A Fast And Simple Algorithm for the Maximum Flow Problem

by

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Abstract

We present a simple $O(nm + n^2 \log U)$ sequential algorithm for the maximum flow problem on a network with n nodes, m arcs, and integer arc capacities bounded by U. Under the practical assumption that U is polynomially bounded in n , our algorithm runs in time $O(nm + n^2 \log n)$. This result improves the previous best bound of $O(nm \log (n^2/m))$, obtained by Goldberg and Tarjan , by a factor of log n for networks that are both non-sparse and non-dense without using any complex data structures.

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The maximum flow problem is one of the most fundamental problems in network flow theory and has been investigated extensively. This problem was first formulated by Fulkerson and Dantzig [1955] and Dantzig and Fulkerson [1956], and solved by Ford and Fulkerson [1956] using their well-known augmenting path algorithm. Since then, a number of algorithms have been developed for this problem; some of them are tabulated below. In the table, n is the number of nodes, m is the number of arcs, and U is an upper bound on the integral arc capacities. The algorithms whose time bounds involve U assume integral capacities, whereas others run on arbitrary rational or real capacities.

#	Due to	Running Time
1	Ford and Fulkerson [1956]	$O(nm\ U)$
2	Edmonds and Karp [1972]	$O(nm^2)$
3	Dinic [1970]	$O(n^2m)$
4	Karzanov [1974]	$O(n^{3)}$
5	Cherkasky [1977]	$O(n^2m^{1/2})$
6	Malhotra, Kumar and Maheshwari [1978]	$O(n^3)$
7	Galil [1980]	$O(n^{5/3}m^{2/3})$
8	Galil and Naamad [1980]; Shiloach [1978]	$O(nm \log^2 n)$
9	Shiloach and Vishkin [1982]	$O(n^3)$
10	Sleator and Tarjan [1983]	$O(nm \log n)$
11	Tarjan [1984]	$O(n^3)$
12	Gabow [1985]	$O(nm \log U)$
13	Goldberg [1985]	$O(n^3)$
14	Goldberg and Tarjan [1986]	$O(nm \log (n^2/m))$
15	Cheriyan and Maheshwari [1987]	$O(n^2m^{1/2})$
16	Ahuja and Orlin [1987]	$O(nm + n^2 \log U)$
17	Ahuja, Orlin and Tarjan [1987]	$O\left(nm \log\left(\frac{n \log U}{m \log \log U} + 2\right)\right)$

Table 1. Running times of the maximum flow algorithms.

Edmonds and Karp [1972] showed that the Ford and Fulkerson [1956] algorithm runs in time $O(nm^2)$ if flows are augmented along shortest paths from source to sink. Independently, Dinic [1970] introduced the concept of shortest path networks, called *layered* networks, and obtained an $O(n^2m)$ algorithm. This bound was improved to $O(n^3)$ by Karzanov [1974] who introduced the concept of *preflows* in a layered network. A *preflow* is similar to a flow except that the amount flowing into a node may exceed the amount flowing out of a node. Since then, researchers have improved the complexity of Dinic's algorithm for sparse networks by devising sophisticated data structures. Among these contributions, Sleator and Tarjan's [1983] dynamic tree data structure is the most attractive from a worst case point of view.

The algorithms of Goldberg [1985] and of Goldberg and Tarjan [1986] were a novel departure from these approaches in the sense that they do not construct layered networks. Their method maintains a preflow, as per Karzanov, and proceeds by pushing flows to nodes estimated to be closer to the sink. To estimate which nodes are closer to the sink, it maintains a distance label for each node that is a lower bound on the length of a shortest augmenting path to the sink. Distance labels are a better computational device than layered networks because the labels are simpler to understand, easier to manipulate, and easier to use in a parallel algorithm. Moreover, by cleverly using the dynamic tree data structure, Goldberg and Tarjan obtained the best computational complexity for sparse as well as dense networks. (For applications of distance labels to augmenting path algorithms, see Orlin and Ahuja [1987].)

For problems with arc capacities polynomially bounded in n , our maximum flow algorithm is an improvement of Goldberg and Tarjan's algorithm and uses concepts of scaling introduced by Edmonds and Karp [1972] for the minimum cost flow problem and later extended by Gabow [1985] for other network optimization problems. The bottleneck operation in the straightforward implementation of Goldberg and Tarjan's algorithm is the number of *non-saturating pushes* which is $O(n^3)$. However, they reduce the computational time to $O(nm \log (n^2/m))$ by a clever application of the dynamic tree data structure. We show that the number of non-saturating pushes can be reduced to $O(n^2 \log U)$ by using *excess scaling*. Our algorithm modifies the Goldberg-Tarjan algorithm as follows. It performs $\log U$ scaling iterations; each scaling iteration requires $O(n^2)$ non-saturating pushes if we push flows from nodes with *sufficiently large* excesses to nodes with *sufficiently small* excesses while never allowing the excesses to become *too large*. The computational time of our algorithm is $O(nm + n^2 \log U)$.

Under the reasonable assumption that $U = O(n^{0(1)})$ (i.e., it is polynomial in n) , our algorithm runs in time $O(nm + n^2 \log n)$. On networks that are both non-dense and non-sparse, i.e., $m = \theta(n^{1+\varepsilon})$ for some ε with $0 < \varepsilon < 1$, our algorithm runs in time $O(nm)$, which improves Goldberg and Tarjan's bound of $O(nm \log (n^2/m))$ on such networks by a factor of $\log n$. Moreover, our algorithm is easier to implement and should be more efficient in practice, since it requires only elementary data structures with little computational overheads.

