# Optimality of Affine Policies in Multi-stage Robust Optimization 

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#### Abstract

In this paper, we show the optimality of a certain class of disturbance-affine control policies in the context of one-dimensional, constrained, multi-stage robust optimization. Our results cover the finite horizon case, with minimax (worst-case) objective, and convex state costs plus linear control costs. We develop a new proof methodology, which explores the relationship between the geometrical properties of the feasible set of solutions and the structure of the objective function. Apart from providing an elegant and conceptually simple proof technique, the approach also entails very fast algorithms for the case of piecewise affine state costs, which we explore in connection with a classical inventory management application.


## 1 Introduction.

Multi-stage optimization problems under uncertainty have been prevalent in numerous fields of science and engineering, and have elicited interest from diverse research communities, on both a theoretical and a practical level. Several solution approaches have been proposed, with various degrees of generality, tractability, and performance guarantees. Some of the most successful ones include exact and approximate dynamic programming, stochastic programming, sampling-based methods, and, more recently, robust and adaptive optimization, which is the focus of the present paper.

The topics of robust optimization and robust control have been studied, under different names, by a variety of academic groups, mostly in operations research (Ben-Tal and Nemirovski [1999], Ben-Tal et al. [2002], Ben-Tal and Nemirovski [2002], Bertsimas and Sim [2003], Bertsimas and Sim [2004], Bertsimas et al. [2004]) and control theory (Bertsekas and Rhodes [1971], Fan et al. [1991], El-Ghaoui et al. [1998], Grieder et al. [2003], Bemporad et al. [2003], Kerrigan and Maciejowski [2004], Zhou and Doyle [1998], Dullerud and Paganini [2005]), with considerable effort put into justifying the assumptions and general modeling philosophy. As such, the goal of the current paper will not be to justify the use of robust (and, more generally, distribution-free) techniques. Rather, we will take the modeling approach as a given, and investigate tractability and performance issues in the context of a certain class of optimization problems. More precisely, we will be concerned with the following multi-stage decision problem:

Problem 1.1. Consider the following one-dimensional, discrete, linear, time-varying dynamical system:

$$
\begin{equation*}
x_{k+1}=\alpha_{k} \cdot x_{k}+\beta_{k} \cdot u_{k}+\gamma_{k} \cdot w_{k} \tag{1}
\end{equation*}
$$

where $\alpha_{k}, \beta_{k}, \gamma_{k} \neq 0$ are known scalars, and the initial state $x_{1} \in \mathbf{R}$ is specified. The random disturbances $w_{k}$ are unknown, but bounded:

$$
\begin{equation*}
w_{k} \in \mathcal{W}_{k} \stackrel{\text { def }}{=}\left[\underline{w}_{k}, \bar{w}_{k}\right] . \tag{2}
\end{equation*}
$$

We would like to find a sequence of robust controllers $\left\{u_{k}\right\}$, obeying certain constraints:

$$
\begin{equation*}
u_{k} \in\left[L_{k}, U_{k}\right] \tag{3}
\end{equation*}
$$

[^0]( $L_{k}, U_{k}$ are known and fixed), and optimizing the following min-max cost function over a finite horizon $1, \ldots, T$ :
\[

$$
\begin{align*}
& J_{m M} \stackrel{\text { def }}{=} \min _{u_{1}}\left[c_{1} \cdot u_{1}+\max _{w_{1}}\left[h_{1}\left(x_{2}\right)+\cdots+\min _{u_{k}}\left[c_{k} \cdot u_{k}+\max _{w_{k}}\left[h_{k}\left(x_{k+1}\right)+\ldots\right.\right.\right.\right. \\
&\left.\left.\left.+\min _{u_{T}}\left(c_{T} \cdot u_{T}+\max _{w_{T}} h_{T}\left(x_{T+1}\right)\right) \cdots\right] \cdots\right]\right] \tag{4}
\end{align*}
$$
\]

