Sensitivity Analysis for Stochastic User Equilibrium Network Flows : A Dual Approach

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Abstract

Recently extensive studies have been conducted on the sensitivity analysis for the Wardropian equilibrium modeling of traffic networks. Here we present a method for sensitivity analysis for network flows at stochastic user equilibrium. Our method is developed from a dual formulation of the stochastic user equilibrium analysis. By adopting Dial's algorithm for stochastic traffic assignment, we are able to formulate a computationally efficient link-based algorithm for the sensitivity analysis. Since the Wardropian equilibrium in a traffic network is an extreme case of stochastic user equilibrium with $\theta \to \infty$, θ being a dispersion parameter in the expected utility function for stochastic route choice, the method presented here can also be used for the sensitivity analysis for the Wardropian equilibrium by taking θ large enough.

Introduction

Sensitivity analysis for traffic network equilibrium problems is important for two reasons. The first is that many parameters in functions (such as in link performance functions) underlying a mathematical model for equilibrium analysis are likely incurring some uncertainty, thus sensitivity analysis is important for validation of the mathematical model. The second is that sensitivity analysis can be applied to a variety of optimal design and control problems in traffic networks. See, for example, the works of Kim and Suh, (1990), Yang et al. (1994), Yang and Lam (1996), Yang (1997), Yang and Bell (1997), for such applications.

To our knowledge, the research works up to now concerning the method and application of sensitivity analysis are all for the Wardropian equilibrium or user equilibrium, at which the travel time on all used paths is equal for each origin destination (OD) pair, and also less than or equal to the travel time that would be experienced by a single vehicle on any unused path (Sheffi, 1985).

A well known method for sensitivity analysis for traffic network user equilibrium was developed by Tobin and Friesz (1988). Some method (e.g., Qiu and Magnanti, 1989) of sensitivity analysis for general variational inequality can also be used for Wardropian equilibrium model of traffic networks. Sensitivity analysis for some extension of the Wardropian equilibrium model, for instance a model with elastic travel demand, has also been developed and applied for solving congestion pricing and network design problems (Yang, 1997).

Since the stochastic user equilibrium (SUE), at which no motorist can improve his or her stochastically "perceived" travel time by unilaterally changing routes, is a more general and to some extent more realistic model for traffic network analysis (Sheffi, 1985), sensitivity analysis for the stochastic user equilibrium model is also an important problem.

In this paper we present a method for sensitivity analysis for the stochastic user equilibrium model of traffic networks. The main problem of sensitivity analysis here is the computation of the derivatives of link costs and flows with respect to some uncertainty parameters in link cost functions and in OD demands. Our method is developed from a dual formulation of the stochastic user equilibrium analysis, which was first presented by Daganzo (1982).

By adopting Dial's algorithm (Dial, 1971) for stochastic traffic assignment, we are able to formulate a computationally efficient link-based algorithm for the sensitivity analysis.

We note that a ''path choice entropy decomposition'' technique has been recently developed by Akamatsu (1997), which can be used for efficient computation of expected minimum cost for a stochastic diver. Our work has been much stimulated by Akamatsu's result.

In this paper the stochastic user equilibrium will be formulated based on multinominal logit-based model (Sheffi, 1985).

In the next section we briefly review this logit-based model and its dual mathematical programming formulation. In Section 2 the sensitivity analysis method is formulated.

A very simple procedure based on Dial's algorithm will be provided for computing the derivatives of link costs and flows with respect to some uncertainty parameters in link cost functions and in OD demands. A reader is refered to Dial (1971), Sheffi (1985) or Akamatsu (1997) for exposition of Dial's algorithm.

Two numerical examples are provided in Section 3 for demonstrating the correctness and implementability of our method. Section 4 gives a simple explanation regarding the simplicity of our method.

1 Stochastic User Equilibrium and the Dual Mathematical Program Formulation

A list of notations used in this paper are as follows.

notations:

- $\mathrm{N}=\{i,j,\cdots\}$: set of nodes
- $\mathrm{A}=\{ij,\cdots\}$: set of links
- $\mathrm{w}=\{rs,\cdots\}$: set of OD pairs
- $q_{rs}(\gamma_{rs})$: OD demand, $rs \in W$, where γ_{rs} is an uncertainty parameter
- $\mathrm{q}=(q_{rs})_{rs\in W}$ and $\gamma=(\gamma_{rs})_{rs\in W}$ denote the vectors of all OD demands and their uncetainty parameters, respectively
- $\mathrm{R}_{rs}=\{\mathrm{k,p},\cdots\}$: set of paths connecting rs
- h_k^{rs} : flow on path k with origin r and destination s
- P_k^{rs} : probability that a traveler from r to s chooses path k
- P_{ij}^{rs} : probability that a traveler from r to s traces link ij
- X_{ij} : link flow, for ij $\in$ A
- $\mathrm{t}_{ij}(\mathrm{X}_{ij},\epsilon_{ij})$: differentiable cost function of link ij with respect to flow X_{IJ}, and parameter ϵ_{ij}.

  It is assumed that t_{ij} is strongly monotone with respect to X_{ij}.