1. Notation

Let G = (N, A) be a directed network with a positive integer capacity u_{ij} for every arc $(i, j) \in A$. Let $n = |N|$ and $m = |A|$. The source *s* and sink *t* are two distinguished nodes of the network. We assume without loss of generality that the network does not contain multiple arcs and that there are no arcs directed into the source or directed from the sink. It is also assumed that for every arc $(i, j) \in A$, an arc (j, i) is also contained in A, possibly with zero capacity. We further assume that none of the paths from source to sink has infinite capacity as such a path can be easily detected in O(m) time. Observe that if the network contains some infinite capacity arcs but no infinite capacity path, then the capacity of such arcs can be replaced by

$$\sum_{\{(i,j) \in A:\ u_{ij} < \infty\}} u_{ij} .$$

We therefore assume that all arcs have finite capacity.

Let $U = \max_{(s,j) \in A} \{u_{sj}\}$.

A *flow* is a function $x : A \rightarrow \mathbf{R}$ satisfying

$$\sum_{\{j:\ (j,i) \in A\}} x_{ji} - \sum_{\{j:\ (i,j) \in A\}} x_{ij} = 0 \text{ , for all } i \in N - \{s, t\}, \tag{1}$$

$$\sum_{\{j:\ (j,t) \in A\}} x_{jt} = v , \tag{2}$$

$$0 \leq x_{ij} \leq u_{ij} , \quad \text{for all } (i, j) \in A , \tag{3}$$

for some $v \geq 0$. The maximum flow problem is to determine a flow x for which v is maximized.

A *preflow* x is a function $x: A \rightarrow \mathbf{R}$ which satisfies (2), (3), and the following relaxation of (1):

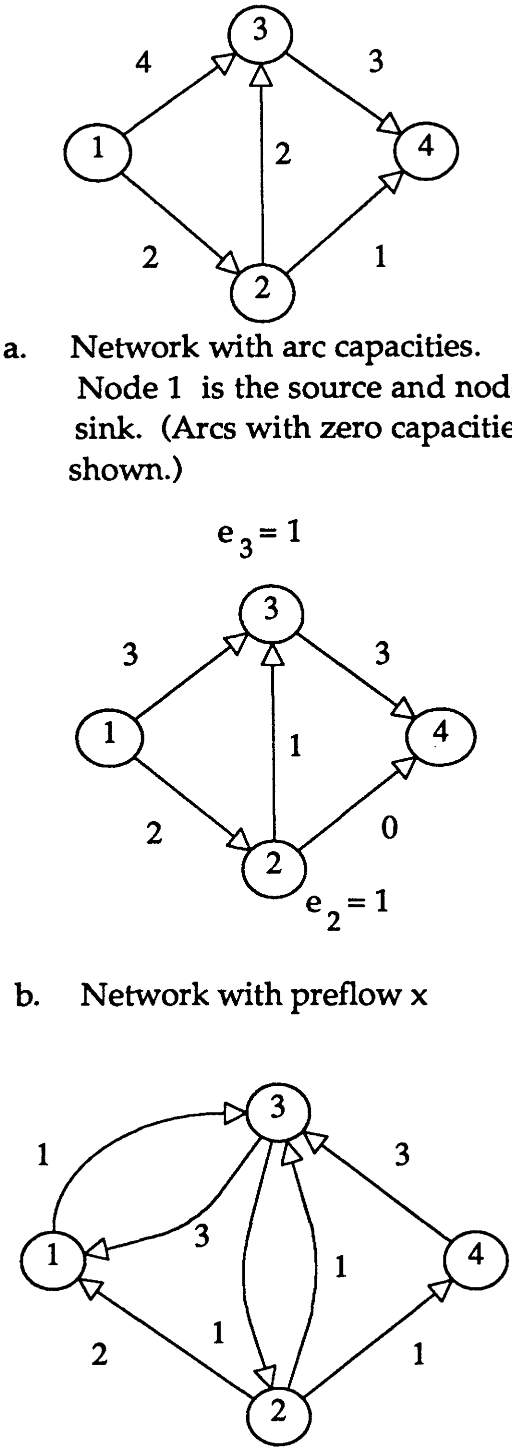
$$\sum_{\{j:\ (j,i) \in A\}} x_{ji} - \sum_{\{j:\ (i,j) \in A\}} x_{ij} \geq 0 \ , \text{ for all } i \in N - \{s, t\}. \qquad (4)$$

The algorithms described in this paper maintain a preflow at each intermediate stage.

For a given preflow x, we define for each node $i \in N - \{s, t\}$, the *excess*

$$e_i = \sum_{\{j:\ (j,i) \in A\}} x_{ji} - \sum_{\{j:\ (i,j) \in A\}} x_{ij} \quad .$$

A node with positive excess is referred to as an *active node*. We define the excess of the source and sink nodes to be zero; consequently, these nodes are never active. The *residual capacity* of any arc $(i, j) \in A$ with respect to a given preflow x is given by $r_{ij} = u_{ij} - x_{ij} + x_{ji}$. The residual capacity of arc (i, j) represents the maximum additional flow that can be sent from node i to node j using the arcs (i, j) and (j, i). The network consisting only of arcs with positive residual capacities is referred to as the *residual network*. Figure 1 illustrates these definitions.



a. Network with arc capacities. Node 1 is the source and node 4 is the sink. (Arcs with zero capacities are not shown.)

b. Network with preflow x

c. The residual network with residual arc capacities

Figure 1. Illustrations of a preflow and the residual network

We define the *arc adjacency list* A(i) of a node $i \in N$ as the set of arcs directed out of the node i, i.e., $A(i) := \{(i, k) \in A : k \in N\}$. Note that our adjacency list is a set of arcs rather than the more conventional definition of the list as a set of nodes.