where the functions $h_{k}: \mathbf{R} \rightarrow \mathbf{R}$ are known, convex and coercive, and $c_{k} \geq 0$ are fixed and known.
The problem corresponds to a situation in which, at every time step $k$, the decision maker has to compute a control action $u_{k}$, in such a way that certain constraints (3) are obeyed, and a cost penalizing both the state $\left(h_{k}\left(x_{k+1}\right)\right)$ and the control $\left(c_{k} \cdot u_{k}\right)$ is minimized. The uncertainty, $w_{k}$, always acts so as to maximize the costs, hence the problem solved by the decision maker corresponds to a worst-case scenario (a minimization of the maximum possible cost). Examples of such problems include the case of quadratic state costs, $h_{k}\left(x_{k+1}\right)=$ $r_{k+1} \cdot x_{k+1}^{2},\left(r_{k+1} \geq 0\right)$, as well as norm-1 or norm- $\infty \operatorname{costs}, h_{k}\left(x_{k+1}\right)=r_{k+1} \cdot\left|x_{k+1}\right|,\left(r_{k+1} \geq 0\right)$, all of which have been studied extensively in the literature in the unconstrained case (see, for example, Zhou and Doyle [1998], and Dullerud and Paganini [2005]).

The solution to Problem 1.1 could be obtained using a "classical" Dynamic Programming formulation (see Bertsekas [2001]), in which the optimal policies $u_{k}^{*}\left(x_{k}\right)$ and the optimal value functions $J_{k}^{*}\left(x_{k}\right)$ are computed backwards in time, starting at the end of the planning horizon, $k=T$. The resulting policies are piecewise affine in the state $x_{t}$, and have properties that are well known and documented in the literature, dating back to Scarf et al. [1958].

In the current paper, we would like to study the performance of a new class of policies, where instead of regarding the controllers $u_{k}$ as functions of the state $x_{k}$, one seeks direct parameterizations in the observed disturbances:

$$
\begin{equation*}
u_{k}: \mathcal{W}_{1} \times \mathcal{W}_{2} \times \cdots \times \mathcal{W}_{k-1} \rightarrow \mathbf{R} \tag{5}
\end{equation*}
$$

In this framework, we would require that constraint (3) should be robustly feasible:

$$
\begin{equation*}
u_{k}\left(\boldsymbol{w}^{k}\right) \in\left[L_{k}, U_{k}\right], \quad \forall \boldsymbol{w}^{k} \stackrel{\text { def }}{=}\left(w_{1}, \ldots, w_{k-1}\right) \in \mathcal{W}_{1} \times \cdots \times \mathcal{W}_{k-1} \tag{6}
\end{equation*}
$$

Note that if we insisted on this category of parameterizations, then we would have to consider a new state for the system, $\boldsymbol{X}_{k}$, which would include all the past-observed disturbances, $\boldsymbol{w}^{k}$. Furthermore, while $x_{k}$ summarizes all the past information needed to make an optimal decision at stage $k$, the same would not necessarily be true for $\boldsymbol{w}^{k}$, so we may want to include even more information in $\boldsymbol{X}_{k}$ (for example, the previous controls $\left\{u_{t}\right\}_{1 \leq t<k}$ or the previous states $\left\{x_{t}\right\}_{1 \leq t<k}$ or some combination thereof). Compared with the original, compact state formulation, $x_{k}$, the new state $\boldsymbol{X}_{k}$ would become much larger, and solving the Dynamic Program with state variable $\boldsymbol{X}_{k}$ would produce exactly the same optimal objective function value $J_{m M}$. Therefore, one would be tempted to ask what the benefit for introducing such a complicated state might be.

The hope is that, by considering policies over a larger state, one might be able to obtain simpler functional forms, for instance, affine policies. These have a very compact representation, since only the coefficients of the parameterization are needed, and, for certain classes of convex costs $h_{k}(\cdot)$, there may be efficient procedures available for computing them.

This approach is also not new in the literature. It has been originally advocated in the context of stochastic programming (Rockafellar and Wets [1978]), then in robust optimization (Ben-Tal et al. [2004]), and extended to linear systems theory (Ben-Tal et al. [2005a], Ben-Tal et al. [2006]), with notable contributions from researchers in robust model predictive control and receding horizon control (see Bemporad et al. [2003], Kerrigan and Maciejowski [2004], Löfberg [2003], Skaf and Boyd [2008], and references therein). In all the papers, which usually deal with the more general case of multi-dimensional linear systems, the authors show how the reformulation can be done, and how the corresponding affine policies can be found by solving specific types of optimization problems, which vary from linear and quadratic programs (Ben-Tal et al. [2005a], Kerrigan and Maciejowski [2004], Kerrigan and Maciejowski [2003]) to conic and semi-definite (Ben-Tal et al. [2005a], Löfberg [2003], Bertsimas and Brown [2007], Grieder et al. [2003]), or even multi-parametric, linear or quadratic programs (Bemporad et al. [2003]). The first steps towards analyzing the properties of
such parameterizations were made in Kerrigan and Maciejowski [2004], where the authors show that, under suitable conditions, the resulting affine parameterization has certain desirable system theoretic properties (stability and robust invariance). Another notable contribution was made by Goulart and Kerrigan [2005], who prove that the class of affine disturbance feedback policies is equivalent to the class of affine state feedback policies with memory of prior states, thus subsuming the well known classes of open-loop and prestabilizing control policies. However, to the best of our knowledge, apart from these theoretical advances, there has been very little progress in proving results about the quality of the objective function values resulting from the use of such parameterizations.