  For given ϵ_{ij}, the inverse of the cost function is denoted as $\mathrm{X}_{ij}(\mathrm{t}_{ij},\ \epsilon_{ij})$, which is also strongly monotone in t_{ij}
- $(\mathrm{X}_{ij})_{tij}$: partial derivative of X_{ij} with respect to t_{ij}
- $(\mathrm{X}_{ij})_{\epsilon ij}$: partial derivative of X_{ij} with respect to ϵ_{ij}
- $\mathrm{x}=(\mathrm{X}_{ij})_{ij\in A}$ · $\mathrm{t}=(\mathrm{t}_{ij})_{ij\in A}$ and $\epsilon=(\epsilon_{ij})_{ij\in A}$ denote the vectors of all link flows,

link costs and uncertainty parameters, respectively

- $\delta^{rs}_{ij,k} = \begin{cases} 1 & \text{if ij is a link on path k;} \\ 0 & \text{otherwise.} \end{cases}$

- $\delta^{rs}_{ij,gh} = \begin{cases} 1 & \text{if } ij = gh \in A; \\ 0 & \text{otherwise.} \end{cases}$

- $C^{rs}_k = \Sigma_{ij \in A} t_{ij} \delta^{rs}_{ij,k}$: the total cost of traveling on a path $k \in R_{rs}$
- θ : a dispersion parameter in SUE
- For simplicity, summation notations $\Sigma_{k \in R_{rs}}$, $\Sigma_{p \in R_{rs}}$, $\Sigma_{rs \in W}$ will be abbreviated as Σ_k, Σ_p, Σ_{rs}, respectively.

In a multinominal logit-based stochastic user equilibrium (SUE), the "expected utility" of traveling on path $k \in R_{rs}$ is given by $U^{rs}_k = -\theta c^{rs}_k$, where θ is a unit scaling parameter, see, e.g., Chapter 10 of Sheffi (1985). For a traveler on OD pair $rs \in W$, the probability P^{rs}_k at which the path k is chosen is given by

$$P^{rs}_k = \frac{\exp(-\theta c^{rs}_k)}{\Sigma_p \exp(-\theta c^{rs}_p)}, \ k \in R_{rs}. \tag{1}$$

At stochastic user equilibrium, the path flows are

$$h^{rs}_k = q_{rs} \frac{\exp(-\theta c^{rs}_k)}{\Sigma_p \exp(-\theta c^{rs}_p)}, \ k \in R_{rs}. \tag{2}$$

θ can be understood as a dispersion parameter indicating how precisely a driver can correctly choose the shortest routes; the higher the θ, the higher the probability that a driver chooses shortest routes.

From (2) it can be derived that the Wardropian equilibrium is a special case of SUE when we take $\theta \rightarrow \infty$.

The link flows are

$$X_{ij} = \sum_{rs}\sum_{k} h^{rs}_k \delta^{rs}_{ij,k} = \sum_{rs} q_{rs} \frac{\Sigma_k \exp(-\theta C^{rs}_k)\ \theta\ \delta^{rs}_{ij,k}}{\Sigma_p \exp(-\theta C^{rs}_p)}, \ ij \in A. \tag{3}$$

As was shown by Daganzo (1982) (p. 346, the Extremal Equivalence Theorem), the stochastic user equilibrium is achieved if and only if $t=(t_{ij})_{ij \in A}$, is a minimizing point of the function Z,

$$Z(t,\epsilon) = \sum_{ij} \int_{tij(0,\epsilon_{ij})}^{tij} x_{ij}(\nu,\epsilon_{ij}) d\nu - \sum_{rs} q_{rs} S_{rs}(c^{rs}(t)), \tag{4}$$

where

$$S_{rs}(C^{rs}(t)) = -\frac{1}{\theta}\ln\sum_k \exp(-\theta c_k^{rs}) \tag{5}$$

is the expected minimum cost perceived by a traveler from r to s.

In fact, the minimizing condition for this unconstrained program is as follows

$$\begin{aligned}
&\frac{\partial Z}{\partial t_{ij}} \\
&= x_{ij}(t_{ij}, \epsilon_{ij}) - \sum_{rs} q_{rs}\sum_k \frac{\partial S_{rs}}{\partial c_k^{rs}} \frac{\partial c_k^{rs}}{\partial t_{ij}} \\
&= x_{ij}(t_{ij}, \epsilon_{ij}) - \sum_{rs} q_{rs}\sum_k \frac{\partial S_{rs}}{\partial c_k^{rs}} \delta_{ij,k}^{rs} \\
&= x_{ij}(t_{ij}, \epsilon_{ij}) - \sum_{rs} q_{rs} \frac{\sum_k \exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}}{\sum_k \exp(-\theta c_p^{rs})} \\
&= 0,\ ij \in A.
\end{aligned} \tag{6}$$

It implies that

$$= x_{ij}(t_{ij}, \epsilon_{ij}) - \sum_{rs} q_{rs} \frac{\sum_k \exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}}{\sum_k \exp(-\theta c_p^{rs})},\ ij \in A,$$

which are the link flows in a stochastic user equilibrium state.

