each force can deploy a maximum number of live units alongside its dead, while neither force knows which of its visible opponents is live or dead. For example, logistics or command and control capabilities can only support $B_{\max}$ (resp. $R_{\max}$) active combatants at a time. In such a situation Blue, for example, initially sees $R_{\max}$ Red units, all live, but thereafter sees $R_{\max} + R_0 - R$ units ($R_{\max}$ live plus $R_0 - R$ killed). Later, after R passes below $R_{\max}$, Blue sees R_0 units of which R are live. Thus the dynamics are initially

$$\begin{aligned}\frac{dB}{dt} &= -rR_{\max}\frac{B_{\max}}{B_{\max}+B_0-B},\\ \frac{dR}{dt} &= -bB_{\max}\frac{R_{\max}}{R_{\max}+R_0-R},\end{aligned} \tag{28}$$

from the initial values R_0, B_0 until (again, without loss of generality) $B = B_{\max}$ and $R = R_1$. The state equation is then

$$r(R_{\max}+R_0)(R_1-R_0)+\frac{1}{2}r(R_0^2-R_1^2) = b(B_{\max}+B_0)(B_{\max}-B_0)+\frac{1}{2}b(B_0^2-B_{\max}^2). \tag{29}$$

This is a quadratic equation, but there is no need to solve for R_1. During the next stage, which ceases when $R = R_{\max}$ and $B = B_1 < B_{\max}$,

$$\begin{aligned}\frac{dB}{dt} &= -rR_{\max}\frac{B}{B_0},\\ \frac{dR}{dt} &= -bB\frac{R_{\max}}{R_{\max}+R_0-R},\end{aligned} \tag{30}$$

and the state equation is

$$r(R_{\max}+R_0)(R_{\max}-R_1)+\frac{1}{2}r(R_1^2-R_{\max}^2) = bB_0(B_1-B_{\max}). \tag{31}$$

The final stage is a simple unaimed-fire linear law, and eliminating B_1 and R_1 we find

$$\frac{rR_0^2}{bB_0^2} = \frac{1+2\mu-\mu^2}{1+2\nu-\nu^2}. \tag{32}$$

It is interesting to compare (32) – corresponding to no BDA – with (26) where BDA is perfect. The lack of BDA is seen in the additional ones in the fraction, whose effect is to mitigate any asymmetry in proportions of forces able to deploy. For example, suppose Blue is initially able to only deploy a quarter of its forces (that is, $\mu = 0.25$), while Red is able to deploy half ($\nu = 0.5$). Then the final fraction in (26) is 0.58, while that in (32) is approximately 0.82. In other words, reduced TC decreases the effect of tactical advantage.

In the asymmetric TC case where, say, Red has perfect TC but Blue has none, the parity equation becomes

$$\frac{rR_0^2}{bB_0^2} = \frac{2\mu - \mu^2}{1 + 2\nu - \nu^2}. \tag{33}$$

If Red has no deployment constraint, $\nu = 1$, this is simply (19). If Blue also has no deployment constraint so that $\mu = 1$, it is (10). Examining parity conditions (26), (32), (33), we note that whereas the Blue (resp. Red) deployment parameter appears in the numerator (resp. denominator), the presence or absence of Blue (resp. Red) TC appears as a 0 or 1 in the *denominator* (resp. numerator).

4.3 Loss-tolerance

First we consider high loss-tolerance, which means that withdrawal levels (of βB_0 for Blue, ρR_0 for Red) are reached only in the final stage of the engagement, when deployment constraints no longer apply. In this section when we consider the absence of TC, we use the extreme lack of BDA model corresponding to equation (32). Then the parity equations are

$$\frac{rR_0^2}{bB_0^2} = \frac{2\mu - \mu^2 - \beta^2}{2\nu - \nu^2 - \rho^2} \tag{34}$$

for the case of perfect TC,

$$\frac{rR_0^2}{bB_0^2} = \frac{1 + 2\mu - \mu^2 - 2\beta}{1 + 2\nu - \nu^2 - 2\rho} \tag{35}$$

for the case of absent TC, and

$$\frac{rR_0^2}{bB_0^2} = \frac{2\mu - \mu^2 - \beta^2}{1 + 2\nu - \nu^2 - 2\rho} \tag{36}$$

when Red has TC but Blue has none.

