

UCLA

Recent Work

Title

Group Decisions With Multiple Criteria

Permalink

<https://escholarship.org/uc/item/41z1n2gn>

Authors

Baucells, Manel
Sarin, Rakesh K.

Publication Date

2002-01-09

Group Decisions with Multiple Criteria*

Manel Baucells[†] Rakesh K. Sarin[‡]

January 9, 2002

Abstract

We consider a decision problem where a group of individuals evaluates multi-attribute alternatives. We explore the minimal required agreements that are sufficient to specify the group utility function. A surprising result is that, under some conditions, a bilateral agreement among pairs of individuals on a single attribute is sufficient to derive the multi-attribute group utility. The bilateral agreement between a pair of individuals could be on the weight of an attribute, on an attribute evaluation function, or on willingness to pay.

We investigate cases in which each individual's utility function is either additive or multiplicative. In the additive case, the group utility can be represented as the weighted sum of group attribute weights and group attribute evaluation functions. In the multiplicative case, the group utility takes a more complex form.

1 Introduction

In this paper, we focus on group decisions where a group of individuals or a committee collectively shares the responsibility for choosing among alternative proposals for action. Individual or committee members may have different views on the relative merit of each proposal. Therefore, the problem boils down to how one should

*The authors are thankful to the seminar participants at Duke University, The Wharton School, INSEAD (Singapore), and Informs Hawaii for several helpful suggestions.

[†]IESE Business School and the Fuqua School of Business

[‡]The Anderson School at UCLA

aggregate the views/preferences of the committee members to arrive at a preferred decision.

Arrow (1963) and Sen (1970) have shown that, in general, there is no procedure for combining individual rankings into a group ranking without violating some rather reasonable assumptions. Interpersonal comparison of preferences is the Achilles Heel in escaping from the trap that Arrow's Impossibility Theorem has so eloquently laid out. In the spirit of Keeney and Raiffa (1976), we will assume that individuals are willing to perform interpersonal comparisons of utility or welfare.

Our contribution is to provide explicit ways to elicit the utility comparisons to derive the group utility. We show that it is possible to restrict these utility comparisons at a pair level. Our strategy is to seek compromise and agreement on a chosen parameter from the two individuals comprising the pair. Thus, for example, two individuals who attach different importance weights to an attribute of interest are asked to compromise and to come up with an agreement on their common weight for the attribute in question. We show that a bilateral agreement, when elicited in a systematic way, is sufficient to arrive at the joint utility for the pair of individuals. We will make it clear how some chosen bilateral agreements define the "pair utility" and how $(n - 1)$ pair utilities determine the group utility, where n is the number of individuals in the group. Therefore, at least in theory, there is never a need to put more than two individuals together at a time to derive the group utility from individual utilities. We use the typical divide and conquer strategy that is so often employed in decision analysis to decompose and simplify a complex problem. In simple terms, our approach can be described as follows.

Given the individual utility function u_1, \dots, u_n , a series of bilateral agreements not to exceed $(n - 1)$ in total, are used to derive the group utility function u_N . Given a particular set of alternatives, the alternative with the highest value of u_N (or the highest expected value of u_N) is chosen.

Our interpretation of individual utilities are individual preferences that satisfy the usual rationality requirements (von Neumann-Morgenstern utility). For organizational decisions, an individual may represent a department such as Marketing or R&D. The individual would presumably reflect the preferences of the department. Marketing, for example, may believe that an early entry to market (attribute: time to market entry) is more important, but R&D may believe that the features of the

introduced product (attribute: product features) is a more important attribute. The group utility (or company utility in this case) is derived by forming bilateral agreements that seek a compromise on the attribute weight. Thus, depending on the context of the decision, individual utility may reflect an impact on oneself, or an impact on the department or constituency that the individual represents. Keeney and Raiffa (1976) and Sen (1977) have noted that there is no single group decision problem; instead, there are several group decision problems each requiring a different interpretation of individual utilities and of the aggregation procedure that is used to derive the group utility. Our approach is consistent with these alternative group decision problems. We, however, require that individuals are able to specify their own utilities or the utilities of the constituency they represent precisely and that a given pair of individuals can reach an agreement on some chosen parameter (e.g., the weight of an attribute or the utility of an outcome). Technically, we have assumed that individual preferences as well as pair preferences are complete. The latter assumption implies that bilateral agreements can always be reached. Coalition ($n \geq 3$) or group preferences are not assumed to be complete; instead, these are *derived* to be complete.

Consider a simple example with $n = 3$. Suppose $u_i(x^0) = 0$, $u_i(x^*) = 1$, $i = 1$ to 3, where x^0 and x^* are respectively the least preferred and the most preferred consequences. For an intermediate consequence, x , let $u_1(x) = 0.25$ and $u_2(x) = 0.45$. To reach a bilateral agreement, Individuals 1 and 2 must agree on a “pair utility” for x in the range $(0.25, 0.45)$ that represents the pair preference, say $u_{12}(x) = 0.35$. The bilateral agreement between Individuals 1 and 2 means that they have jointly agreed that x is indifferent to a lottery that yields x^* with a 0.35 chance and x^0 with a 0.65 chance, even though their individual indifference probabilities were 0.25 and 0.45 respectively. We will show that once an agreement is reached on the utility of a single outcome, the complete agreement between Individuals 1 and 2 can be inferred regarding all other possible outcomes. For multidimensional outcomes, x , the indifference probability may be difficult to elicit. We will show that under some simplifying assumptions on the structure of utility function, a bilateral agreement on one parameter (e.g., weight of an attribute) may be sufficient to determine the joint utility for the pair of individuals.

Returning to our example, now suppose $u_3(x) = 0.35$. We now need either a

bilateral agreement between Individuals 1 and 3 or between Individuals 2 and 3, but not both. Let us suppose $u_{13}(x) = 0.3$. So for $n = 3$, we have two bilateral agreements: one between Individuals 1 and 2 and the other between Individuals 1 and 3. These two bilateral agreements are sufficient to derive the group utility. In our example, u_{123} has to be $\frac{1}{3}u_1 + \frac{1}{3}u_2 + \frac{1}{3}u_3$. The key assumption that permits us to derive the entire pair utility function from the bilateral agreement on the utility of a single consequence, and the group utility function from $(n - 1)$ pair utility functions, is the Extended Pareto Principle. Simply stated, the Extended Pareto Principle requires that if a coalition, A , prefers x to y and another disjoint coalition, B , also prefers x to y , then the coalition $A \cup B$ must prefer x to y . It is trivial to see that for $n = 2$ ($A = \{1\}, B = \{2\}$) the Extended Pareto Principle particularizes to the well known Pareto Principle that is the cornerstone of most social welfare theory. In Section 2, we summarize some key results that show how bilateral agreements among $(n - 1)$ pairs of individuals are enough to derive the group utility function. The theoretical support for the results in this section is based on Baucells and Shapley (2000).

In the remainder of this paper, we consider the multiattribute decision problem where the consequences or the outcomes of a decision are measured using multiple attributes. Examples include location of facilities, selection of a candidate and choice of a new product or process technology (for numerous applications see Keeney and Raiffa, 1976). In Section 3, we assume that the individual multiattribute utility functions are additive (i.e., they can be expressed as the weighted sum of single-attribute evaluation functions (attribute utilities)). In Subsection 3.1, we examine the special situation where all members of the committee agree on single-attribute evaluation functions, but differ in the weights given to each attribute (homogeneous attribute evaluation). In Subsection 3.2, we treat the general case where the individuals differ on both the single-attribute evaluation functions as well as on the weights (heterogeneous attribute evaluation). A surprising result is that individuals' weights and attribute utilities can be used to derive group weights for the attributes and group attribute utilities. Thus, group utility can be expressed as a weighted sum of group attribute utilities. These group attribute utilities are themselves weighted sums of the individual attribute utilities. Clearly, a series of bilateral agreements are needed to derive group utility from individual utilities. Interestingly, agreements regarding

all attribute weights and attribute utilities is not necessary. In fact, an agreement on the weight of any one chosen attribute, or on the trade-off between two attributes, may be sufficient to derive the multiattribute pair utility. For example, suppose time to market entry, product features, product performance, product reliability and cost are relevant attributes for a new product introduction decision. Marketing is willing to pay ten million dollars to accelerate market entry by six months; whereas R&D prefers that the additional resources be spent on enhancing product features and is willing to pay only five million dollars to facilitate early market entry. A bilateral agreement between the two departments may be to pay eight million dollars for an earlier introduction of the product. Given individual utilities of Marketing and R&D, the one bilateral agreement on trade-offs is sufficient to derive their joint utility. Consistency with the Extended Pareto Rule and the additive structure of the multiattribute utility ensures that the agreement on the trade-offs among the other remaining attributes (say cost and product features) cannot be arbitrary.

In Section 4, we consider the case where individual multiattribute utilities are multiplicative. In the homogeneous case (individuals agree on attribute utilities, but not on weights), the group utility takes the multiplicative form and the group multiplicative scaling constant is a weighted sum of individuals' multiplicative constants. In the heterogeneous case (individuals disagree on both attribute utilities and on weights), a group utility can be obtained through bilateral agreements. The structure of the group utility, however, takes a more complex form and does not simplify to an easy-to-interpret multiplicative form.

If a large number of individuals are involved in a group decision, pairwise comparisons may be too onerous. Section 5 shows how our framework can be extended to consider coalition agreements. Finally, in Section 6, we discuss some possible extensions of the present work. We begin by laying out the formal framework for our approach.

2 The Formal Framework

2.1 Notation

Consider a set of individuals $N = \{1, 2, \dots, n\}$, with $n \geq 2$, who must jointly choose from a set of alternatives. The outcome of a decision is evaluated on m attributes

and therefore an outcome $x = (x_1, \dots, x_m)$ is a point in the outcome space $X = X_1 \times X_2 \times \dots \times X_m$, where X_a , $a = 1$ to m , is the set of possible outcomes on the a^{th} attribute. We assume X is finite. Each individual i 's utility u_i is assumed to satisfy von Neumann-Morgenstern (1947) rationality requirements, $u_i : x \rightarrow \text{Re}$. The choice set \tilde{X} from which the individuals and the group choose a preferred course of action is the set of all probability distributions over X . We will use symbols x, y, z , etc. to denote both the deterministic outcomes as well as probability distributions over X . The expected value of u_i is used to determine individual i 's preference over probability distributions or lotteries over X . Though not necessary, we will often assume that there is the least preferred consequence, x^0 , and the most preferred consequence, x^* , for all individuals and we can therefore set $u_i(x^0) = 0$, $u_i(x^*) = 1$, and $i = 1$ to n .

It is clear that we have assumed that the preference relation \succsim_i for individual i is complete and satisfies von Neumann-Morgenstern axioms. A coalition S is a subset of N . We will assume that preferences of each coalition $S \subseteq N$ also satisfy von Neumann-Morgenstern axioms, except completeness ($x \succsim y$ or $y \succsim x$) is replaced by the much weaker condition of reflexivity ($x \succsim x$). Thus, a coalition will have a partial ordering or incomplete preferences over X . Incomplete preferences merely imply that the coalition is sometimes unable to express the direction of preference for certain pairs of alternatives. Though we do not assume from the outset that group preferences are complete, the assumption of completeness of individual and pair preferences together with the Extended Pareto Principle will ensure that group preferences are also complete. For a more thorough discussion and development of incomplete preferences see Aumann (1962; 1964), Dubra and Ok (1999), Sen (1970) and Baucells and Shapley (2000).