In a compact vector expression, (6) is rewritten as

$$\nabla_t Z = 0. \tag{7}$$

It can be shown that the Hessian of Z

$$\nabla_t^2 Z = \left(\frac{\partial^2 Z}{\partial t_{ij}\partial t_{gh}}\right)_{ij,gh} \tag{8}$$

is a positive definite matrix. This implies that Z is convex and the minimum point is unique. In fact, it is trivial to show that the Hessian of the first term of Z

$$\nabla_T^2 \left(\sum_{ij} \int_{t_{ij}(0,\in IJ)}^{t_{ij}} x_{ij}(\upsilon, \in_{ij})d\upsilon\right) = \text{diag}\left((x_{ij})_{tij}\right)_{ij} \tag{9}$$

is positive definite, since each diagonal entry is positive from the assumption that x_{ij} is strongly monotone in t_{ij}. Where diag( ) denotes a diagonal matrix with corresponding diagonal entries. It is well known (see, e.g., p. 278, Sheffi, 1985) that $s_{rs}(c^{rs})$ is concave with respect to c^{rs}. As c^{rs} is a vector with components which are linear conbinations of t_{ij}, it is thus shown that the function $-\sum_{rs} q_{rs} s_{rs}(c^{rs}(t))$ is convex with respect to t_{ij},or equivalently, its Hessian with respect to t_{ij} is semi-positive definite. It then follows that the Hessian of Z

$$\nabla_t^2 Z = \mathrm{diag}(((X_{ij})_{t_{ij}})_{ij} + \nabla_t^2 (-\sum_{rs} q_{rs} S_{rs}(c^{rs}(t))) \tag{10}$$

is a positive definite matrix.

2 Sensitivity Analysis of Stochastic User Equilibrium of Traffic Nerworks

The problem of sensitivity analysis treated here is the computation of the changes of costs t_{ij} and flows x_{ij} caused by small perturbation of ϵ_{ij} and γ_{rs}, or more precisely, the corresponding partial derivatives, at an equilibrium state. For simplicitity of description, we shall first consider in Section 2.1 the case $\gamma_{rs} \equiv 0$, and the problem is thus the one for computing $\frac{\partial t_{ij}}{\partial \epsilon_{ij}}$ and $\frac{\partial t_{ij}}{\partial \epsilon_{ij}}$, with q_{rs} being constants. A method for computing $\frac{\partial t_{ij}}{\partial \gamma_{rs}}$ and $\frac{\partial x_{ij}}{\partial \gamma_{rs}}$ will be provided in Section 2.2.

Note: In this paper we make a different use of the notations $\frac{\partial x_{ij}}{\partial x_{gh}}$ and $(x_{ij})_{\epsilon gh}$ as follows. While $(x_{ij})_{\epsilon gh}$ denotes the derivative of x_{ij} as an explicit function in ϵ_{gh} and thus $(x_{ij})_{\epsilon gh} = 0$ if $ij \neq gh$ (we refer to such derivative as an "apparent" derivative in the following); $\frac{\partial t_{ij}}{\partial \epsilon_{ij}}$ denotes the "true" partial derivative which, by definition, is the ratio of change of x_{ij} with respect to an infinitesimal change of ϵ_{gh} in the manner governed by the equilibrium conditions, therefore in general $\frac{\partial x_{ij}}{\partial \epsilon_{ij}} \neq 0$ even if $ij \neq gh$.

The difference between the notations $\frac{\partial t_{ij}}{\partial \epsilon_{ij}}$ and $(t_{ij})_{\epsilon gh}$ is similar.

2.1 Computing $\frac{\partial t_{ij}}{\partial \epsilon_{ij}}$ and $\frac{\partial x_{ij}}{\partial \epsilon_{ij}}$

The start point of our approach is to take

$$\frac{\partial Z}{\partial t_{ij}} = x_{ij(t_{ij}\epsilon_{ij})} - \sum_{rs} q_{rs} \frac{\Sigma_k exp(-\theta c_k^{rs}) \delta_{ij,k}^{rs}}{\Sigma_p exp(-\theta c_p^{rs})}$$

as functions with arguments t_{ij} and ϵ_{ij}, $ij \in$ A. The equations $\frac{\partial Z}{\partial t_{ij}} = 0$, $ij \in A$ (6) define t_{ij} as implicit functions in variables ϵ_{ij}.

The derivatives of t_{ij} and x_{ij} with respect to ϵ_{ij} are computed as follows. At the equilibrium, from (7) we have

$$\nabla_\epsilon(\nabla_t Z) = 0, \text{ or equivalently,}$$

$$(\nabla_t^2 Z)\frac{\partial t}{\partial \epsilon} + (\nabla_t Z)_x (x)_\epsilon = 0, \tag{11}$$

where

$$\frac{\partial t}{\partial \epsilon} = \left(\frac{\partial t_{ij}}{\partial \epsilon_{gh}}\right)_{ij,gh}$$

is the matrix of partial derivatives of t_{ij} with respect to ϵ_{gh} ;

$$(\nabla_t Z)_x = \left((\frac{\partial Z}{\partial t_{ij}})_{x_{gh}} \right)_{ij,gh} = I$$

is the matrix of the apparent partial derivatives of $\frac{\partial Z}{\partial t_{ij}}$ with respect to x_{gh} , where x_{gh} are explicit variables appearing in $\frac{\partial Z}{\partial t_{ij}}$,which is actually a unit matrix; and

$$(x)_\epsilon = ((x_{ij})_{\epsilon gh})_{ij,gh} = \text{diag}\ ((x_{ij})_{\epsilon gh})_{ij}$$

is the matrix of apparent partial derivatives of x_{ij} as an explicit function with respect to $\epsilon_{gh}, ij, gh \in A$.

