

Exponential-Size Neighborhoods for the Pickup-and-Delivery Traveling Salesman Problem

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Neighborhood search is a cornerstone of state-of-the-art traveling salesman and vehicle routing metaheuristics. While neighborhood exploration procedures are well developed for problems with individual services, their counterparts for one-to-one pickup-and-delivery problems have been more scarcely studied. A direct extension of classic neighborhoods is often inefficient or complex due to the necessity of jointly considering service pairs. To circumvent these issues, we introduce major improvements to existing neighborhood searches for the pickup-and-delivery traveling salesman problem and new large neighborhoods. We show that the classical RELOCATE-PAIR neighborhood can be fully explored in $\mathcal{O}(n^2)$ instead of $\mathcal{O}(n^3)$ time. We adapt the 4-OPT and BALAS-SIMONETTI neighborhoods to consider precedence constraints. Moreover, we introduce an exponential-size neighborhood called 2k-OPT, which includes all solutions generated by multiple nested 2-OPTS and can be searched in $\mathcal{O}(n^2)$ time using dynamic programming. We conduct extensive computational experiments, highlighting the significant contribution of these new neighborhoods and speed-up strategies within two classical metaheuristics. Notably, our approach permits to repeatedly solve small pickup-and-delivery problem instances to optimality or near-optimality within milliseconds, and therefore it represents a valuable tool for time-critical applications such as meal delivery or mobility on demand.

Key words: Local search, Large neighborhood search, Dynamic programming, Computational complexity, Pickup-and-delivery problem

1. Introduction

Neighborhood search holds a central place in all modern metaheuristics for routing problems (Applegate et al. 2006, Vidal et al. 2013, Laporte et al. 2014), to such an extent that all current

state-of-the-art metaheuristics (e.g., Nagata and Kobayashi 2013, Vidal et al. 2014, Subramanian et al. 2013, Christiaens and Vanden Berghe 2020) rely on efficient local or large neighborhood search algorithms for achieving high quality solutions. One general thumb rule for success in local-search based metaheuristics is to search *large and fast*. Indeed, larger neighborhoods generally permit further solution refinements, whereas faster exploration techniques permit more restarts (i.e., from initial solutions generated by crossover or perturbation) to diversify the search. Finding a good compromise between neighborhood size and exploration effort is a challenging task, which often involves experimenting with various neighborhoods, pruning rules, and speed-up techniques.

Traveling salesman and vehicle routing problem variants are typically classified in accordance to their service types (Vidal et al. 2020). In one-to-many-to-one (1-M-1) problems (including one-to-many 1-M and many-to-one M-1 as a special case), each service originates or returns to the depot. Effective local searches in that setting are based on simple exchanges or relocations of customer visits or replacements of arcs. Many neighborhoods for these problems (such as 2-OPT, SWAP and RELOCATE) contain $\mathcal{O}(n^2)$ solutions in their classical form before any other neighborhood restriction. In contrast, in one-to-one pickup-and-delivery problems, every pickup service is paired with its associated delivery, and moves often need joint optimizations of *visit pairs* to remain feasible, leading to neighborhoods which are one order of magnitude more complex (e.g., containing $\mathcal{O}(n^3)$ solutions) and whose complete exploration can be challenging for large-scale instances.

In this article, we introduce new neighborhood searches for one-to-one pickup-and-delivery problems. More precisely, we focus on the pickup-and-delivery traveling salesman problem (PDTSP), which is the canonical problem in this family, and make the following contributions:

- We design an efficient evaluation strategy which permits to explore the RELOCATE PAIR neighborhood in $\mathcal{O}(n^2)$ instead of $\mathcal{O}(n^3)$ time.
- We introduce a neighborhood of exponential size called 2k-OPT. This neighborhood contains all solutions formed with *nested* 2-OPTS and can be explored in $\mathcal{O}(n^2)$ time. This neighborhood overcomes the natural tendency of 2-OPT to create infeasible solutions due to route reversals and precedence constraints in pickup-and-delivery problems.
- We adapt the restricted 4-OPT neighborhood of Glover (1996) and the neighborhood of Balas and Simonetti (2001) to the PDTSP.
- We integrate all these neighborhoods within a sophisticated local search and evolutionary metaheuristic: the hybrid genetic search (HGS) of Vidal et al. (2012, 2014). As visible in our computational experiments on the PDTSP, the proposed solution approach performs remarkably well on the 163 classical instances from Renaud et al. (2000) and Dumitrescu et al. (2010), identifying better or equal solutions than all previous algorithms in all cases. We also conduct sensitivity analyses on the individual contributions of each neighborhood. Based on

these experiments, the efficient RELOCATE PAIR, the OR-OPT and the adapted 4-OPT are highlighted as the most important search operators for this problem.

2. Related Literature

Neighborhood search. Let X be the set of all feasible solutions of a given optimization problem. Formally, a neighborhood is a mapping $N : X \rightarrow 2^X$ associating to each solution x a set of neighbors $N(x) \subseteq X$. Most vehicle routing neighborhoods are built on the definition of a *move* operator (e.g., a RELOCATE, which consists in changing the position of a service in the routing plan) and contain all solutions reachable from X by a single move. Based on this definition, a local-search procedure iteratively explores the neighborhood of an incumbent solution to find an improved one, replaces the incumbent solution with this solution, and repeats this process until no further improvement exists. The final solution is called a local optimum.

Local searches for 1-M-1 problems typically rely on k -OPT, RELOCATE and SWAP moves and some of their extensions. A k -OPT move consists of replacing up to k arcs in the solution. RELOCATE consists of moving a customer visit to another position, whereas SWAP exchanges the positions of two customer visits. There are $\mathcal{O}(k!n^k)$ possible k -OPT and $\mathcal{O}(n^2)$ possible RELOCATE and SWAP moves. In contrast, 1-1 pickup-and-delivery problems involve paired services. Neighborhood searches are more complex in this setting since the locations of the pickups and of their corresponding deliveries need to be jointly optimized. Single-visit RELOCATE moves, for example, often generate infeasible solutions violating precedence or capacity constraints. As a consequence, it is common to explore the RELOCATE-PAIR neighborhood, which consists of jointly relocating a pickup and its associated delivery. However, an enumerative exploration of this neighborhood takes $\mathcal{O}(n^3)$ time.

Various speed-up techniques can be used to improve neighborhood search. The well-known Lin and Kernighan (1973) algorithm and its implementation by Helsgaun (2000, 2009) combines filtering mechanisms and bounds to perform an effective search of k -OPT of order up to $k = 5$ for the traveling salesman problem. Other simple neighborhood restriction strategies consist in restricting the moves to “close” locations (Johnson and Mcgeoch 1997, Toth and Vigo 2003). In a different fashion, Taillard and Helsgaun (2019) firstly generate a pool of initial solutions and restrict the subsequent search to the edges belonging at least to one of these initial solutions. Efficient search strategies and spatial data structures may also be exploited to achieve significant speedups (Bentley 1992, Glover 1996, De Berg et al. 2020, Lancia and Dalpasso 2019).

Some families of neighborhoods qualified as *very large* contain an exponential number of solutions but can be efficiently searched (Ahuja et al. 2002). This is often done by exploiting combinatorial subproblems, e.g., shortest paths, assignment or network flows (De Franceschi et al. 2006, Hintsch and Irnich 2018, Vidal 2017, Capua et al. 2018) and dynamic programming

techniques (Deineko and Woeginger 2000, Balas and Simonetti 2001, Congram et al. 2002). Balas and Simonetti (2001) exploit the availability of a linear-time dynamic programming algorithm for a restricted variant of the traveling salesman problem (TSP) to design a very large neighborhood. Given an incumbent solution represented as a permutation σ and a fixed value k , this neighborhood contains all permutations π of σ such that π fulfills $\pi(1) = 1$ and $\pi(i) \leq \pi(j)$ for all $i, j \in \{1, \dots, n\}$ such that $\sigma(i) + k \leq \sigma(j)$. In other words, if j is located at least k positions after i in the original permutation σ , then j must appear after i in permutation π . This neighborhood was later exploited for several variants of vehicle routing and arc routing problems (Irñich 2008, Vidal 2017, Gschwind and Drexl 2019, Toffolo et al. 2019). Finally, ejection chain neighborhoods consist of exploring chained customer relocations. Some exponential-size neighborhoods of this class can be reduced to the search of a negative-cost cycle or path in an auxiliary graph (Glover and Rego 2006, Vidal 2017). It is also noteworthy that efficient search procedures for exponential-size neighborhoods exist for the TSP, but that they are theoretically impossible for other problems such as the quadratic assignment problem unless P=NP (see Deineko and Woeginger 2000).

The pickup-and-delivery TSP. The PDTSP is the canonical form of a single-vehicle one-to-one pickup-and-delivery problem. This problem is highly relevant in robotics, production and cutting as well as transportation logistics. It has become the focus of renewed research due to its direct applications to mobility-on-demand and meal-delivery services (see, e.g., O’Neil 2018, Yıldız and Savelsbergh 2019). Formally, the PDTSP can be defined over a graph $G = (V, E)$ in which the vertices $V = 0 \cup P \cup D$ represent service locations and the edges in E indicate possible trips between locations. Let $P = \{1, \dots, n\}$ and $D = \{n+1, \dots, 2n\}$ represent pickup and delivery vertices, respectively. There are n pickup-and-delivery requests $(i, n+i)$ for $i \in \{1, \dots, n\}$, each one associated to a pickup location $i \in P$ and a delivery location $n+i \in D$. Let $N = 2n + 1$ be the number of visits in the problem. Vertex 0 represents an origin location, from which each vehicle should start and return. Finally, each edge $(i, j) \in E$ is associated to a travel cost c_{ij} . The goal of the PDTSP is to find a tour of minimum total distance, starting and ending at 0, in such a way that each pickup and delivery location is visited once, and that each pickup precedes its corresponding delivery in the tour.

The PDTSP is formulated as a binary integer program in Equations (1) to (6). This model and notations originates from Ruland (1994) and Dumitrescu et al. (2010). The binary decision variable x_{ij} takes value 1 if and only if the vehicle travels on edge $(i, j) \in E$. For each subset of vertices

$S \subseteq V$, we define the cut set $\delta(S) = \{(i, j) \in E : i \in S, j \notin S \text{ or } i \notin S, j \in S\}$. Moreover, $\delta(i)$ is used instead of $\delta(\{i\})$ for unit sets and $x(E') = \sum_{(i,j) \in E'} x_{ij}$ for any subset $E' \subseteq E$.

$$\text{minimize} \quad \sum_{(i,j) \in E} c_{ij} x_{ij} \quad (1)$$

$$\text{subject to} \quad x_{0,2n+1} = 1 \quad (2)$$

$$x(\delta(i)) = 2 \quad \forall i \in V \quad (3)$$

$$x(\delta(S)) \geq 2 \quad \forall S \subseteq V, 3 \leq |S| \leq |V|/2 \quad (4)$$

$$x(\delta(S)) \geq 4 \quad \forall S \in \mathcal{U} \quad (5)$$

$$x_{ij} \in \{0, 1\} \quad \forall (i, j) \in E. \quad (6)$$

In this formulation, \mathcal{U} is the collection of all subsets $S \subset V$ such that $3 \leq |S| \leq |V| - 2$, $0 \in S$, $2n + 1 \notin S$, and there exists $i \in P$ such that $i \notin S$ and $n+i \in S$. Objective (1) minimizes the total travel distance. Constraint (2) permits to model the solution as a flow circulation. Constraints (3) and (4) are the standard degree and subtour elimination constraints, respectively. Finally, Constraints (5) enforce every pickup location $i \in P$ to be visited before its corresponding delivery location $n+i$.

Despite the availability of sophisticated integer programming solvers, an exact solution of the PDTSP remains impractical beyond a few dozens of pickup-and-delivery requests (Dumitrescu et al. 2010). This is in stark contrast with the classical TSP, for which instances with thousands of nodes can be routinely solved to proven optimally (Applegate et al. 2009). As a consequence, metaheuristics remain the most suitable alternative to solve larger instances. Psaraftis (1983), Savelsbergh (1990) and Healy and Moll (1995) were among the first to study construction and local improvement heuristics for this problem. Renaud et al. (2000) introduced a construction and local search algorithm, which was later refined by adding seven different perturbation strategies to escape local optima in Renaud et al. (2002). Later on, Dumitrescu et al. (2010) used a large neighborhood search (LNS) to generate initial upper bounds for their branch-and-cut algorithm. Veenstra et al. (2017) designed another LNS with five different destruction operators to address the PDTSP and its generalization with handling costs. To our knowledge, this method is regarded as the current state-of-the-art for the PDTSP.

Finally, several PDTSP variants with side constraints have been introduced to accommodate diverse requirements of practical applications, e.g., time windows, capacity constraints, handling costs, and loading constraints. These variants are thoroughly surveyed in Parragh et al. (2008b,a). LIFO and FIFO loading constraints, in particular, have been the focus of several studies (Carrabs et al. 2007, Erdogan et al. 2009, Cordeau et al. 2012, Pollaris et al. 2015). From a methodological standpoint, LIFO or FIFO can restrict the number of feasible solutions, but also enhance

neighborhood searches as they uniquely determine a delivery position within the tour from its pickup position. Overall, despite decades of research, significant progress remains needed regarding efficient neighborhood exploration procedures for the unconstrained PDTSP.

3. Neighborhood Exploration Methodology

We propose to rely on six neighborhoods: RELOCATE PAIR, 2-OPT, OR-OPT, 2 k -OPT, 4-OPT, and BALAS-SIMONETTI. We divide these neighborhoods into two groups according to the way they operate. The first group contains the first three neighborhoods. A common characteristic of these neighborhoods is that it is possible to decompose their exploration, e.g., per pickup-and-delivery pair. Therefore, improving moves can be applied as soon as they are detected. These three neighborhoods are well known, yet we propose a new exploration strategy for RELOCATE PAIR which takes $\mathcal{O}(n^2)$ instead of $\mathcal{O}(n^3)$ time. The second group contains the last three neighborhoods, with exploration procedures based on dynamic programming (DP). A complete DP is typically needed to identify the best move. These neighborhoods are either new (2 k -OPT), or adaptations of TSP neighborhoods (4-OPT and BALAS-SIMONETTI). The 2 k -OPT neighborhood consists of combining nested 2-OPT nested moves in $\mathcal{O}(n^2)$ operations, while the adaptations of 4-OPT and BALAS-SIMONETTI neighborhoods allow to evaluate a large set of feasible PDTSP moves in $\mathcal{O}(n^2)$ and $\mathcal{O}(k^2 2^{k-2} n)$ time, respectively.

All these neighborhoods are restricted to feasible solutions satisfying the precedence constraints between pickups and deliveries. We describe each neighborhood search algorithm in a dedicated subsection. Each search procedure is applied on a feasible incumbent solution σ , represented as a visit sequence σ such that $\sigma(0) = \sigma(2n+1) = 0$ and $\sigma(i) \in \{1, \dots, 2n\}$ for all $i \in \{1, \dots, 2n\}$. In this text, we will also refer to $\sigma_{[i,j]}$ as the subsequence of consecutive visits $(\sigma(i), \dots, \sigma(j))$.

