# A NOTE ON THE OPTIMAL POSITIONING OF SERVICE UNITS


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In this paper we address the problem of locating $p$ mobile service units in an $n$-dimensional space minimizing the expected response time. It is shown that an optimal solution to the problem is a degenerate distribution for the service, concluding that it is optimal to park the $p$ units. This extends previous results in the literature for location on a segment.


In continuous facility location problems, one seeks the location for $p$ facilities, minimizing some function of the distances to a given set of demand points.

We assume that the distribution of the demand is given by a probability measure $\mu$ on the family of Borels in $IR^n$, thus including, as particular instances, cases in which the demand is concentrated at a finite set of points (see, e.g., Francis et al. 1993), or is continuously distributed (see, e.g., Carrizosa et al. 1995).

Let $p$ be the number of facilities to be located. Suppose that there exists a Borel-measurable function $\phi$ such that, if servers are actually positioned at $x_1, \ldots, x_p$, the average response time to a call is given by

$$\int \phi(a, x_1, \ldots, x_p)\, d\mu(a).$$

Typically $\phi$ will be a function of the distances $d(x_j, a)$ from the unit at $x_j$ to the call at $a$, where $d(x_j, a) = \gamma(a - x_j)$ for some norm or gauge, see Durier and Michelot (1985), in $IR^n$.

Simple examples of $\phi$ are

$$\phi(a, x_1, \ldots, x_p) = \min_{1 \leqslant j \leqslant p} d(x_j, a), \tag{1}$$

[the unit closest to $a$ is dispatched] or

$$\phi(a, x_1, \ldots, x_p) = \frac{1}{p} \sum\nolimits_{1 \leqslant j \leqslant p} d(x_j, a), \tag{2}$$

[the dispatched unit is selected at random].

See also Kaufman and Plastria (1988) and Hooker and Garfinkel (1989) for other allocation scheme leading to different functions $\phi$.

If the $p$ facilities (units) to be located had fixed positions, such optimal positions would be obtained by solving the optimization problem

$$\min_{(x_1, \ldots, x_p)} \int \phi(a, x_1, \ldots, x_p)\, d\mu(a), \tag{3}$$

which reduces respectively to the multi-Weber problem (Rosing 1992) and the multifacility location problem (Michelot 1987) when $\phi$ is given by 1 and 2.

On the other hand, if the units are allowed to patrol, then the decision variable should be the joint probability measure $\nu$ of the distribution of the $p$ units through $IR^{np}$, and the problem to be solved would be

$$\min_{\nu \in \mathcal{F}} \int \left\{ \int \phi(a, x_1, \ldots, x_p)\, d\mu(a) \right\} d\nu(x_1, \ldots, x_p), \tag{4}$$

where $\mathcal{F}$ is the set of probability measures over the Borel sets of $IR^{np}$ for which the integral above is well defined.

We show in the next section that an optimal $\nu \in \mathcal{F}$ is degenerate at a point, (i.e., it is optimal to park the $p$ mobile units), thus Problem 4 is reduced to the formulation given by 3.

This implies that, although beneficial for other aspects, (e.g., crime prevention for police patrol units), patrolling always has a negative impact on the average response time to call.

## 1. OPTIMAL POSITIONING

The relationship among the optimal solutions of the problems 4 and 3 is given in the following theorem.

**Theorem 1.1.** *If $(x_1^*, \ldots, x_p^*)$ is an optimal solution of the Problem* 3, *then the probability measure $\nu^*$ degenerate at $(x_1^*, \ldots, x_p^*)$ is an optimal solution of the Problem* 4.

**Proof.** Let $(x_1^*, \ldots, x_p^*)$ be an optimal solution of the Problem 3. Then it verifies that


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$$\int \phi(a, x_1^*, \ldots, x_p^*)\, d\mu(a)$$

$$\leqslant \int \phi(a, x_1, \ldots, x_p)\, d\mu(a)$$

for any $(x_1, \ldots, x_p) \in IR^{np}$. Thus, for any $\nu \in \mathscr{F}$,

$$\iint \phi(a, x_1, \ldots, x_p)\, d\mu(a)\, d\nu^*(x_1, \ldots, x_p)$$

$$= \int \phi(a, x_1^*, \ldots, x_p^*)\, d\mu(a)$$

$$= \iint \phi(a, x_1^*, \ldots, x_p^*)\, d\mu(a)\, d\nu(x_1, \ldots, x_p)$$

$$\leqslant \iint \phi(a, x_1, \ldots, x_p)\, d\mu(a)\, d\nu(x_1, \ldots, x_p)$$

and consequently $\nu^*$ is an optimal solution of the Problem 4. □

The previous theorem is a generalization of the results given in Anderson and Fontenot (1992), Larson and Odoni (1981), and Levine (1986). Larson and Odoni (1981), showed that if the demand is uniformly distributed over a segment an optimum solution for Problem 4 is a probability measure degenerate at a point. Thus, if one wished to locate a service which could be mobile, it is preferable to fix its position to maintain it patrolling. Levine (1986) obtained the same conclusion for the case of a general distribution of the demand over a segment, but however he did not observe that the optimum solution is the median of the distribution. Anderson and Fontenot (1992), showed that if the demand is distributed over a segment and one wishes to locate a fixed number of facilities and the allocation rule is given by the function $\phi$ defined in 1, then the solution fixes the position of the services, in the special case of one unit, this means that the unit should be located at the median of the demand.

Our result shows that this policy (i.e., fixing the position of units instead of letting them patrol) is optimal for a much broader class of problems and dispatching rules than those previously addressed in the literature.

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