A *distance function* $d : N \rightarrow Z^+$ for a preflow x is a function from the set of nodes to the non-negative integers . We say that a distance function d is *valid* if it also satisfies the following two conditions:

C1. $d(t) = 0$;

C2. $d(i) \leq d(j) + 1$, for every arc $(i, j) \in A$ with $r_{ij} > 0$.

Our algorithm maintains a valid distance function at each iteration. We also refer to d(i) as the *distance label* of node i. It is easy to demonstrate using induction that d(i) is a lower bound on the length of the shortest path from i to t in the residual network. Let $i = i_1 - i_2 - \ldots - i_k - i_{k+1} = t$ be any path of length k in the residual network from node i to the sink. Then from condition C2 we have, $d(i) = d(i_1) \leq d(i_2) + 1, d(i_2) \leq d(i_3) + 1, \ldots, d(i_k) \leq d(i_{k+1}) + 1 = 1$. This yields $d(i) \leq k$ for *any* path of length k in the residual network and, hence, must also hold for the shortest path too. If for each i, the distance label d(i) *equals* the minimum length of any path from i to t in the residual network, then we call the distance label *exact*. For example, in Figure 1(c), $d = (0, 0, 0, 0)$ is a valid distance label, though $d = (3, 1, 2, 0)$ represents the exact distance labels.

An arc (i, j) in the residual network is called *admissible* if it satisfies $d(i) = d(j) + 1$. An arc which is not admissible is called an *inadmissible* arc. The algorithms discussed in this paper push flow only on admissible arcs.

All logarithms in this paper are assumed to be of base 2 unless stated otherwise.

2. Preflow-Push Algorithms

The *preflow-push algorithms* for the maximum flow problem maintain a preflow at every step and proceed by pushing the node excesses closer to the sink. The first preflow-push algorithm is due to Karzanov [1974]. Tarjan [1984] has suggested a simplified version of this algorithm. The recent algorithms of Goldberg [1985] and Goldberg and Tarjan [1986] are based on ideas similar to those presented in Tarjan [1984], but use distance labels to direct flows closer to the sink instead of constructing layered networks. We refer to their algorithm as the *(distance-directed) preflow-push* algorithm. In this section, we review the basic features of their algorithm, which for the sake of brevity, we shall simply refer to as the preflow-push algorithm. Here we describe the 1-phase version of the preflow-push algorithm presented by Goldberg [1987]. The results in this section are due to Goldberg and Tarjan [1986].

All operations of the preflow-push algorithm are performed using only local information. At each iteration of the algorithm (except at the initialization and at the termination) the network contains at least one active node, i.e., a non-source and non-sink node with positive excess. The goal of each iterative step is to choose some active node and to send its excess "closer" to the sink, with closer being judged with respect to the current distance labels. If excess flow at this node can not be sent to nodes with smaller distance labels, then the distance label of the node is increased. The algorithm terminates when the network contains no active nodes. The preflow-push algorithm uses the following subroutines:

PRE-PROCESS. On each arc $(s, j) \in A(s)$, send u_{sj} units of flow. Let $d(s) = n$ and $d(t) = 0$. Let $d(i) = 1$ for each $i \neq s$ or t. (Alternatively, any valid labeling can be used, e.g., the distance label for each node $i \neq s, t$ can be determined by a backward breadth first search on the residual network starting at node t.)

SELECT. Select an active node i.

PUSH(i). Select an admissible arc (i, j) in $A(i)$. Send $\delta = \min \{e_i, r_{ij}\}$ units of flow from node i to j.

We say that a push of flow on arc (i, j) is *saturating* if $\delta = r_{ij}$, and *non-saturating* otherwise.

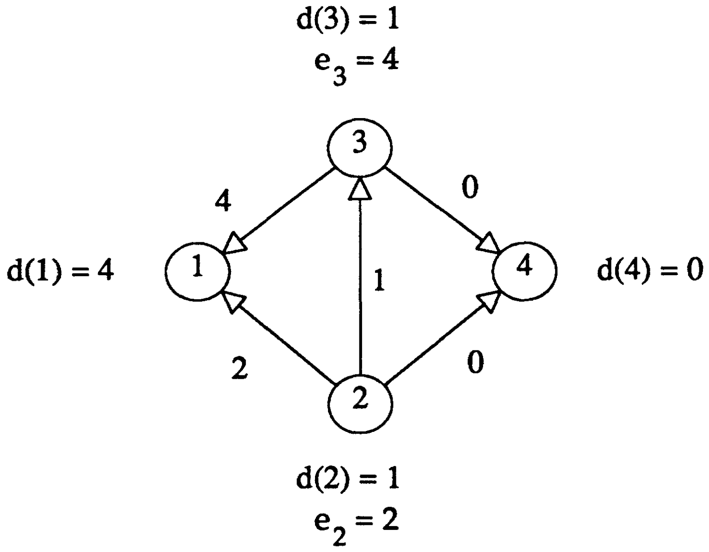
RELABEL(i). Replace $d(i)$ by $\min\{ d(j) + 1 : (i, j) \in A(i) \text{ and } r_{ij} > 0 \}$.

This step is called a *relabel* step. The result of the relabel step is to create at least one admissible arc on which further pushes can be performed.