3.1. Relocate Pair Neighborhood

The RELOCATE PAIR neighborhood is defined by a *move* operation which consists in relocating a pickup and delivery pair $(x, x+n)$ from its current positions i and j to new positions i' and j' such that $i' < j'$. Our search procedure iterates over the n pickup and delivery pairs $(x, n+x)$ in random order. For each such pair, it firstly calculates in $\mathcal{O}(1)$ operations the removal cost:

$$\Delta_{\text{REM}} = \begin{cases} c_{\sigma(i-1), \sigma(j+1)} - c_{\sigma(i-1), \sigma(i)} - c_{\sigma(i), \sigma(j)} - c_{\sigma(j), \sigma(j+1)} & \text{if } j = i + 1 \\ c_{\sigma(i-1), \sigma(i+1)} - c_{\sigma(i-1), \sigma(i)} - c_{\sigma(i), \sigma(i+1)} + c_{\sigma(j-1), \sigma(j+1)} - c_{\sigma(j-1), \sigma(j)} - c_{\sigma(j), \sigma(j+1)} & \text{otherwise.} \end{cases} \quad (7)$$

Then, it searches for the best insertion positions i' and j' within the route π after removal (such that $\pi(0) = \pi(2n-1) = 0$, and $\pi(i)$ for $i \in \{1, \dots, 2n-2\}$ represent customer visits). As illustrated in Figure 1, two possibilities need to be considered.

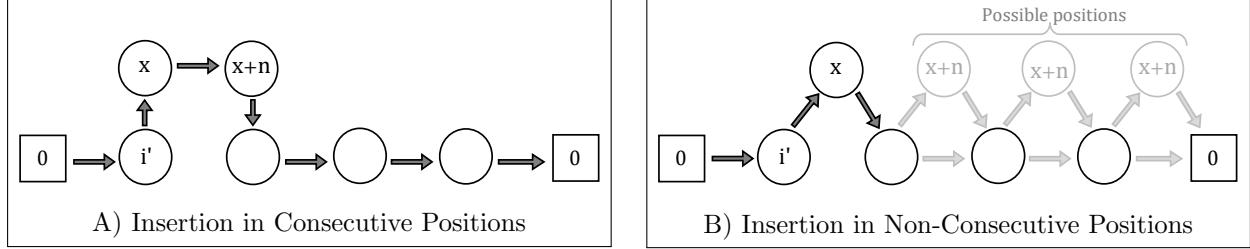


Figure 1 Illustration of the Relocate Pair neighborhood

Case A). The best insertion cost in **consecutive** positions $\Delta_{\text{ADD}}^{\text{C}}$ can be found in $\mathcal{O}(n)$ time by simple inspection, using the following equation:

$$\Delta_{\text{ADD}}^{\text{C}} = \min_{i' \in \{0, \dots, 2n-2\}} \{c_{\pi(i'),x} + c_{x,n+x} + c_{n+x,\pi(i'+1)} - c_{\pi(i'),\pi(i'+1)}\}. \quad (8)$$

Case B). The best insertion cost in **non-consecutive** positions $\Delta_{\text{ADD}}^{\text{NC}}$ is calculated as:

$$\Delta_{\text{ADD}}^{\text{NC}} = \min_{i' \in \{0, \dots, 2n-3\}} \left\{ c_{\pi(i'),x} + c_{x,\pi(i'+1)} - c_{\pi(i'),\pi(i'+1)} + \min_{j' \in \{i'+1, \dots, 2n-2\}} \{c_{\pi(j'),n+x} + c_{n+x,\pi(j'+1)} - c_{\pi(j'),\pi(j'+1)}\} \right\}. \quad (9)$$

A direct application of this equation takes $\mathcal{O}(n^2)$ time for each pair $(x, x+n)$, leading to a complete neighborhood evaluation in $\mathcal{O}(n^3)$ time. To avoid this high computational effort, we observe that:

$$\Delta_{\text{ADD}}^{\text{NC}} = \min_{i'} \{\Delta(i') + \Phi(i')\}, \text{ with } \begin{cases} \Delta(i') = c_{\pi(i'),x} + c_{x,\pi(i'+1)} - c_{\pi(i'),\pi(i'+1)} \\ \Phi(i') = \min_{j' \in \{i'+1, \dots, 2n-2\}} \{c_{\pi(j'),n+x} + c_{n+x,\pi(j'+1)} - c_{\pi(j'),\pi(j'+1)}\}, \end{cases} \quad (10)$$

and that all the values $\Phi(i')$ can be calculated by backward recursion, in $\mathcal{O}(n)$ time overall:

$$\Phi(i') = \begin{cases} c_{\pi(2n-2),n+x} + c_{n+x,0} - c_{\pi(2n-2),0} & \text{if } i' = 2n-3 \\ \min\{\Phi(i'+1), c_{\pi(i'+1),n+x} + c_{n+x,\pi(i'+2)} - c_{\pi(i'+1),\pi(i'+2)}\} & \text{otherwise.} \end{cases} \quad (11)$$

As a consequence $\Delta_{\text{ADD}}^{\text{C}}$ and $\Delta_{\text{ADD}}^{\text{NC}}$ are calculated in $\mathcal{O}(n)$ time. The best RELOCATE PAIR move for the pickup and delivery pair $(x, x+n)$ has a cost of $\Delta = \Delta_{\text{REM}} + \min\{\Delta_{\text{ADD}}^{\text{C}}, \Delta_{\text{ADD}}^{\text{NC}}\}$ and is applied if $\Delta < 0$. Overall, running this search procedure for all pickup and delivery pairs requires $\mathcal{O}(n \times n) = \mathcal{O}(n^2)$ time and $\mathcal{O}(n)$ space.

3.2. 2-Opt Neighborhood

The 2-OPT neighborhood is defined by a move which replaces two edges with two new ones. Given the current solution σ and two positions i and j such that $i+2 < j$, a 2-OPT move on these positions reverses the visit sequence $\sigma_{[i+1,j-1]}$ and replaces arcs $(\sigma(i), \sigma(i+1))$ and $(\sigma(j-1), \sigma(j))$ by arcs $(\sigma(i), \sigma(j-1))$ and $(\sigma(i+1), \sigma(j))$. This leads to a solution improvement when the following cost difference is negative:

$$\Delta_{\text{2OPT}}(i, j) = c_{\sigma(i),\sigma(j-1)} + c_{\sigma(i+1),\sigma(j)} - c_{\sigma(i),\sigma(i+1)} - c_{\sigma(j-1),\sigma(j)}. \quad (12)$$

Evaluating all 2-OPTS takes $\mathcal{O}(n^2)$ time. However, since any 2-OPT for positions i and j reverses the visit sequence $\sigma_{[i+1,j-1]}$, it leads to an infeasible solution whenever there exists a pickup and its associated delivery within the subsequence. These infeasible cases can be efficiently filtered out during the search when iterating over i and j , by stopping the inner loop whenever position j corresponds to a delivery and the associated pickup belongs to $\sigma_{[i+1,j-1]}$.

3.3. Or-Opt Neighborhood

Each OR-OPT move (Or 1976) consists in relocating a sequence of visits $\sigma_{[i,j]}$ at a different position k . There are $\mathcal{O}(n^3)$ such moves. Since each OR-OPT move breaks three edges at most, OR-OPT is effectively a subset of 3-OPT.

To cut down the computational complexity of neighborhood exploration, we opt to restrict this neighborhood to sequences $\sigma_{[i,j]}$ no longer than a constant k_{OR} , and we allow joint sequence relocations and reversals. This restriction is meaningful since relocations of long sequences tend to be frequently infeasible. Moreover, with this limitation, the evaluation of all moves can be done by simple enumeration in $\mathcal{O}(k_{\text{OR}}n^2)$ time.

3.4. 2k-Opt Neighborhood

The 2-OPT neighborhood is widely used for the classical TSP, but its efficiency for the PDTSP is drastically limited since it reverses a visit sequence and therefore often violates precedence constraints. For example, Figure 2 represents a fragment of solution (route). In this fragment, all pickups (respectively, deliveries) are paired with deliveries (respectively, pickups) located outside of the sequence, except for one pickup-and-delivery pair denoted as x and $x+n$. Assuming that distances are Euclidean, the two crossings induced by the current solution visibly lead to extra distance. Moreover, there exists two 2-OPT moves capable of uncrossing the solution, but each of these moves, in isolation, would reverse the order of x and $x+n$ and render the solution infeasible due to precedence constraints. Obviously, the solution, in such a case, is to perform both 2-OPT moves simultaneously. As seen in the following, this is a special case of the 2k-OPT neighborhood, and we propose dynamic programming algorithms to search this neighborhood efficiently.

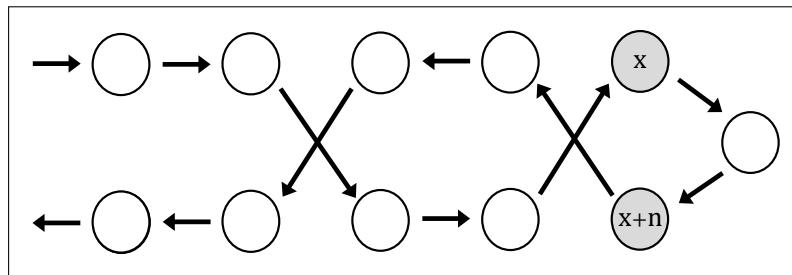


Figure 2 Example of a setting in which a 2k-Opt move is essential to improve distance.

We will demonstrate that the best 2k-OPT move can be found by dynamic programming in $\mathcal{O}(n^2)$ operations. For any pair of indices (i, j) such that $1 \leq i \leq j \leq 2n$, let $F(i, j)$ be the cost of the best combination of 2-OPT in the visit sequence $\sigma_{[i,j]}$. In a similar fashion, let $R(i, j)$ be the cost of the best combination of 2-OPT within the reversed sequence $\pi_{[i,j]} = (\sigma(j) \dots, \sigma(i))$. Evaluating $F(i, j)$ can be done by induction, selecting the best out of three cases:

- 1) Searching for the best 2-OPT combination in $\sigma_{[i+1,j]}$;
- 2) Searching for the best 2-OPT combination in $\sigma_{[i,j-1]}$;
- 3) Applying 2-OPT(i, j) and searching for additional moves in the reversed sequence $\pi_{[i+1,j-1]}$.

This policy can be represented by the following equations, in which $\Delta_{\text{2OPT}}(i, j)$ represents the cost of a 2-OPT between i and j as defined in Equation (12).

$$F(i, j) = \begin{cases} 0, & \text{if } j \leq i + 1 \\ \min \left\{ \begin{array}{l} F(i+1, j), \\ F(i, j-1), \\ \Delta_{\text{2OPT}}(i, j) + R(i+1, j-1) \end{array} \right\}, & \text{otherwise} \end{cases} \quad (13)$$

In a similar fashion, the best policy for $R(i, j)$ can be calculated as:

$$R(i, j) = \begin{cases} \infty, & \text{if } \sigma(i) + n \in \sigma_{[i+1,j]} \text{ or } \sigma(j) - n \in \sigma_{[i,j-1]} \\ 0, & \text{if } j = i \\ 0, & \text{if } j = i + 1 \text{ and } \sigma(j) \neq n + \sigma(i) \\ \infty, & \text{if } j = i + 1 \text{ and } \sigma(j) = n + \sigma(i) \\ \min \left\{ \begin{array}{l} R(i+1, j), \\ R(i, j-1), \\ \Delta_{\text{2OPT}}(i, j) + F(i+1, j-1) \end{array} \right\}, & \text{otherwise.} \end{cases} \quad (14)$$

In this recursion, the first statement eliminates cases in which the node $\sigma(i)$ is paired with a delivery in $\sigma_{[i+1,j]}$, or in which $\sigma(j)$ is paired with a pickup in $\sigma_{[i,j-1]}$. Indeed, regardless of the internal 2-OPT choices, $\sigma(j)$ and $\sigma(i)$ are guaranteed to be respectively the first and the last visited. The next three statements cover the cases of sequences containing one or two visits, and the final statement performs the recursive calls in a similar fashion as Equation (13).

The number of possible states of $F(i, j)$ and $R(i, j)$ is in $\mathcal{O}(n^2)$ and the evaluation of the best cost for each state is done in constant time through Equations (13–14). Overall, a direct application of these equations gives an algorithm to find the best 2k-OPT move in $\mathcal{O}(n^2)$ time and space.

3.5. 4-Opt Neighborhood

Moves of a higher order such as 4-OPT have the potential to lead to structurally different solutions that are not attainable with simpler moves. However, a complete exploration of this neighborhood is prohibitively long. Some previous works have opted to explore only a small fraction of this neighborhood. Renaud et al. (2000, 2002), for example, explored a subset of $\mathcal{O}(n^2)$ moves.

We take a fairly different approach to explore this neighborhood, and instead rely on an extension of the search strategy of Glover (1996) originally designed for the TSP. As discussed in this section, this allows exploring a subset of the 4-OPT neighborhood with up to $\Theta(n^4)$ moves in $\mathcal{O}(n^2)$ time. This search strategy can be described as a search for an improving alternating cycle in an auxiliary graph. Here we will adopt a direct presentation in a recursive form. We focus on the 4-OPT moves that are obtained as a combination of a pair of *connecting* or *disconnecting* 2-OPT moves, referred as 2-OPT-C and 2-OPT-D throughout this section. As listed in Figure 3, six possible configurations exist.

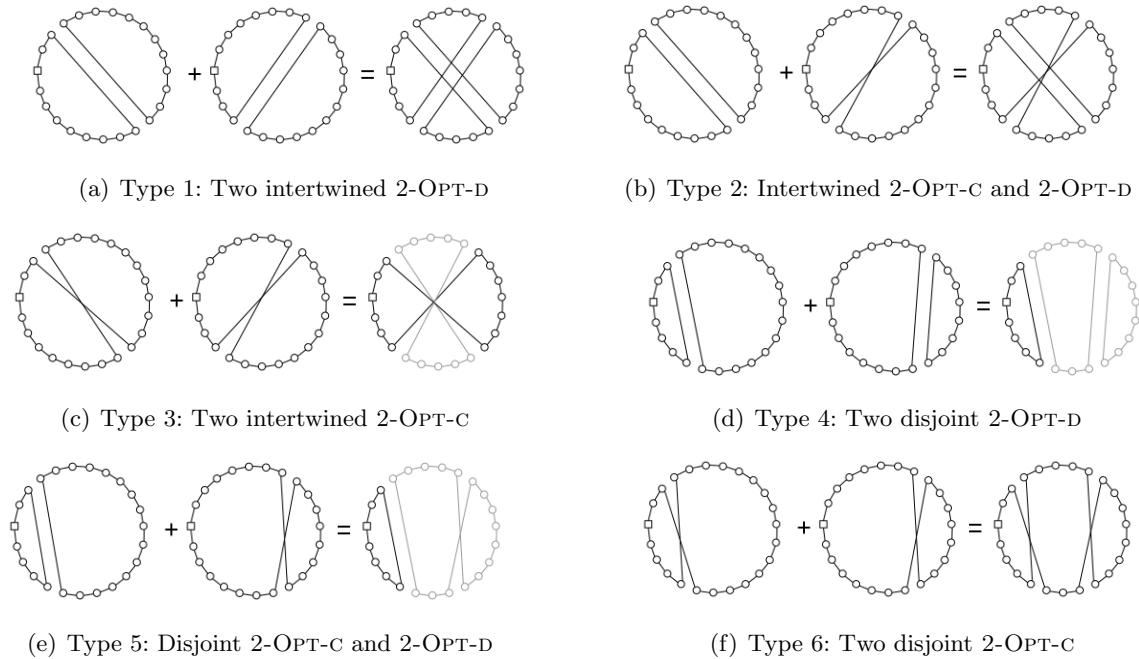


Figure 3 4-Opt moves obtained as a combination of 2-Opt-c and 2-Opt-d.