Our main result, summarized in Theorem 3.1 of Section 3, is that, for Problem 1.1 stated above, affine policies of the form (5) are, in fact, optimal. Furthermore, we are able to prove that a certain (affine) relaxation of the state costs is also possible, without any loss of optimality, which gives rise to very efficient algorithms for computing the optimal affine policies when the state costs $h_{k}(\cdot)$ are piece-wise affine. To the best of our knowledge, this is the first result of its kind, and it provides intuition and motivation for the widespread advocation of such policies in both theory and applications. Our theoretical results are tight (if the conditions in Problem 1.1 are slightly perturbed, then simple counterexamples for Theorem 3.1 can be found), and the proof of the theorem itself is atypical, consisting of a forward induction and making use of polyhedral geometry to construct the optimal affine policies. Thus, we are able to gain insight into the structure and properties of these policies, which we explore in connection with a classical inventory management problem.

The paper is organized as follows. Section 2 presents an overview of the Dynamic Programming formulation in state variable $x_{k}$, extracting the optimal policies $u_{k}^{*}\left(x_{k}\right)$ and optimal value functions $J_{k}^{*}\left(x_{k}\right)$, as well as some of their properties. Section 3 contains our main result, and briefly discusses some immediate extensions and computational implications. In Section 4, we introduce the constructive proof for building the affine control policies and the affine cost relaxations, and present counterexamples that prevent a generalization of the results. In Section 5, we explore our results in connection with a classical inventory management problem. Section 6 presents our conclusions and directions for future research.

## 2 Dynamic Programming Solution.

As already mentioned in the introduction, the solution to Problem 1.1 can be obtained using a "classical" Dynamic Programming formulation [Bertsekas, 2001], in which the state is taken to be $x_{k}$, and the optimal policies $u_{k}^{*}\left(x_{k}\right)$ and optimal value functions $J_{k}^{*}\left(x_{k}\right)$ are computed starting at the end of the planning horizon, $k=T$, and moving backwards in time. In this section, we will briefly outline the solution of the Dynamic Program for our problem, and will state some of the key properties that will be used throughout the rest of the paper. For completeness, a full proof of the results is included in Section 7.1 of the Appendix.

In order to simplify the notation, we remark that, since the constraints on the controls $u_{k}$ and the bounds on the disturbances $w_{k}$ are time-varying, and independent for different time-periods, we can restrict attention, without loss of generality ${ }^{1}$, to a system with $\alpha_{k}=\beta_{k}=\gamma_{k}=1$. With this simplification, the problem that we would like to solve is the following:
(DP)

$$
\left.\left.\begin{array}{rl}
J_{m M} \stackrel{\text { def }}{=} & \min _{u_{1}}\left[c_{1} \cdot u_{1}+\max _{w_{1}}[ \right.
\end{array} h_{1}\left(x_{2}\right)+\cdots+\min _{u_{k}}\left[c_{k} \cdot u_{k}+\max _{w_{k}}\left[h_{k}\left(x_{k+1}\right)+\ldots\right] \text { } \quad+\min _{u_{T}}\left(c_{T} \cdot u_{T}+\max _{w_{T}} h_{T}\left(x_{T+1}\right)\right) \ldots\right] \ldots\right]\right] \quad \begin{aligned}
\text { s.t. } & x_{k+1}=x_{k}+u_{k}+w_{k} \\
& L_{k} \leq u_{k} \leq U_{k} \\
& w_{k} \in \mathcal{W}_{k}=\left[\underline{w}_{k}, \bar{w}_{k}\right] . \quad \forall k \in\{1,2, \ldots, T\}
\end{aligned}
$$