2.2 The Extended Pareto Principle

The Pareto Principle states that if each individual prefers x to y , then the group must prefer x to y . A natural extension of this principle is to require that if each disjoint coalition A and B prefers x to y , then the coalition $A \cup B$ must prefer x to y . Thus if $\{1, 2\}$ and $\{3, 4, 5\}$ each prefer x to y , then the combined subgroup $\{1, 2, 3, 4, 5\}$ must prefer x to y . We use the following definition of the Extended Pareto Rule (EPR).

A collection of preferences \succsim_S , $S \subseteq N$, satisfies the Extended Pareto Rule (EPR)

if for all disjoint coalitions A and B , and for all $x, y \in \tilde{X}$,

$$x \succsim_A y, x \succsim_B y \implies x \succsim_{A \cup B} y, \text{ and} \quad (1)$$

$$x \succ_A y, x \succsim_B y \implies x \succ_{A \cup B} y. \quad (2)$$

It is clear that for $n = 2$, the EPR reduces to exactly the well-known Pareto Rule. An implication of the EPR is that if we break a group into subgroups and if each subgroup prefers x to y , then the group as a whole should prefer x to y . We will use the EPR to derive group preferences from pair preferences.

2.3 Pair Preferences

Consider two individuals i and j who are endowed with utility functions u_i and u_j , respectively. The joint utility of the pair $\{i, j\}$, denoted u_{ij} , is the convex combination of u_i and u_j if the pair preference is complete and the Extended Pareto Rule holds. The following theorem, when applied to a pair of individuals, provides the desired result.

Theorem 1 *Let A and B be two disjoint coalitions with complete preferences. If the preference for the coalition $A \cup B$ is complete and the Extended Pareto Rule holds, then there exists $0 < \alpha < 1$ such that $\succsim_{A \cup B}$ is represented by $u_{A \cup B} = \alpha u_A + (1 - \alpha)u_B$.*

Proof. This is a well-known result after Harsanyi (1955)'s seminal work. A proof is given in Baucells and Shapley (2000). ■

An immediate implication of Theorem 1 is that $u_{ij} = \alpha_i^j u_i + (1 - \alpha_i^j)u_j$ for some $\alpha_i^j \in (0, 1)$. The parameter α_i^j is to be determined through a *bilateral agreement* between individuals i and j . We reiterate that a bilateral agreement requires that the two individuals are able to perform interpersonal comparisons of utility. We now present an example of how α_i^j can be elicited. Clearly, other procedures are possible.

Choose a consequence x so that $u_i(x) \neq u_j(x)$ and without loss of generality let $u_i(x) < u_j(x)$. The usual von Neumann-Morgenstern interpretation of utility implies that if individual i is indifferent between x for sure and the binary lottery yielding x^* (the best outcome) with a probability p_i and yielding x^0 (the worst outcome) with a probability $(1 - p_i)$, then $u_i(x) = p_i$. Similar interpretation holds for determining $u_j(x) = p_j$. To elicit $u_{ij}(x)$ through a bilateral agreement, find a probability

$p_{ij} \in (p_i, p_j)$ so that, as a pair, i and j are indifferent between x for sure and the lottery yielding x^* with a p_{ij} chance and x^0 with a $(1 - p_{ij})$ chance. A compromise between individuals i and j is needed to reach the common indifference probability, p_{ij} . We note that p_{ij} cannot take the extreme values of the interval because such an assignment ($p_{ij} = p_i$ or $p_{ij} = p_j$) would violate strong condition 2 of the Extended Pareto Rule. To derive α_i^j from p_{ij} , observe that

$$p_{ij} = u_{ij}(x) = \alpha_i^j u_i(x) + (1 - \alpha_i^j) u_j(x),$$

so that

$$\alpha_i^j = \frac{p_j - p_{ij}}{p_j - p_i}. \quad (3)$$

Once α_i^j is determined using (3), $u_{ij}(x)$, $x \in X$ is completely specified. A compromise or bilateral agreement on the utility of one chosen consequence, therefore, determines the entire utility function for the pair (i, j) . It may not be immediately clear why a compromise on the utility of one outcome predetermines the compromise on the utility of other outcomes. We show by means of an example that the Extended Pareto Principle will be violated if the compromise on the utility of another outcome y is not restricted in accord with the original compromise on the utility of the outcome x .

Consider a setting with four monetary consequences of \$0, \$200, \$500 and \$1,000. The individual utilities for these outcomes, normalized so that $u_{12}(0) = 0$ and $u_{12}(1000) = 1$, are given below.

outcome	\$0	\$200	\$500	\$1,000
u_1	0	0.20	0.50	1
u_2	0	0.40	0.75	1

Assume that Individuals 1 and 2 agree that $u_{12}(500) = 0.55$. Using (3) yields $\alpha_1^2 = 0.8$. Now, suppose we choose the outcome \$200 and seek a compromise to determine $u_{12}(200)$. Notice that in order to be consistent with the original compromise ($u_{12}(500) = 0.55$), $u_{12}(200)$ must be $0.24 = 0.8u_1(200) + 0.2u_2(200)$: any other agreement, say $u_{12}(200) = 0.32$, produces a violation of the Pareto Principle. To show this, we will construct two lotteries L_1 and L_2 such that $L_1 \succ_1 L_2$, $L_1 \succ_2 L_2$, but

$L_2 \succ_{12} L_1$ - a violation of the Extended Pareto Principle. The lotteries L_1 and L_2 and the corresponding expected utilities are given below.

	\$0	\$200	\$500	\$1,000		EU_1	EU_2	EU_{12}
L_1	40%	0%	60%	0%	L_1	0.30	0.45	0.33
L_2	13%	75%	0%	12%	L_2	0.27	0.42	0.36

Thus, in the above example both Individuals 1 and 2 prefer L_1 to L_2 . If, however, we choose $u_{12}(200) = 0.32$ and $u_{12}(500) = 0.55$, then the pair $\{1, 2\}$ prefers L_2 to L_1 . The violation of the Pareto Principle vanishes if $u_{12}(200)$ is chosen to be 0.24 as implied by the original agreement on $u_{12}(500)$.

In theory, agreement on the utility of *one* outcome is sufficient to completely specify the pair utility. In practice, however, one should seek compromises on additional points and revisit the agreements to ensure consistency. Further, the ranking of alternatives implied by the pair utility should be displayed. Such a ranking may make it vivid whether either individual has unknowingly given up too much in the compromise. If the individuals do not find the ranking satisfactory, the agreement point should be revisited.

So far, we have shown how the pair utility u_{ij} can be derived from a bilateral agreement. We now discuss how the utility of a three-person coalition can be obtained.

2.4 Three Individuals And “No Arbitrage” In The Utility Comparison Rates

A bilateral agreement between Individuals 1 and 2 yields u_{12} . Similarly, a bilateral agreement between Individuals 2 and 3 yields u_{23} . We now show that these two bilateral agreements are sufficient to derive u_{123} (group utility for the case $n = 3$). We could of course choose either u_{12} and u_{13} or u_{13} and u_{23} to derive u_{123} . Recall that $u_{ij} = \alpha_i^j u_i + (1 - \alpha_i^j) u_j$. Alternatively, we may write $u_{ij} = (u_i + \delta_i^j u_j) / (1 + \delta_i^j)$, where the “utility comparison rate” $\delta_i^j = (1 - \alpha_i^j) / \alpha_i^j \in (0, \infty)$. In the alternative expression for u_{ij} , the parameter δ_i^j is the utility comparison rate between i and j : δ_i^j units of i 's utility are expressed in the same units as one unit of j 's utility. The order of the subscripts and superscripts is important because $\delta_j^i = 1 / \delta_i^j$. Henceforth, we use δ_i^j to denote the utility comparison rate associated with a bilateral agreement between individuals i and j .

For a geometric illustration of the assertion that u_{12} and u_{23} are sufficient to determine u_{123} , refer to Figure 1. If the outcomes in X are finite, then u_i can be thought of a vector of utilities or a point in a vector space. Under this interpretation, note that u_{12} is a convex combination of u_1 and u_2 and, therefore, lies on the line segment connecting u_1 and u_2 . Similarly, u_{23} lies on the line segment connecting u_2 and u_3 . Now apply Theorem 1 to the partition $\{12, 3\}$. So u_{123} must lie somewhere on the line segment connecting u_{12} and u_3 . Apply Theorem 1 again to the partition $\{1, 23\}$ to conclude that u_{123} must lie somewhere on the line segment connecting u_1 and u_{23} . There is only one point that lies on both the line segments $\overline{u_{12}u_3}$ and $\overline{u_1u_{23}}$ - the point u_{123} where these two line segments intersect. Thus from the two bilateral agreements u_{12} and u_{23} , a complete preference \succsim_{123} with utility u_{123} emerges. Figure 1 also illustrates that there is a unique utility candidate for the remaining bilateral agreement u_{13} - the point where the line connecting u_2 and u_{123} intersects line segment $\overline{u_1u_3}$. Otherwise, the Extended Pareto Rule applied to $\{2, 13\}$ would not hold.

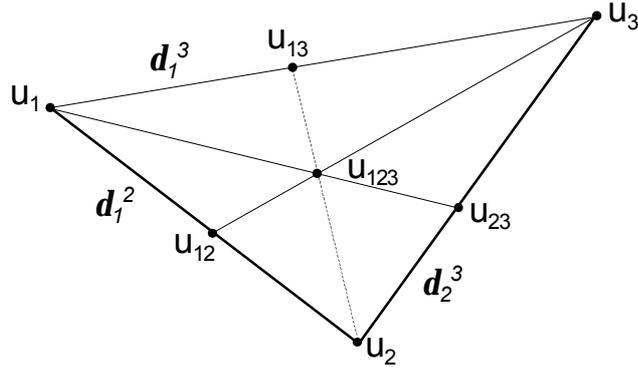


Figure 1: δ_1^3 follows from δ_1^2 and δ_2^3 as $\delta_1^2\delta_2^3$.

The following result provides the utilities that obtain from this construction.

Theorem 2 Consider bilateral agreements u_{12} and u_{23} with utility comparison rates δ_1^2 and δ_2^3 , respectively. If the Extended Pareto Rule holds, then \succsim_{123} is complete and has utility $u_{123} \equiv (u_1 + \delta_1^2 u_2 + \delta_1^3 u_3)/(1 + \delta_1^2 + \delta_1^3)$. Moreover, if \succsim_{13} is a complete preference, then it has utility $u_{13} \equiv (u_1 + \delta_1^3 u_3)/(1 + \delta_1^3)$, where $\delta_1^3 \equiv \delta_1^2 \delta_2^3$.

Proof. See Baucells and Shapley (2000). ■

The utility comparison rates that stem from this geometrical construction have an interesting interpretation in terms of “no arbitrage.” Thus, if δ_1^2 and δ_2^3 are the utility comparison rates of the bilateral agreements u_{12} and u_{23} (δ_1^2 “utils” of Individual 1 are comparable to one “util” of Individual 2; and δ_2^3 “utils” of Individual 2 are comparable to one “util” of Individual 3), then it should be the case that $\delta_1^2\delta_2^3$ “utils” of Individual 1 are comparable to one “util” of Individual 3. Similar to “no arbitrage” in currency exchange rates, the natural utility comparison rate between Individuals 1 and 3 is $\delta_1^3 = \delta_1^2\delta_2^3$.