Only three of these configurations (Types 1, 2, and 6) produce connected solutions:

- Type-1 moves are obtained by intertwining two 2-OPT-D moves. Such moves are commonly used as a perturbation operator and called “double bridge” in the TSP and vehicle routing literature (Subramanian et al. 2013).

- Type-2 moves are obtained by intertwining a 2-OPT-C move with a 2-OPT-D move. This type will be divided into two cases (Types 2A and 2B) depending on whether 2-OPT-D is applied first or second. Some visit sequences are reversed when applying this move.
- Type-6 moves consist of two disjoint 2-OPT-C moves, and represent a special case of the 2K-OPT neighborhood described in Section 3.4. For this reason, we focus exclusively on Type-1 and Type-2 moves.

Each 4-OPT of a given type is characterized by four indices $0 \leq i_1 < i_2 < j_1 < j_2 \leq 2n$ representing the removal of edges $e_{i_1} = (\sigma(i_1), \sigma(i_1+1))$, $e_{i_2} = (\sigma(i_2), \sigma(i_2+1))$, $e_{j_1} = (\sigma(j_1), \sigma(j_1+1))$ and $e_{j_2} = (\sigma(j_2), \sigma(j_2+1))$. After removal, the incumbent solution σ is split into five visit sequences: $\pi_1 = \sigma_{[0, i_1]}$, $\pi_2 = \sigma_{[i_1+1, i_2]}$, $\pi_3 = \sigma_{[i_2+1, j_1]}$, $\pi_4 = \sigma_{[j_1+1, j_2]}$ and $\pi_5 = \sigma_{[j_2+1, 2n]}$. During reconstruction, the extremities of edge e_{i_1} are reconnected with those of edge e_{j_1} . There are two possible ways to proceed: 2-OPT-C(i_1, j_1) creates edges $(\sigma(i_1), \sigma(j_1))$ and $(\sigma(i_1+1), \sigma(j_1+1))$, whereas 2-OPT-D(i_1, j_1) creates edges $(\sigma(i_1), \sigma(j_1+1))$ and $(\sigma(i_1+1), \sigma(j_1))$. Likewise, the extremities of edges e_{i_2} and e_{j_2} can be reconnected in two possible ways. This leads to the following sequence orientations and routes for each move type (Figure 4):

- Type 1: 2-OPT-D(i_1, j_1) and 2-OPT-D(i_2, j_2), producing route $\pi_1 \oplus \pi_4 \oplus \pi_3 \oplus \pi_2 \oplus \pi_5$;
- Type 2A: 2-OPT-C(i_1, j_1) and 2-OPT-D(i_2, j_2), producing route $\pi_1 \oplus r(\pi_3) \oplus r(\pi_4) \oplus \pi_2 \oplus \pi_5$;
- Type 2B: 2-OPT-D(i_1, j_1) and 2-OPT-C(i_2, j_2), producing route $\pi_1 \oplus \pi_4 \oplus r(\pi_2) \oplus r(\pi_3) \oplus \pi_5$;

where $r()$ represents the reversal of a sequence and \oplus is the operation of concatenation of two sequences.

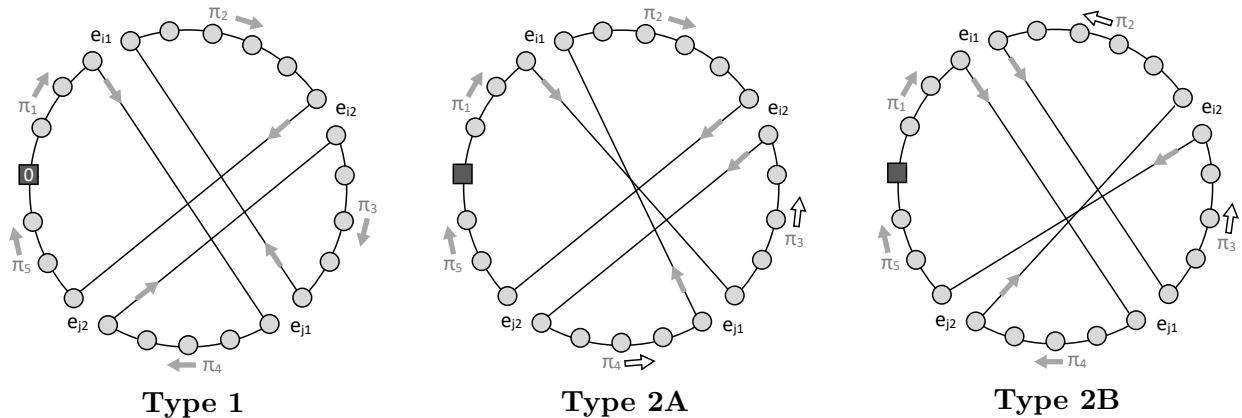


Figure 4 Different reconstructions and sequence orders corresponding to different types of 4-opt moves

The cost of each 2-OPT-D and 2-OPT-C move can be calculated as:

$$\Delta_{2\text{OPT}}^D(i, j) = c_{\sigma(i), \sigma(j+1)} + c_{\sigma(i+1), \sigma(j)} - c_{\sigma(i), \sigma(i+1)} - c_{\sigma(j), \sigma(j+1)}. \quad (15)$$

$$\Delta_{2\text{OPT}}^C(i, j) = c_{\sigma(i), \sigma(j)} + c_{\sigma(i+1), \sigma(j+1)} - c_{\sigma(i), \sigma(i+1)} - c_{\sigma(j), \sigma(j+1)}, \quad (16)$$

and the cost of the best 4-OPT move for each fixed edge pair i_2 and j_2 can be calculated as:

$$\Delta_{4\text{OPT}}^{\text{TYPE-1}}(i_2, j_2) = \Delta_{2\text{OPT}}^D(i_2, j_2) + \min_{i_1 \in \{0, \dots, i_2-1\}} \left\{ \min_{j_1 \in \{i_2+1, \dots, j_2-1\}} \{\Delta_{2\text{OPT}}^D(i_1, j_1)\} \right\} \quad (17)$$

$$\Delta_{4\text{OPT}}^{\text{TYPE-2A}}(i_2, j_2) = \Delta_{2\text{OPT}}^C(i_2, j_2) + \min_{i_1 \in \{0, \dots, i_2-1\}} \left\{ \min_{j_1 \in \{i_2+1, \dots, j_2-1\}} \{\Delta_{2\text{OPT}}^D(i_1, j_1)\} \right\} \quad (18)$$

$$\Delta_{4\text{OPT}}^{\text{TYPE-2B}}(i_2, j_2) = \Delta_{2\text{OPT}}^D(i_2, j_2) + \min_{i_1 \in \{0, \dots, i_2-1\}} \left\{ \min_{j_1 \in \{i_2+1, \dots, j_2-1\}} \{\Delta_{2\text{OPT}}^C(i_1, j_1)\} \right\}. \quad (19)$$

The efficient evaluation of the best 4-OPT move for each fixed edge pair i_2 and j_2 depends on our capacity to efficiently evaluate the following quantities for $X \in \{C, D\}$:

$$\Phi^X[i_2, j_2] = \min_{i_1 \in \{0, \dots, i_2-1\}} \left\{ \min_{j_1 \in \{i_2+1, \dots, j_2-1\}} \Delta_{2\text{OPT}}^X(i_1, j_1) \right\}. \quad (20)$$

By reverting the order of the minimums, observe that:

$$\Phi^X[i_2, j_2] = \min_{j_1 \in \{i_2+1, \dots, j_2-1\}} \left\{ \min_{i_1 \in \{0, \dots, i_2-1\}} \Delta_{2\text{OPT}}^X(i_1, j_1) \right\} = \min_{j_1 \in \{i_2+1, \dots, j_2-1\}} \Phi_{\text{SUB}}^X[i_2, j_1], \quad (21)$$

$$\text{where } \Phi_{\text{SUB}}^X[i_2, j_1] = \min_{i_1 \in \{0, \dots, i_2-1\}} \Delta_{2\text{OPT}}^X(i_1, j_1). \quad (22)$$

Finally, observe that the $\mathcal{O}(n^2)$ auxiliary variables Φ_{SUB}^X and Φ^X can be calculated in $\mathcal{O}(n^2)$ time with the following recursions:

$$\Phi_{\text{SUB}}^X[i_2, j_1] = \begin{cases} \Delta_{2\text{OPT}}^X(0, j_1) & \text{if } i_2 = 1 \\ \min\{\Phi_{\text{SUB}}^X[i_2 - 1, j_1], \Delta_{2\text{OPT}}^X(i_2 - 1, j_1)\} & \text{otherwise.} \end{cases} \quad (23)$$

$$\Phi^X[i_2, j_2] = \begin{cases} \Phi_{\text{SUB}}^X[i_2, i_2 + 1] & \text{if } j_2 = i_2 + 2 \\ \min\{\Phi^X[i_2, j_2 - 1], \Phi_{\text{SUB}}^X[i_2, j_2 - 1]\} & \text{otherwise.} \end{cases} \quad (24)$$

Therefore, the values of these auxiliary variables can be preprocessed in $\mathcal{O}(n^2)$ time. After doing this, each evaluation of Equation (20) takes constant time, leading to a neighborhood exploration in $\mathcal{O}(n^2)$ time. The complete evaluation process is presented in Algorithm 1. For brevity, this pseudo-code only presents the calculation of the best-move cost, but the positions associated with the best move can be likewise recorded.

Finally, to satisfy precedence constraints between pickup-and-delivery pairs, we include a feasibility check (Line 17 of Algorithm 1) which evaluates whether the best-found move of Type-1, Type-2A and Type-2B for each given i_2 and j_2 is feasible. For each move type, the conditions listed in Table 1 are checked in $\mathcal{O}(1)$ time, using auxiliary variables which can be preprocessed in $\mathcal{O}(n^2)$ time prior to move evaluations, and which state, for each sequence $\pi = \sigma_{[i,j]}$, whether this sequence is reversible (in which case $\text{REV}(\pi) = \text{TRUE}$) and which give the largest index $\text{LAST}(\pi)$ of a pickup node that does not belong to π but whose associated delivery belongs to it.

```

1  $\Delta_{\text{BEST}} \leftarrow 0$ 
2 for  $j_1 \in \{2, \dots, 2n - 1\}$  do
3   for  $i_2 \in \{1, \dots, j_1 - 1\}$  do
4     for  $X \in \{\text{C}, \text{D}\}$  do
5        $\Phi_{\text{SUB}}^X[i_2, j_1] \leftarrow \begin{cases} \Delta_{\text{2OPT}}^X(0, j_1) & \text{if } i_2 = 1 \\ \min\{\Phi_{\text{SUB}}^X[i_2 - 1, j_1], \Delta_{\text{2OPT}}^X(i_2 - 1, j_1)\} & \text{otherwise.} \end{cases}$ 
6     end
7   end
8 end
9 for  $i_2 \in \{1, \dots, 2n - 2\}$  do
10  for  $j_2 \in \{i_2 + 2, \dots, 2n\}$  do
11    for  $X \in \{\text{C}, \text{D}\}$  do
12       $\Phi^X[i_2, j_2] \leftarrow \begin{cases} \Phi_{\text{SUB}}^X[i_2, i_2 + 1] & \text{if } j_2 = i_2 + 2 \\ \min\{\Phi^X[i_2, j_2 - 1], \Phi_{\text{SUB}}^X[i_2, j_2 - 1]\} & \text{otherwise.} \end{cases}$ 
13    end
14     $\Delta_{\text{4OPT}}^{\text{TYPE-1}}(i_2, j_2) \leftarrow \Delta_{\text{2OPT}}^{\text{D}}(i_2, j_2) + \Phi^{\text{D}}[i_2, j_2]$ 
15     $\Delta_{\text{4OPT}}^{\text{TYPE-2A}}(i_2, j_2) \leftarrow \Delta_{\text{2OPT}}^{\text{C}}(i_2, j_2) + \Phi^{\text{D}}[i_2, j_2]$ 
16     $\Delta_{\text{4OPT}}^{\text{TYPE-2B}}(i_2, j_2) \leftarrow \Delta_{\text{2OPT}}^{\text{D}}(i_2, j_2) + \Phi^{\text{C}}[i_2, j_2]$ 
17    Check precedence constraints feasibility
18    Record the overall best feasible improving move
19  end
20 end

```

Algorithm 1: Efficient exploration of the 4-OPT neighborhood**Table 1 Feasibility conditions for 4-Opt moves subject to pickup-and-delivery constraints**

	Feasibility Condition	Efficient Evaluation
Type 1	π_4 cannot contain deliveries whose pickups are located in π_2 or π_3 π_3 cannot contain deliveries whose pickups are located in π_2	$\text{LAST}(\pi_4) < \pi_1 $ $\text{LAST}(\pi_3) < \pi_1 $
Type 2A	π_3 and π_4 cannot contain deliveries whose pickups are located in π_2 π_3 and π_4 should be reversible	$\max\{\text{LAST}(\pi_3), \text{LAST}(\pi_4)\} < \pi_1 $ $\text{REV}(\pi_3) \wedge \text{REV}(\pi_4)$
Type 2B	π_4 cannot contain deliveries whose pickups are located in π_2 or π_3 π_2 and π_3 should be reversible	$\text{LAST}(\pi_4) < \pi_1 $ $\text{REV}(\pi_3) \wedge \text{REV}(\pi_4)$

The evaluation of the REV and LAST values for all sequences $\sigma_{[i,j]}$ can be done in $\mathcal{O}(n^2)$ time using the following simple recursive calculation:

$$\text{REV}(\sigma_{[i,j]}) = \begin{cases} \text{TRUE} & \text{if } i = j \\ \text{REV}(\sigma_{[i,j-1]}) \wedge (\sigma(j) - n \notin \sigma_{[i,j-1]}) & \text{otherwise,} \end{cases} \quad (25)$$

$$\text{LAST}(\sigma_{[i,j]}) = \begin{cases} \text{PICK}(\sigma_{[i,j]}, j) & \text{if } i = j \\ \max\{\text{LAST}(\sigma_{[i,j-1]}), \text{PICK}(\sigma_{[i,j]}, j)\} & \text{otherwise,} \end{cases} \quad (26)$$

where $\text{PICK}(\pi, j)$ returns the position of the pickup service associated to a delivery $\sigma(j)$ if this position does not belong to the subsequence π , and -1 otherwise. This function also returns -1 if $\sigma(j)$ is a pickup.