[^1]The corresponding Bellman recursion for $(D P)$ can then be written as follows:

$$
J_{k}^{*}\left(x_{k}\right) \stackrel{\text { def }}{=} \min _{L_{k} \leq u_{k} \leq U_{k}}\left[c_{k} \cdot u_{k}+\max _{w_{k} \in \mathcal{W}_{k}}\left[h_{k}\left(x_{k}+u_{k}+w_{k}\right)+J_{k+1}^{*}\left(x_{k}+u_{k}+w_{k}\right)\right]\right]
$$

where $J_{T+1}^{*}\left(x_{T+1}\right) \equiv 0$. By defining:

$$
\begin{align*}
y_{k} & \stackrel{\text { def }}{=} x_{k}+u_{k} \\
g_{k}\left(y_{k}\right) & \stackrel{\text { def }}{=} \max _{w_{k} \in \mathcal{W}_{k}}\left[h_{k}\left(y_{k}+w_{k}\right)+J_{k+1}^{*}\left(y_{k}+w_{k}\right)\right] \tag{7}
\end{align*}
$$

we obtain the following solution to the Bellman recursion (see Section 7.1 in the Appendix for the derivation):

$$
\begin{align*}
& u_{k}^{*}\left(x_{k}\right)= \begin{cases}U_{k}, & \text { if } x_{k}<\underline{y}_{k}-U_{k} \\
-x_{k}+y^{*}, & \text { otherwise } \\
L_{k}, & \text { if } x_{k}>\bar{y}_{k}-L_{k}\end{cases}  \tag{8}\\
& J_{k}^{*}\left(x_{k}\right)=c_{k} \cdot u_{k}^{*}\left(x_{k}\right)+g_{k}\left(x_{k}+u_{k}^{*}\left(x_{k}\right)\right)= \begin{cases}c_{k} \cdot U_{k}+g_{k}\left(x_{k}+U_{k}\right), & \text { if } x_{k}<\underline{y}_{k}-U_{k} \\
c_{k} \cdot\left(y^{*}-x_{k}\right)+g_{k}\left(y^{*}\right), & \text { otherwise } \\
c_{k} \cdot L_{k}+g_{k}\left(x_{k}+L_{k}\right), & \text { if } x_{k}>\bar{y}_{k}-L_{k}\end{cases} \tag{9}
\end{align*}
$$

where $y^{*} \in\left[\underline{y}_{k}, \bar{y}_{k}\right]$, and $\left[\underline{y}_{k}, \bar{y}_{k}\right]$ represents the (compact) set of minimizers of the convex function $c_{k} \cdot y+g_{k}(y)$. A typical example of the optimal control law and the optimal value function is shown in Figure 1.


Figure 1: Optimal control law $u_{k}^{*}\left(x_{k}\right)$ and optimal value function $J_{k}^{*}\left(x_{k}\right)$ at time $k$.

The main properties of the solution that will be relevant for our later treatment are listed below:
P1 The optimal control law $u_{k}^{*}\left(x_{k}\right)$ is piecewise affine, with 3 pieces, continuous and non-increasing.
P2 The optimal value function, $J_{k}^{*}\left(x_{k}\right)$, and the function $g_{k}\left(y_{k}\right)$ are convex.
P3 The difference in the values of the optimal control law at two distinct arguments $s \leq t$ always satisfies: $u_{k}^{*}(s)-u_{k}^{*}(t)=-f_{k} \cdot(s-t)$, for some $f_{k} \in[0,1]$.
$\mathbf{P} 4$ The optimal value function, $J_{k}^{*}\left(x_{k}\right)$, has a subgradient at most $-c_{k}$ for $x_{k}<\underline{y}_{k}-U_{k}$, exactly $-c_{k}$ in the interval $\left[\underline{y}_{k}-U_{k}, \bar{y}_{k}-L_{k}\right]$, and at least $-c_{k}$ for $x_{k}>\bar{y}_{k}-L_{k}$.

P5 The function $g_{k}\left(y_{k}\right)$ is decreasing, with a subgradient at most $-c_{k}$, in the interval $\left(-\infty, \underline{y}_{k}\right]$, and is increasing, with a sugradient at least $-c_{k}$, in the interval $\left[\bar{y}_{k}, \infty\right)$.