The generalization of Theorem 2 to $n = 4$ case requires a nontrivial extension that uses Desargues’ Theorem. This result is illustrated in the Appendix. The subsequent generalization to $n > 4$ is then straightforward. The idea is to establish at least one comparison channel between each pair of individuals. Moreover, the “no arbitrage” condition indicates that a chain of bilateral agreements that “cycles” (starts and finishes in the same individual) contains redundancies. If we view the individuals as the nodes of a graph and the bilateral agreements as the edges, then these two conditions express that the bilateral agreements form a connected and acyclic graph, i.e., a spanning tree.

2.5 Group Preferences Based On Bilateral Agreements

For a group of n individuals, the group utility, u_N , is obtained by eliciting $(n - 1)$ bilateral agreements. For an example, suppose one obtains $u_{12}, u_{13}, \dots, u_{1n}$. We now know $\delta_1^2, \delta_1^3, \dots, \delta_1^n$; that is, the utility comparison rates between Individual 1 and Individual i , $i = 2$ to n . The EPR permits us to write

$$u_N = (u_1 + \delta_1^2 u_2 + \dots + \delta_1^n u_n) / (1 + \delta_1^2 + \dots + \delta_1^n). \quad (4)$$

There are many possible ways (spanning trees) to obtain $(n - 1)$ bilateral agreements. For example, another possibility is to obtain $u_{12}, u_{23}, \dots, u_{(n-1)n}$. Such a comparison or spanning tree will yield $\delta_1^2, \delta_2^3, \dots, \delta_{(n-1)}^n$. We can easily obtain δ_1^i in (4) by the multiplication $\delta_1^2 \times \delta_2^3 \times \dots \times \delta_{(i-1)}^i$. In general, however, we need to first specify the spanning tree. It is an empirical question whether the choice of a spanning tree systematically influences the derived utility comparison rates or utility weights. We assume that the decision context and practical considerations will dictate the choice of an appropriate spanning tree.

Equipped with a spanning tree \mathcal{T} on the set of individuals,¹ consider complete preferences or bilateral agreements for all pairs in \mathcal{T} and let δ_{ij}^j , $ij \in \mathcal{T}$, be the corresponding utility comparison rates. To obtain the weights λ_i in that apply to u_i in the expression for the group (or coalition) utility, we have to multiply the δ_{ij}^j along the branches of the tree that connect some chosen base agent and individual i . Formally, choose an arbitrary *base agent*,² say $i = 1$, as the “root” of the tree and let $\delta_1^1 \equiv 1$. For $i \neq 1$, let \mathcal{P}_1^i be the collection of pairs in the unique path between 1 and i in \mathcal{T} , and define

$$\lambda_i \equiv \delta_1^i = \prod_{jk \in \mathcal{P}_1^i} \delta_j^k \quad (5)$$

The following theorem provides the main result upon which the subsequent developments are based.

Theorem 3 *Assume that any three utilities are linearly independent and that every pair ij in N has complete (but undetermined) preferences. Let \mathcal{T} be a spanning tree of bilateral agreements (determined pair preferences) and let $(\lambda_1, \dots, \lambda_n)$ be given as in (5). Then, the Extended Pareto Rule holds if and only if for all coalition S in N , \succsim_S is complete and has utility*

$$u_S \equiv \frac{\sum_{i \in S} \lambda_i u_i}{\sum_{i \in S} \lambda_i}. \quad (6)$$

Proof. We have provided the intuition of the proof. See Baucells and Shapley (2000) for the formal details. ■

The interest of Theorem 3 resides in its usefulness to devise a practical procedure for preference aggregation. We will exploit several nice features of Theorem 3. First, it just requires the determination of $n - 1$ parameters, namely, the $n - 1$ utility exchange rates between pairs of individuals. Second, the choice of the $n - 1$

¹Given a set N , a collection of pairs \mathcal{T} is a *spanning tree* of n if and only if there is a unique path (sequence of pairs) in \mathcal{T} connecting any two agents in n . It follows that \mathcal{T} contains precisely $n - 1$ pairs and has no cycles.

²A moment’s reflection reveals that a different choice of base agent, say $i^* \neq 1$, would produce weights $\lambda_i^* = \delta_{i^*}^1 \lambda_i$, $i \in n$. Because the utility representation of the preference of a coalition S that we seek is $u_S \equiv \sum_{i \in S} \lambda_i u_i / \sum_{i \in S} \lambda_i$, the choice of base agent is immaterial: the factor $\delta_{i^*,1}$ would produce a re-scaled version of the same utility function.

pairs is rather flexible in that we can choose any $n - 1$ pairs that form a spanning tree. Additional criteria may be used so that a particular spanning tree offers more advantages. Third, simple manipulations allow us to obtain an alternative representation of the group utility as a weighted sum of utilities of disjoint coalitions. The resulting utility comparison rates between disjoint coalitions enable us to apply our results from bilateral agreements to agreements between two disjoint coalitions (see Section 5).

3 Multiattribute Group Decisions: The Additive Case

We now examine the case where the consequence of a decision is evaluated on multiple attributes. We assume that the members of the group have agreed on the attributes X_1, \dots, X_m and each individual is able to specify his own utility function for these multiple attributes. An arbitrary multidimensional consequence is denoted $x = (x_1, \dots, x_m)$, and x_a^* and x_a^0 are respectively the best and the worst consequences on attribute a that are agreed upon by all individuals. Our theory permits that the worst and the best levels of an attribute could be different for different individuals. We further assume that each individual's utility function over the multiple attributes is additively separable. Thus, we can write

$$u_i(x) = \sum_{a=1}^m \beta_{i,a} v_{i,a}(x), \quad i = 1 \text{ to } n, \quad (7)$$

where $\beta_{i,a}$ is the importance weight (scaling constant) that individual i attaches to attribute a , $0 < \beta_{i,a} < 1$, $\sum_{a=1}^m \beta_{i,a} = 1$ and $v_{i,a}$ is individual i 's component utility function (attribute evaluation function) for attribute a which, is normalized by $v_{i,a}(x_a^*) = 1$, $v_{i,a}(x_a^0) = 0$, $i = 1$ to n , $a = 1$ to m .

3.1 Homogeneous Attribute Evaluation

We begin by studying a special case where all individuals agree on how each attribute ranks the alternatives. Thus, $v_{i,a}(x) = v_a(x)$, for all $i = 1$ to n . The individuals differ, however, on the particular weights to be associated with each attribute. Thus, (7)

particularizes into

$$u_i(x) = \beta_{i,1}v_1(x) + \beta_{i,2}v_2(x) + \dots + \beta_{i,m}v_m(x), \quad (8)$$

where $v_a(x)$ is the common evaluation of alternative x with respect to attribute $a = 1$ to m . Our concern here is to find a consensus in the trade-off between the attributes. In other words, find a collection of weights that expresses a group decision rule.

Having a common ordering of the alternatives by attribute is not as unrealistic as it may appear at first. For example, these orderings could be given by independent experts. The role of the committee is not to question such opinions, but to judge the relative importance of the different attributes (environmental impact, cost, health and safety, etc.). We make this assumption to make the reader acquainted with our aggregation rule based on single attribute comparisons. Later we will show that our results generalize to the case of heterogeneous attribute evaluation functions.

From our previous theory, we know that we need $(n - 1)$ bilateral agreements to determine the group utility. The advantage here is that instead of bilateral agreements based on indifference probability judgements, we can now use compromises on weights assigned to attributes. It suffices that a pair of individuals compromise on the weight given to *a single attribute*. The agreement on just one weight will extend to the other attributes. Finally, our Extended Pareto Rule allows us to form a group preference based on $n - 1$ such bilateral agreements, provided they form a spanning tree of N .

The following result ties the bilateral agreement in terms of weights $\beta_{i,a}$ and $\beta_{j,a}$ (the weights given by individuals i and j to attribute a) to the utility weight α_i^j and the utility comparison rate δ_i^j . To avoid degeneracy, it is required that $\beta_{i,a} \neq \beta_{j,a}$, say $\beta_{i,a} < \beta_{j,a}$. We assume throughout that any three utilities are linearly independent, so that one such attribute exists for every pair of individuals.

Theorem 4 *For a given pair of individuals i and j consider an attribute a such that $\beta_{i,a} < \beta_{j,a}$ and let $\beta_{ij,a} \in (\beta_{i,a}, \beta_{j,a})$ be a compromise between Individuals i and j (i.e., as a pair, Individuals i and j agree that a pair preference should use weight $\beta_{ij,a}$ to determine the importance of attribute a). Under this condition, the pair preference is complete and given by $u_{ij} = \alpha_i^j u_i + (1 - \alpha_i^j) u_j$, where the utility weight α_i^j and the corresponding utility exchange rate δ_i^j are given by*

$$\alpha_i^j = \frac{\beta_{j,a} - \beta_{ij,a}}{\beta_{j,a} - \beta_{i,a}} \quad \text{and} \quad \delta_i^j = \frac{\beta_{ij,a} - \beta_{i,a}}{\beta_{j,a} - \beta_{ij,a}}. \quad (9)$$

Proof. Noting that the pair utility u_{ij} is a linear combination of u_i and u_j and exploiting expression (8) produce

$$\begin{aligned} u_{ij} &= \alpha_i^j u_i + (1 - \alpha_i^j) u_j = \alpha_i^j [\beta_{i,1} v_1 + \dots + \beta_{i,m} v_m] + (1 - \alpha_i^j) [\beta_{j,1} v_1 + \dots + \beta_{j,m} v_m] = \\ &= [\alpha_i^j \beta_{i,1} + (1 - \alpha_i^j) \beta_{j,1}] v_1 + \dots + [\alpha_i^j \beta_{i,m} + (1 - \alpha_i^j) \beta_{j,m}] v_m. \end{aligned}$$

Because the weight assigned to attribute a is $\alpha_i^j \beta_{i,a} + (1 - \alpha_i^j) \beta_{j,a}$, which we know is agreed to be $\beta_{ij,a}$, the expression for α_i^j and $\delta_i^j = (1 - \alpha_i^j) / \alpha_i^j$ follows. ■

Thus, the aggregation procedure is particularly simple: $n - 1$ pairs of individuals forming a spanning tree shall agree on how to compare the weights given to one of the attributes. Once we have the $n - 1$ bilateral agreements and the corresponding utility comparison rates δ_i^j for $ij \in \mathcal{T}$ given by (9), we calculate the weights as in (5). The group utility is readily available as in (6). Because every individual knows how attribute a trades off with the rest of attributes in their overall utility, individuals i and j should be aware that the pair weight $\beta_{ij,a}$ supposes a dual agreement, one regarding the weight on attribute a , and another regarding the relative weights that i and j will receive. This, in turn, determines an implicit agreement in all the other attribute weights $b \neq a$ given by $\beta_{ij,b} = \alpha_i^j \beta_{i,b} + (1 - \alpha_i^j) \beta_{j,b}$.