Overall, we achieve a complete neighborhood exploration in $\mathcal{O}(n^2)$ time and space including feasibility checks. However, note that the feasibility is only evaluated for the best move of Type-1, Type-2A, and Type-2B for the pair i_2 and j_2 . If this best move is infeasible, then no other move associated with this pair is applied. This assumption seems necessary to avoid a high computation effort in the feasibility checking process.

3.6. Balas-Simonetti Neighborhood

Balas and Simonetti (2001) studied a TSP variant with additional constraints on the relative customer positions. This problem can be solved in polynomial time, and it leads to an efficient neighborhood search for the classical TSP. It is also possible to extend the same principle for the PDTSP, as discussed in this section.

Let k be a positive integer, and let σ be an incumbent solution. The $BS(k)$ neighborhood for the PDTSP is formed of all solutions π satisfying the following two constraints:

$$\pi(i) < \pi(i + n) \quad \forall i \in \{1, \dots, n\}, \quad (27)$$

$$\pi(i) < \pi(j) \quad \forall i \in V, j \in V : \sigma(i) + k \leq \sigma(j). \quad (28)$$

In other words, it contains all permutations π respecting precedence constraints between pickups and deliveries (Equation 27) such that no service i occurs after another service j in π if service j was planned k positions later than i in σ (Equation 28). Parameter k therefore represents a degree of flexibility from the incumbent solution σ .

In a similar way as in Balas and Simonetti (2001), the search for a best permutation π satisfying Constraints (27–28) reduces to a shortest path problem in an auxiliary acyclic graph $G^* = (V^*, E^*)$. This graph is arranged into layers in such a way that only successive layers are connected by arcs. There are $2n + 2$ layers in the case of the PDTSP. The first and last layers correspond to the departure and arrival at the depot 0 and contain a single node. The other layers correspond to the $2n$ visits in π . Each such layer contains several nodes that represent combinations of anticipated and delayed visits to clients in relation to σ . More specifically, each node in V^* represents a quadruplet $(i, v, S^-(\sigma, i), S^+(\sigma, i))$, where i is the current position in π ; v is the visit (depot, pickup or delivery) allocated to position i of π ; $S^-(\sigma, i)$ is the set of visits that are allocated at position i or later in σ and allocated at position i or before in π ; and, finally, $S^+(\sigma, i)$ is the set of visits allocated before position i in σ that will be allocated at position i or after in π .

Observe that sets $S^-(\sigma, i)$ and $S^+(\sigma, i)$ give enough information to restrict the search to solutions satisfying Constraints (27) and (28). Constraint (27) is ensured by restricting the search to nodes such that $S^-(\sigma, i)$ does not contain any delivery whose associated pickup belongs to $S^+(\sigma, i) \cup v$ and $S^+(\sigma, i)$ does not contain any pickup whose associated delivery belongs to $S^-(\sigma, i) \cup v$. Moreover, Constraint (28) is ensured by restricting the search to nodes such that all visits in $S^+(\sigma, i)$ are located no more than k positions earlier than v in σ .

Figure 5 displays G^* on a simple example with three pickup-and-delivery pairs ($n = 3$), starting with solution $\sigma = (0, 1, 2, 2 + n, 1 + n, 3, 3 + n, 0)$ with $k = 3$. The nodes in gray represent states that are eliminated due to the pickup-and-delivery precedence constraints (Condition 27). For each node, we indicate the service associated with it (i.e., v in the quadruplet), and on the left of the

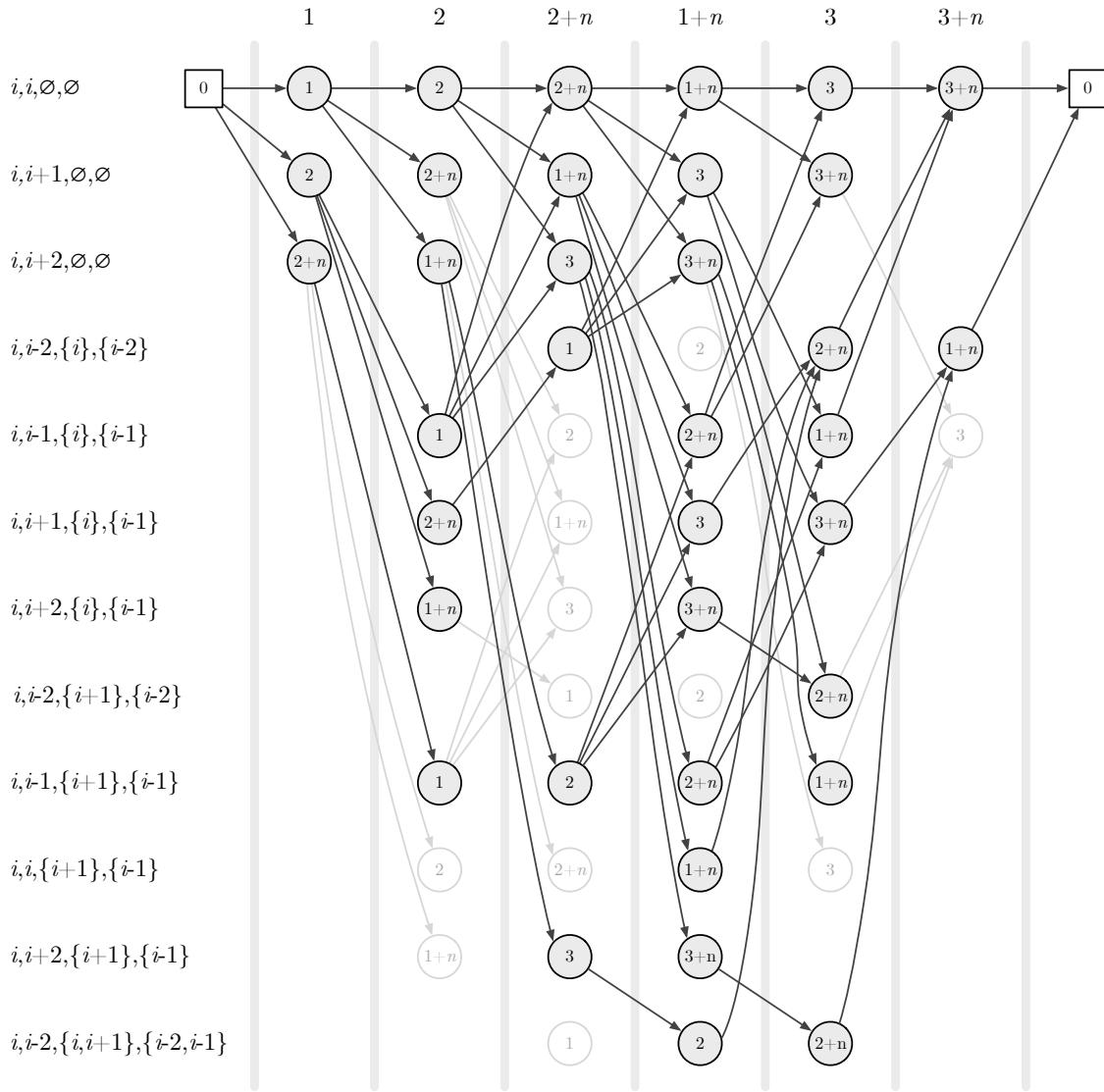


Figure 5 Graph G^* with $k = 3$ for $\sigma = (0, 1, 2, 2 + n, 1 + n, 3, 3 + n, 0)$

figure, we list the possible $S^-(\sigma, i)$ and $S^+(\sigma, i)$ states. The first and last layers represent the depot. The second layer, for example, contains a node corresponding to a direct visit to either 1, 2, or $2 + n$. The third layer contains nodes representing visits to 1, 2, $1 + n$, or $2 + n$ with different combinations of anticipated and delayed visits.

Forming this graph and finding the shortest path is done in $\mathcal{O}(k^2 2^{k-2} n)$ time and $\mathcal{O}(k 2^{k-2} n)$ space. In other words, the complexity of this neighborhood exploration is linear in the number of pickup-and-delivery pairs and exponential in k . Notably, when k is fixed, we obtain a polynomial-time algorithm to locate the best solution in this large neighborhood.

4. Computational Experiments

Most of the neighborhoods presented in this paper are either new or explored with new search strategies. The overarching goal of our experimental campaign is to measure the performance and impact of these methodological building blocks. To be representative of a broad range of methods, we conduct experiments with two structurally different heuristics: the ruin-and-recreate (RR) algorithm of Veenstra et al. (2017), which is the state-of-the-art for the PDTSP to date, and the hybrid genetic search (HGS) of Vidal et al. (2012, 2014), a classical population-based metaheuristic which has demonstrated good performance on a large family of node and arc routing problems (see, e.g., Vidal et al. 2016, Vidal 2017). Notably, the principle that we proposed for an efficient exploration of the RELOCATE PAIR neighborhood (in Section 3.1) can also be applied to efficiently evaluate candidate insertions in the context of a large neighborhood search such as RR. As this permits considerable reductions of complexity and computational effort or, alternatively, an increased number of iterations evaluated in a given time, we will also present comparative analyses demonstrating the impact of this improvement in the RR methodology.

Our experiments are conducted on three main groups of instances: (i) classic PDTSP instances with 11 to 493 visits, (ii) new challenging instances derived from the X set of Uchoa et al. (2017) with 101 to 1001 visits, and (iii) the “Grubhub” instance set (available at <https://github.com/grubhub/tsppdlib>) connected to meal-delivery applications. The latter instances only include up to 31 visits, but nonetheless remain challenging for exact methods. Due to their repeated solution in a delivery-on-demand setting, it is essential to consistently solve them to near-optimality within the shortest possible time, ideally a few milliseconds.

In Section 4.1, we describe the local search and metaheuristic frameworks used in our experiments. In Section 4.2, we detail the experimental setup, benchmark instances and parameter-calibration methodology. In Section 4.3, we compare the performance of the considered metaheuristics as well as the impact of the fast evaluation of insertions in the RR. In Section 4.4, we measure the sensitivity of HGS to changes in the choices of neighborhoods. Finally, in Section 4.5 we report results on the Grubhub data sets, which require very short solution times.

4.1. Search Strategies

We first describe the local search built on the proposed neighborhoods, and then proceed with a description of the RR and HGS metaheuristics. Unless specified, these metaheuristics were used with their original parameters listed in Vidal et al. (2012) and Veenstra et al. (2017).

4.1.1. Local Search. The proposed local search algorithm is built on the six neighborhoods presented in Section 3. As illustrated in Algorithm 2, the exploration of the neighborhoods is divided into two alternating phases. In a first phase, it explores the RELOCATE PAIR, 2-OPT and OR-OPT neighborhoods, dividing the evaluation of the moves among the different pickup-and-delivery pairs $(x, x + n)$ and applying the best move for each such pair. Next, it performs a second-phase search that consists of iteratively finding and applying the best 2K-OPT, 4-OPT or BALAS-SIMONETTI move. The search stops whenever all those neighborhoods have been explored without improvement.

```

1  $S \leftarrow$  Initial Solution
2 while Improvements Found do
3   for all pickup-and-delivery requests  $(x, x + n)$  in random order       $\triangleright n$  iterations
4     do
5       Find and apply, if improving, the best move on  $S$  among:
6         • all RELOCATE PAIR moves for  $(x, x + n)$ ;           $\triangleright \mathcal{O}(n)$  time
7         • all 2-OPT moves starting at the position of  $x$  or  $x + n$ ;     $\triangleright \mathcal{O}(n)$  time
8         • all OR-OPT moves starting at the position of  $x$  or  $x + n$ .     $\triangleright \mathcal{O}(k_{\text{OR}}n)$  time
9     end
10    Find and apply, if improving, the best move on  $S$  in 2K-OPT; 4-OPT;
11        and BALAS-SIMONETTI neighborhoods.            $\triangleright \mathcal{O}(k^2 2^{k-2} n + n^2)$  time
12  end
13 Return  $S$ 
```

Algorithm 2: General structure of the local search

4.1.2. Ruin-and-Recreate. The RR algorithm of Veenstra et al. (2017) is based on a simple iterative search process. It starts from an initial solution built with the basic greedy insertion procedure of Ropke and Pisinger (2006), which consists of shuffling the order of the pickup-and-delivery pairs and inserting each of them in their best position. Then, it iteratively destroys and reconstructs the incumbent solution. Four destruction operators are used.

- **RANDOM REMOVAL:** Remove q random pickup-and-delivery services.
- **WORST REMOVAL:** Calculate the gain associated with the removal of each pickup-and-delivery service. List the services by decreasing gain. Remove q random services, giving a higher probability to elements that are early in the list.

- **BLOCK REMOVAL:** Iteratively select a random pickup-and-delivery service $(x, x+n)$. Remove all the services containing a pickup or a delivery in the sequence starting at x and finishing at $x+n$. Stop as soon as q services have been removed.

The number of pickup-and-delivery services q selected for removal is each time randomly selected, with uniform probability, between $\min\{30, 0.20n\}$ and $\min\{50, 0.55n\}$. Reconstruction is done with the same greedy insertion heuristic as in the initialization phase, considering only the removed services. The resulting solution is accepted as a new incumbent solution in case of improvement, and otherwise subject to a probabilistic acceptance using Metropolis criterion (as in the simulated annealing algorithm).

In Veenstra et al. (2017), each evaluation of the best insertion position for a pickup-and-delivery pair is done by enumeration (on the pickup and delivery insertion position) and therefore takes $\mathcal{O}(n^2)$ time, leading to a total complexity of $\mathcal{O}(qn^2)$ for each reconstruction phase. Notably, applying the same strategy as in the RELOCATE PAIR through Equations (8–11) permits to cut down computational complexity to $\mathcal{O}(qn)$ time for the complete reconstruction phase. This significantly speeds up the overall RR approach since the reconstruction operator is the main bottleneck of the method. We will show the impact of this improvement in Section 4.3.

4.1.3. Hybrid Genetic Search. Our HGS for the PDTSP follows the original search strategy of Vidal et al. (2012) and Vidal (2022) with some adaptations and simplifications. Since a solution of the PDTSP consists of a single tour, the algorithm maintains a single population of feasible solutions represented as permutations of the pickup and delivery vertices. To produce each new solution, our HGS selects two parents by binary tournament, applies a crossover and a mutation operator to generate a new child, which is then improved using the proposed local search (see Section 4.1.1) and included in the population. To reduce the computational effort, the complete local search with the large neighborhoods is only used for a random subset of the generated individuals selected with probability p_{LARGE} . For the other individuals, the local search relies exclusively on RELOCATE PAIR, 2-OPT and OR-OPT (i.e., Algorithm 2 without Line 7).