From (11) it can be derived that

$$\begin{aligned}(\frac{\partial t}{\partial \epsilon}) = (\frac{\partial t_{ij}}{\partial \epsilon_{gh}})_{ij,gh} &= -(\nabla_t^2 Z)^{-1}(\nabla_t Z)_x (x)_\epsilon \\ &= -(\frac{\partial^2 Z}{\partial t_{ij} \partial t_{gh}})^{-1}_{ij,gh}\ \text{diag}\ ((x_{ij})_{\epsilon ij})_{ij}.\end{aligned} \tag{12}$$

Now for each $ij \in A$, x_{ij} is a function in a free variable ϵ_{ij} and an intermediate variable t_{ij} dependant on all ϵ_{gh}, $gh \in A$, thus we have

$$(\frac{\partial t}{\partial \epsilon}) = (\frac{\partial t_{ij}}{\partial \epsilon_{gh}})_{ij,gh} = \left((x_{ij})_{\epsilon_{ij}} \delta_{ij,gh} + (x_{ij})_{tij} \frac{\partial t_{ij}}{\partial \epsilon_{gh}} \right)_{ij,gh}. \tag{13}$$

The terms $(x_{ij})_{tij}, (x_{ij})_{\epsilon ij}$ are directly computable from the explicit cost functions $t_{ij}(x_{ij}, \epsilon_{ij})$.. For example, if

$$t_{ij}(x_{ij}, \epsilon_{ij}). = \alpha_{ij} + (\beta_{ij} + \epsilon_{ij}) x_{ij}^4,$$

a function equivalent to the BPR (the US Berea of Public Roads, see, e.g., Sheffi, 1985) link cost function with uncertainty parameter ϵ_{ij}, then

$$(x_{ij})_{tij} = \frac{1}{4\beta_{ij} x_{ij}^3}, \text{ at } \epsilon_{ij} = 0,$$

and

$$(x_{ij})_{tij} = \frac{-x_{ij}}{4\beta_{ij}}, \text{ at } \epsilon_{ij} = 0,$$

The part in the computation that appears to be difficult is that for computing $\frac{\partial^2 Z}{\partial t_{ij} \partial t_{gh}}$, which however can be efficiently computed by the following method based on Dial's traffic assignment algorithm.

Let us expend $\frac{\partial^2 Z}{\partial t_{ij} \partial t_{gh}}$ as

$$\frac{\partial^2 Z}{\partial t_{ij} \partial t_{gh}}$$

$$= (x_{ij})_{tij}\delta_{ij,gh} - \Sigma_{rs}q_{rs}\Big[\frac{-\theta\Sigma_k exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}\delta_{gh,k}^{rs}}{\Sigma_p exp(-\theta c^{rs})}$$

$$-\frac{-\theta\Sigma_k exp(-\theta c_k^{rs})\delta_{ij,k}^{rs})(\Sigma_l exp(-\theta c_l^{rs})\delta_{gh,l}^{rs})}{\Sigma_p exp(-\theta c_p^{rs}))^2}\Big] \quad (14)$$

By applying Dial's algorithm, Akamatsu (1997) observed that

$$\Sigma_p exp(-\theta c_p^{rs}) = exp(-\theta S_{rs})$$

can be computed without enumerating all the paths for an OD pair rs. Triggered by this observation, we worked out that

$$(\text{i})\ \frac{\Sigma_k exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}}{\Sigma_k exp(-\theta c_k^{rs})}$$

and

$$(\text{ii})\ \frac{\Sigma_k exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}\delta_{gh,k}^{rs}}{\Sigma_p exp(-\theta c_k^{rs})}$$

can also be computed in a very simple manner by using Dial's algorithm. And therefore $\frac{\partial^2 Z}{\partial t_{ij}\partial t_{gh}}$ can be efficiently computed.

To compute the formula in (i), note that at a stochastic user equilibrium state the fraction of the number of travelers from r to s who use link ij *is*

$$\frac{x_{ij}^{rs}}{q_{rs}} = \frac{\Sigma_k exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}}{\Sigma_k exp(-\theta c_p^{rs})} \quad (15)$$

This can be trivially derived from the formulas $\frac{x_k^{rs}}{q_{rs}} = \frac{exp(-\theta c_k^{rs})}{\Sigma_p exp(-\theta c_p^{rs})}$ and $x_{ij}^{rs} = \Sigma_k x_k^{rs}\delta_{ij,k}^{rs}$. Hereafter we denote $\frac{x_{ij}^{rs}}{q_{rs}}$ by the symbol P_{ij}^{rs} :

$$P_{ij}^{rs} = \frac{x_{ij}^{rs}}{q_{rs}} = \frac{\Sigma_k exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}}{\Sigma_k exp(-\theta c_k^{rs})} \quad (16)$$

Assuming that all the link costs are fixed, x_{ij}^{rs} can be directly computed by running Dial's traffic assignment algorithm. In fact, for each link ij, in Dial's algorithm, x_{ij}^{rs} are computed for all the OD pairs rs in a link-based manner and the link flow is obtained as $x_{ij}^{rs} = \Sigma_{rs}x_{ij}^{rs}$. See, e.g., Dial (1971), Sheffi (1985) or Akamatsu (1997) for details. This implies that

$$\frac{\Sigma_k exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}}{\Sigma_k exp(-\theta c_k^{rs})} = P_{ij}^{rs}$$

can be efficiently computed in a link-based manner by Dial's algorithm.

For computing the terms in (ii), let x_{ij-gh}^{rs} denote the number of travelers from r to s who traces link ij prior to link gh later, and $x_{ij\bar{-}gh}^{rs}$ denote the number of travelers from r to s who traces link gh prior to link ij. Note that either x_{ij-gh} or $^{rs}x_{ij-ij}$ is zero if only efficient paths are accounted for in the Dial's algorithm. By an efficient or reasonable path is meant a path that does not backtack, see, e.g., p. 89, Dial (1971).