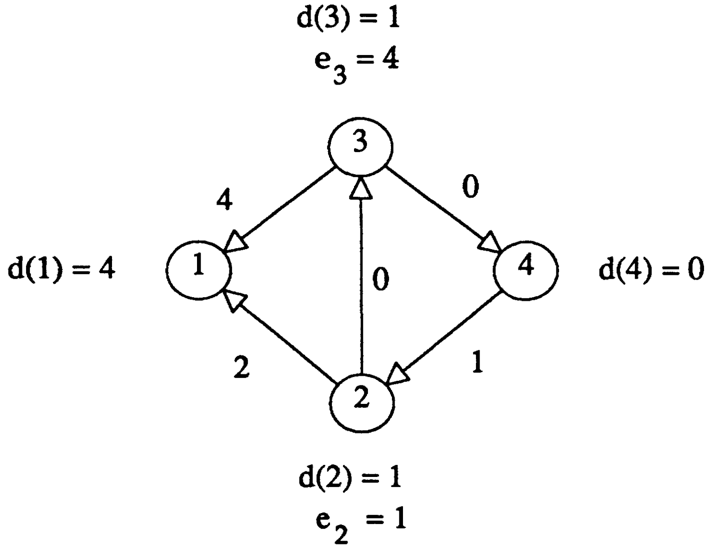
The generic version of the preflow-push algorithm is given below.

```
algorithm PREFLOW-PUSH;
begin
    PRE-PROCESS;
    while there is an active node do begin
        SELECT {let i denote the node selected};
        if there is an admissible arc in A(i) then PUSH(i)
        else RELABEL(i);
    end;
end;
```

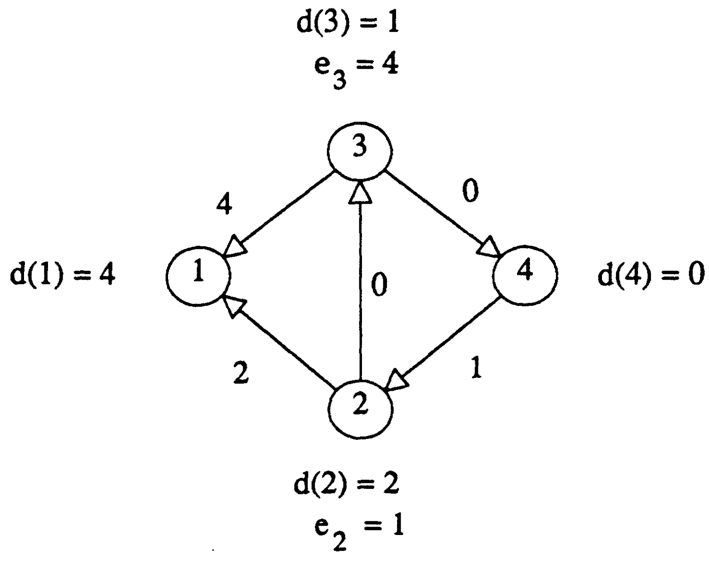
Figure 2 illustrates the steps PUSH(i) and RELABEL(i) as applied to the network in Figure 1(a). The number beside each arc represents its residual capacity. Figure 2(a) specifies the residual network after the PRE-PROCESS step. The SELECT step selects node 2 for examination. Since arc (2, 4) has residual capacity $r_{24} = 1$ and $d(2) = d(4) + 1$, the algorithm performs a saturating push of value $\delta = \min\{2, 1\}$ units. The push reduces the excess of node 2 to 1. Arc (2, 4) is deleted from the residual network and arc (4, 2) is added to the residual network. Since node 2 is still an active node, it can be selected again for further pushes. The arcs (2, 3) and (2, 1) have positive residual capacities, but they do not satisfy the distance condition. Hence the algorithm performs RELABEL(2), and gives node 2 a new distance $d'(2) = \min \{d(3) + 1, d(1) + 1\} = \min \{2, 5\} = 2$.



(a) The residual network after the pre-processing step.



(b) After the execution of step PUSH(2).



(c) After the execution of step RELABEL(2).

Figure 2. Illustrations of Push and Relabel steps.

The pre-process step accomplishes several important tasks. First, it causes the nodes adjacent to s to have positive excess, so that we can subsequently execute the select step. Second, by saturating arcs incident to s , the feasibility of setting d(s) = n is immediate. Third, since the distance label d(s) = n is a lower bound on the length of the minimum path from s to t , there is no path from s to t . Further, since distance labels are non–decreasing (see Lemma 1 to follow), we are also guaranteed that in subsequent iterations the residual network will never contain a directed path from s to t, and so there can never be any need to push flow from s again.

In our improvement of the preflow-push algorithm, we need a few of the results given in Goldberg and Tarjan [1986]. We include some of their proofs in order to make this presentation more self-contained.

Lemma 1. The generic preflow-push algorithm maintains valid distance labels at each step. Moreover, at each relabel step the distance label of some node strictly increases.

Proof. First note that the pre-process step constructs valid distance labels. Assume inductively that the distance function is valid prior to an operation, i.e., it satisfies the validity conditions C1 and C2 . A push operation on the arc (i, j) may create an additional arc (j, i) with $r_{ji} > 0$, and an additional condition $d(j) \le d(i) + 1$ needs to be satisfied. This validity condition remains satisfied since $d(i) = d(j) + 1$ by the property of the push operation. A push operation on arc (i, j) might delete this arc from the residual network, but this does not affect the validity of the distance function. During a relabel step, the new distance label of node i is $d'(i) = \min\{d(j) + 1: (i, j) \in A(i) \text{ and } r_{ij} > 0\}$, which is again consistent with the validity conditions. The relabel step is performed when there is no arc $(i, j) \in A(i)$ with $d(i) = d(j) + 1$ and $r_{ij} > 0$. Hence, $d(i) < \min \{d(j) + 1: (i, j) \in A(i) \text{ and } r_{ij} > 0\} = d'(i)$, thereby proving the second part of the lemma. ■

Lemma 2. At any stage of the preflow–push algorithm, for each node i with positive excess, there is a directed path from i to node s in the residual network.