We use the linear order crossover (LOX – Falkenauer and Bouffoix 1991) illustrated in Figure 6. This operator consists of copying a random sequence of consecutive visits from the first parent into the child and completing the missing visits from the left to the right based on the visit order of the second parent. The crossover is followed by a mutation which consists of finding and applying the best 4-OPT move of Type 1, regardless of its feasibility. These two operations may lead to precedence-constraints violations. Therefore, in a last step, HGS applies the best RELOCATE PAIR move for each pickup-and-delivery pair violating precedence constraints to obtain the best feasible reinsertion positions and achieve feasibility. This mutation operator successfully introduces promising edges and contributes to the diversification of the search.



Figure 6 Illustration of the LOX crossover

As usual in HGS, we evaluate the *biased fitness* $BF(\sigma)$ of each solution as a linear combination of its rank in terms of cost and its rank in terms of contribution to the population diversity.

$$BF(\sigma) = R_c(\sigma) + \left(1 - \frac{\mu^{\text{ELITE}}}{|\mathcal{P}|}\right) \times R_d(\sigma) \quad (29)$$

This fitness measure is used for parents and survivors selections. The parameter μ^{ELITE} represents a number of elite individuals which are guaranteed to survive. It has been set to $\mu^{\text{ELITE}} = 1$ in our experiments to promote a diversified search and preserve the best solution. $R_c(\sigma)$ represents the rank of solution σ within the population \mathcal{P} in terms of objective value, and $R_d(\sigma)$ represents its rank in terms of diversity contribution. The diversity contribution of each solution is computed as its average distance to the two closest other solutions in \mathcal{P} . To calculate the distance $\delta(\sigma, \sigma')$ between two solutions σ and σ' , we use the Jaccard distance (Levandowsky and Winter 1971) between the sets $E(\sigma)$ and $E(\sigma')$ containing the pairs of consecutive visit locations within each solution (i.e., considering two visits in the same location as identical):

$$\delta(\sigma, \sigma') = \frac{|E(\sigma) \cup E(\sigma')| - |E(\sigma) \cap E(\sigma')|}{|E(\sigma) \cup E(\sigma')|}. \quad (30)$$

This choice of distance metric is due to the fact that PDTSP applications and datasets commonly include consecutive pickups or deliveries from different services at the same locations. It avoids distance over-estimations for pairs of symmetrical solutions with different visit permutations in the same location.

Finally, the population is managed to contain between μ and $\mu + \lambda$ solutions. Initially, it contains random solutions generated by the same randomized greedy algorithm as in Section 4.1.2. Each time the maximum population size is attained, the method iteratively removes the worst solution in terms of biased fitness until reaching the initial population size of μ .

4.2. Experimental Setup

All our computational experiments were conducted on a single thread of an AMD Rome 7532 2.40GHz CPU running Linux CentOS 8.4. The algorithms were implemented in C++, and compiled with G++ v7.4.0 using option -O3.

The classical benchmark instances of the PDTSP are classified into three sets. The first two classes of instances (RBO) were introduced in Renaud et al. (2000) and the third class (DRCL)

was introduced in Dumitrescu et al. (2010). The instances of the second class are very peculiar since, by design, the optimal PDTSP solution on each instance matches that of the TSP, while the instances of the third class are fairly small. For these reasons, we focus our studies on the instances of the first class, also including computational results for the other classes in the electronic companion. The first class includes 108 instances containing 51 to 493 visits. These instances are derived from 36 TSPLIB data sets (Reinelt 1991) and subdivided into three groups called A, B, and C. For each TSP data set, three PDTSP instances were generated by randomly picking a delivery point for each (pickup) vertex. In groups A and B, one random delivery point was selected for each pickup among the five and ten closest vertices, respectively. In group C, there is no such limitation, and therefore the delivery vertex was freely selected among all other vertices. To obtain a wider diversity of challenging data sets, we also derived 300 new instances for the PDTSP from the X instance set of Uchoa et al. (2017). To do so, we retained the coordinates of the depot and customers for each of the 100 original instances, and we created pickup and delivery pairs using the same process as previously described, thereby creating three instances (A, B and C). We use the same rounding conventions as in the set X for distance calculations, i.e., we round the distance values to the nearest integer.

Calibration of the neighborhood search. Our neighborhood search includes three main parameters: the probability of using the large neighborhoods (p_{LARGE}), the maximum size of the OR-OPT sequences (k_{OR}), and the BALAS-SIMONETTI parameter (k_{BS}). To calibrate these parameters, a full factorial experiment would take excessive computational effort. Therefore, we conducted preliminary analyses to identify a suitable level for p_{LARGE} , leading us to a value of $p_{\text{LARGE}} = 0.1$. Next, we calibrated k_{OR} and k_{BS} , and lastly we reevaluated our choice of p_{LARGE} to make sure that it remained meaningful. For this calibration, we used the 300 new instances of set X and ran HGS up until a fixed time limit of $T_{\text{MAX}} = N$ seconds, where N represents the total number of visits in the instance. We performed ten runs with different random seeds for each instance and algorithm configuration. Table 2 reports the performance of each configuration, measured as an average error gap over all instances and runs. For each instance, the gap is calculated relatively to the best-known solution (BKS) collected from all runs as $\text{Gap}(\%) = 100 \times (z_{\text{SOL}} - z_{\text{BKS}})/z_{\text{BKS}}$. The table also includes a color scale to highlight better configurations, and the overall best configuration is marked in boldface.

As visible in these experiments, parameter k_{OR} has a noticeable impact on the performance of the overall method. This parameter limits the size of the OR-OPT neighborhood, which is especially important for the PDP as it permits structural changes to the solutions without reverting visit sequences. However, the exploration of this neighborhood is a time bottleneck when k_{OR} is large. Our experiments show that suitable values for k_{OR} are located in the $\{20, \dots, 50\}$ range. In contrast,

Table 2 Calibration of the local search parameters

	$k_{BS} = 1$	$k_{BS} = 2$	$k_{BS} = 3$	$k_{BS} = 4$	$k_{BS} = 5$	$k_{BS} = 6$
$k_{OR} = 1$	0.675	0.667	0.680	0.688	0.686	0.696
$k_{OR} = 2$	0.590	0.592	0.591	0.598	0.588	0.586
$k_{OR} = 3$	0.523	0.531	0.507	0.540	0.529	0.544
$k_{OR} = 5$	0.489	0.486	0.479	0.486	0.491	0.501
$k_{OR} = 7$	0.453	0.448	0.451	0.455	0.458	0.457
$k_{OR} = 10$	0.428	0.439	0.437	0.423	0.438	0.427
$k_{OR} = 15$	0.416	0.411	0.415	0.422	0.415	0.421
$k_{OR} = 20$	0.416	0.412	0.408	0.424	0.396	0.409
$k_{OR} = 30$	0.418	0.415	0.395	0.408	0.409	0.418
$k_{OR} = 40$	0.407	0.420	0.410	0.409	0.408	0.420
$k_{OR} = 50$	0.402	0.420	0.411	0.415	0.420	0.415

parameter k_{BS} has a more limited impact, though excessively small or large values (e.g., 1 or 6) also lead to a performance deterioration. Based on these results, we adopted as *reference configuration* the values $(k_{OR}, k_{BS}) = (30, 3)$, which achieved the best performance. As displayed in Table 3, we also ran pairwise Wilcoxon tests to compare the results of the reference configuration with those of the 65 other configurations. At a significance level of 0.05/65 (using Bonferroni correction to account for multiple testing), the reference configuration is significantly better than any other configuration with $k_{OR} \leq 10$. Finally, as seen on Table 4, re-calibrating p_{LARGE} based on the reference values of (k_{OR}, k_{BS}) confirmed our initial choice of $p_{LARGE} = 0.1$.

Table 3 p-values of paired-samples Wilcoxon tests relative to the reference configuration

	$k_{BS} = 1$	$k_{BS} = 2$	$k_{BS} = 3$	$k_{BS} = 4$	$k_{BS} = 5$	$k_{BS} = 6$
$k_{OR} = 1$	1.43×10^{-153}	8.48×10^{-149}	1.00×10^{-155}	9.69×10^{-162}	4.60×10^{-156}	8.88×10^{-153}
$k_{OR} = 2$	2.44×10^{-93}	2.29×10^{-108}	3.10×10^{-92}	2.20×10^{-101}	1.88×10^{-88}	7.82×10^{-92}
$k_{OR} = 3$	4.48×10^{-50}	3.61×10^{-59}	2.61×10^{-43}	2.19×10^{-63}	1.92×10^{-55}	1.62×10^{-66}
$k_{OR} = 5$	2.39×10^{-30}	1.71×10^{-28}	7.75×10^{-27}	1.92×10^{-31}	1.73×10^{-34}	2.72×10^{-38}
$k_{OR} = 7$	7.08×10^{-16}	2.15×10^{-15}	2.19×10^{-14}	7.60×10^{-14}	3.25×10^{-15}	7.47×10^{-17}
$k_{OR} = 10$	1.31×10^{-5}	2.39×10^{-8}	5.02×10^{-9}	1.31×10^{-5}	1.36×10^{-9}	1.40×10^{-5}
$k_{OR} = 15$	1.85×10^{-3}	7.71×10^{-3}	7.32×10^{-4}	5.34×10^{-5}	3.50×10^{-3}	3.77×10^{-4}
$k_{OR} = 20$	1.42×10^{-2}	2.83×10^{-2}	1.69×10^{-1}	8.13×10^{-5}	9.12×10^{-1}	5.60×10^{-2}
$k_{OR} = 30$	9.52×10^{-3}	5.85×10^{-2}	—	7.94×10^{-2}	1.45×10^{-1}	8.23×10^{-3}
$k_{OR} = 40$	2.88×10^{-1}	7.11×10^{-4}	6.89×10^{-2}	2.91×10^{-2}	2.56×10^{-2}	1.24×10^{-2}
$k_{OR} = 50$	4.74×10^{-2}	2.88×10^{-3}	7.08×10^{-2}	1.04×10^{-2}	7.66×10^{-4}	3.76×10^{-4}

Table 4 Final calibration of the p_{Large} parameter

p_{LARGE}	k_{OR}	k_{BS}	Gap(%)	p-value
0.00	30	3	0.423	3.23×10^{-3}
0.05	30	3	0.428	5.38×10^{-4}
0.10	30	3	0.395	—
0.20	30	3	0.422	2.98×10^{-3}
0.30	30	3	0.427	1.98×10^{-4}
0.50	30	3	0.434	3.02×10^{-6}
1.00	30	3	0.490	1.89×10^{-24}

4.3. Performance Comparisons

The goal of this section is to establish a comparative analysis of the different PDTSP solution methods. We use new seed values to avoid any bias due to possible calibration overfit. As we re-implemented the RR approach, we can establish a comparison with the same computational environment and termination criterion. We therefore use a common time limit of $T_{\text{MAX}} = N$ seconds for each instance. The cooling schedule of the RR method has also been calibrated to follow an exponential decay from the initial temperature to the final temperature within the allotted time. Finally, we confront the new experimental results with those reported from previous studies, which serve as a baseline (i.e., to compare our re-implementation of RR with the original results and previous approaches). Overall, we report the results of the following heuristics:

- RBO-Best: Best results from Renaud et al. (2000) and Renaud et al. (2002), directly provided to us by the authors. We note that these solution values were registered from multiple runs (136 and 168, respectively) using different parameter values.
- RR-VRVC: Results of the RR algorithm of Veenstra et al. (2017) over ten runs, as reported by the authors. Experiments conducted on an Intel Core i3-2120 @3.3GHz CPU.
- RR and RR-fast: Results of our re-implementation of the RR algorithm over ten runs, without or with the speed-up strategy for the reconstruction step, respectively, using $T_{\text{MAX}} = N$.
- HGS: Results of the hybrid genetic search over ten runs, using $T_{\text{MAX}} = N$.

The CPU used in Veenstra et al. (2017) is estimated to be $1.41 \times$ slower than ours according to Passmark single-thread benchmark. Therefore, we divide the CPU time values of RR-VRVC by this coefficient to draw a reliable comparison.

Tables 5 and 6 compare the performance of all of these methods on the classical instances and the new ones derived from set X, respectively. These tables list, when available, the best solutions from RBO as well as the average percentage gap (Avg) relative to the BKS over ten runs for each instance, the best percentage gap (Best), and the average (scaled) computational time in seconds ($T(s)$) for each other method. The gap is calculated as $\text{Gap}(\%) = 100 \times (z_{\text{SOL}} - z_{\text{BKS}})/z_{\text{BKS}}$ for each instance and solution. In Table 5, the BKS have been collected from all previous works by Renaud et al. (2000, 2002), Dumitrescu et al. (2010) and Veenstra et al. (2017), therefore negative values represent improvements over the previous BKSs. In Table 6, as no previous results are available, the BKSs correspond to the best solutions ever found by any method on any run, including the calibration experiments. The results of these tables are aggregated per range of instance size and instance type, and the best average solution quality is indicated on each line. The detailed solution values for all instances are provided in the Electronic Companion. Moreover, the source code and scripts needed to run these experiments, the new instances, as well as the solutions for each instance are openly accessible at <https://github.com/vidalt/PDTSP>.

Table 5 Performance comparison on the instances of Class 1

N	Type	RBO	RR-VRVC			RR			RR-fast			HGS		
		Best	Avg	Best	T(s)	Avg	Best	T(s)	Avg	Best	T(s)	Avg	Best	T(s)
50-99	A	0.015	0.222	0.000	17.38	0.005	-0.020	87.55	-0.008	-0.020	87.55	-0.023	-0.023	87.55
	B	0.000	0.128	0.027	18.54	0.022	0.011	87.55	0.009	-0.014	87.55	-0.014	-0.014	87.55
	C	0.000	0.060	0.007	19.73	0.007	0.007	87.55	0.007	0.007	87.55	0.007	0.007	87.55
100-199	A	0.401	0.353	0.016	83.59	0.000	-0.132	146.00	-0.120	-0.169	146.00	-0.172	-0.172	146.00
	B	0.488	0.260	0.002	87.05	0.008	-0.064	146.00	-0.056	-0.096	146.00	-0.098	-0.098	146.00
	C	0.482	0.366	0.000	93.67	0.168	0.012	146.00	0.027	-0.061	146.00	-0.074	-0.074	146.00
200-299	A	1.296	0.824	0.104	364.28	0.508	0.101	254.60	0.014	-0.413	254.60	-0.580	-0.580	254.60
	B	1.118	0.643	0.000	381.73	0.070	-0.481	254.60	-0.396	-0.561	254.60	-0.605	-0.625	254.60
	C	1.618	0.691	0.000	414.33	0.099	-0.179	254.60	-0.229	-0.460	254.60	-0.516	-0.580	254.60
300-499	A	2.974	0.852	0.000	1015.46	0.438	-0.247	417.67	-0.391	-0.613	417.67	-0.844	-1.030	417.67
	B	2.778	0.863	0.000	1027.61	0.543	-0.043	417.67	-0.099	-0.487	417.67	-0.847	-1.070	417.67
	C	1.698	1.159	0.025	1126.87	0.940	-0.331	417.67	-0.189	-0.635	417.67	-1.249	-1.484	417.67
All	A	0.836	0.462	0.021	257.66	0.145	-0.084	188.50	-0.112	-0.231	188.50	-0.295	-0.326	188.50
	B	0.808	0.373	0.009	263.81	0.110	-0.096	188.50	-0.090	-0.201	188.50	-0.268	-0.307	188.50
	C	0.695	0.450	0.006	287.81	0.238	-0.073	188.50	-0.051	-0.192	188.50	-0.307	-0.355	188.50
Average		0.780	0.428	0.012	269.76	0.164	-0.084	188.50	-0.085	-0.208	188.50	-0.290	-0.329	188.50

From these results, we first observe that our re-implementation of the RR algorithm has found solutions of a similar or better quality than the original RR-VRVC. As seen in Table 5, for all instances with 200 or more services, RR finds solutions of a better quality than RR-VRVC in a shorter amount of time. Generally, the use of the termination criterion and cooling schedule proportional to the size of the instances is beneficial for the method.