We can combine these as

$$\frac{rR_0^2}{bB_0^2} = \frac{1 - \delta_R + 2\mu - \mu^2 - 2\beta}{1 - \delta_B + 2\nu - \nu^2 - 2\rho} \tag{37}$$

where δ_R denotes the entire presence ($\delta_R = 1$) or absence ($\delta_R = 0$) of Red TC, and likewise for Blue.

When at least one side's loss-tolerance is low, so that withdrawal levels are reached before deployment constraints, the engagement does not go through all the stages of the cases above. Looking only at perfect TC, suppose first that loss-tolerance is high (relative to deployment) on one side but low on the other — without loss of generality, let $\beta B_0 > B_{\max}$ but $\rho R_0 < R_{\max}$. Then (34) is replaced by

$$\frac{rR_0^2}{bB_0^2} = \frac{2\mu(1 - \beta)}{2\nu - \nu^2 - \rho^2}. \tag{38}$$

When loss-tolerance is low on both sides, $\beta B_0 > B_{\max}$ and $\rho R_0 > R_{\max}$, we have

$$\frac{rR_0^2}{bB_0^2} = \frac{2\mu(1 - \beta)}{2\nu(1 - \rho)}. \tag{39}$$

It is straightforward to combine low or mixed loss-tolerance with absent or mixed TC; we do not give details.

4.4 Casualties

Our focus has been determining the victor, which is dictated by the parity conditions in (37)–(39). While the parity condition is arguably the most important output of Lanchesterian analysis, other metrics are also informative. These include the number of casualties suffered by the victor and the time until the battle ends. For example a Blue commander might choose to avoid a direct confrontation with Red, even if Blue can theoretically defeat Red in the battle, because Blue's projected casualties are too high.

In this section we present results for the number of casualties in the high loss-tolerance setting ($\mu > \beta$ and $\nu > \rho$). We examine the complete model that captures both deployment and loss-tolerance and analyze the perfect TC and absent TC cases separately.

We assume Blue wins the battle, so that (37) implies

$$\frac{rR_0^2}{bB_0^2} < \frac{1 - \delta_R + 2\mu - \mu^2 - 2\beta}{1 - \delta_B + 2\nu - \nu^2 - 2\rho}. \tag{40}$$

We define B_F and R_F as the final force levels at the end of the battle, and hence $B_0 - B_F$ and $R_0 - R_F$ are the casualties. By assumption $R_F = \rho R_0$ and $B_F > \beta B_0$.

For both perfect and absent TC, the results depend upon whether Blue reaches its deployment constraint before winning (B_F vs $B_{\max}$). If Blue does hit the deployment constraint ($B_F \leq B_{\max}$), we can further examine whether Red or Blue reaches the deployment constraint first; however, the results are the same for these two subscenarios.

Section 4.4.1 presents the casualties for the perfect TC case and Section 4.4.2 contains the analogous results for the absent TC case. Section 4.4.3 concludes with numerical illustrations.

4.4.1 Perfect TC

Since Blue wins the battle, equation (34) implies the following condition must hold throughout this section

$$\frac{rR_0^2}{bB_0^2} < \frac{2\mu - \mu^2 - \beta^2}{2\nu - \nu^2 - \rho^2} \tag{41}$$

By assumption $\mu > \beta$ and $\nu > \rho$, and hence both the numerator and denominator on the right-hand side of (41) are positive.

The final Blue force level B_F depends upon whether Blue reaches the deployment constraint $B_{\max}$ before Red reaches its withdrawal proportion ρ.