Adopting the notation defined by (16), P_{gh}^{js} is the fraction of the number of travelers from j to s who trace link gh. By viewing x_{ij}^{rs} as an OD demand from j to s, we have

$$\frac{x_{ij-gh}^{rs}}{x_{ij}^{rs}} = P_{gh}^{js} = \frac{\Sigma_k exp(-\theta c^{js})\delta_{gh,k}^{js}}{\Sigma_k exp(-\theta c^{js})} \qquad (17)$$

Suppose that I_{js} is a unit OD demand from j to s, by running Dial's algorithm, the marginal flow I_{gh}^{js} caused by this demand on link gh is obtained, which is equal to P_{gh}^{js}. Thus P_{gh}^{js} can be computed by Dial's algorithm. Note that

$$\frac{x_{uj-gh}^{rs}}{x_{ij}^{rs}} = P_{gh}^{js} = \frac{x_{ij-gh}^{rs}}{x_{ij}^{rs}} \qquad (18)$$

thus an P_{gh}^{js} need be calculated only if some $x_{ij}^{rs} \neq 0$, and it can be used for the assignment of other non zero x_{uj}^{us} on link gh. It is therefore concluded that $x_{ij-gh}^{rs} = x_{ij}^{rs}P_{gh}^{js}$ can be computed efficiently by Dial's algorithm. Similarly we can efficiently compute

$$x_{gh-ij}^{rs} = x_{gh}^{rs}P_{ij}^{rs}.$$

Let $x_{ij,gh} = x_{gh,ij}$ denote the number of travelers who use both link ij and link gh, without concideration of prionity then we have

$$x_{ij,gh}^{rs} = x_{gh,ij}^{rs} = \begin{cases} x_{ij-gh}^{rs} + x_{gh-ij}^{rs}, & ij \neq gh, \\ x_{ij}^{rs}, & ij = gh \end{cases} \qquad (19)$$

Now it is clear that

$$\frac{\Sigma_k exp(-\theta c_k^{js})\delta_{gh,k}^{rs}\delta_{gh,k}^{rs}}{\Sigma_k exp(-\theta c_k^{js})} = \frac{x_{ij,gh}^{rs}}{q_{rs}}. \qquad (20)$$

Therefore the left term can be efficiently computed.

To summarize, $\dfrac{\partial^2 Z}{\partial t_{ij}\partial t_{gh}}, ij, gh \in A$, can be computed in an efficient link-based manner by using Dial's traffic assignment algorithm. An outline of the computation procedure is as follows.

Step 1. Compute the SUE by the *method of successive average* (MSA) which uses repeatedly Dial's algorithm until the link flows converge, (see, e.g., Sheffi (1985), p.

324), and as a result obtain the link flows x_{ij}^{rs} and the link costs t_{ij}.

Step 2. Compute x_{ij}, $rs \in W$, $ij \in A$, by running Dial's algorithm once.

Step 3. For each node pair js, if there is some $x_{ij}^{rs} \neq 0$, then compute P_{gh}^{js} for all links gh; P_{gh}^{js} can be computed by running Dial's algorithm for assigning a virtual unit OD demand 1_{js} on link gh.

Step 4.

$$x_{ij-gh}^{rs} = x_{ij}^{rs} P_{gh}^{js},\ x_{gh-ij}^{rs} = x_{gh}^{rs} P_{ij}^{hs}$$

and

$$x_{ij,gh}^{rs} = x_{gh,ij}^{rs} = \begin{cases} x_{ij-gh}^{rs} + x_{gh-ij}^{rs}, ij \neq gh, \\ x_{ij,}^{rs} \qquad ij = gh \end{cases} \tag{19}$$

Step 5. Compute

$$\frac{\Sigma_k exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}}{\Sigma_k exp(-\theta c_k^{rs})} = \frac{x_{ij,}^{rs}}{q_{rs}}$$

and

$$\frac{\Sigma_k exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}\delta_{gh,k}^{rs}}{\Sigma_k exp(-\theta c_k^{rs})} = \frac{x_{ij,gh}^{rs}}{q_{rs}}$$

Step 6. Compute(14).

2.2 Computing $\frac{\partial t_{ij}}{\partial \gamma_{rs}}$ and $\frac{\partial x_{ij}}{\partial \gamma_{rs}}$

The same ideas and methods as exposited in Sectoin 2.1 can be used for computing $\frac{\partial t_{ij}}{\partial \gamma_{rs}}$ and $\frac{\partial x_{ij}}{\partial \gamma_{rs}}$. What to be modified is that the variables γ_{rs}, along with t_{ij}, will be taken as arguments of the functions

$$\frac{\partial Z}{\partial t_{ij}} = x_{ij}(t_{ij}, \epsilon_{ij}) - \Sigma_{rs} q_{rs}(\gamma_{rs}) \frac{\Sigma_k exp(-\theta c_k^{rs})\delta_{\epsilon ij,k}^{rs}}{\Sigma_p exp(-\theta c_p^{rs})}, \tag{21}$$

where ϵ_{ij} are now fixed constants. At the equilibrium, from (7) we have

$$\nabla_\gamma(\nabla_t Z) = 0, \text{ or equivalently}$$

$$(\nabla_t^2 Z)\left(\frac{\partial t}{\partial r}\right) + (\nabla_t Z)_q (q)_r = 0. \tag{22}$$