The RR solution approach is further improved by relying on the fast evaluation strategy for the pickup-and-delivery insertions (RR-fast). The increased speed of these evaluations permits to do significantly more iterations, leading to an average gap of -0.085% for RR-fast compared to 0.164% for RR on the classic instances, and 0.837% compared to 1.695% on the X instances. The significance of these solution-quality improvement is confirmed by pairwise Wilcoxon tests, with p-values of 4.90×10^{-60} and 2.06×10^{-286} on the classic and X instance sets, respectively. Notably, this means that the proposed mechanisms for evaluating insertions are not only useful from a complexity viewpoint, but also from a practical perspective.

Overall, considering all methods, the HGS with the proposed local search achieves the best performance with an average gap of -0.290% on the classical instances, and 0.395% on the X instances. The significance of these improvements over RR-fast is also confirmed by pairwise Wilcoxon tests, with respective p-values of 4.24×10^{-67} and 2.67×10^{-289} . Out of the 128 instances from Renaud et al. (2000), it found or improved the best-known solution in 126 cases (i.e., all BKS except those of KROD99C and PR107B, reported only in RBO). Moreover, among these 126 cases, the BKS has been matched on all ten runs in 82 cases, whereas the average solution quality of HGS was better than the BKS for the remaining 44 cases. Finally, the 28 known optimal solutions from Dumitrescu et al. (2010) have been found on all runs.

Table 6 Performance comparison on the instances of Class X

N	Type	RR			RR-fast			HGS		
		Avg	Best	T(s)	Avg	Best	T(s)	Avg	Best	T(s)
100-199	A	0.251	0.132	147.57	0.166	0.072	147.58	0.000	0.000	147.57
	B	0.167	0.044	147.57	0.063	0.010	147.58	0.001	0.000	147.57
	C	0.276	0.156	147.57	0.146	0.093	147.57	0.013	0.003	147.57
200-299	A	0.967	0.381	248.55	0.423	0.174	248.55	0.025	0.005	248.55
	B	0.904	0.386	248.55	0.421	0.165	248.55	0.034	0.009	248.55
	C	1.181	0.429	248.55	0.438	0.166	248.55	0.067	0.008	248.55
300-399	A	1.245	0.594	341.67	0.513	0.166	341.67	0.045	0.010	341.67
	B	1.531	0.671	341.67	0.617	0.321	341.67	0.171	0.057	341.67
	C	1.954	1.023	341.67	0.989	0.348	341.67	0.370	0.151	341.67
400-499	A	2.126	1.292	444.60	1.172	0.495	444.60	0.140	0.021	444.60
	B	2.035	0.852	444.60	0.997	0.479	444.60	0.229	0.058	444.60
	C	2.521	1.267	444.60	1.626	0.855	444.60	0.619	0.180	444.60
500-599	A	2.585	1.764	548.56	1.447	0.931	548.56	0.517	0.183	548.56
	B	2.790	1.752	548.56	1.323	0.519	548.57	0.605	0.241	548.56
	C	2.966	1.930	548.56	1.603	0.934	548.56	0.880	0.200	548.56
600-699	A	2.667	1.657	648.34	1.351	0.671	648.33	0.443	0.150	648.33
	B	2.922	1.929	648.34	1.496	0.781	648.33	0.657	0.224	648.33
	C	3.277	2.358	648.34	1.859	0.896	648.33	1.294	0.663	648.33
700-799	A	2.765	1.720	741.01	1.404	0.805	741.00	0.874	0.311	741.00
	B	3.259	2.317	741.01	1.800	0.904	741.02	1.005	0.534	741.00
	C	3.261	1.847	741.01	1.667	0.827	741.00	1.342	0.643	741.00
800-899	A	3.045	2.158	847.01	1.530	0.978	847.00	0.893	0.253	847.00
	B	3.365	2.263	847.01	1.755	0.896	847.00	1.173	0.483	847.00
	C	3.454	2.325	847.01	1.680	0.472	847.00	1.899	1.100	847.00
900-1001	A	3.268	2.258	957.41	1.359	0.541	957.40	1.012	0.349	957.40
	B	3.284	2.282	957.42	1.363	0.563	957.40	1.246	0.623	957.40
	C	2.616	1.161	957.42	0.991	0.065	957.40	2.016	1.022	957.40
ALL	A	1.569	0.934	412.80	0.777	0.386	412.80	0.256	0.081	412.80
	B	1.655	0.942	412.80	0.788	0.364	412.80	0.343	0.144	412.80
	C	1.860	1.031	412.80	0.944	0.413	412.80	0.587	0.257	412.80
Average		1.695	0.969	412.80	0.837	0.388	412.80	0.395	0.161	412.80

Solution-quality differences are more marked for larger instances with many services. For example, an improvement of 3.768% over previous known results have been achieved for instance PR439C. HGS produces superior results for the wide majority of the instances, though on a few of the largest instances of set X (class C with more than 800 visits), the simpler RR-fast strategy seems to perform better. This is likely due to the fact that, on such instances, the crossover mechanism of HGS creates many disruptions in the new solutions, which are not fully re-optimized in a single local-search descent. In contrast, the ruin-and-recreate mechanism is more localized. This effect is also exacerbated by the precedence constraints implied by the pickups and deliveries, which limit the amount of feasible moves.

4.4. Impact of the Neighborhoods

To understand the contribution of each of the six neighborhoods used in the local search of HGS, we conduct an ablation study which consists of removing one neighborhood at a time and comparing the performance of the resulting approach to our baseline results, using the same benchmark instances and stopping criterion as in Section 2. Table 7 displays summarized results for this experiment over the instances of set X. It indicates, for each algorithm variant, the average percentage gap over ten runs relative to the BKS, as well as the p-value of a pairwise Wilcoxon test between the considered configuration and the baseline without any ablation.

Table 7 Ablation study on the impact of the different neighborhoods

Configuration	Average Gap(%)	p-value
Baseline	0.395	—
No RELOCATE PAIR	0.463	5.64×10^{-21}
No 2-OPT	0.399	8.91×10^{-1}
No 2K-OPT	0.405	6.00×10^{-1}
No OR-OPT	0.793	2.84×10^{-216}
No 4-OPT	0.409	7.26×10^{-2}
No BALAS-SIMONETTI	0.416	5.16×10^{-2}
No Large Neighborhoods	0.423	3.23×10^{-3}

As seen in this experiment, the baseline method with all the neighborhoods achieved better average gaps than the other configurations. In terms of their individual impact, canonical neighborhoods such as OR-OPT and RELOCATE PAIR are indispensable to achieve good solutions. This is confirmed by pairwise Wilcoxon tests comparing the results of the configurations obtained by deactivating these neighborhoods with those of the baseline.

The RELOCATE PAIR, 2-OPT, and OR-OPT neighborhoods are explored in the first stage in our local search (Lines 3 to 6 of Algorithm 2) as it is possible to decompose their exploration and rapidly apply improving moves. In contrast, the large neighborhoods are explored less frequently since they require a complete quadratic-time exploration. Their main goal is to help escaping local minima and sub-optimal structures that may otherwise persist in the solutions. These distinct roles are clearly visible as we monitor the local search. As seen in Table EC.5 (in the Electronic Companion), a local search on an instance of Class X applies on average 21.48 RELOCATE PAIR, 2.95 2-OPT, and 16.43 OR-OPT moves, compared to only 1.71 2K-OPT, 1.86 4-OPT and 1.45 BALAS-SIMONETTI moves. Still, a Wilcoxon test demonstrates that jointly deactivating all the large neighborhoods leads to results that are significantly worse. These observations are also in line with other studies on variable neighborhood search and large neighborhood search, which have shown that a diversity neighborhood structures permits to effectively escape low-quality local minima, since a local minimum of a specific neighborhood may not be a minimum of other ones (Hansen et al. 2019, Pisinger and Ropke 2019).

4.5. Speed evaluation on Grubhub data

Finally, we analyze the performance of the complete HGS method on the Grubhub data set, which contains instances with $n \in \{2, \dots, 15\}$ pickup-and-delivery pairs and therefore $N \in \{5, \dots, 31\}$ visits. These instances involve open tours (i.e., the return to the origin point is not counted). There are ten instances for each size, totaling 140 instances. Solution time is critical since these instances relate to a real-time meal delivery application. Ideally, it should be no greater than a fraction of a second. For this reason, we use a short termination criterion of 100 successive iterations without improvement.

O’Neil (2018) and O’Neil and Hoffman (2019) have proposed different solution techniques to solve these instances to near optimality. The authors suggested using constraint programming, decision diagrams, and integer programming to achieve optimality and terminate the search quickly. Based on their experiments, they proposed a configuration called “Ruland+CP+AP”, which consists of using constraint programming as a hot start for a MIP solver based on the formulation from Ruland and Rodin (1997). They conducted experiments on an Intel Core i7-7300U @2.6GHz CPU and measured the time taken to (i) attain a solution within 10% of the optimum, (ii) attain a solution within 5% of the optimum, (iii) attain the optimal solution, (iv) prove optimality.

As we rely on heuristics, we do not prove optimality, but we can nonetheless measure in a similar fashion the time to achieve (i), (ii), and (iii). Figures 7 and 8 compare the CPU time of HGS and Ruland+CP+AP to achieve those targets. For consistency, since Ruland+CP+AP performed a single run per instance, we used the first run of HGS to create the figures. The CPU time of Ruland+CP+AP was also scaled by a factor 1.086 to account for CPU speed differences. For each group of ten instances of each given size, the first figure reports the minimum, average, and maximum time needed to attain each target. The second figure depicts the average solution time needed to attain each target as a stacked bar plot. In both cases, CPU times are represented on a logarithmic scale. Finally, Tables EC.6 and EC.7 (in the Electronic Companion) provide the detailed time and solution values for each instance, considering the ten runs of HGS.

As observed in these experiments, the proposed HGS with local search attained the known optimal solution for all instances and runs (i.e., on 1400 executions overall). This confirms the performance and robustness of the approach. Most notably, solutions within 5% of the optimum are typically found within a fraction of a millisecond, and optimal solutions are found in milliseconds, for all instances, as visible in Figure 7. This is a remarkable capability, given that instances with 15 pickup-and-delivery services (i.e., 31 visits) are far from trivial, with already $30! / 2^{15} \approx 8.09 \times 10^{27}$ possible solutions. In comparison, the Ruland+CP+AP approach based on constraint programming and integer programming requires dozens of seconds on average to achieve the same targets.

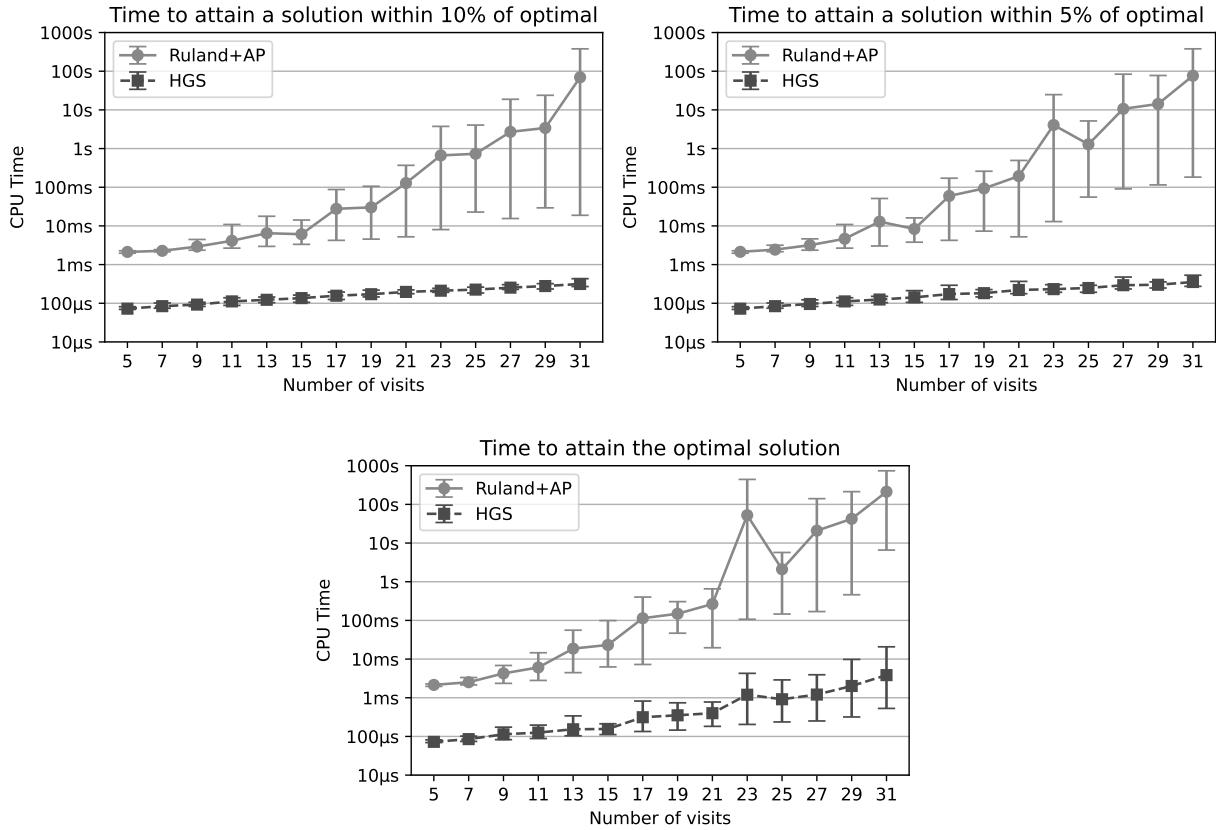


Figure 7 Minimum, average and maximum computational effort to attain solutions within 10%, 5% and 0% of the optimum

We also measured the number of local search runs needed to attain the same targets. Interestingly, we observed that a single local search run was sufficient to attain the optimal solution in 61.79% of the cases. Similarly, 99.93% of the optimal solutions were found when generating the initial population, whereas the remaining 0.07% solutions were found during the evolution of the genetic algorithm on these instances. This indicates that the proposed local search effectively locates high-quality PDTSP solutions and that it represents a meaningful alternative for applications requiring very short response time.