1. Blue does not reaches its deployment constraint ($B_F > B_{\max}$) if and only if

$$\frac{rR_0^2}{bB_0^2} < \frac{2\mu(1-\mu)}{2\nu - \nu^2 - \rho^2}. \tag{42}$$

   Blue's final force level is

$$B_F = B_0 \left(1 - \frac{1}{2\mu} \frac{rR_0^2}{bB_0^2}(2\nu - \nu^2 - \rho^2)\right). \tag{43}$$

2. Blue reaches its deployment constraint ($B_F \leq B_{\max}$) if and only if

$$\frac{rR_0^2}{bB_0^2} \geq \frac{2\mu(1-\mu)}{2\nu - \nu^2 - \rho^2}, \tag{44}$$

   Blue's final force level is

$$B_F = B_0 \sqrt{\mu^2 - \left(\frac{rR_0^2}{bB_0^2}(2\nu - \nu^2 - \rho^2) - 2\mu(1-\mu)\right)}. \tag{45}$$

The steps to derive the final force levels are similar to the logic required to move from equation (22) to (26). We sketch the steps here for scenario 1, when Blue does not reach its deployment constraint. We first solve for Blue's force level when Red reaches its deployment constraint $R_{\max}$. We denote this level B_1 and it satisfies a similar state equation to (23)

$$bB_{\max}(B_0 - B_1) = rR_{\max}(R_0 - R_{\max}). \tag{46}$$

After solving for B_1, the state equation for B_F is similar to (25)

$$bB_{\max}(B_1 - B_F) = \frac{1}{2}r(R_{\max}^2 - \rho^2 R_0^2). \tag{47}$$

Solving for B_F via (46)–(47) yields (43). Requiring $B_F > B_{\max}$ generates condition (42).

4.4.2 Absent TC

Blue achieves victory according to condition (35 if and only if

$$\frac{rR_0^2}{bB_0^2} < \frac{1 + 2\mu - \mu^2 - 2\beta}{1 + 2\nu - \nu^2 - 2\rho}. \tag{48}$$

We assume condition (48) holds. Blue's final force level appears below.

1. Blue does not reaches its deployment constraint if and only if

$$\frac{rR_0^2}{bB_0^2} < \frac{1 - \mu^2}{1 + 2\nu - \nu^2 - 2\rho}. \tag{49}$$

   The final final force level is

$$B_F = B_0 \left(1 + \mu - \sqrt{\mu^2 + \frac{rR_0^2}{bB_0^2}(1 + 2\nu - \nu^2 - 2\rho)} \right). \tag{50}$$