Substituting the expressions of u_i as a weighted sum of attribute evaluation functions and grouping terms allow us to express the group utility u_N as a weighted sum of attribute evaluation functions with weights given by $\beta_{N,a} = \sum_{i=1}^n \lambda_i \beta_{i,a} / \sum_{i=1}^n \lambda_i$. Thus,

$$\begin{aligned} u_N(x) &= [\lambda_1 u_1(x) + \lambda_2 u_2(x) + \dots + \lambda_n u_n(x)] / \sum_{i=1}^n \lambda_i \\ &= \beta_{N,1} v_1(x) + \beta_{N,2} v_2(x) + \dots + \beta_{N,m} v_m(x). \end{aligned}$$

Here, we have a choice of both a spanning tree and, for each pair of individuals, a particular attribute to compare. In a given instance, there may be one attribute on which it is easier to compromise, such as a monetary consequence. If different individuals have a better understanding of different attributes, then we could design the spanning tree of bilateral agreements in such a way that every pair of individuals compare the weights of a well understood attribute. The following example may help clarify the previous discussion.

Example 5 Consider $n = 4$ individuals and $m = 3$ attributes, with homogeneous attribute evaluation functions.

The individual utilities are a weighted sum of the three attribute evaluation functions as follows.

$$\begin{aligned}
u_1 &= 0.1v_1 + 0.3v_2 + 0.6v_3 \\
u_2 &= 0.3v_1 + 0.4v_2 + 0.3v_3 \\
u_3 &= 0.2v_1 + 0.4v_2 + 0.4v_3 \\
u_4 &= 0.2v_1 + 0.6v_2 + 0.2v_3
\end{aligned}$$

Suppose pairs $\{1, 2\}$ and $\{1, 3\}$ feel comfortable in compromising on Attribute 1; whereas, pair $\{3, 4\}$ prefers to compromise on Attribute 3. We could use spanning tree $\mathcal{T} = \{12, 13, 34\}$, and Attributes 1, 1, and 3 in the respective bilateral agreements. Suppose that the bilateral agreements are $\beta_{12,1} = 0.22 \in (0.1, 0.3)$, $\beta_{13,1} = 0.12 \in (0.1, 0.2)$ and $\beta_{34,3} = 0.25 \in (0.2, 0.4)$. The pair utility u_{12} would be $u_{12} = 0.4u_1 + 0.6u_3 = (u_1 + 1.5u_3)/2.5$, where $\alpha_1^2 = (\beta_{2,1} - \beta_{12,1})/(\beta_{2,1} - \beta_{1,1}) = 0.4$ and $\delta_1^2 = (\beta_{12,1} - \beta_{1,1})/(\beta_{2,1} - \beta_{12,1}) = 3/2$ follows from (9). Similarly, $u_{13} = 0.8u_1 + 0.2u_3 = (u_1 + 0.25u_3)/(1.25)$, where $\alpha_1^3 = (\beta_{3,1} - \beta_{13,1})/(\beta_{3,1} - \beta_{1,1}) = 0.8$ and $\delta_1^3 = (\beta_{13,1} - \beta_{1,1})/(\beta_{3,1} - \beta_{13,1}) = 0.25$. Regarding u_{34} , we have that $\alpha_3^4 = 0.25$ and $\delta_3^4 = 3$. Finally, the weights λ_i that produce the group utility are given from (5) as follows.

$$\lambda_1 = 1; \lambda_2 = \delta_1^2 = 3/2; \lambda_3 = \delta_1^3 = 1/4; \text{ and } \lambda_4 = \delta_1^4 = \delta_1^3 \delta_3^4 = 3/4,$$

and the group attribute weights are given by $\beta_{N,1} = \sum_{i=1}^n \lambda_i \beta_{i,a} / \sum_{i=1}^n \lambda_i$. Thus,

$$\beta_{N,1} = 0.2143; \beta_{N,2} = 0.4143; \beta_{N,3} = 0.3714,$$

and

$$\begin{aligned}
u_{123} &= (u_1 + 3u_2/2 + u_3/4 + 3u_4/4)/(1 + 3/2 + 1/4 + 3/4) \\
&= 0.2143v_1 + 0.4143v_2 + 0.3714v_3.
\end{aligned}$$

If the individuals are not satisfied with these weights or the ranking implied by them, they may revisit their original bilateral agreements. Again, such an iterative approach is common in the applications of multiattribute utility theory.

3.2 Heterogeneous Attribute Evaluation

In the previous section, we had assumed that for any given attribute each individual agrees on its component utility function (attribute evaluation), but may differ from

other individuals on the weight assigned to the attribute. We now examine the case where both the weight and the component utility function associated with an attribute may be different across individuals. In this case,

$$u_N(x) = \sum_{i=1}^n \lambda_i \sum_{a=1}^m \beta_{i,a} v_{i,a}(x), \quad (10)$$

if we assume that each individual has a multiattribute additive utility function and the group utility is linear in individual utilities. Though we do not make the latter assumption from the outset, through $(n - 1)$ bilateral agreements we are always able to produce the linear group utility $u_N \equiv \sum_{i \in N} \lambda_i u_i$, where for each i , $u_i = \sum \beta_{i,a} v_{i,a}$. Since these weights and component utilities differ across individuals, some compromise is needed to obtain the pair utility u_{ij} . A compromise directly on the utilities that the two individuals assign to a multidimensional consequence may be difficult. We assume that individuals have a good and clear understanding of their preferences (i.e., each individual is clear about his own weights and his own component utilities), but may not be able to directly provide a holistic preference for the complex multidimensional consequence.

Fortunately, the pair preference u_{ij} can be constructed by seeking a compromise on a part of the problem (e.g., on the weight that individuals i and j assign to an attribute a). The Extended Pareto Principle then restricts the compromises for the other parts of the problem. Thus, if two individuals compromise on the weight of an attribute, their implied agreements on the weights on the other remaining attributes can be displayed and they can be given an opportunity to revisit their original compromise. At the expense of belaboring the argument, we again state that a compromise more in favor of Individual 1 on one attribute weight, but more in favor of Individual 2 on the other attribute weight is not permitted. For the two attribute case, it is trivial to see that if $\beta_{1,1} = 0.8$, $\beta_{2,1} = 0.6$, then we cannot have $\beta_{12,1} = 0.75$ (compromise closer to Individual 1) and simultaneously have $\beta_{12,2} = 0.35$ (compromise closer to Individual 2 as $\beta_{1,2} = 0.2$, $\beta_{2,2} = 0.4$). The two weights $\beta_{12,1}$ and $\beta_{12,2}$ must add to 1 and therefore $\beta_{12,2}$ has to be 0.25 if $\beta_{12,1}$ was agreed to be 0.75. So, there is no free lunch here and the individuals must understand the full implication of their compromise.

In general, the Extended Pareto Rule restricts the compromises on other parts of the problem if a compromise on one part of the problem is reached. Suppose two

individuals compromise half way, each on their willingness to pay to improve the environment. Their compromise on the willingness to pay for improving health and safety can now be derived, but it would not be half way in general. Thus, the implied compromises are not proportionately the same as the original compromise, but they can be derived.

We will discuss three different methods for eliciting bilateral agreements to specify pair utilities which, in turn, are used to derive the group utility. The group utility so constructed will have $n \times m$ terms (see (10)). It is interesting to examine whether the group utility can be represented as the weighted sum of group attribute utilities (i.e., $u_N = \beta_1 v_1 + \dots + \beta_n v_m$, where β_a is the “group attribute weight” for the attribute a and v_a is the “group attribute utility” for the attribute a). Surprisingly, as Theorem 7 shows, we are indeed able to express the group utility as the weighted sum of group attribute utilities. From the narrow perspective of simply being able to rank alternatives, it may not seem important how the group utility is represented so long as it can somehow be assessed. For practical considerations, however, the decomposition of the group utility into “group attribute weights” and “group attribute utilities” may facilitate communication and implementation. For example, a higher level policy maker who has delegated the responsibility of the decision to the committee may wish to interpret the group attribute weights as the weights he will assign to the attributes if he were to make the decision on his own.

Before we describe our formal development, an example with two individuals and two attributes illustrates our strategy for constructing the group utility. In our framework, the EPR is used to obtain the group utility from the pairwise utilities. Imagine that our two-individual, two-attribute problem is conceived as a four-individual aggregation problem where Individual $1a$ reflects the preference of Individual 1 on attribute a in the original problem, Individual $1b$ reflects the preference of Individual 1 on attribute b , and so on for Individuals $2a$ and $2b$. We now need three pairwise comparisons to obtain the group utility of the modified problem. The bilateral agreements between $1a$ and $1b$ and similarly between $2a$ and $2b$ are immediate as we know their respective weights for attributes a and b . Actually, the bilateral agreement between hypothetical Individuals $1a$ ($2a$) and $1b$ ($2b$) simply means how the real Individual 1 (2) trades off attribute a with attribute b . So we only need to elicit one compromise say between $1a$ and $2a$ or we could have chosen a compromise between

1*b* and 2*b*. This compromise between the two hypothetical Individuals 1*a* and 2*a* (or 1*b* and 2*b*) is actually a compromise between the real Individuals 1 and 2 on attribute *a* (or attribute *b*). Unlike our real *n* person problem, some pair preferences cannot be elicited in any practical way. For example, a compromise between hypothetical Individuals 1*a* and 2*b* is rather nonsensical. In other words, for our hypothetically enlarged problem, the choice of spanning tree is limited. In the original problem - if there were to be truly four real individuals - any spanning tree can be used. But this causes no real problem either in theory or in practice. This is because we only need $(n - 1)$ and not $nC_2 = n(n - 1)/2$ pair preferences to construct the group utility.

Basically, we need to impose the Extended Pareto Rule to a larger set of preferences, namely, \succsim_Q , where $Q \subseteq N \times M$ (N is the set of individuals $\{1 \dots n\}$ and M is the set of attributes $\{1 \dots m\}$). Intuitively, we may think that each individual has multiple preferences, one for each attribute and that individual aggregates these preferences using weights $\beta_{i,a}$. Thus, we may enlarge the set of “basic” individuals to $N \times M$, and any such subset is endowed with a preference. The enlarged coalition $N \times M$ corresponds to the group utility, and the enlarged coalition $\{i\} \times M$ corresponds to individual *i*, with utility u_i . Similarly, $v_{i,a}$, the attribute evaluation function of individual *i*, represents the preference of the coalition $\{i\} \times \{a\}$; and the group attribute evaluation function v_a is the preference of the coalition $N \times \{a\}$.

Application of the Extended Pareto Rule to all the disjoint subsets of $N \times M$ permits us to increase the number of linearity conditions. In particular, for all consequences *x* the following relationships hold

$$u_{ij} = \alpha_i^j u_i + (1 - \alpha_i^j) u_j, \quad \text{for some } \alpha_i^j \in (0, 1) \quad (11)$$

$$v_{ij,a} = \alpha_{i,a}^{j,a} v_{i,a} + (1 - \alpha_{i,a}^{j,a}) v_{j,a}, \quad \text{for some } \alpha_{i,a}^{j,a} \in (0, 1) \quad (12)$$

$$u_{ij} = \sum_{a=1}^m \beta_{ij,a} v_{ij,a}, \quad \text{for some } \beta_{ij,a} \text{ with } \sum_{a=1}^m \beta_{ij,a} = 1. \quad (13)$$

As before, α_i^j and the utility comparison rate $\delta_i^j = (1 - \alpha_i^j)/\alpha_i^j$ determine pair utility $u_{ij} = \alpha_i^j u_i + (1 - \alpha_i^j) u_j$. The weights $\alpha_{i,a}^{j,a}$ represent the trade-off between $v_{i,a}$ and $v_{j,a}$ to form the pair attribute evaluation function $v_{ij,a} = \alpha_{i,a}^{j,a} v_{i,a} + (1 - \alpha_{i,a}^{j,a}) v_{j,a}$, where $v_{ij,a}$ is the utility of the coalition $\{i, j\} \times \{a\}$, expressed as the weighted sum of the utilities of $\{i\} \times \{a\}$ and $\{j\} \times \{a\}$. Finally, Corollary 9 allows us to express u_{ij} as the weighted sum of utilities that can be formed in any partition of $\{i, j\} \times M$.