5. Conclusions

In this paper, we have introduced efficient search strategies and new large neighborhoods for the pickup-and-delivery traveling salesman problem. Our study permitted significant complexity reductions for classical neighborhoods such as RELOCATE PAIR as well as the efficient exploration of large neighborhoods such as 2K-OPT, 4-OPT and BALAS-SIMONETTI. We conducted extensive computational experiments, demonstrating that the use of these neighborhoods in the classical HGS metaheuristic leads to a robust search algorithm that produces almost systematically the

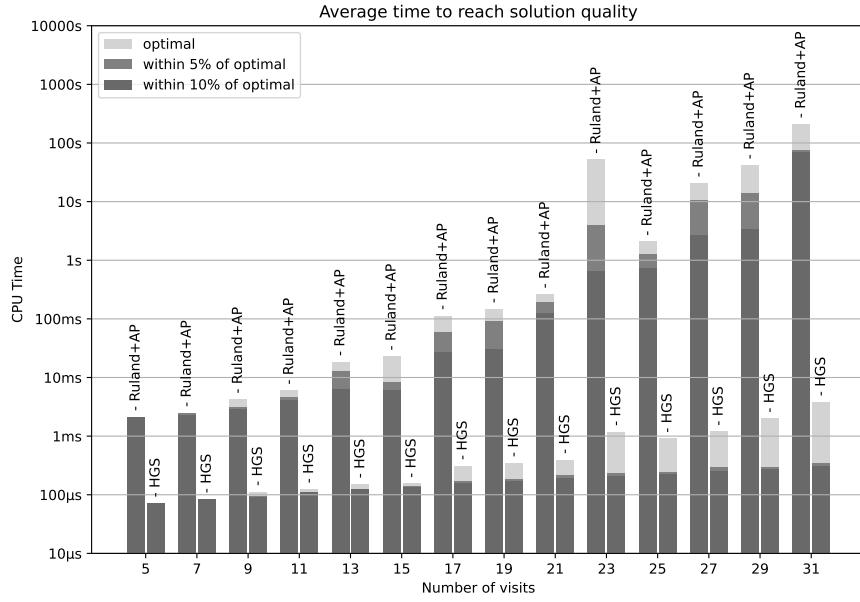


Figure 8 Distribution of the CPU time needed to locate solutions within 10%, 5% and 0% of the optimum

best-known and optimal solutions for all the considered instances. Our methodology also permitted significant speed-ups and improvements over a ruin-and-recreate algorithm previously proposed by Veenstra et al. (2017). Our search approach is especially useful in applications where time is critical, e.g., meal-delivery and mobility-on-demand, as it permits us to repeatedly achieve optimal or near-optimal solutions for small problems within milliseconds.

The research perspectives are numerous. First of all, research can be pursued on some of the neighborhood structures discussed in this paper, such as 2k-OPT and 4-OPT. Since the evaluation of these neighborhoods is grounded on dynamic programming, identifying the families of problem attributes (Vidal et al. 2013) for which an efficient (sub-quadratic) search is achievable is an important and challenging research question. Studies on incremental solution strategies and decompositions for the dynamic programs could also lead to substantial improvements and give more flexibility on move acceptance policies.

More generally, research should be pursued on fundamental neighborhood structures for canonical problems in the vehicle routing family. Many studies have been focused on high-level search strategies (often new metaheuristic concepts) over the past two decades, but improvements along this line see diminishing returns nowadays. The core component of state-of-the-art metaheuristics is often a low-complexity neighborhood search, used for systematic solution improvement. However, the best possible complexity for many useful neighborhoods still remains an open research question, and regular breakthroughs are still being made (De Berg et al. 2020). Generally, we believe

that a disciplined study of neighborhood searches, grounded on computational complexity theory, combinatorial optimization, and possibly machine learning techniques (Arnold et al. 2021, Karimi-Mamaghan et al. 2022, Lancia and Dalpasso 2019), still has potential for significant breakthrough and paradigm shifts.

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Electronic Companion: Exponential-Size Neighborhoods for the Pickup-and-Delivery Traveling Salesman Problem

This Electronic Companion provides additional detailed results. The instances of Class 1 and 2 have been introduced in Renaud et al. (2000), whereas the instances of Class 3 come from Dumitrescu et al. (2010). The Grubhub data sets have been introduced in O’Neil (2018) and O’Neil and Hoffman (2019) and are accessible at <https://github.com/grubhub/tsppdlib>. Finally, the instances of Class X have been introduced in the present study.

Tables EC.1–EC.4 and EC.6 provide detailed results over ten runs for all classes of instances. The format of the tables is identical to that of Table 5 in the main paper, except that Best and Avg solution values as well as CPU times ($T(s)$) are provided for each separate instance. Known optimal solutions from Dumitrescu et al. (2010) are also marked with an *. As discussed in the paper, CPU times have been scaled to obtain an equivalent time on an AMD Rome 7532 2.40GHz. Table EC.5 reports the average number of moves of each type applied during a local search on the different instances subgroups. Finally, Table EC.7 gives the detailed data used to generate Figures 7 and 8, based on the the first run of HGS. It compares the time taken by Ruland+CP+AP and HGS to attain a solution-quality target of 10%, 5%, and 0% from the known optimal solution.

Table EC.3 Detailed results on the instances of Class 2

Instance	N	BKS	RBO/BC10			RR-VRVC			RR			RR-FAST			HGS				
			Best	Avg	Best	T(s)	Avg	Best	T(s)	Avg	Best	T(s)	Avg	Best	T(s)	Avg	Best	T(s)	
N101P1	101	799*	799	799.00	799.00	26.38	799.00	799.00	101.00	799.00	799.00	101.00	799.00	799.00	101.00	799.00	799.00	101.00	
N101P2	101	729*	729	729.00	729.00	26.95	729.00	729.00	101.00	729.00	729.00	101.00	729.00	729.00	101.00	729.00	729.00	101.00	
N101P3	101	748*	748	748.00	748.00	26.88	748.00	748.00	101.00	748.00	748.00	101.00	748.00	748.00	101.00	748.00	748.00	101.00	
N101P4	101	807*	807	807.00	807.00	26.88	807.00	807.00	101.00	807.00	807.00	101.00	807.00	807.00	101.00	807.00	807.00	101.00	
N101P5	101	783*	783	783.00	783.00	26.88	783.00	783.00	101.00	783.00	783.00	101.00	783.00	783.00	101.00	783.00	783.00	101.00	
N101P6	101	755*	755	755.00	755.00	26.95	755.00	755.00	101.00	755.00	755.00	101.00	755.00	755.00	101.00	755.00	755.00	101.00	
N101P7	101	767*	767	767.00	767.00	27.09	767.00	767.00	101.00	767.00	767.00	101.00	767.00	767.00	101.00	767.00	767.00	101.00	
N101P8	101	762*	762	762.00	762.00	26.88	762.00	762.00	101.00	762.00	762.00	101.00	762.00	762.00	101.00	762.00	762.00	101.00	
N101P9	101	766*	766	766.00	766.00	26.81	766.00	766.00	101.00	766.00	766.00	101.00	766.00	766.00	101.00	766.00	766.00	101.00	
N101P10	101	754*	754	754.00	754.00	27.02	754.00	754.00	101.00	754.00	754.00	101.00	754.00	754.00	101.00	754.00	754.00	101.00	
N201P1	201	1039*	1039	1039.00	1039.00	183.97	1039.00	1039.00	201.00	1039.00	1039.00	201.00	1039.00	1039.00	201.00	1039.00	1039.00	201.00	
N201P2	201	1086*	1086	1086.00	1086.00	183.26	1086.00	1086.00	201.00	1086.00	1086.00	201.00	1086.00	1086.00	201.00	1086.00	1086.00	201.00	
N201P3	201	1070*	1070	1072.00	1070.00	186.17	1070.00	1070.00	201.00	1070.00	1070.00	201.00	1070.00	1070.00	201.00	1070.00	1070.00	201.00	
N201P4	201	1050*	1050	1050.00	1050.00	183.97	1050.00	1050.00	201.00	1050.00	1050.00	201.00	1050.00	1050.00	201.00	1050.00	1050.00	201.00	
N201P5	201	1052*	1052	1052.00	1052.00	184.54	1052.00	1052.00	201.00	1052.00	1052.00	201.00	1052.00	1052.00	201.00	1052.00	1052.00	201.00	
N201P6	201	1059*	1059	1059.00	1059.00	185.82	1059.00	1059.00	201.00	1059.00	1059.00	201.00	1059.00	1059.00	201.00	1059.00	1059.00	201.00	
N201P7	201	1036*	1036	1036.00	1036.00	185.11	1036.00	1036.00	201.00	1036.00	1036.00	201.00	1036.00	1036.00	201.00	1036.00	1036.00	201.00	
N201P8	201	1079*	1079	1079.00	1079.00	184.75	1079.00	1079.00	201.00	1079.00	1079.00	201.00	1079.00	1079.00	201.00	1079.00	1079.00	201.00	
N201P9	201	1050*	1050	1050.00	1050.00	183.83	1050.00	1050.00	201.00	1050.00	1050.00	201.00	1050.00	1050.00	201.00	1050.00	1050.00	201.00	
N201P10	201	1085*	1085	1085.00	1085.00	184.47	1085.00	1085.00	201.00	1085.00	1085.00	201.00	1085.00	1085.00	201.00	1085.00	1085.00	201.00	
Average			913.80	913.80	913.90	913.80	105.73	913.80	913.80	151.00	913.80	913.80	151.00	913.80	913.80	151.00	913.80	913.80	151.00

Table EC.5 Average number of moves per local search on the instances of Class X

<i>N</i>	Type	RELOC	2-OPT	OR-OPT	2K-OPT	4-OPT	B&S
100-199	A	9.02	1.34	8.23	1.42	1.64	1.30
	B	11.51	1.77	9.90	1.52	1.66	1.34
	C	15.66	2.75	12.68	1.64	1.70	1.40
200-299	A	14.54	1.88	12.16	1.56	1.83	1.39
	B	18.36	2.45	14.37	1.67	1.82	1.44
	C	27.33	4.23	19.73	1.88	1.91	1.54
300-399	A	18.1	2.25	15.03	1.63	1.96	1.45
	B	22.76	2.93	17.43	1.76	1.92	1.49
	C	36.43	5.22	25.31	2.05	2.05	1.60
400-499	A	23.94	2.86	19.45	1.75	2.10	1.52
	B	29.62	3.66	22.09	1.91	2.05	1.57
	C	50.78	7.05	34.84	2.31	2.24	1.71
500-599	A	32.58	3.60	25.38	1.88	2.29	1.59
	B	43.36	4.85	31.12	2.14	2.29	1.68
	C	67.00	8.77	44.3	2.56	2.44	1.80
600-699	A	37.60	4.00	28.62	1.97	2.38	1.62
	B	49.34	5.35	34.76	2.23	2.42	1.72
	C	85.28	10.66	54.02	2.80	2.63	1.89
700-799	A	47.66	5.07	35.62	2.14	2.52	1.71
	B	57.60	6.28	40.01	2.35	2.48	1.81
	C	90.55	11.37	58.12	2.86	2.65	1.91
800-899	A	55.52	5.80	41.82	2.21	2.73	1.78
	B	64.05	6.99	44.96	2.43	2.58	1.82
	C	114.78	14.11	72.6	3.10	2.93	2.03
900-1001	A	60.93	6.31	45.62	2.30	2.83	1.81
	B	74.02	7.89	51.49	2.52	2.72	1.88
	C	141.16	16.60	86.24	3.39	3.12	2.12
ALL	A	15.71	1.99	13.1	1.56	1.84	1.39
	B	19.92	2.61	15.47	1.69	1.83	1.44
	C	29.90	4.45	21.35	1.90	1.92	1.53
Average		21.48	2.95	16.43	1.71	1.86	1.45

Table EC.6: Detailed results on the Grubhub instances

Instance	N	BKS	HGS		
			Avg	Best	T(s)
grubhub-02-0	5	3214*	3214	3214	4.49×10^{-3}
grubhub-02-1	5	2736*	2736	2736	4.53×10^{-3}
grubhub-02-2	5	2558*	2558	2558	4.49×10^{-3}
grubhub-02-3	5	1382*	1382	1382	4.42×10^{-3}
grubhub-02-4	5	2467*	2467	2467	4.37×10^{-3}
grubhub-02-5	5	3099*	3099	3099	4.46×10^{-3}
grubhub-02-6	5	1778*	1778	1778	4.38×10^{-3}
grubhub-02-7	5	2206*	2206	2206	4.44×10^{-3}
grubhub-02-8	5	1275*	1275	1275	4.40×10^{-3}
grubhub-02-9	5	2121*	2121	2121	4.42×10^{-3}
grubhub-03-0	7	2802*	2802	2802	5.71×10^{-3}
grubhub-03-1	7	3318*	3318	3318	8.01×10^{-3}
grubhub-03-2	7	3050*	3050	3050	5.24×10^{-3}
grubhub-03-3	7	3129*	3129	3129	5.57×10^{-3}
grubhub-03-4	7	2714*	2714	2714	5.23×10^{-3}
grubhub-03-5	7	4993*	4993	4993	5.22×10^{-3}
grubhub-03-6	7	3128*	3128	3128	5.25×10^{-3}
grubhub-03-7	7	3653*	3653	3653	7.31×10^{-3}
grubhub-03-8	7	2419*	2419	2419	6.88×10^{-3}
grubhub-03-9	7	4744*	4744	4744	6.78×10^{-3}
grubhub-04-0	9	4810*	4810	4810	6.27×10^{-3}
grubhub-04-1	9	3041*	3041	3041	6.79×10^{-3}
grubhub-04-2	9	3959*	3959	3959	6.28×10^{-3}
grubhub-04-3	9	4320*	4320	4320	6.18×10^{-3}
grubhub-04-4	9	4718*	4718	4718	8.58×10^{-3}
grubhub-04-5	9	4791*	4791	4791	6.90×10^{-3}
grubhub-04-6	9	4020*	4020	4020	6.43×10^{-3}
grubhub-04-7	9	3987*	3987	3987	7.21×10^{-3}
grubhub-04-8	9	4361*	4361	4361	6.29×10^{-3}
grubhub-04-9	9	4320*	4320	4320	7.27×10^{-3}
grubhub-05-0	11	4259*	4259	4259	7.51×10^{-3}
grubhub-05-1	11	5168*	5168	5168	1.08×10^{-2}
grubhub-05-2	11	4351*	4351	4351	9.83×10^{-3}
grubhub-05-3	11	4318*	4318	4318	1.04×10^{-2}
grubhub-05-4	11	4664*	4664	4664	1.10×10^{-2}
grubhub-05-5	11	4447*	4447	4447	1.04×10^{-2}
grubhub-05-6	11	5824*	5824	5824	1.06×10^{-2}
grubhub-05-7	11	4919*	4919	4919	8.18×10^{-3}
grubhub-05-8	11	4354*	4354	4354	7.17×10^{-3}
grubhub-05-9	11	6001*	6001	6001	7.73×10^{-3}
grubhub-06-0	13	4613*	4613	4613	8.54×10^{-3}
grubhub-06-1	13	5116*	5116	5116	1.17×10^{-2}
grubhub-06-2	13	6147*	6147	6147	1.23×10^{-2}
grubhub-06-3	13	4853*	4853	4853	8.57×10^{-3}
grubhub-06-4	13	5038*	5038	5038	1.22×10^{-2}
grubhub-06-5	13	5628*	5628	5628	9.85×10^{-3}
grubhub-06-6	13	5896*	5896	5896	1.17×10^{-2}
grubhub-06-7	13	6056*	6056	6056	1.27×10^{-2}
grubhub-06-8	13	6037*	6037	6037	1.34×10^{-2}
grubhub-06-9	13	5406*	5406	5406	9.23×10^{-3}
grubhub-07-0	15	5401*	5401	5401	1.64×10^{-2}
grubhub-07-1	15	4897*	4897	4897	1.49×10^{-2}
grubhub-07-2	15	6966*	6966	6966	1.64×10^{-2}
grubhub-07-3	15	7633*	7633	7633	1.20×10^{-2}
grubhub-07-4	15	5139*	5139	5139	9.16×10^{-3}
grubhub-07-5	15	6353*	6353	6353	1.72×10^{-2}
grubhub-07-6	15	5249*	5249	5249	1.10×10^{-2}
grubhub-07-7	15	6522*	6522	6522	1.65×10^{-2}
grubhub-07-8	15	6960*	6960	6960	1.52×10^{-2}
grubhub-07-9	15	5319*	5319	5319	1.48×10^{-2}
grubhub-08-0	17	7394*	7394	7394	1.80×10^{-2}
grubhub-08-1	17	7539*	7539	7539	1.38×10^{-2}