2. Blue reaches its deployment constraint if and only if

$$\frac{rR_0^2}{bB_0^2} \geq \frac{1-\mu^2}{1+2\nu-\nu^2-2\rho}. \tag{51}$$

Blue's final force level is

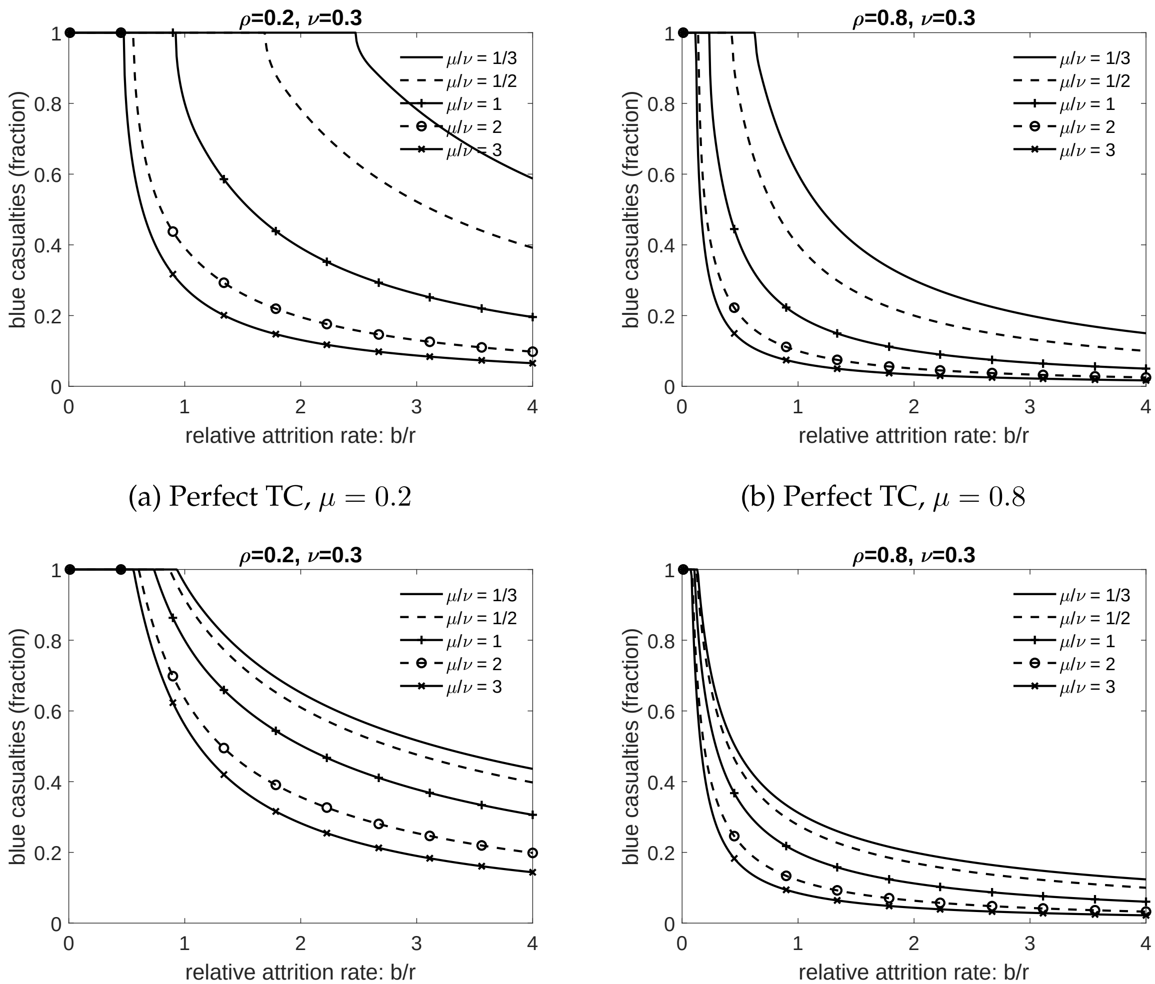
$$B_F = B_0\left(\mu - \frac{1}{2}\left(\frac{rR_0^2}{bB_0^2}(1+2\nu-\nu^2-2\rho)-(1-\mu^2)\right)\right). \tag{52}$$

The details to derive the above final force levels are similar to the steps required to move from (28) to (32).

4.4.3 Numerical Illustrations

Figure 2 plots the fraction of Blue casualties ($\frac{B_0-B_F}{B_0}$) against Blue attrition rate b. We fix $R_0 = B_0$, $r = 1$, $\beta = 0$, $\nu = 0.3$, and vary μ, ρ, and TC across the curves and panels. The top row corresponds to the full TC case from Section 4.4.1 and the bottom row corresponds to the absent TC case from Section 4.4.2. The results in Sections 4.4.1–4.4.2 are only for the high loss-tolerance situation when $\rho < \nu$. Only the left column of Figure 2 satisfies this high loss-tolerance criteria, however it is straightforward to derive final force levels for the low loss-tolerance settings similar to Sections 4.4.1–4.4.2.

Figure 2 reveals that all the parameters have a significant impact on the results. Increasing the attrition rate b and deployment μ can decrease Blue casualties substantially. Any action Blue takes to decrease Red's loss-tolerance (e.g., lower Red morale) also has an impact on Blue casualties. TC has the most interesting relationship with Blue casualties, as Blue's preference for perfect TC vs. absent TC depends upon the situation. Comparing the top row of Figure 2 to the bottom, we see the absent TC curves are more tightly bunched. This implies that when Blue has the tactical advantage (larger μ and/or b), then Blue prefers perfect TC so that Blue can exploit its tactical superiority. However, when Red has the advantage (smaller μ and/or b) Blue prefers absent TC. The absence of TC negates some of Red's advantage and provides more opportunity for Blue to win the battle and suffer fewer casualties.