One such partition takes the form $\{i, j\} \times \{a\}$, for $a \in M$ and produces (13).

Before proceeding with different ways to elicit the previous parameters, we now present a result that links their values leaving only one degree of freedom for each pair of individuals, and $n - 1$ degrees of freedom for the group. The intuition behind this result comes from our idea of the spanning tree. If we view $v_{i,a}$ as the nodes of the spanning tree, then the fact that u_i is a linear combination of the $v_{i,a}$ for all $a \in M$ tells us that all the bilateral agreements between the nodes $v_{i,a}$ and $v_{i,b}$ are already in place with $\delta_{i,a}^{i,b} = \beta_{i,b}/\beta_{i,a}$. Thus, once we establish one link between i and j , then our consistency condition already fills the gaps and establishes all the utility comparison rates between individuals and between attribute evaluation functions. Having imposed the Extended Pareto Rule to the attribute evaluation function will allow us to determine some additional parameters that are linked to α_i^j .

Lemma 6 *If the Extended Pareto Rule holds among the disjoint subsets of $N \times M$, then the parameters $\delta_i^j = (1 - \alpha_i^j)/\alpha_i^j$, $\delta_{i,a}^{j,a} = (1 - \alpha_{i,a}^{j,a})/\alpha_{i,a}^{j,a}$, and $\beta_{ij,a}$ are related as follows*

$$\delta_{i,a}^{j,a} = \frac{\beta_{j,a}}{\beta_{i,a}} \delta_i^j \quad (14)$$

$$\beta_{ij,a} = \alpha_i^j \beta_{i,a} + (1 - \alpha_i^j) \beta_{j,a} \quad (15)$$

for all $i, j \in N$ and all $a \in M$. Moreover

$$v_{ij,a} = \frac{\alpha_i^j \beta_{i,a} v_{i,a} + (1 - \alpha_i^j) \beta_{j,a} v_{j,a}}{\beta_{ij,a}}. \quad (16)$$

Proof. Using (11), (12) and (13), we write

$$\begin{aligned} u_{ij} &= \sum_{a=1}^m \beta_{ij,a} v_{ij,a} = \sum_{a=1}^m \beta_{ij,a} [\alpha_{i,a}^{j,a} v_{i,a} + (1 - \alpha_{i,a}^{j,a}) v_{j,a}] \\ &= \alpha_i^j u_i + (1 - \alpha_i^j) u_j = \alpha_i^j \sum_{a=1}^m \beta_{i,a} v_{i,a} + (1 - \alpha_i^j) \sum_{a=1}^m \beta_{j,a} v_{j,a}. \end{aligned}$$

Because this holds for all $x \in X$, linear independence of the attribute utilities only holds if the coefficients of the $v_{i,a}$'s are the same. Evaluate this expression at alternatives $(x_1^o, \dots, x_{a-1}^o, x_a^*, x_{a+1}^o, \dots, x_m^o)$ for all $a \in M$ to arrive at

$$\begin{aligned} \beta_{ij,a} \alpha_{i,a}^{j,a} &= \beta_{i,a} \alpha_i^j \\ \beta_{ij,a} (1 - \alpha_{i,a}^{j,a}) &= \beta_{j,a} (1 - \alpha_i^j) \end{aligned}$$

Equation (14) follows from dividing one expression by the other, and (15) from adding both equations. The last expression follows from replacing $\alpha_{i,a}^{j,a} = \beta_{i,a}\alpha_i^j/\beta_{ij,a}$ in (12) and using (15). ■

With this result in mind we now present three alternative ways to elicit bilateral agreements among $(n - 1)$ pairs of individuals.

3.2.1 Method I: Compromise On Attribute Evaluation Functions

For a given pair of individuals i and j , and some choice of an attribute a , let $v_{i,a}$ and $v_{j,a}$ be the corresponding attribute evaluation functions. Next, we fix alternatives $x^h = (x_1^o, \dots, x_{a-1}^o, x_a^*, x_{a+1}^o, \dots, x_m^o)$ and $x^\ell = (x_1^o, \dots, x_{a-1}^o, x_a^o, x_{a+1}^o, \dots, x_m^o)$ and find some intermediate alternative $x = (x_1^o, \dots, x_{a-1}^o, x_a, x_{a+1}^o, \dots, x_m^o)$ such that $v_{i,a}(x) \neq v_{j,a}(x)$. Thus, $u_i(x) = \beta_{i,a}v_{i,a}(x_a)$, $u_i(x^h) = \beta_{i,a}$ and $u_i(x^\ell) = 0$ so that individual i is indifferent between x and a probability mixture $p_i = v_{i,a}(x)$ between x^h and x^ℓ . Then, we ask individuals i and j to find a compromise probability $p_{ij} \in (p_i, p_j)$. This says that $u_{ij}(x) = \beta_{ij,a}p_{ij}$, or

$$v_{ij,a}(x) = p_{ij} = \alpha_{i,a}^{j,a}p_i + (1 - \alpha_{i,a}^{j,a})p_j = \alpha_{i,a}^{j,a}v_{i,a}(x) + (1 - \alpha_{i,a}^{j,a})v_{j,a}(x)$$

so that

$$\alpha_{i,a}^{j,a} = \frac{p_j - p_{ij}}{p_j - p_i} \quad \text{and} \quad \delta_{i,a}^{j,a} = \frac{p_{ij} - p_i}{p_j - p_i}. \quad (17)$$

Using (14) and (15) we derive the value of the other parameters. Specifically we obtain that

$$u_{ij} \equiv \alpha_i^j u_i + (1 - \alpha_i^j) u_j,$$

where $\alpha_i^j = 1/(1 + \delta_i^j)$, $\delta_i^j = \beta_{i,a}\delta_{i,a}^{j,a}/\beta_{j,a}$. Alternatively, the group utility is represented by

$$u_{ij} \equiv \sum_{a=1}^m \beta_{ij,a} v_{ij,a},$$

where $\beta_{ij,a} = \alpha_i^j \beta_{i,a} + (1 - \alpha_i^j) \beta_{j,a}$. Finally, for all $b \in M$, $v_{ij,b}$ can be expressed as in (16). This possibility of finding pair attribute evaluation functions is the result that later we will extend to u_N .

3.2.2 Method II: Compromise On Attribute Weights

An alternative way to elicit the bilateral agreement is by means of comparing the weights given to the different attributes, as in the homogeneous case. For a given pair of individuals i and j and some choice of an attribute a , let $\beta_{i,a}$ and $\beta_{j,a}$ be the corresponding weights given to attribute a . Recall that $\beta_{i,a}$ is interpreted as the utility of alternative $x^h = (x_1^o, \dots, x_{a-1}^o, x_a^*, x_{a+1}^o, \dots, x_m^o)$. Alternatively, individual i is indifferent between x^h and a mixture that with probability $\beta_{i,a}$ produces x^* or x^o . If $\beta_{i,a} \neq \beta_{j,a}$, then we ask the pair to reach a bilateral agreement $\beta_{ij,a} \in (\beta_{i,a}, \beta_{j,a})$ that will be interpreted as the pair utility of x^h , or

$$u_{ij}(x^h) = \beta_{ij,a} = \alpha_i^j u_i(x^h) + (1 - \alpha_i^j) u_j(x^h) = \alpha_i^j \beta_{i,a} + (1 - \alpha_i^j) \beta_{j,a}$$

so that the same expression as in (9) results

$$\alpha_i^j = \frac{\beta_{j,a} - \beta_{ij,a}}{\beta_{j,a} - \beta_{i,a}} \quad \text{and} \quad \delta_i^j = \frac{\beta_{ij,a} - \beta_{i,a}}{\beta_{j,a} - \beta_{ij,a}}. \quad (18)$$

Using α_i^j , δ_i^j , (15), and (16), we can compute $\beta_{ij,b}$ and $v_{ij,b}$ for all $b \in M$, and write u_{ij} as a linear combination of $v_{ij,b}$.

3.2.3 Method III: Compromise On The Willingness To Pay

A common procedure widely used in multiattribute applications to elicit the individual weights $\beta_{i,a}$ is the following. Assuming attribute $a = 1$ is the most important, we seek x_1 so that

$$(x_1, x_2^o, x_3, \dots, x_m) \sim_i (x_1^o, x_2^*, x_3, \dots, x_m)$$

This implies that $\beta_{i,1} v_{i,1}(x_1) = \beta_{i,2}$. Now, we seek x'_1 so that

$$(x'_1, x_2, x_3^o, x_4, \dots, x_m) \sim_i (x_1^o, x_2, x_3^*, x_4, \dots, x_m),$$

which implies $\beta_{i,1} v_{i,1}(x'_1) = \beta_{i,3}$, and so on. This method relates all the attribute weights to $\beta_{i,1}$, and we elicit $\beta_{i,1}$ by finding the probability p so that a mixture of x^* and x^o is indifferent to $(x_1^*, x_2^o, \dots, x_m^o)$. This implies that $\beta_{i,1} = p$. The idea is to get one weight by the lottery method and the others by trade-offs. Notice that if the first attribute is money, then we derive $\beta_{i,1}$ by finding out how much individual i is willing

to pay to raise the value of x_2 from x_2^o to x_2^* , and so on. Let x_1^i be i 's willingness to pay, or

$$(x_1^i, x_2^o, x_3, \dots, x_m) \sim_i (x_1^o, x_2^*, x_3, \dots, x_m)$$

Similarly, let x_1^j be j 's willingness to pay. If $x_1^i \neq x_1^j$, then we can ask the pair to reach a compromise and seek a pair willingness to pay $x_1^{ij} \in (x_1^i, x_1^j)$. The next step, of course, is to find the α_i^j and δ_i^j that follow from x_1^{ij} . They are given by:

$$\alpha_i^j = \frac{\beta_{j,1}[v_{j,1}(x_1^j) - v_{j,1}(x_1^{ij})]}{\beta_{i,1}[v_{i,1}(x_1^{ij}) - v_{i,1}(x_1^i)] + \beta_{j,1}[v_{j,1}(x_1^j) - v_{j,1}(x_1^{ij})]}, \text{ and} \quad (19)$$

$$\delta_i^j = \frac{\beta_{i,1}[v_{i,1}(x_1^{ij}) - v_{i,1}(x_1^i)]}{\beta_{j,1}[v_{j,1}(x_1^j) - v_{j,1}(x_1^{ij})]}.$$

To derive these expressions, just notice that the agreement in willingness to pay x_1^{ij} says

$$(x_1^{ij}, x_2^o, x_3, \dots, x_m) \sim_{ij} (x_1^o, x_2^*, x_3, \dots, x_m), \text{ or}$$

$$u_{ij}(x_1^{ij}, x_2^o, x_3, \dots, x_m) = u_{ij}(x_1^o, x_2^*, x_3, \dots, x_m).$$

Using $u_{ij} = \alpha_i^j u_i + (1 - \alpha_i^j) u_j$, and cancelling the terms involving x_3, \dots, x_m , produces

$$\alpha_i^j \beta_{i,1} v_{i,1}(x_1^{ij}) + (1 - \alpha_i^j) \beta_{j,1} v_{j,1}(x_1^{ij}) = \alpha_i^j \beta_{i,2} + (1 - \alpha_i^j) \beta_{j,2}$$

and the following alternative expression for α_i^j ,

$$\alpha_i^j = \frac{\beta_{j,2} - \beta_{j,1} v_{j,1}(x_1^{ij})}{\beta_{j,2} - \beta_{i,2} + \beta_{i,1} v_{i,1}(x_1^{ij}) - \beta_{j,1} v_{j,1}(x_1^{ij})}.$$

Plugging $\beta_{i,2} = \beta_{i,1} v_{i,1}(x_1^i)$ and $\beta_{j,2} = \beta_{j,1} v_{j,1}(x_1^j)$ yields (19).