Proof. By the flow decomposition theory of Ford and Fulkerson [1962] , any preflow x can be decomposed with respect to the original network G into the sum of non-negative flows along (i) paths from s to t, (ii) paths from s to active nodes, and (iii) flows around directed cycles. Let i be an active node relative to the preflow x in G. Then there must be a path P from s to i in the flow decomposition of x , since paths from s to t and flows around cycles do not contribute to the excess at node i. Then the reversal of P (P with the orientation of each arc reversed) is in the residual network , and hence there is a path from i to s in the residual network. ■

Corollary 1. For each node $i \in N$, $d(i) < 2n$.

Proof. The last time node i was relabeled, it had a positive excess, and hence the residual network contained a path of length at most $n - 1$ from i to s. The fact that $d(s) = n$ and condition C2 imply that $d(i) \leq d(s) + n - 1 < 2n$. ■

Lemma 2 also implies that a relabel step never minimizes over an empty set.

Corollary 2. The number of relabel steps is less than $2n^2$.

Proof. Each relabel step increases the distance label of a node by at least one, and by Corollary 1 no node can be relabeled more than 2n times. ■

Corollary 3. The number of saturating pushes is no more than nm .

Proof. Suppose that arc (i, j) becomes saturated at some iteration (at which $d(i) = d(j) + 1$). Then no more flow can be sent on (i, j) until flow is sent back from j to i, at which time $d'(j) = d'(i) + 1 \geq d(i) + 1 = d(j) + 2$; this flow change cannot occur until d(j) increases by at least 2. Thus by Corollary 1, arc (i, j) can become saturated at most n times, and the total number of arc saturations is no more than nm. (Recall that we assume that (i, j) and (j, i) are both in A , so the number of arcs in the residual network is no more than m .) ■

Lemma 3. The number of non-saturating pushes is at most $2n^2m$.

Proof. See Goldberg and Tarjan [1986] . ■

Lemma 4. The algorithm terminates with a maximum flow.

Proof. When the algorithm terminates, each node in $N - \{s, t\}$ has zero excess; so the final preflow is a feasible flow. Further, since the distance labels satisfy conditions C1 and C2 and $d(s) = n$, it follows that upon termination, the residual network contains no directed path from s to t . This condition is the classical termination criterion for the maximum flow algorithm of Ford and Fulkerson [1962] . ■

The bottleneck operation in many preflow based algorithms, such as the algorithms due to Karzanov [1974] , Tarjan [1984], and Goldberg and Tarjan [1986], is the number of non-saturating pushes. A partial explanation of why the number of non-saturating pushes dominates the number of saturating pushes is as follows: The saturating pushes cause structural changes -- they delete saturated arcs from the residual network. This observation leads to a bound of $O(nm)$ on the number of saturating pushes --no matter in which order they are performed. The non-saturating pushes do not change the structure of the residual network and seem more difficult to bound. Goldberg [1985] showed that the number of non-saturating pushes is $O(n^3)$ when nodes are examined in a first-in-first-out order. Goldberg and Tarjan [1986] reduced the running time of their network flow algorithm by using dynamic trees to reduce the average time per non-saturating push. Cheriyan and Maheshwari [1987] showed that the number of non-saturating pushes can be decreased to $O(n^2 m^{1/2})$ if flow is always pushed from a node with highest distance label, and they showed that this bound is tight. In the next section, we show that by using scaling, we can dramatically reduce the number of non-saturating pushes to $O(n^2 \log U)$. We have recently discovered a new scaling algorithm which further reduces the number of non-saturating pushes to $O\left(\frac{n^2 \log U}{\log \log U}\right)$. This result is presented in Ahuja, Orlin and Tarjan [1987].

3. The Scaling Algorithm

Our maximum flow algorithm improves the generic preflow-push algorithm of Section 2 by using "excess scaling" to reduce the number of non-saturating pushes from $O(n^2m)$ to $O(n^2 \log U)$. The basic idea is to push flow from active nodes with *sufficiently large* excesses to nodes with *sufficiently small* excesses while never letting the excesses become *too large.*

The algorithm performs $K = \lceil \log U \rceil + 1$ scaling iterations. For a scaling iteration, the *excess-dominator* is defined to be the least integer Δ that is a power of 2 and satisfies $e_i \leq \Delta$ for all $i \in N$. Further , a new scaling iteration is considered to have begun whenever Δ decreases by a factor of 2. In a scaling iteration we guarantee that each non-saturating push sends at least $\Delta/2$ units of flow and that the excess-dominator does not increase. To ensure that each non-saturating push has value at least $\Delta/2$, we consider only nodes with excess more than $\Delta/2$; and among these nodes with large excess, we select a node with minimum distance label. This choice ensures that the flow will be sent to a node with small excess. We show that after at most $8n^2$ non-saturating pushes, the excess–dominator decreases by a factor of at least 2, and a new scaling iteration begins. After at most K scaling iterations, all node excesses drop to zero and we obtain a maximum flow.