(a) Perfect TC, $\mu = 0.2$

(b) Perfect TC, $\mu = 0.8$

(c) Absent TC, $\mu = 0.2$

(d) Absent TC, $\mu = 0.8$

Figure 2: The number of Blue casualties relative to initial force level ($\frac{B_0 - B_F}{B_0}$) as a function of the relative attrition coefficient b/r. $R_0 = B_0$, $\beta = 0$, $\nu = 0.3$. Each curve corresponds to a fixed ratio $\frac{\mu}{\nu} \in \{\frac{1}{3}, \frac{1}{2}, 1, 2, 3\}$. Each column corresponds to a different $\rho \in \{0.2, 0.8\}$. Top row: perfect TC, Bottom row: absent TC.

5 Analysis

In the simple aimed-fire Lanchester model, each side has only two parameters, its initial numbers B_0 (resp. R_0) and its unit effectiveness (kill rate) b (resp. r). The Square Law can be framed as a statement about the relative values of small proportional increases in b and B_0, deduced from the parity equation: an increase in B_0 by a factor $1 + x/100$ (that is, giving Blue x% additional initial units) is equivalent to an increase in b by a factor of approximately $1 + 2x/100$ (that is, giving Blue $2x$% better individual effectiveness). We can write this as a statement about logarithmic derivatives: in the parity equation,

$$d_b := \frac{d(\log rR_0^2)}{d(\log b)} = 1, \tag{53}$$

$$d_{B_0} := \frac{d(\log rR_0^2)}{d(\log B_0)} = 2. \tag{54}$$

That is, Blue prefers by a factor of two a small proportional increase in initial force size to the same proportional increase in kill rate.

We frame the results of the previous two sections in a similar way. The aim is to understand the trade-offs among force size, unit kill-rate, TC, tactical deployment capability and combat loss-tolerance.

We present a series of propositions about these trade-offs which apply for general values of the parameters. We provide one proposition for the asymmetric case of Section 3 before turning to the symmetric case of Section 4. Unless otherwise stated, there are no restrictions on the values the model parameters can take, and hence our results are quite general. All results apply assuming other parameters are held constant.

First we look at the trade-off between per-unit **kill-rate** b or **total force** B_0 against continuously-variable **TC** for the asymmetric case.

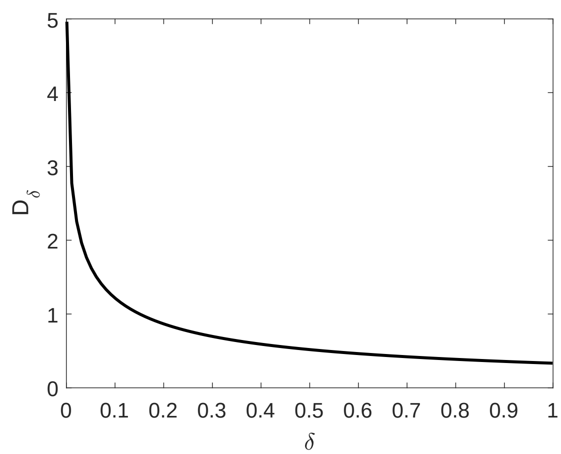
Proposition 1. *For the asymmetric case of Section 3 with total Blue deployment ($\mu = 1$) and loss-tolerance ($\beta = 0$), Blue prefers proportional improvements in its kill-rate or numbers to im-*

provements in its TC, unless TC is almost entirely absent.

Proof. Here we are comparing absolute increases in δ with proportional increases in b and B_0, so that, for example, an increase from $\delta = 0.2$ to $\delta = 0.3$ is being compared with a 10% increase in b or B_0 (not a 50% increase). We begin by computing, from the parity equation (13),

$$D_\delta := \frac{d(\log rR_0^2)}{d\delta} = \frac{1}{\delta}\left\{\left(1 + \frac{\delta \log \delta}{1-\delta}\right)^{-1} - \frac{1+\delta}{1-\delta}\right\}. \tag{55}$$

This is not very intuitive, so we plot its numerical values:



This figure is analogous to Fig. 2(a) of Kress and MacKay (2014), but now with that paper's parameter $I_0/P = 1$. The crucial point is that $D_\delta < 2$ for all $\delta > 0.03$ and $D_\delta < 1$ for all $\delta > 0.15$, so that Blue prefers improvements in force size to equivalent improvements in its TC whenever TC is greater than 0.03, and also prefers improvements in kill rate when TC is greater than 0.15. □

The remaining proofs pertain to the symmetric case in Section 4 and primarily utilize equations (34)–(36). We first examine Blue's trade-off of **kill-rate** against **deployment**, when Blue's loss-tolerance is total or very high (β small or zero). TC is binary, and may be either absent or complete for each of Blue and Red, but the comparison in Blue's trade-off is dependent only on *Red's* TC.