Again, using α_i^j , δ_i^j , (15), and (16), we can compute $\beta_{ij,b}$ and $v_{ij,b}$ for all $b \in M$, and write u_{ij} as a linear combination of the $v_{ij,b}$.

3.2.4 The Main Result

After repeating any of these procedures for $n - 1$ pairs forming a spanning tree, the final results are weights α_i^j and utility exchange rates δ_i^j , for $ij \in \mathcal{T}$. We use (5) to calculate $\lambda_i = \delta_1^i$. Finally, we simplify the notation by writing $v_a \equiv v_{N,a}$ and $\beta_a \equiv \beta_{N,a}$. The result we obtain is:

Theorem 7 Let λ_i be the weights obtained as in (5) by means of $n - 1$ bilateral agreements forming a spanning tree of N . Then, group preference is complete and is represented by

$$u_N \equiv \frac{\sum_{i=1}^n \lambda_i u_i}{\sum_{i=1}^n \lambda_i}. \quad (20)$$

Alternatively, the group utility is equally represented by

$$u_N \equiv \sum_{a=1}^m \beta_a v_a, \quad (21)$$

where $\beta_a = \sum_{i=1}^n \lambda_i \beta_{i,a} / \sum_{i=1}^n \lambda_i$ and each v_a is a group attribute preference given by

$$v_a \equiv \frac{\sum_{i=1}^n \lambda_i \beta_{i,a} v_{i,a}}{\sum_{i=1}^n \lambda_i \beta_{i,a}}. \quad (22)$$

Proof. (20) is just a direct application of Theorem 3. By substituting (7) into (20), we obtain a sum of $n \times m$ terms of the form $v_{i,a}$, with weights of the form $\lambda_i \times \beta_{i,a}$. (21) follows from forming a partition of $N \times M$ in sets $A_a = N \times \{a\}$, for $a \in M$, and applying Corollary 9. Indeed, $\delta_{1,1}^{A_a} = \sum_{i=1}^n \lambda_i \beta_{i,a} / (\lambda_1 \beta_{1,1})$. Here the subscript (1, 1) indicates that the utilities are expressed in units of $v_{1,1}$. By normalizing so that $\sum_{a=1}^m \beta_a = 1$, we obtain

$$\beta_a = \frac{\delta_{1,1}^{A_a}}{\sum_{a=1}^m \delta_{1,1}^{A_a}} = \frac{\delta_{1,1}^{A_a}}{\sum_{a=1}^m \sum_{i=1}^n \lambda_i \beta_{i,a}} = \frac{\sum_{i=1}^n \lambda_i \beta_{i,a}}{\sum_{i=1}^n \lambda_i \sum_{a=1}^m \beta_{i,a}} = \frac{\sum_{i=1}^n \lambda_i \beta_{i,a}}{\sum_{i=1}^n \lambda_i}.$$

The attribute evaluation function v_a corresponds to the utility of the ‘‘coalition’’ $A_a = N \times \{a\}$. To express v_a in terms of the $v_{i,a}$ ’s suffices to break A_a into their components $\{i\} \times \{a\}$. Then, $\delta_{1,a}^{i,a} = (\lambda_i \beta_{i,a}) / (\lambda_1 \beta_{1,1})$. Of course, the denominator cancels out once we write

$$v_a = \frac{\sum_{i=1}^n \delta_{1,a}^{i,a} v_{i,a}}{\sum_{i=1}^n \delta_{1,a}^{i,a}} = \frac{\sum_{i=1}^n \lambda_i \beta_{i,a} v_{i,a}}{\sum_{i=1}^n \lambda_i \beta_{i,a}}.$$

■

3.2.5 An Example

To illustrate the use of the three elicitation methods, we consider an example with $n = 2$ individuals and $m = 3$ attributes. The three attributes of an alternative

$x = (x_1, x_2, x_3)$ may represent a monetary outcome (say a discounted cash flow) x_1 , some indicator of environmental impact x_2 , and some measure of health and safety x_3 . Let the worst outcome be $x^o = (0, 0, 0)$ and the best be $x^* = (100, 100, 100)$. The individual utilities are additive, separable in the following attribute evaluation functions

$$\begin{aligned} u_1(x) &= 0.4 \frac{1 - e^{-x_1/100}}{1 - e^{-1}} + 0.35 \frac{x_2}{100} + 0.25 \frac{x_3}{100} \\ u_2(x) &= 0.5 \frac{1 - e^{-x_1/200}}{1 - e^{-1/2}} + 0.3 \frac{x_2}{100} + 0.2 \frac{x_3}{100} \end{aligned}$$

Notice that $v_{i,a}(x^o) = 0$ and $v_{i,a}(x^*) = 1$. The first component reveals that both individuals are risk averse with respect to monetary consequences with exponential utility functions. Regarding the other two attributes, we assume for simplicity that their attribute evaluation function is linear in x_2 and x_3 .

To elicit a bilateral agreement applying **Method I**, we choose the monetary attribute x_1 . Accordingly, we let $x^h = (100, 0, 0)$, $x^\ell = (0, 0, 0)$ and fix some intermediate alternative, say $x = (40, 0, 0)$. Thus, $v_{1,1}(x) = 0.52 \neq v_{2,1}(x) = 0.46$ so that Individual 1 is indifferent between x and a probability mixture $p_1 = 0.52$ between x^h and x^ℓ . $p_2 = 0.46$ has the same interpretation for Individual 2. Now, Individuals 1 and 2 elicit a bilateral agreement by finding a compromise probability $p_{12} \in (0.46, 0.52)$, say $p_{12} = 0.51$. Using (17) we find that

$$\alpha_{1,1}^{2,1} = \frac{p_2 - p_{12}}{p_2 - p_1} = \frac{0.46 - 0.51}{0.46 - 0.52} = 0.81 \quad \text{and} \quad \delta_{1,1}^{2,1} = \frac{0.51 - 0.52}{0.46 - 0.51} = 0.23$$

From $\delta_{1,1}^{2,1}$, we derive $\delta_1^2 = \beta_{1,1} \delta_{1,1}^{2,1} / \beta_{2,1} = 0.4 \times 0.23 / 0.5 = 0.19$ and $\alpha_1^2 = 1 / (1 + \delta_1^2) = 0.84$. This allows us to write the group utility as

$$u_{12}(x) = 0.84u_1(x) + 0.16u_2(x).$$

Alternatively, the group utility can be represented by

$$u_{12} \equiv \sum_{a=1}^m \beta_{12,a} v_a = 0.42v_1 + 0.34v_2 + 0.24v_3$$

where $\beta_{12,a} = \alpha_1^2 \beta_{1,a} + (1 - \alpha_1^2) \beta_{2,a}$ and $v_a \equiv v_{12,a} = \sum_{i=1}^n \lambda_i \beta_{i,a} v_{i,a} / \sum_{i=1}^n \lambda_i \beta_{i,a}$.

Because $\lambda_1 = 1$ and $\lambda_2 = \delta_1^2 = 0.19$, we have that

$$\begin{aligned} v_1 &= \frac{1 \times 0.4 \times v_{1,1} + 0.19 \times 0.5 \times v_{2,1}}{0.4 + 0.19 \times 0.5} = 0.81v_{1,1} + 0.19v_{2,1} \\ v_2 &= \frac{1 \times 0.35 \times v_{1,2} + 0.19 \times 0.3 \times v_{2,2}}{0.35 + 0.19 \times 0.3} = 0.86v_{1,2} + 0.14v_{2,2} \\ v_3 &= \frac{1 \times 0.25 \times v_{1,3} + 0.19 \times 0.2 \times v_{2,3}}{0.25 + 0.19 \times 0.2} = 0.87v_{1,3} + 0.13v_{2,3} \end{aligned}$$

Alternatively, Individuals 1 and 2 could use **Method II** and compare the weights given to some attribute, say $a = 2$. In this case, $\beta_{1,2} = 0.35$ and $\beta_{2,2} = 0.3$ are the weights, and a bilateral agreement is some $\beta_{12,2} \in (0.3, 0.35)$. If this bilateral agreement is to be consistent with the previous one, then $\beta_{12,2} = 0.342$ so that

$$\alpha_1^2 = \frac{\beta_{2,2} - \beta_{12,2}}{\beta_{2,2} - \beta_{1,2}} = 0.84 \quad \text{and} \quad \delta_1^2 = \frac{\beta_{12,2} - \beta_{1,2}}{\beta_{2,2} - \beta_{12,2}} = 0.19.$$

Finally, **Method III** is based on willingness to pay. Thus, we begin finding the monetary outcomes that Individuals 1 and 2 are willing to pay to move x_2 from 0, its worst level, to 100, its best level. By setting

$$(x_1^i, 0, x_3) \sim_i (0, 100, x_3)$$

we find that $x_1^1 = 80.5$ and $x_1^2 = 53.8$. Now a bilateral agreement takes the form of a compromise in this willingness to pay, $x_1^{12} \in (53.8, 80.5)$. If this bilateral agreement is to be consistent with the previous one, then $x_1^{12} = 74.4$ and (19) produces

$$\begin{aligned} \alpha_1^2 &= \frac{\beta_{2,1}[v_{2,1}(x_1^2) - v_{2,1}(x_1^{12})]}{\beta_{1,1}[v_{1,1}(x_1^{12}) - v_{1,1}(x_1^1)] + \beta_{2,1}[v_{2,1}(x_1^2) - v_{2,1}(x_1^{12})]} = 0.84, \text{ and} \\ \delta_1^2 &= \frac{\beta_{1,1}[v_{1,1}(x_1^{12}) - v_{1,1}(x_1^1)]}{\beta_{2,1}[v_{2,1}(x_1^j) - v_{2,1}(x_1^{12})]} = \frac{0.4 [0.83 - 0.875]}{0.5 [0.6 - 0.79]} = 0.19. \end{aligned}$$

Of course, in both Method II and Method III the group utility can also be expressed in terms of group attribute evaluation functions v_a upon computing $\beta_{12,a}$ and using Equation (22).

4 Multiattribute Group Decisions: The Multiplicative Case

In the previous section, we showed that the group utility can be represented as a weighted sum of group weights and group attribute evaluation functions provided that

each individual's multiattribute utility function is additive. Further, we showed that $n - 1$ bilateral agreements are sufficient to derive the group utility. We now consider the case in which each individual's multiattribute utility function is multiplicative (Keeney and Raiffa, 1976) and individuals still wish to abide by the Extended Pareto Rule.