As it is clear from (21) that

$$(\nabla_t Z)_q = -\left(\frac{\Sigma_k exp(-\theta c_k^{rs})\delta_{\epsilon ij,k}^{rs}}{\Sigma_p exp(-\theta c_k^{rs})}\right)_{ij,rs}$$

$$= -\left(\frac{x_{ij}^{rs}}{q_{rs}}\right)_{ij,rs} = -\left(P_{ij}^{rs}\right)_{ij,rs},$$

we have

$$\left(\frac{\partial t}{\partial \gamma}\right) = \left(\frac{\partial t_{ij}}{\partial \gamma_{rs}}\right)_{ij,rs} = -(\nabla_t^2 Z)^{-1}(\nabla_t Z)_q(q)_\gamma$$

$$= \left(\frac{\partial^2 Z}{\partial t_{ij}\partial t_{gh}}\right)_{ij,gh}\left(P_{ij}^{rs}\right)_{ij,rs} \operatorname{diag}\left(\left(q_{rs}\right)_{\gamma rs}\right)_{rs}, \qquad (23)$$

and

$$\left(\frac{\partial t}{\partial \gamma}\right) = \left(\frac{\partial t_{ij}}{\partial \gamma_{rs}}\right)_{ij,rs} = \left((x_{ij})_{tij}\frac{\partial t_{ij}}{\partial \gamma_{rs}}\right)_{ij,rs} \qquad (24)$$

Since$(\nabla_t^2 Z)$ and $\left(P_{ij}^{rs}\right)_{ij,rs}$ can be computed by the algorithm developed in Section 2.1, assuming that $(q_{rs})_{\gamma rs}$ can be computed directly from given functions $q_{rs}(\gamma_{rs})$ (for instance, if $q_{rs}(\gamma_{rs}) = q_{rs}^0 + \gamma_{rs} q_{rs}^0$ being a constant, then $(q_{rs})_{\gamma rs} = 1$), $\left(\frac{\partial t}{\partial \gamma}\right)$ and $\left(\frac{\partial x}{\partial \gamma}\right)$ can be efficiently computed.

3 Numerical Examples

[**Example 1**] At first a network with 5 nodes and 6 links, as shown in Figure 1, is used for illustrating our method. This network is equivalent to the one used in Friesz and Tobin (1988), where nodes 4 and 5 do not exist. The reason for our modification is that in our algorithm a link should be uniquely characterized by a head and a tail node. This is also a basic requirement for implementing Dial's algorithm. See, Dial (1971) or Sheffi (1985). The link cost functions are of the form

$$t_{ij} = (\alpha_{ij} + \epsilon_{ij}^{(1)}) + (\beta_{ij} + \epsilon_{ij}^{(2)})x_{ij}^4 \qquad (25)$$

where ϵ_{ij} and ϵ_{ij} are distinct uncertainty parameters in the cost functions. Though in Section 2 we only considered explicitly one uncertainty variable (ϵ_{ij}) in a cost function, that formulation does not lose generality because when we compute the derivative with respect to one uncertainty variable, the rest are fixed to be 0. The values of the parameters of the functions are

$$\alpha_{12} = 4, \quad \alpha_{14} = \alpha_{42} = 10, \quad \alpha_{23} = 1, \quad \alpha_{25} = \alpha_{52} = 15;$$

$$\beta_{12} = 1, \quad \beta_{14} = \beta_{42} = 2.5, \quad \beta_{23} = 30, \quad \beta_{25} = \beta_{52} = 0.5.$$

The single traffic demand is

$$q_{13} = 10 + \gamma_{13}.$$

With dispersion parameter $\theta = 0.001$, the link flows at the stochastic equilibrium are $x_{12} = 5.8442$, $x_{14} = x_{42} = 4.1558$, $x_{23} = 3.1598$, $x_{53} = 6.8402$.

The Hessian matrix is

$$\nabla_t^2 Z = \begin{bmatrix} 0.0037 & -0.0024 & 0.0000 & 0.0000 & -0.0024 & 0.0000 \\ -0.0024 & 0.0038 & 0.0000 & 0.0000 & 0.0024 & 0.0000 \\ 0.0000 & 0.0000 & 0.0024 & -0.0022 & 0.0000 & -0.0022 \\ 0.0000 & 0.0000 & -0.0022 & 0.0037 & 0.0000 & 0.0022 \\ -0.0024 & 0.0024 & 0.0000 & 0.0000 & 0.0038 & 0.0000 \\ 0.0000 & 0.0000 & -0.0022 & 0.0022 & 0.0000 & 0.0037 \end{bmatrix}.$$

The derivatives of the first four distinct link flows $x_{12}, x_{14}, x_{23}, x_{25}$ with respect to $\epsilon_{12}^{(1)}$, $\epsilon_{12}^{(2)}$ and γ_{13} are

$$\begin{bmatrix} \frac{\partial x_{12}}{\partial \epsilon_{12}^{(1)}} & \frac{\partial x_{12}}{\partial \epsilon_{12}^{(2)}} & \frac{\partial x_{12}}{\partial \gamma_{13}} \\ \frac{\partial x_{14}}{\partial \epsilon_{12}^{(1)}} & \frac{\partial x_{14}}{\partial \epsilon_{12}^{(2)}} & \frac{\partial x_{14}}{\partial \gamma_{13}} \\ \frac{\partial x_{23}}{\partial \epsilon_{12}^{(1)}} & \frac{\partial x_{23}}{\partial \epsilon_{12}^{(2)}} & \frac{\partial x_{23}}{\partial \gamma_{13}} \\ \frac{\partial x_{25}}{\partial \epsilon_{12}^{(1)}} & \frac{\partial x_{25}}{\partial \epsilon_{12}^{(2)}} & \frac{\partial x_{25}}{\partial \gamma_{13}} \end{bmatrix} = \begin{bmatrix} -0.0004 & -0.4409 & 0.6335 \\ 0.0004 & 0.4409 & 0.3665 \\ 0 & 0 & 0.258 \\ 0 & 0 & 0.742 \end{bmatrix}.$$

Table I shows comparisons of estimated perturbed flows using a linear approximation based on the derivatives with the actual solutions re-computed by the method of successive average (MSA).