In order to select an active node with excess more than $\Delta/2$ and with a minimum distance label among such nodes, we maintain the lists $\text{LIST}(r) = \{ i \in N : e_i > \Delta/2 \text{ and } d(i) = r\}$ for each $r = 1, \ldots, 2n-1$. These lists can be maintained in the form of either linked stacks or linked queues (see, for example, Aho, Hopcroft and Ullman [1974]), which enables insertion and deletion of elements in O(1) time. The variable *level* indicates the smallest index r for which LIST(r) is non-empty.

As per Goldberg and Tarjan, we use the following data structure to efficiently select the eligible arc for pushing flow out of a node. We maintain with each node i the list, A(i), of arcs directed out of it. Arcs in each list can be arranged arbitrarily, but the order, once decided, remains unchanged throughout the algorithm. A special arc named *null* is appended to the end of each list. Each node i has a *current arc* (i, j) which is the current candidate for pushing flow out of i. Initially, the current arc of node i is the first arc in its arc list. This list is examined sequentially, and whenever the current arc is found to be inadmissible for pushing flow, the next arc in the arc list is made the current arc. When the arc list has been completely examined, the null arc is reached. At this time, the node is relabeled and the current arc is again set to the first arc in the arc list.

The algorithm can be formally described as follows:

```
algorithm MAX-FLOW;
begin
    PRE-PROCESS;
    K : = 1 + ⌈log U⌉ ;
    for k : = 1 to K do begin
        $\Delta = 2^{K-k}$
        for each $i \in N$ do if $e_i > \Delta/2$ then add i to LIST(d(i));
        level : = 1 ;
        while level < 2n do
            if LIST(level) = ø then level : = level + 1
            else begin
                select a node i from LIST(level);
                PUSH/RELABEL(i);
            end;
    end;
end;
```

```
procedure PUSH/RELABEL(i);
begin
    found : = false ;
    let (i, j) be the current arc of node i;
    while found = false and (i, j) ≠ null do
        if d(i) = d(j) + 1 and $r_{ij} > 0$ then found : = true
        else replace the current arc of node i by the next arc (i, j);

    if found = true then begin {found an admissible arc}
        push min {$e_i$ , $r_{ij}$ , $\Delta - e_j$} units of flow on arc (i, j) ;
        update the residual capacity $r_{ij}$ and the excesses $e_i$ and $e_j$ ;
        if (the updated excess) $e_i \leq \Delta/2$ , then delete node i from LIST(d(i));
        if j ≠ s or t and (the updated excess) $e_j > \Delta/2$ , then add node j to
            LIST(d(j)) and set level : = level – 1;
    end
    else begin {finished arc list of node i}
        delete node i from LIST(d(i));
        update d(i) : = min{d(j) + 1 ; (i, j) ∈ A(i) and $r_{ij} > 0$} ;
        add node i to LIST(d(i)) and set the current arc of node i to the
          first arc of A(i);
    end;
end;
```

4. Complexity of the Algorithm

In this section , we show that the distance–directed preflow-push algorithm with excess scaling correctly computes a maximum flow in $O(nm + n^2 \log U)$ time.

Lemma 5. The scaling algorithm satisfies the following two conditions:

C3. Each non-saturating push from a node i to a node j sends at least $\Delta/2$ units of flow.

C4. No excess increases above Δ (i.e., the excess-dominator does not increase subsequent to a push).

Proof. For every push on arc (i, j) we have $e_i > \Delta/2$ and $e_j \le \Delta/2$, since node i is a node with smallest distance label among nodes whose excess is more than $\Delta/2$, and $d(j) = d(i) - 1 < d(i)$ by the property of the push operation. Hence , by sending $\min \{e_i , r_{ij} , \Delta - e_j\} \ge \min \{\Delta/2 , r_{ij}\}$ units of flow, we ensure that in a non-saturating push the algorithm sends at least $\Delta/2$ units of flow. Further, the push operation increases the excess at node j only. Let e'_j be the excess at node j after the push. Then $e'_j = e_j + \min \{ e_i , r_{ij} , \Delta - e_j\} \le e_j + \Delta - e_j \le \Delta$. All node excesses thus remain less than or equal to Δ . ■

While there are other ways of ensuring that the algorithm always satisfies the properties stated in the conditions C3 and C4, pushing flow from a node with excess greater than $\Delta/2$ and with minimum distance among such nodes is a simple and efficient way of enforcing these conditions.

With properties C3 and C4, the push operation may be viewed as a kind of "restrained greedy approach." Property C3 ensures that the push from i to j is sufficiently large to be effective. Property C4 ensures that the maximum excess never exceeds Δ during an iteration. In particular, rather than greedily getting rid of all its excess, node i shows some restraint so as to prevent e_j from exceeding Δ. Keeping the maximum excess lower may be very useful in practice as well as in theory. Its major impact is to "encourage" flow excesses to be distributed fairly equally in the network. This distribution of flows should make it easier for nodes to send flow towards the sink. This may also be important because of the following consideration: suppose several nodes send flow to a single node j creating a large excess. It is likely that node j would not be able to send the accumulated flow closer to the sink, in which case its distance label would increase and much of its excess would have to be returned. This phenomenon is prevented by maintaining condition C4.

Lemma 6. If each push satisfies conditions C3 and C4, then the number of non-saturating pushes per scaling iteration is at most $8n^2$.

Proof. Consider the potential function $F = \sum_{i \in N} e_i \, d(i)/\Delta$.