## 3 Optimality of Affine Policies in $\boldsymbol{w}^{k}$.

In this section, we introduce our main contribution, namely a proof that policies that are affine in the disturbances $\boldsymbol{w}^{k}$ are, in fact, optimal for problem $(D P)$. Using the same notation as in Section 2, we can summarize our main result in the following theorem:
Theorem 3.1. For every time step $k=1, \ldots, T$, the following quantities exist:

- An affine control policy:

$$
\begin{equation*}
q_{k}\left(\boldsymbol{w}^{k}\right) \stackrel{\text { def }}{=} q_{k, 0}+\sum_{t=1}^{k-1} q_{k, t} \cdot w_{t} \tag{10}
\end{equation*}
$$

- An affine running cost:

$$
\begin{equation*}
z_{k}\left(\boldsymbol{w}^{k+1}\right) \stackrel{\text { def }}{=} z_{k, 0}+\sum_{t=1}^{k} z_{k, t} \cdot w_{t} \tag{11}
\end{equation*}
$$

such that the following properties are obeyed:

$$
\begin{array}{ll}
L_{k} \leq q_{k}\left(\boldsymbol{w}^{k}\right) \leq U_{k}, & \forall \boldsymbol{w}^{k} \in \mathcal{W}_{1} \times \cdots \times \mathcal{W}_{k-1} \\
z_{k}\left(\boldsymbol{w}^{k+1}\right) \geq h_{k}\left(x_{1}+\sum_{t=1}^{k}\left(q_{t}\left(\boldsymbol{w}^{t}\right)+w_{t}\right)\right), & \forall \boldsymbol{w}^{k+1} \in \mathcal{W}_{1} \times \cdots \times \mathcal{W}_{k} \\
J_{m M}=\max _{w_{1}, \ldots, w_{k}}\left[\sum_{t=1}^{k}\left(c_{t} \cdot q_{t}\left(\boldsymbol{w}^{t}\right)+z_{t}\left(\boldsymbol{w}^{t+1}\right)\right)+J_{k+1}^{*}\left(x_{1}+\sum_{t=1}^{k}\left(q_{t}\left(\boldsymbol{w}^{t}\right)+w_{t}\right)\right)\right] . \tag{14}
\end{array}
$$

Let us interpret the main statements and results in the theorem. Equation (12) confirms the existence of an affine policy $q_{k}\left(\boldsymbol{w}^{k}\right)$ that is robustly feasible, i.e. that obeys the control constraints, no matter what the realization of the disturbances may be. Equation (13) states the existence of an affine cost $z_{k}\left(\boldsymbol{w}^{k+1}\right)$ that is always larger than the convex state cost $h_{k}\left(x_{k+1}\right)$ incurred when the affine policies $\left\{q_{t}(\cdot)\right\}_{1 \leq t \leq k}$ are used. Equation (14) guarantees that, despite using the (suboptimal) affine control law $q_{k}\left(\boldsymbol{w}^{k}\right)$, and incurring a (potentially larger) affine stage cost $z_{k}\left(\boldsymbol{w}^{k+1}\right)$, the overall objective function value $J_{m M}$ is, in fact, not increased. This translates in the following two main results:

- Existential result. Affine policies $q_{k}\left(\boldsymbol{w}^{k}\right)$ are, in fact, optimal for Problem 1.1.
- Computational result. When the convex costs $h_{k}\left(x_{k+1}\right)$ are piecewise affine, the optimal affine policies $\left\{q_{k}\left(\boldsymbol{w}^{k}\right)\right\}_{1 \leq k \leq T}$ can be computed by solving a Linear Programming problem.
To see why the second implication would hold, suppose that $h_{k}\left(x_{k+1}\right)$ is the maximum of $m_{k}$ affine functions:

$$
h_{k}\left(x_{k+1}\right)=\max \left(p_{k}^{i} \cdot x_{k+1}+p_{k, 0}^{i}\right), i \in\left\{1, \ldots, m_{k}\right\} .
$$

Then an optimal affine policy $q_{k}\left(\boldsymbol{w}^{k}\right)$ can be obtained by solving the following optimization problem (see Ben-Tal et al. [2005a,b]):

$$
\min _{J ;\left\{q_{k, t}\right\} ;\left\{z_{k, t}\right\}}
$$

$$
\text { s.t. } \quad \forall \boldsymbol{w} \in \mathcal{W}_{1} \times \mathcal{W}_{2} \times \cdots \times \mathcal{W}_{T}, \quad \forall k=1, \ldots, T:
$$

$$
\begin{align*}
\forall i \in\left\{1, \ldots, m_{k}\right\}: \quad & z_{k, 0}+\sum_{t=1}^{k} z_{k, t} \cdot w_{t} \geq p_{k}^{i} \cdot\left[x_{1}+\sum_{t=1}^{k}\left(q_{t, 0}+\sum_{\tau=1}^{t-1} q_{t, \tau} \cdot w_{\tau}+w_{t}\right)\right]+p_{k, 0}^{i}  \tag{AARC}\\
& L_{k} \leq q_{k, 0}+\sum_{t=1}^{k-1} q_{k, t} \cdot w_{t} \leq U_{k}
\end{align*}
$$