(continued on next page)

Table EC.6: Detailed results on the Grubhub instances

Instance	N	BKS	HGS		
			Avg	Best	T(s)
grubhub-08-2	17	5450*	5450	5450	1.82×10^{-2}
grubhub-08-3	17	7694*	7694	7694	1.88×10^{-2}
grubhub-08-4	17	6963*	6963	6963	1.51×10^{-2}
grubhub-08-5	17	6632*	6632	6632	1.58×10^{-2}
grubhub-08-6	17	7318*	7318	7318	1.66×10^{-2}
grubhub-08-7	17	7281*	7281	7281	1.80×10^{-2}
grubhub-08-8	17	6390*	6390	6390	1.20×10^{-2}
grubhub-08-9	17	6684*	6684	6684	1.89×10^{-2}
grubhub-09-0	19	7607*	7607	7607	1.46×10^{-2}
grubhub-09-1	19	4800*	4800	4800	1.69×10^{-2}
grubhub-09-2	19	6820*	6820	6820	1.89×10^{-2}
grubhub-09-3	19	7841*	7841	7841	1.38×10^{-2}
grubhub-09-4	19	7078*	7078	7078	1.66×10^{-2}
grubhub-09-5	19	6598*	6598	6598	2.10×10^{-2}
grubhub-09-6	19	6097*	6097	6097	1.68×10^{-2}
grubhub-09-7	19	7187*	7187	7187	2.19×10^{-2}
grubhub-09-8	19	6333*	6333	6333	1.59×10^{-2}
grubhub-09-9	19	7130*	7130	7130	2.28×10^{-2}
grubhub-10-0	21	7881*	7881	7881	2.34×10^{-2}
grubhub-10-1	21	7954*	7954	7954	2.47×10^{-2}
grubhub-10-2	21	6430*	6430	6430	2.41×10^{-2}
grubhub-10-3	21	6245*	6245	6245	2.55×10^{-2}
grubhub-10-4	21	7589*	7589	7589	2.43×10^{-2}
grubhub-10-5	21	7773*	7773	7773	2.27×10^{-2}
grubhub-10-6	21	6049*	6049	6049	2.37×10^{-2}
grubhub-10-7	21	8061*	8061	8061	2.42×10^{-2}
grubhub-10-8	21	6848*	6848	6848	2.51×10^{-2}
grubhub-10-9	21	7232*	7232	7232	2.54×10^{-2}
grubhub-11-0	23	8637*	8637	8637	2.72×10^{-2}
grubhub-11-1	23	7456*	7456	7456	2.42×10^{-2}
grubhub-11-2	23	9716*	9716	9716	2.77×10^{-2}
grubhub-11-3	23	9117*	9117	9117	2.67×10^{-2}
grubhub-11-4	23	7693*	7693	7693	2.95×10^{-2}
grubhub-11-5	23	8419*	8419	8419	3.08×10^{-2}
grubhub-11-6	23	7683*	7683	7683	2.40×10^{-2}
grubhub-11-7	23	6783*	6783	6783	2.78×10^{-2}
grubhub-11-8	23	7565*	7565	7565	2.55×10^{-2}
grubhub-11-9	23	7342*	7342	7342	3.25×10^{-2}
grubhub-12-0	25	8622*	8622	8622	3.04×10^{-2}
grubhub-12-1	25	9446*	9446	9446	2.32×10^{-2}
grubhub-12-2	25	6764*	6764	6764	2.94×10^{-2}
grubhub-12-3	25	8035*	8035	8035	3.27×10^{-2}
grubhub-12-4	25	9325*	9325	9325	2.75×10^{-2}
grubhub-12-5	25	9269*	9269	9269	2.75×10^{-2}
grubhub-12-6	25	8805*	8805	8805	3.15×10^{-2}
grubhub-12-7	25	7460*	7460	7460	3.25×10^{-2}
grubhub-12-8	25	8030*	8030	8030	3.19×10^{-2}
grubhub-12-9	25	8989*	8989	8989	3.23×10^{-2}
grubhub-13-0	27	9285*	9285	9285	3.47×10^{-2}
grubhub-13-1	27	9484*	9484	9484	3.62×10^{-2}
grubhub-13-2	27	9391*	9391	9391	3.88×10^{-2}
grubhub-13-3	27	8109*	8109	8109	3.52×10^{-2}
grubhub-13-4	27	7555*	7555	7555	2.57×10^{-2}
grubhub-13-5	27	9368*	9368	9368	3.26×10^{-2}
grubhub-13-6	27	10036*	10036	10036	3.15×10^{-2}
grubhub-13-7	27	8976*	8976	8976	3.76×10^{-2}
grubhub-13-8	27	9643*	9643	9643	3.79×10^{-2}
grubhub-13-9	27	8937*	8937	8937	3.67×10^{-2}
grubhub-14-0	29	9464*	9464	9464	4.04×10^{-2}
grubhub-14-1	29	9288*	9288	9288	3.82×10^{-2}
grubhub-14-2	29	11592*	11592	11592	3.60×10^{-2}
grubhub-14-3	29	10681*	10681	10681	4.36×10^{-2}

(continued on next page)

Table EC.7: Time-to-target results on Grubhub instances

Instance	N	T(s) of Ruland+AP			T(s) of HGS		
		10%	5%	0%	10%	5%	0%
grubhub-11-7	23	3.74	3.74	3.74	2.22×10^{-4}	2.78×10^{-4}	4.06×10^{-4}
grubhub-11-8	23	5.63×10^{-1}	5.63×10^{-1}	5.63×10^{-1}	1.92×10^{-4}	1.92×10^{-4}	9.87×10^{-4}
grubhub-11-9	23	1.35	2.48×10^1	4.42×10^2	2.25×10^{-4}	3.02×10^{-4}	5.95×10^{-4}
grubhub-12-0	25	4.05	4.05	4.89	2.35×10^{-4}	2.52×10^{-4}	4.43×10^{-4}
grubhub-12-1	25	3.51×10^{-1}	4.62×10^{-1}	4.62×10^{-1}	1.84×10^{-4}	1.92×10^{-4}	2.67×10^{-4}
grubhub-12-2	25	5.50×10^{-2}	1.06×10^{-1}	3.11×10^{-1}	2.14×10^{-4}	2.14×10^{-4}	2.21×10^{-3}
grubhub-12-3	25	9.15×10^{-1}	1.04	1.25	2.18×10^{-4}	2.83×10^{-4}	5.13×10^{-4}
grubhub-12-4	25	1.07×10^{-1}	1.43×10^{-1}	1.46×10^{-1}	2.10×10^{-4}	2.10×10^{-4}	2.37×10^{-4}
grubhub-12-5	25	6.75×10^{-1}	6.75×10^{-1}	6.94×10^{-1}	2.61×10^{-4}	2.61×10^{-4}	3.84×10^{-4}
grubhub-12-6	25	2.28×10^{-2}	5.56×10^{-2}	3.03×10^{-1}	2.26×10^{-4}	2.44×10^{-4}	6.41×10^{-4}
grubhub-12-7	25	2.60×10^{-2}	1.07×10^{-1}	3.17	2.28×10^{-4}	2.50×10^{-4}	9.57×10^{-4}
grubhub-12-8	25	1.88×10^{-1}	5.17	5.70	2.34×10^{-4}	2.91×10^{-4}	2.89×10^{-3}
grubhub-12-9	25	9.39×10^{-1}	1.05	4.04	2.43×10^{-4}	2.81×10^{-4}	6.05×10^{-4}
grubhub-13-0	27	2.70×10^{-2}	5.88×10^{-1}	7.45	2.34×10^{-4}	2.34×10^{-4}	1.53×10^{-3}
grubhub-13-1	27	1.22×10^{-1}	3.29	3.48	2.28×10^{-4}	3.01×10^{-4}	1.72×10^{-3}
grubhub-13-2	27	1.88×10^1	8.39×10^1	1.41×10^2	3.03×10^{-4}	4.76×10^{-4}	1.42×10^{-3}
grubhub-13-3	27	4.79×10^{-1}	4.79×10^{-1}	2.35	2.60×10^{-4}	2.95×10^{-4}	1.29×10^{-3}
grubhub-13-4	27	1.55×10^{-2}	1.55×10^{-1}	1.89×10^{-1}	2.51×10^{-4}	2.51×10^{-4}	2.51×10^{-4}
grubhub-13-5	27	1.21×10^{-1}	1.41×10^{-1}	2.60×10^{-1}	2.51×10^{-4}	2.51×10^{-4}	3.24×10^{-4}
grubhub-13-6	27	1.69×10^{-1}	1.69×10^{-1}	1.69×10^{-1}	2.37×10^{-4}	2.37×10^{-4}	2.67×10^{-4}
grubhub-13-7	27	7.36×10^{-2}	1.05×10^1	1.59×10^1	2.46×10^{-4}	2.46×10^{-4}	7.48×10^{-4}
grubhub-13-8	27	1.64×10^{-2}	9.07×10^{-2}	2.77×10^1	2.43×10^{-4}	3.09×10^{-4}	6.12×10^{-4}
grubhub-13-9	27	7.23	7.23	1.08×10^1	2.81×10^{-4}	3.30×10^{-4}	3.93×10^{-3}
grubhub-14-0	29	2.38×10^1	4.80×10^1	1.66×10^2	2.48×10^{-4}	2.48×10^{-4}	9.82×10^{-3}
grubhub-14-1	29	3.67×10^{-1}	3.67×10^{-1}	4.61×10^{-1}	2.34×10^{-4}	2.82×10^{-4}	3.18×10^{-4}
grubhub-14-2	29	2.93×10^{-2}	4.60×10^{-1}	1.90	2.67×10^{-4}	2.67×10^{-4}	1.07×10^{-3}
grubhub-14-3	29	8.20	8.20	2.45×10^1	3.21×10^{-4}	3.21×10^{-4}	8.04×10^{-4}
grubhub-14-4	29	3.77×10^{-1}	3.77×10^{-1}	8.47	2.87×10^{-4}	3.21×10^{-4}	7.04×10^{-4}
grubhub-14-5	29	3.54×10^{-2}	1.15×10^{-1}	5.38×10^{-1}	2.90×10^{-4}	3.36×10^{-4}	3.99×10^{-4}
grubhub-14-6	29	7.45×10^{-1}	5.83	5.83	2.62×10^{-4}	2.62×10^{-4}	4.60×10^{-4}
grubhub-14-7	29	2.12×10^{-1}	9.33×10^{-1}	9.94×10^{-1}	2.87×10^{-4}	3.30×10^{-4}	1.67×10^{-3}
grubhub-14-8	29	1.16×10^{-1}	3.38×10^{-1}	4.62×10^{-1}	2.87×10^{-4}	2.87×10^{-4}	4.82×10^{-4}
grubhub-14-9	29	1.45×10^{-1}	7.76×10^1	2.13×10^2	3.03×10^{-4}	3.57×10^{-4}	4.44×10^{-3}
grubhub-15-0	31	6.58	6.58	6.59	2.74×10^{-4}	2.74×10^{-4}	5.30×10^{-4}
grubhub-15-1	31	3.79×10^2	3.79×10^2	7.38×10^2	3.23×10^{-4}	5.31×10^{-4}	2.75×10^{-3}
grubhub-15-2	31	1.40×10^2	1.40×10^2	2.23×10^2	3.45×10^{-4}	3.45×10^{-4}	2.98×10^{-3}
grubhub-15-3	31	1.10×10^2	1.44×10^2	4.33×10^2	4.32×10^{-4}	4.32×10^{-4}	9.06×10^{-4}
grubhub-15-4	31	1.26×10^1	2.40×10^1	2.40×10^1	2.83×10^{-4}	2.83×10^{-4}	7.58×10^{-4}
grubhub-15-5	31	1.88×10^{-2}	1.83×10^{-1}	4.34×10^1	3.03×10^{-4}	3.03×10^{-4}	4.30×10^{-3}
grubhub-15-6	31	1.81×10^{-1}	1.20	8.69	3.15×10^{-4}	3.32×10^{-4}	2.08×10^{-2}
grubhub-15-7	31	2.21×10^1	3.33×10^1	4.24×10^2	3.13×10^{-4}	4.45×10^{-4}	8.28×10^{-4}
grubhub-15-8	31	4.31	8.60	2.17×10^1	2.71×10^{-4}	3.05×10^{-4}	5.39×10^{-4}
grubhub-15-9	31	2.21×10^1	2.77×10^1	2.02×10^2	2.92×10^{-4}	2.92×10^{-4}	4.14×10^{-3}
Average		5.52	7.64	2.36×10^1	1.73×10^{-4}	1.88×10^{-4}	7.83×10^{-4}