By taking $\theta = 1$, the link flows at the equilibrium are
$x_{12}=5.9998$, $x_{14}=x_{42}=4.002$, $x_{23}=3.0002$, $x_{25}=x_{53}=6.9998$.
And the derivatives are

$$\begin{bmatrix} \frac{\partial x_{12}}{\partial \epsilon_{12}^{(1)}} & \frac{\partial x_{12}}{\partial \epsilon_{12}^{(2)}} & \frac{\partial x_{12}}{\partial \gamma_{13}} \\ \frac{\partial x_{14}}{\partial \epsilon_{12}^{(1)}} & \frac{\partial x_{14}}{\partial \epsilon_{12}^{(2)}} & \frac{\partial x_{14}}{\partial \gamma_{13}} \\ \frac{\partial x_{23}}{\partial \epsilon_{12}^{(1)}} & \frac{\partial x_{23}}{\partial \epsilon_{12}^{(2)}} & \frac{\partial x_{23}}{\partial \gamma_{13}} \\ \frac{\partial x_{25}}{\partial \epsilon_{12}^{(1)}} & \frac{\partial x_{25}}{\partial \epsilon_{12}^{(2)}} & \frac{\partial x_{25}}{\partial \gamma_{13}} \end{bmatrix} = \begin{bmatrix} -0.0005 & -0.6041 & 0.5975 \\ 0.0005 & 0.6044 & 0.4032 \\ 0 & 0.0001 & 0.2977 \\ 0 & 0.0003 & 0.7031 \end{bmatrix}.$$

This matrix agrees with that computed by Tobin and Friesz for Wardropian equilibrium up to a maximum error 0.001 in the entries in the last column, see formula (69) in Tobin and Friesz (1988).

[Example 2] For verifying the correctness of our method in complicated networks, a network with 20 nodes and 35 links, as shown in Figure 2, is considered. The link cost functions are of the form

$$t_{ij} = (\alpha_{ij} + \epsilon_{ij}^{(1)})\left(1 + (\beta_{ij} + \epsilon_{ij}^{(2)})\left(\frac{x_i}{c_{ij}}\right)\right)$$

Table I : Perturbed flows corresponding to therr distinct parameter uncertainties, respectively, in Exmaple 1; $\theta = 0.001$

link	unperturbed	perturbed	with $\epsilon_{12}^{(1)} = 5$
(i-j)	flows	actual	estimated
(1-2)	5.8442	5.8423	5.8423
(1-4)	4.1558	4.1577	4.1577
(2-3)	3.1597	3.1597	3.1597
(2-5)	6.8403	6.8403	6.8403

link	unperturbed	perturbed	with $\epsilon_{12}^{(2)} = 0.4$
(i-j)	flows	actual	estimated
(1-2)	5.8442	5.6887	5.6678
(1-4)	4.1558	4.3113	4.3322
(2-3)	3.1597	3.1597	3.1597
(2-5)	6.8403	6.8403	6.8403

link	unperturbed	perturbed	with $\gamma_{13} = 0.4$
(i-j)	flows	actual	estimated
(1-2)	5.8442	6.0969	6.0976
(1-4)	4.1558	4.3031	4.3024
(2-3)	3.1597	3.264	3.2629
(2-5)	6.8403	7.136	7.1371

where c_{ij} is the "practical capacity" of link ij, see p. 358, Sheffi (1985). The unperturbed parameters are assumed as in Table II. Multiple OD demands are supposed to be as in Table III. The dispersion parameter is set as $\theta = 0.5$. Suppose the network incurs simultaneously the following perturbation of parameters

$$\epsilon_{1\,2}^{(1)} = 1, \ \epsilon_{10\,6}^{(1)} = 1, \ \epsilon_{14\,15}^{(1)} = 1;$$

$$\epsilon_{5\,6}^{(2)} = -0.03, \ \epsilon_{7\,11}^{(2)} = -0.03, \ \epsilon_{15\,16}^{(2)} = -0.04;$$

$$\gamma_{1\,20} = 30, \ \gamma_{9\,4} = 30,$$

(with the rest perturbation parameters being zero), a comparison of the linearly approximated perturbed flows estimated based on derivatives and the actual solutions recomputed by the MSA algorithm is shown in Table IV. The "actual change" term in the table equals the actual perturbed flow minus the unpertubed one, the "estimation error" equals the the actual perturbed flow minus the estimated one. The estimation errors are small relative to the flow changes.

Note that in this example the parameter perturbations are considered simultaneously. Good estimation results could also be obtained for flows perturbed by separate parameter uncertainty, which are not presented here for save of space.