For notation simplicity, we describe a case with $m = 2$ attributes. The extension to $m > 2$ attributes is straightforward. Let the individual utilities take the following representation

$$u_i(x) = \beta_{i,1}v_{i,1}(x) + \beta_{i,2}v_{i,2}(x) + k_iv_{i,1}(x)v_{i,2}(x) \quad (23)$$

In this multiplicative form, $v_{i,1}$ and $v_{i,2}$ take values between 0 and 1 and we have normalized the scales so that $k_i = (1 - \beta_{i,1} - \beta_{i,2})$, where k_i is the multiplicative constant. As in the additive case when we interpreted the attribute evaluation functions $v_{i,k}$ as utility functions associated with an attribute-agent within individual i , we now interpret the multiplicative term as a third attribute-agent that takes care of the complementarity effects ($k > 0$) or substitution effects ($k < 0$) between attributes. Thus, it is like having a third attribute whose evaluation function is limited to a constant times the product of the previous two evaluation functions.

We begin by considering the **homogenous attribute evaluation** case: all the individuals agree on how each attribute ranks the consequences, and disagree on the particular weights to be associated with each attribute. Thus, (23) particularizes into

$$u_i(x) = \beta_{i,1}v_1(x) + \beta_{i,2}v_2(x) + (1 - \beta_{i,1} - \beta_{i,2})v_1(x)v_2(x)$$

where $v_a(x) = v_{i,a}(x)$, $i = 1$ to n , is the common component utility function for attribute a .

For a given pair of individuals, say $\{1, 2\}$, we can obtain a compromise on the attribute weight of an attribute a . Thus, $\beta_{12,a} \in (\beta_{1,a}, \beta_{2,a})$ is elicited from Individuals 1 and 2 and this bilateral agreement is sufficient to compute the utility exchange rate between Individuals 1 and 2. The compromise on the weights of the other remaining attributes will be implied by the elicited compromise on the weight of attribute a . Thus, a single compromise permits us to compute the weights that the pair associates with the attributes in consideration; hence, the multiplication

constant for the pair can be easily computed. Keeney and Raiffa (1976) have noted the close similarity of the additive and the multiplicative multiattribute models. The multiplicative constant is calculated using a simple equation once all of the attribute weights become available. More formally,

Theorem 8 *For a given pair of individuals, say $i = 1, 2$, consider one attribute $a \in M$ such that $\beta_{1,a} < \beta_{2,a}$ and let $\beta_{12,a} \in (\beta_{1,a}, \beta_{2,a})$ be a compromise between Individuals 1 and 2 (i.e., as a pair, Individuals 1 and 2 agree that a pair preference should use weight $\beta_{12,a}$ to determine the importance of attribute a). Under this condition, the pair preference is complete and is given by $u_{12} = \alpha_1^2 u_1 + (1 - \alpha_1^2) u_2$, where the utility weight α_1^2 and the corresponding utility exchange rate δ_1^2 are given by*

$$\alpha_1^2 = \frac{\beta_{2,a} - \beta_{12,a}}{\beta_{2,a} - \beta_{1,a}} \quad \text{and} \quad \delta_1^2 = \frac{\beta_{12,a} - \beta_{1,a}}{\beta_{2,a} - \beta_{12,a}}. \quad (24)$$

Proof. Noting that the pair utility u_{12} is a linear combination of u_1 and u_2 and exploiting the expression (8) produces

$$\begin{aligned} u_{12} &= \alpha_1^2 u_1 + (1 - \alpha_1^2) u_2 \\ &= \alpha_1^2 [\beta_{1,1} v_1 + \beta_{1,2} v_2 + \kappa_1 v_1 v_2] + (1 - \alpha_1^2) [\beta_{2,1} v_1 + \beta_{2,2} v_2 + \kappa_2 v_1 v_2] \\ &= [\alpha_1^2 \beta_{1,1} + (1 - \alpha_1^2) \beta_{2,1}] v_1 + [\alpha_1^2 \beta_{1,2} + (1 - \alpha_1^2) \beta_{2,2}] v_2 + [\alpha_1^2 \kappa_1 + (1 - \alpha_1^2) \kappa_2] v_1 v_2 \\ &= \beta_{12,1} v_1 + \beta_{12,2} v_2 + \kappa_{12} v_1 v_2. \end{aligned}$$

Thus, the weight assigned to attribute $a \in M$ is $\alpha_1^2 \beta_{1,a} + (1 - \alpha_1^2) \beta_{2,a}$, which we know is agreed to be $\beta_{12,a}$. The expression for α_1^2 and $\delta_1^2 = (1 - \alpha_1^2) / \alpha_1^2$ follows. ■

Using Theorem 8, we can obtain the utility for any given pair of individuals by seeking their compromise on the weights of a chosen attribute. Now, the aggregation procedure is particularly simple: $n - 1$ pairs of individuals forming a spanning tree agree on how to compare one of the attributes. Once we have the $n - 1$ bilateral agreements and the corresponding utility comparison rates δ_i^j for $ij \in \mathcal{T}$, we calculate the group utility weights as in (5). The group utility is easily derived. The final group weight for a given attribute is then $\beta_{N,a} = \sum_{i=1}^n \lambda_i \beta_{i,a} / \sum_{i=1}^n \lambda_i$, the group multiplicative constant becomes $\kappa_N = \sum_{i=1}^n \lambda_i \kappa_i / \sum_{i=1}^n \lambda_i$ and

$$\begin{aligned} u_N(x) &= [\lambda_1 u_1(x) + \lambda_2 u_2(x) + \dots + \lambda_n u_n(x)] / \sum_{i=1}^n \lambda_i = \\ &= \beta_{N,1} v_1(x) + \beta_{N,2} v_2(x) + \kappa_N v_1(x) v_2(x). \end{aligned}$$

In the **heterogeneous attribute evaluation** case (individuals use different attribute evaluation functions an arbitrary weights), we can apply the general theory already discussed Section 2 and obtain a group utility that is a weighted sum of individuals' utilities, or

$$\begin{aligned} u_n &= \lambda_1 u_1 + \dots + \lambda_n u_n = \\ &= \lambda_1 [\beta_{1,1} v_{1,1} + \beta_{1,2} v_{i,2} + k_1 v_{1,1} v_{1,2}] + \dots + \lambda_n [\beta_{n,1} v_{n,1} + \beta_{n,2} v_{n,2} + k_n v_{n,1} v_{n,2}]. \end{aligned} \tag{25}$$

However, the decomposition in group attribute evaluation functions that we obtained in (21) and (22) is not possible. Specifically, the group utility cannot be written as $u_N = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_1 v_2$ for group attribute evaluation functions v_1 and v_2 , which are a weighed sum of the individual attribute functions. Assuming that this were possible, the idea would be to expand the expression $u_N = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_1 v_2$ and compare it with (25) to identify the coefficients. However, this expansion produces terms of the form $v_{i,1} v_{i,2}$ as in (25), along with terms of the form $v_{i,1} v_{j,2}$ for $i \neq j$, but these terms do not appear in (25). Thus, in the heterogeneous case, we cannot express the group utility in the form of the original multiplicative form (23) where the parameters now refer to the group rather than the individuals. We can, however, obtain the group utility from the pairwise bilateral agreements, which in turn can be used to produce the group's ranking of alternatives.

5 Coalition Agreements

So far, we have emphasized bilateral agreements and the resulting pairwise utilities as the building blocks for the group utility. If a large number of individuals are involved in a group decision, pairwise comparisons may be too onerous. A more practical procedure may be to divide the group into small coalitions A_1, A_2, \dots, A_q . Then, representatives of each coalition will meet and try to reach an agreement. This is analogous to multiattribute utility theory where trade-offs are generally sought among individual attributes, but sometimes also among subgroups of attributes.

Because our setting treats individuals and coalitions alike, once we have a partition of the group and utilities for each of these partitions, we can proceed as if these coalitions were the "original" individuals. For example, let u_A and u_B be the utilities of two disjoint coalitions A and B . Both u_A and u_B are weighted averages of the

individual utilities of the members of A and B , respectively. Now, instead of some pair agreement between some $i \in A$ and some $j \in B$ of the form $u_{ij} = (u_i + \delta_i^j u_j)/(1 + \delta_i^j)$, let the agreement take place at a coalition level. Thus, two representatives from A and B will now try to reach a bilateral agreement at the level of coalitions of the form $u_{A \cup B} = (u_A + \delta_A^B u_B)/(1 + \delta_A^B)$. It is immediately clear that if $q - 1$ such agreements take place among a spanning tree of the partition A_1, A_2, \dots, A_q of N , then group utility u_N is obtained as before.

The question remains of how such group agreements relate to the hypothetical pair agreements that individuals i and j could have made on their own and that produced the same group utility $u_{A \cup B}$. First, the weight of i in u_A is $\lambda_i / \sum_{k \in A} \lambda_k$ and the weight of j in u_B is $\lambda_j / \sum_{k \in B} \lambda_k$. Because the utility comparison rate between u_A and u_B is δ_A^B , or

$$u_{A \cup B} = \frac{u_A + \delta_A^B u_B}{1 + \delta_A^B} = \frac{\sum_{k \in A} \lambda_k u_k / \sum_{k \in A} \lambda_k + \delta_A^B \sum_{k \in B} \lambda_k u_k / \sum_{k \in B} \lambda_k}{1 + \delta_A^B}$$

the ratio of weights between u_i and u_j in $u_{A \cup B}$ is

$$\delta_i^j = \delta_A^B \frac{\lambda_j / \sum_{k \in B} \lambda_k}{\lambda_i / \sum_{k \in A} \lambda_k}.$$

Of course, we don't have to restrict ourselves to one partition. We could imagine that an intermediate partition may form (filtrations of N), and agreements are formed at higher levels. We now see a picture where individuals may act as portrayers of their own preferences or become representatives of some coalition (say the different departments in a corporation). In the latter case, the individual should reflect the coalition preferences. We leave the investigation of a more general hierarchical coalition structure, appropriate for organizational decision making, as a future research question.

Conversely, having obtained a group utility u_N , we could derive the hypothetical utility comparison rates between disjoint coalitions implied by the bilateral agreements. Simple manipulations of expression (6) in Theorem 3 allow us to obtain an alternative representations of the group utility as a weighted sum of utilities of disjoint coalitions.

Proposition 9 *Let u_N be a group utility as in (6). If A and B are disjoint coalitions such that $A \cup B = N$, then $\delta_A^B \equiv (\sum_{i \in B} \lambda_i) / (\sum_{i \in A} \lambda_i)$ is the utility comparison rate*

between coalitions A and B , and $u_{AUB} = (u_A + \delta_A^B u_B)/(1 + \delta_A^B)$. If A_1, A_2, \dots, A_q is a partition of N and let $\delta_1^{A_i} \equiv (\sum_{i \in A_i} \lambda_i)$, then $u_N = \sum_{i=1}^q \delta_1^{A_i} u_{A_i} / \sum_{i=1}^q \delta_1^{A_i}$.