The initial value of F at the beginning of Δ-scaling phase is bounded by $2n^2$ because e_i is bounded by Δ and d(i) is bounded by 2n. When the algorithm examines node i, one of the following two cases must apply:

Case 1. The algorithm is unable to find an arc along which flow can be pushed. In this case no arc (i, j) satisfies $d(i) = d(j) + 1$ and $r_{ij} > 0$ and the distance label of node i goes up by $\varepsilon \geq 1$ units. This increases F by at most ε units. Since the total increase in d(i) throughout the running of the algorithm for each i is bounded by 2n, the total increase in F due to relabelings of nodes is bounded by $2n^2$ in the scaling phase (Actually, the increase in F due to node relabelings is at most $2n^2$ over all scaling phases).

Case 2. The algorithm is able to identify an arc on which flow can be pushed and so it performs either a saturating or a non-saturating push. In either case, F decreases. A non-saturating push on arc (i, j) sends at least $\Delta/2$ units of flow from node i to node j and since $d(j) = d(i) - 1$, this decreases F by at least $\frac{1}{2}$ units . As the initial value of F for a scaling phase plus the increases in F sum to at most $4n^2$, this case can not occur more than $8n^2$ times. ■

Theorem 1. The scaling algorithm performs $O(n^2 \log U)$ non-saturating pushes.

Proof. The initial value of the excess-dominator Δ is $2^{\lceil \log U \rceil} \leq 2U$. By Lemma 6 , the value of the excess-dominator decreases by a factor of 2 within $8n^2$ non-saturating pushes and a new scaling iteration begins. After $1 + \lceil \log U \rceil$ such scaling iterations, $\Delta < 1$; and by the integrality of the flows $e_i = 0$ for all $i \in N - \{s, t\}$. The algorithm thus obtains a feasible flow, which by Lemma 4 must be a maximum flow. ■

Theorem 2. The complexity of the maximum flow scaling algorithm is $O(nm + n^2 \log U)$.

Proof: The complexity of the algorithm depends upon the number of executions of the **while** loop in the main program. In each such execution either a PUSH/RELABEL(i) step is performed or the value of the variable **level** increases. Each execution of the procedure PUSH/RELABEL(i) results in one of the following outcomes:

Case 1. A push is performed. Since the number of saturating pushes is $O(nm)$ and the number of non-saturating pushes is $O(n^2 \log U)$, this case occurs $O(nm + n^2 \log U)$ times.

Case 2. The distance label of node i goes up. By Corollary 1, this outcome can occur $O(n)$ times for each node i and $O(n^2)$ in total.

Thus the algorithm calls the procedure PUSH/RELABEL(i) $O(nm + n^2 \log U)$ times. The effort needed to find an arc to perform the push operation is $O(1)$ plus the number of times the current arc of node i is replaced by the next arc in A(i). After $|A(i)|$ such replacements for node i, Case 2 occurs and distance label of node i goes up. Thus, the total effort needed is $\sum_{i \in N} 2n\, |A(i)| = O(nm)$ plus the number of PUSH/RELABEL(i) operations. This is clearly $O(nm + n^2 \log U)$.

Now consider the time needed for relabel operations. Computing the new distance label of node i requires examining arcs in A(i). This yields a total of $\sum_{i \in N} 2n\, |A(i)| = O(nm)$ time for all relabel operations. The lists LIST(r) are stored as linked stacks and queues, hence addition and deletion of any element takes O(1) time. Consequently, updating these lists is not a bottleneck operation.

Finally, we need to bound the number of increases of the variable *level*. In each scaling iteration, *level* is bounded above by $2n - 1$ and bounded below by 1. Hence its number of increases per scaling iteration is bounded by the number of decreases plus $2n$. Further, *level* can decrease only when a push is performed and in such a case it decreases by 1. Hence its increases over all scaling iterations are bounded by the number of pushes plus $2n(1 + \lceil \log U \rceil)$, which is again $O(nm + n^2 \log U)$. ■

5. Refinements

As a practical matter, several modifications of the algorithm might improve its actual execution time without affecting its worst case complexity. We suggest three modifications:

1. Modify the scale factor.
2. Allow some non-saturating pushes of small amount.
3. Try to locate nodes disconnected from the sink.

The first suggestion is to consider the scale factor. The algorithm in the present form uses a scale factor of 2, i.e., it reduces the excess-dominator by a factor 2 in each scaling iteration. In practice, however, some other fixed integer scaling factor $\beta \geq 2$ might yield better results. The excess-dominator will then be the least power of β that is no less than the excess at any node, and property C3 becomes

C3'. Each non-saturating push from a node i to a node j sends at least Δ/β units of flow.

The scaling algorithm presented earlier can easily be altered to incorporate the β scale factor by letting LIST(r) = $\{i \in N : e_i > \Delta/\beta$ and $d(i) = r\}$. The algorithm can be shown to run in $O(nm + \beta n^2 \log_\beta U)$ time. From the worst case point of view any fixed value of β is optimum; the best choice for the value of β in practice should be determined empirically.

The second suggestion focuses on the non-saturating pushes. Our algorithm as stated selects a node with $e_i > \Delta/2$ and performs a saturating or a non-saturating push. We could, however, keep pushing the flow out of this node until either we perform a non-saturating push of value at least $\Delta/2$ or reduce its excess to zero. This variation might produce many saturating pushes from the node and even allow pushes after its excess has decreased below $\Delta/2$. Also, the algorithm as stated sends at least $\Delta/2$ units of flow during every non-saturating push. The same complexity of the algorithm is obtained if for some fixed $r \geq 1$, one out of every $r \geq 1$ non-saturating pushes sends at least $\Delta/2$ units of flow.