Proposition 2. *For the symmetric case of Section 4 with high Blue loss-tolerance ($\beta \approx 0$) and high Red loss-tolerance ($\rho \approx 0$), Blue prefers small proportional increases in kill-rate to small increases in deployed proportion of its force provided the deployed proportion of force is $\mu > 2 - \sqrt{3} = 0.27$ (when Red TC is absent) or $\mu > 2 - \sqrt{2} = 0.59$ (when Red TC is perfect).*

Proof. For this we compute the appropriate derivative from (37),

$$D_\mu := \frac{d(\log r R_0^2)}{d\mu} = \frac{2(1-\mu)}{1 - \delta_R + 2\mu - \mu^2}. \tag{56}$$

Then $D_\mu < 1$ when $\mu > 2 - \sqrt{3 - \delta_R}$. □

Note that a small increase $\mu \mapsto \mu + \zeta$ is identical with a small increase in deployable force to $B_{\max} + \zeta B_0$, with B_0 fixed. Proposition 2 illustrates that Blue deployment is relatively more important when Red has perfect TC.

To treat the trade-off of Blue's **total force** B_0 against **deployable force** $B_{\max}$, again when loss-tolerance is total or very high, we proceed slightly differently, comparing small absolute increases (measured in units of force) in both.

Proposition 3. *Consider the symmetric case of Section 4 with high Blue loss-tolerance ($\beta \approx 0$) and high Red loss-tolerance ($\rho \approx 0$). Absent Red TC, Blue always prefers a small absolute increase in its total force B_0 over a small absolute increase in its deployable force $B_{\max}$. With perfect Red TC, Blue prefers additional total force to additional deployable force provided $\mu > 0.5$.*

Proof. Consider

$$(1 - \delta_R + 2\mu - \mu^2) B_0^2 = (1 - \delta_R) B_0^2 + 2 B_0 B_{\max} - B_{\max}^2, \tag{57}$$

which is the Blue component of the parity conditions in (37), ignoring the constant b. Now make small changes $B_0 \mapsto B_0 + x$, $B_{\max} \mapsto B_{\max} + y$ (equivalent to computing partial derivatives with respect to B_0 and $B_{\max}$). The first-order variation in (57 is

$$(1-\delta_R)(B_0+x)^2 + 2(B_0+x)(B_{\max}+y) - (B_{\max}+y)^2 - ((1-\delta_R)B_0^2 + 2B_0B_{\max} - B_{\max}^2)$$
$$= 2((1-\delta_R)B_0 + B_{\max})x + 2(B_0 - B_{\max})y + \mathcal{O}(x,y)^2. \tag{58}$$

For $\delta_R = 0$ (that is, absent Red TC), $B_0 + B_{\max} > B_0 - B_{\max}$ always, so the coefficient of x is greater than that of y, and an increase in B_0 is more valuable than an increase in $B_{\max}$. For $\delta_R = 1$ (perfect Red TC), the equivalent condition is $B_{\max} > B_0 - B_{\max}$, true only when $2B_{\max} > B_0$ or $\mu > 0.5$. □

It is natural to consider a more practical choice: what happens if additional units become available to Blue (augmenting B_0) when Blue is also in control of its deployed units $B_{\max}$? Should Blue immediately deploy its newly-available units, or hold them in reserve? For this we have

Corollary 1. *Consider the symmetric case of Section 4 with high Blue loss-tolerance ($\beta \approx 0$) and high Red loss-tolerance ($\rho \approx 0$). Suppose Blue has a small number of additional units, and can choose to deploy or reserve them. Then Blue should always choose to deploy them, whatever Red's TC state.*

Proof. Suppose x units become available. Holding them in reserve is $B_0 \mapsto B_0 + x$, $B_{\max} \mapsto B_{\max}$. Deployment is $B_0 \mapsto B_0 + x$, $B_{\max} \mapsto B_{\max} + x$. But the latter is always better, since the coefficient of y in the change (58) is positive, independent of whether δ_R is one or zero. □

Corollary 1 is essentially the longstanding military principle of concentration of force at the decisive point: if Blue has (echoing Ein-A-Tinna) 10 tanks, and one more tank becomes available, then Blue should deploy the tank, if it can, rather than hold it in reserve.