Proof. Notice that

$$u_{AUB} = \frac{\sum_{i \in A} \lambda_i u_i + \sum_{i \in B} \lambda_i u_i}{\sum_{i \in AUB} \lambda_i} = \frac{(\sum_{i \in A} \lambda_i) u_A + (\sum_{i \in B} \lambda_i) u_B}{\sum_{i \in AUB} \lambda_i} = \frac{u_A + \delta_A^B u_B}{1 + \delta_A^B}.$$

■

6 Discussion

The key building block in our theory for aggregating individuals' preferences is the bilateral agreement on a chosen parameter (utility of a consequence, weight of an attribute, willingness to pay, etc.) between two individuals or two coalitions. To reach the bilateral agreement, both parties must compromise. In committee decisions, such compromises are commonplace. Nevertheless, bilateral agreements can be difficult to achieve in some circumstances. An interesting research question is to investigate the case where individuals are unable to reach a complete agreement. Technically, one must relax the completeness assumption at the pair level. It is quite possible that the preferred set of alternatives is narrowed down considerably even when a consensus cannot be reached on the chosen parameter. The presence of a higher level decision maker or an arbitrator helps a great deal in narrowing down the agreement zone. In a practical application of choosing a new product design for introduction where Marketing assigned a greater weight to early market entry, but R&D assigned a greater weight to product features and favored a somewhat delayed market entry, the presence of the Vice President of the division greatly facilitated an agreement on the importance weights assigned to these attributes (see Sarin, 1993).

A related issue is that a compromise on the weight of a single attribute determines the pair utility, and eventually the group utility, through $n - 1$ such bilateral agreements. We caution against using minimal information in a practical application. Instead, we recommend that once a bilateral agreement is reached the full implications of this agreement should be shown to both parties. They must understand the implied compromises on the weights of other attributes and should feel comfortable with the ranking of alternatives that their agreement could produce. This, of course,

is not a new suggestion, as in multiattribute decision analysis it is a routine practice to seek more information than necessary and to use consistency checks to aid the decision maker in the exploration of his preferences.

We note that our approach is equally applicable to decisions under certainty. In this case, individual and group preferences are defined through a measurable value function (Krantz et al., 1971; Dyer and Sarin, 1979a, 1979b). Our theory requires that the preference functions be cardinal (invariant with respect to a positive linear transformation) and thus our approach is not restricted to the von Neumann-Morgenstern utility functions.

Finally, once the Extended Pareto Rule is firmly adhered to, our theory can easily accommodate other forms of multiattribute utility functions (multilinear or more general forms). We believe that the Extended Pareto Rule is a much more appealing normative requirement than the exact form of the multiattribute utility function - additive, multiplicative or some other. It would, however, be interesting to explore the extensions of our theory to cases where the Extended Pareto Rule is relaxed. Such extensions are likely to be nontrivial and would require imposing some constraints on the structure of the group utility.

Several authors have noted that many multicriteria problems are resolved in group settings and that these require aggregating individual weights and preferences (Harte et al. 1996; Limayem and DeSanctis, 2000). Our theory provides a systematic way to elicit group preference from individual preferences. In the spirit of the divide and conquer strategy of decision analysis, we build the group preference through bilateral agreements between two individuals at a time. In multiattribute decisions, the pair of individuals is required to reach an agreement only on the weight of one chosen attribute. Such an agreement specifies the pair utility completely. The group utility is derived from suitably chosen $(n - 1)$ pair utilities. Thus, at most $(n - 1)$ bilateral agreements are needed to completely specify the group utility. An appealing form for the group utility is derived for the case in which individual multiattribute utility functions are additive. In this case, the group utility can be represented as a weighted sum of group attribute weights and group attribute evaluation functions. The group attribute weight for a given attribute is the weighted sum of the individual attribute weights for that attribute. The group attribute evaluation function for a given attribute is the weighted sum of the individual attribute evaluation functions

for that attribute.

When individual multiattribute utility functions are multiplicative, the group utility, in general, does not have an analogous multiplicative representation in terms of group attribute weights and group attribute evaluation functions. It is, however, possible to derive the group utility through $(n - 1)$ bilateral agreements similar to that in the additive case.

In the literature on the aggregation of individual utilities into group utility (Harsanyi 1955; Keeney and Raiffa, 1976; Sen, 1970), there is little guidance provided on how the members of the group who are jointly responsible for making a decision should carry out the interpersonal comparisons of utilities. We have suggested in this paper that bilateral agreements can be fruitfully employed to derive the group utility. Though we cannot avoid the interpersonal comparisons altogether, our theory simplifies such comparisons in two ways: (1) interpersonal comparisons are made only at the pair level, and (2) even at the pair level, the individuals compromise on a chosen parameter, say the weight of an attribute, and the interpersonal comparison is inferred through the compromise reached. Since such compromises are commonplace in group decisions, our results can be useful in practical applications.

Decision analysis has a strong tradition of breaking down complex problems into simple parts and then combining the information collected on these parts to reach a decision. In multiattribute decisions, for example, the multiattribute preference function is built from attribute weights and attribute evaluation functions. The attribute weights, for example, are elicited by restricting attention to trade-offs between two attributes at a time. Independence conditions (utility independence, for example) are used to justify a particular decomposition. In a similar vein, our approach uses the Extended Pareto Rule to build the group preference from individual and pair preferences. Since the aggregation of preferences is at the heart of a group decision problem, we have provided a theoretically sound way to approach the preference aggregation issue at the simplest level. We hope that our work will spark additional theoretical and applied research into the somewhat neglected, but important, area of group decision making.

A Desargues Theorem

Theorem 3 is a non-trivial generalization of Theorem 2. To illustrate one of the difficulties, and its unexpected resolution, consider a case with four individuals. For expository purposes we add a fourth point in Figure 1 corresponding to u_4 , the utility of a new individual viewed as a point in a vector space (see Figure 2). Let $\mathcal{T} = \{12, 23, 34\}$ be the spanning tree of bilateral agreements. The application of Theorem 2 using $u_2, u_3, u_4, \delta_2^3$, and δ_3^4 produces a complete preference u_{234} , with u_{234} as the intersection of u_2u_{34} and $u_{23}u_4$. Because we already u_{123} , we could obtain u_{1234} using the intersection of the lines $\overline{u_{123}u_4}$ and $\overline{u_1u_{234}}$. However, there is a third segment available, namely $\overline{u_{12}u_{34}}$. It is impossible to have consistent and complete preferences unless these three segments are *concurrent*, i.e., they have a common point of intersection. This difficulty can be addressed in geometric terms by means of Desargues' theorem.

Theorem 10 (DESARGUES 1648) *Let p_i and q_i , for $i = 1, 2, 3$ be two sets of independent points in a vector space satisfying $p_i \neq q_i$ ($i = 1, 2, 3$). Then, the segments p_iq_i , $i = 1, 2, 3$ are concurrent if and only if the three points $s_{ij} = p_iq_j \cap q_iq_j$, $1 \leq i < j \leq 3$ are collinear.*

Figure 2 represents Desargues' theorem as applied to

$$\left. \begin{array}{lll} p_1 = u_1 & p_2 = u_{12} & p_3 = u_{123} \\ q_1 = u_{234} & q_2 = u_{34} & q_3 = u_4 \end{array} \right\} \Rightarrow s_{12} = u_2 \quad s_{13} = u_{23} \quad s_{23} = u_3$$

By *EPR*, $s_{13} \in s_{12}s_{23}$ so that the line segments u_1u_{234} , $u_{12}u_{34}$, and $u_{123}u_4$ are concurrent: u_{1234} is well defined. To see that $u_{1234} \in u_{13}u_{24}$, declare $p'_2 = u_{13}$ and $q'_2 = u_{24}$, and maintain the other four points. The desired conclusion follows from $s'_{13} = u_{23} \in u_3u_2 = s'_{12}s'_{23}$.

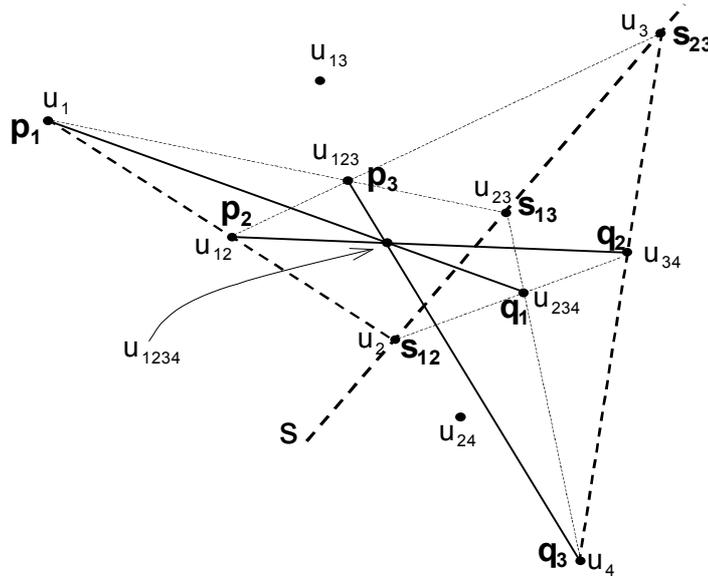


Figure 2: Desargues' Theorem.

References

- [1] Arrow, K. J. 1963. *Social Choice and Individual Values*. Second edition. Wiley, New York, NY.
- [2] Aumann, R. J. 1962. Utility theory without the completeness axiom. *Econometrica*. **30**, 445–462.
- [3] ———. 1964 Utility theory without the completeness axiom—A correction. *Econometrica*. **32**, 210–212.
- [4] Baucells, M., L. S. Shapley. 2000. Multiperson utility. The Anderson School at UCLA.
- [5] Dubra, J., E. A. Ok. 1999. A model of procedural decision making in the presence of risk. Princeton University.
- [6] Dyer, J. S., R. K. Sarin. 1979. Group preference aggregation rules based on strength of preference. *Management Science*. **25** (9), 822–832.
- [7] ———. 1979. Measurable multiattribute value functions. *Operations Research*. **27**, 810–822.
- [8] Harsanyi, J. C. 1955. Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. *Journal of Political Economy*. **63**, 309–321.
- [9] Harte, Johanna M., P. Koele, and G. van Engelenburg. 1996. Estimation of attribute weights in a multiattribute choice situation. *Acta Psychologica*. **93**, 37–55.

- [10] Keeney, R. L., C. Kirkwood. 1975. Group decision making using cardinal social welfare functions. *Management Science*. **22**, 430–437.
- [11] Keeney, R. L., H. Raiffa. 1976. *Decisions with Multiple Objectives*. Wiley, New York.
- [12] Krantz, D. H., R. D. Luce, P. Suppes, A. Tversky. 1971. *Foundations of Measurement*. Volume One. Academic Press, New York, NY.
- [13] Limayem, M., G. DeSanctis. 2000. Providing decisional guidance for multicriteria decision making in groups. *Information Systems Research*. **11** (4), 386-401.
- [14] Sarin, R. K. 1993. A decision analysis approach for coordinating design manufacturing, and marketing in new product decisions. Sarin, R. K., ed. *Perspectives in Operations Management—Essays in Honor of Elwood S. Buffa*. Kluwer. 189-208.
- [15] Sen, A. K. 1970. *Collective Choice and Social Welfare*. Holden-Day, San Francisco, CA.
- [16] ———. 1977. Social choice theory—a re-examination. *Econometrica*. **45**, 53–90.
- [17] von Neumann, J., O. Morgenstern. 1947. *Theory of Games and Economic Behavior*. Second edition. Princeton University Press, Princeton, NJ.