Table II : Network parameters, in Exmaple 2.

link (i-j)	α_{ij}	C_{ij}	β_{ij}
1-2	20	1000	0.15
1-5	18	1500	0.15
2-3	23	500	0.15
2-6	19	500	0.15
3-4	17	500	0.15
3-7	16	500	0.15
4-8	22	500	0.15
5-6	14	1000	0.15
5-9	24	800	0.15
6-2	15	650	0.15
6-7	17	1000	0.15
6-10	20	500	0.15
7-3	18	750	0.15
7-8	13	1000	0.15
7-11	26	500	0.15
8-12	19	1000	0.15
9-10	7	800	0.15
9-13	20	800	0.15
10-6	16	700	0.15
10-11	18	800	0.15
10-14	14	700	0.15
11-7	15	600	0.15
11-12	17	800	0.15
11-15	30	1000	0.15
12-16	38	2000	0.15
13-14	15	500	0.15
13-17	14	600	0.15
14-15	20	700	0.15
14-18	30	1800	0.15
15-16	25	900	0.15
15-19	27	1700	0.15
16-20	10	500	0.15
17-18	9	500	0.15
18-19	20	950	0.15
19-20	16	1000	0.15

Table III : OD demands, in Exmaple 2.

$r \to s$	q_{rs}
$1 \to 12$	$1000 + \gamma_{1\ 12}$
$1 \to 13$	$800 + \gamma_{1\ 13}$
$1 \to 20$	$1500 + \gamma_{1\ 20}$
$2 \to 19$	$500 + \gamma_{2\ 19}$
$3 \to 12$	$450 + \gamma_{3\ 12}$
$4 \to 16$	$400 + \gamma_{4\ 16}$
$5 \to 3$	$300 + \gamma_{5\ 3}$
$6 \to 16$	$400 + \gamma_{6\ 16}$
$9 \to 4$	$800 + \gamma_{9\ 4}$
$10 \to 20$	$500 + \gamma_{10\ 12}$
$13 \to 19$	$400 + \gamma_{13\ 19}$

Table IV: Flows perturbed simultaneously by a set of parameter uncertainties, en Exmaple 2; θ =0.5.

link	unperturbed	perturbed flows		actual	estimation
(i-j)	flows	actual	estimated	change	error
(1-2)	766.4	755.6	756.0	-10.80	-0.41
(1-5)	2533.6	2574.4	2574.0	40.80	0.41
(2-3)	731.4	738.9	738.5	7.46	0.39
(2-6)	837.9	838.0	838.5	0.04	-0.50
(3-4)	835.2	851.6	851.1	16.39	0.41
(3-7)	843.3	846.1	846.4	2.77	-0.31
(4-8)	435.2	421.6	421.1	-13.61	0.41
(5-6)	1341.3	1389.3	1387.2	48.00	2.07
(5-9)	1492.4	1485.2	1486.8	-7.20	-1.67
(6-2)	303.0	321.3	321.0	18.30	0.29
(6-7)	1580.3	1600.8	1599.5	20.55	1.28
(6-10)	942.2	941.1	941.7	-1.14	-0.64
(7-3)	797.0	808.7	809.0	11.70	-0.29
(7-8)	1369.1	1377.1	1379.6	8.06	-2.41
(7-11)	811.2	855.1	850.8	43.88	4.31
(8-12)	1804.2	1798.7	1800.7	-5.54	-2.01
(9-10)	1126.7	1143.6	1144.8	16.90	-1.28
(9-13)	1165.7	1171.6	1172.0	5.90	-0.38
(10-6)	246.2	235.9	236.5	-10.32	-0.63
(10-11)	1138.3	1160.1	1161.2	21.77	-1.10
(10-14)	1184.3	1188.6	1188.8	4.31	-0.19
(11-7)	553.8	594.1	593.5	40.32	0.63
(11-12)	691.7	697.4	696.1	5.65	1.30
(11-15)	704.0	723.7	722.4	19.68	1.28
(12-16)	1046.0	1046.1	1046.8	0.11	-0.71
(13-14)	15.3	16.8	16.8	1.43	-0.04
(13-17)	750.4	754.8	755.2	4.47	-0.34
(14-15)	862.6	844.7	844.5	-17.92	0.24
(14-18)	337.0	360.7	361.2	23.65	-0.47
(15-16)	601.7	609.5	608.6	7.72	0.83
(15-19)	964.9	958.9	958.2	-5.96	0.69
(16-20)	847.7	855.5	855.4	7.83	0.12
(17-18)	750.4	754.8	755.2	4.47	-0.34
(18-19)	1087.4	1115.5	1116.3	28.13	-0.81
(19-20)	1152.3	1174.5	1174.6	22.17	-0.12

4 Discussion

One difficulty in the sensitivity analysis for Wardropian equilibrium is that path flows at the equilibrium are not unique, which can be overcome by taking some representative path flows at extreme points in the constraint sets, see Tobin and Friesz (1988). In the stochastic case, the sensitivity analysis formulation is relatively simple. The reason for this is that in the dual mathematical programming formulation for the stochastic model the flow constraints are not bounding, therefore no boundary conditions need to be considered. Since the Wardropian equilibrium in a traffic network is an extreme case of stochastic user equilibrium with dispersion parameter $\theta \rightarrow \infty$, the method presented here could also be used for the sensitivity analysis for the Wardropian equilibrium by taking θ large enough, as was illustrated by an example (Example 1) in Section 3. Some algorithmic details for implementing our method can be found in Ying and Migagi (2000)

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Figure 1 : Network pf Example 1 .

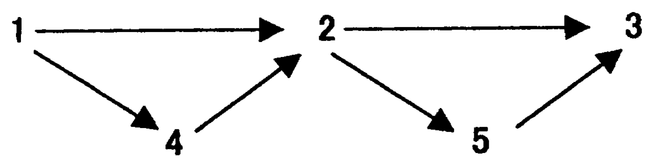

Figure 2 : Network pf Example 2 .