Proposition 3 is more subtle. Suppose Blue has 10 tanks but can deploy only 6. Then, in terms of the battle's final outcome, and whatever Red's TC state, Blue would rather have one additional tank in reserve than be able to deploy one more of its original 10. But if the deployable proportion is less than half and Red has full TC, then the reverse is true: if Blue can deploy (say) only 3 of its 10 tanks, and Red is aiming its fire, then Blue would rather be able to deploy one more tank than have an additional tank in its reserve force. That is, Blue would rather have 4 deployed and 6 in reserve than 3 deployed and 8 in reserve — Blue simply needs more deployed firepower.

Finally we assume perfect TC for both Blue and Red and examine the trade-off of **deployment** against **loss-tolerance**, either in absolute numbers ($B_{\max}$ against $(1-\beta)B_0$) or proportionally (μ against $1-\beta$).

Proposition 4. *For the symmetric case of Section 4 with perfect Blue TC and perfect Red TC, Blue prefers a small increase in deployment to a small increase in loss-tolerance if and only if Blue's initial reserve ($B_0 - B_{\max} = (1-\mu)B_0$) is greater than Blue's withdrawal level (βB_0).*

Proof. The proof requires two separate derivations, but they have the same conclusion. When Blue's loss-tolerance is high — it is willing to continue the engagement until most of its resources are destroyed — the result follows by generalizing the proof of Proposition 3 to the $\beta \neq 0$ case, using (34-36). The condition is $1-\mu > \beta$ or $B_0 - B_{\max} > \beta B_0$. When Blue's loss-tolerance is low, we need instead to consider variations in the numerator of (38) and (39), $2\mu(1-\beta)$, but the condition which results is $1-\beta > \mu$, which is equivalent. □

Propositions 2–4 highlight the importance of Blue having a reasonable level of deployment, especially when Red has perfect TC. Otherwise, Red can effectively pick off Blue forces by aiming its fire at the limited Blue front.

6 Discussion

In this paper we investigated extensions to Lanchester's aimed-fire model and Square Law, quantifying its modification by three effects: unaimed fire, principally in the form of poor targeting capability; the inability to deploy all of a force and thereby bring advantageous numbers to bear; and unwillingness to fight a Lanchestrian battle to annihilation. Our conclusions follow from parity equations, which modify the original Lanchester Square Law (4) by simple functions of the parameters that quantify the three effects. We then presented the implications of these as a series of propositions which affect force planning and operational decision-making.

Starting with the classic Lanchester aimed-fire model, we showed the importance of TC by observing that lack of TC is equivalent to halving the kill-rate (see equation 10). In most scenarios Blue prefers small proportional increases in kill-rate and numbers to small absolute improvements in its TC, deployed proportion of force, and proportion of force it is willing to lose. However, if Blue has low TC or low deployment capability, then Blue prefers to increase those quantities. In particular, when Red has perfect TC Blue needs a moderate deployment level to stand a chance. This result is consistent with the battle of Ein-A-Tinna discussed in the introduction where the Israeli force facing severe deployment restrictions was easily rebuffed by a smaller Syrian force. The comparison of deployment with loss-tolerance is seen in Proposition 4: the higher Blue's willingness to tolerate losses, the more Blue benefits from the ability to deploy most of its resources.

Most broadly, this paper has been about asymmetry in Lanchester combat models – not just in parameter values, but in the dynamics and the conditions which create and constrain them. In real warfare, the gaining of advantage is about both responding to and creating such dynamical asymmetries to one's own advantage. To the extent to which there is truth in the classic 3:1, attacker:defender rule-of-thumb, it is surely in the defender's work to create such asymmetries and the attacker's to mitigate them.

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