The third suggestion recognizes that one potential bottleneck in practice is the number of relabels. In particular, the algorithm "recognizes" that the residual network contains no path from node i to node t only when d(i) exceeds $n-2$. Goldberg [1987] suggested that it may be desirable to occasionally perform a breadth first search so as to make the distance labels exact. He discovered that a judicious use of breadth first search could dramatically speed up the algorithm.

An alternative approach is to keep track of the number n_k of nodes whose distance is k. If n_k decreases to 0 after any relabel for some k, then each node with distance greater than k is disconnected from the sink in the residual network. (Once node j is disconnected from the sink, it stays disconnected since the shortest path from j to t is nondecreasing in length.) We would avoid selecting such nodes until all nodes with positive excess become disconnected from the sink. At this time, the excesses of nodes are sent back to the source. This approach essentially yields the two phase approach to solve the maximum flow problem as outined in Goldberg and Tarjan [1986] . The first phase constructs a maximum preflow which is converted to a maximum flow in the second phase.

6. Future Directions

Our improvement of the distance directed preflow-push algorithm has several advantages over other algorithms for the maximum flow problem. Our algorithm is superior to all previous algorithms for the maximum flow problem under the reasonable assumption that U is polynomially bounded in n. Further, the algorithm utilizes very simple data structures which makes it attractive from an implementation viewpoint.

Our algorithm is computationally attractive from a worst-case perspective even if U is not $O(n^{0(1)})$; that is, if the arc capacities are exponentially large numbers. In this case, the uniform model of computation, in which all arithmetic operations take O(1) steps, is arguably inappropriate. It is more realistic to adopt the logarithmic model of computation (as described by Cook and Reckhow [1973]) which counts the number of *bit* operations. In this model, most arithmetic operations take O(log U) steps rather than O(1) steps. Using the logarithmic model of computation and modifying our algorithm slightly to speed up arithmetic operations on large integers, we claim that our algorithm would run in $O(nm \log n + n^2 \log n \log U)$ time. The corresponding time bound for the Goldberg–Tarjan algorithm is $O(nm \log (n^2/m) \log U)$. Hence, as U becomes exponentially large, our algorithm becomes more and more attractive relative to the Goldberg-Tarjan algorithm. Our results in the logarithmic model of computation will be presented in a future paper.

Our algorithm is a novel approach to combinatorial scaling algorithms. In the previous scaling algorithms developed by Edmonds and Karp [1972], Rock [1980] , and Gabow [1985], scaling involved a sequential approximation of either the cost coefficients or the capacities and right-hand-sides. (e.g., we would first solve the problem with the costs approximated by $C/2^T$ for some integer T. We would then reoptimize so as to solve the problem with C approximated by $C/2^{T-1}$, and then reoptimize for the problem with C approximated by $C/2^{T-2}$, and so forth.) Our scaling method does not fit into this standard framework. Rather, our algorithm works with true data, relaxes the flow conservations constraints and scales the maximum amount of relaxation. The recent cost scaling algorithm of Goldberg and Tarjan [1987] for the minimum cost flow problem is similar in nature -- this algorithm scales the relaxation of the complementary slackness conditions.

The scaling algorithm for the maximum flow problem can be improved further by using more clever rules to push flow or by using dynamic trees. We describe such improvements in Ahuja, Orlin and Tarjan [1987]. We show that by using a larger scale factor and pushing flow from a node with highest distance label among nodes having *sufficiently large* excess, the algorithm runs in $O\left(nm + \frac{n^2 \log U}{\log \log U}\right)$ time (Assume that $U \geq 4$.). Use of the dynamic tree data structure further improves the complexity of this algorithm to $O\left(nm \ \log\left(\frac{n \log U}{m \log \log U} + 2\right)\right)$.

We have also undertaken an extensive empirical study to assess the computational merits of the preflow-push algorithms versus the previous best algorithms, those of Dinic and Karzanov. Our experiments so far suggest that preflow-push algorithms are substantially faster than Dinic's and Karzanov's algorithms.

Our algorithms and those due to Goldberg and Tarjan suggest the superiority of distance label based approaches over the layered network based approaches. The improvements we have obtained do not seem to be possible for the algorithms utilizing layered networks. The distance labels implicitly store dynamically changing layered networks and hence are more powerful. We show the use of distance labels in augmenting path algorithms, capacity scaling algorithms and for unit capacity networks in Orlin and Ahuja [1987].

The maximum flow problem on bipartite networks is an important class of the maximum flow problem (see Gusfield, Martel and Fernandez-Baca [1985]). The bipartite network is a network $G = (N, A)$ such that $N = N_1 \cup N_2$ and $A \subseteq N_1 \times N_2$. Let $n_1 = |N_1|$ and $n_2 = |N_2|$. For cases where $n_1 << n_2$, our maximum flow algorithm can be modified to run in $O(n_1 m + n_1^2 \log U)$, thus resulting in significant speedup over the original algorithm. Our results on bipartite network flows will appear in a future paper jointly with C. Stein and R. Tarjan.

Our maximum flow algorithm is difficult to make "massively parallel" since we push flow from one node at a time. Nevertheless, with $d = \lceil m/n \rceil$ parallel processors we can obtain an $O(n^2 \log Ud)$ time bound. Under the assumption that $U = O(n^{(0(1))})$, the algorithms runs in $O(n^2 \log n)$ time, which is comparable to the best available time bounds obtained by Shiloach and Vishkin [1982] and Goldberg and Tarjan [1986] using *n parallel processors.* Thus, our algorithm has an advantage in situations for which parallel processors are at a premium. Our work on the parallel algorithms will also appear in a future paper.

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