Although Problem $(A A R C)$ is still a semi-infinite LP (due to the requirement of robust constraint feasibility, $\forall \boldsymbol{w})$, since all the constraints are inequalities that are bi-affine in the decision variables and the uncertain quantities, a very compact reformulation of the problem is available. In particular, with a typical constraint in $(A A R C)$ written as:

$$
\lambda_{0}[\boldsymbol{x}]+\sum_{t=1}^{T} \lambda_{t}[\boldsymbol{x}] \cdot w_{t} \leq 0, \quad \forall \boldsymbol{w} \in \mathcal{W}_{1} \times \mathcal{W}_{2} \times \cdots \times \mathcal{W}_{T}
$$

where $\lambda[\boldsymbol{x}]$ denotes that $\lambda$ is an affine function of the decision variables $\boldsymbol{x}$, it can be shown (see Ben-Tal and Nemirovski [2002], Ben-Tal et al. [2004] for details) that the previous condition is equivalent to:

$$
\left\{\begin{array}{l}
\lambda_{0}[\boldsymbol{x}]+\sum_{t=1}^{T}\left(\lambda_{t}[\boldsymbol{x}] \cdot \frac{\underline{w}_{t}+\bar{w}_{t}}{2}+\frac{\bar{w}_{t}-\underline{w}_{t}}{2} \cdot \xi_{t}\right) \leq 0  \tag{15}\\
-\xi_{t} \leq \lambda_{t}[\boldsymbol{x}] \leq \xi_{t}, \quad t=1, \ldots, T
\end{array}\right.
$$

which are linear constraints in the decision variables $\boldsymbol{x}, \boldsymbol{\xi}$. Therefore, $(A A R C)$ can be reformulated as a Linear Program, with $O\left(T^{2} \cdot \max _{k} m_{k}\right)$ variables and $O\left(T^{2} \cdot \max _{k} m_{k}\right)$ constraints, which can be solved very efficiently using commercially available software.

We will conclude our observations by making one last remark related to an immediate extension of the results. Note that in the statement of Problem 1.1, there was no mention about constraints on the states $x_{k}$ of the dynamical system. In particular, one may want to incorporate lower or upper bounds on the states, as well:

$$
\begin{equation*}
L_{k}^{x} \leq x_{k} \leq U_{k}^{x} \tag{16}
\end{equation*}
$$

We claim that, in case the mathematical problem including such constraints remains feasible ${ }^{2}$, then affine policies will, again, be optimal. The reason is that such constraints can always be simulated in our current framework, by adding suitable convex barriers to the stage costs $h_{k}\left(x_{k+1}\right)$. In particular, by considering the modified, convex ${ }^{3}$ stage costs:

$$
\begin{gathered}
\tilde{h}_{k}\left(x_{k+1}\right) \stackrel{\text { def }}{=} h_{k}\left(x_{k+1}\right)+\mathbf{1}_{\left[L_{k+1}^{x}, U_{k+1}^{x}\right]}\left(x_{k+1}\right), \text { where } \\
\mathbf{1}_{\left[L_{k+1}^{x}, U_{k+1}^{x}\right]}\left(x_{k+1}\right) \stackrel{\text { def }}{=} \begin{cases}0, & \text { if } x_{k+1} \in\left[L_{k+1}^{x}, U_{k+1}^{x}\right] \\
\infty, & \text { otherwise },\end{cases}
\end{gathered}
$$

it can be easily seen that the original problem, with convex stage costs $h_{k}(\cdot)$ and state constraints (16), is equivalent to a problem with the modified stage costs $\tilde{h}_{k}(\cdot)$ and no state constraints. And, since affine policies are optimal for the latter problem, the result is immediate. Therefore, our decision to exclude such constraints from the original formulation was made only for sake of brevity and conciseness of the proofs, but without loss of generality.

## 4 Proof of Main Theorem.

The current section contains the proof of Theorem 3.1. Before presenting the details, we would first like to give some intuition behind the strategy of the proof, and introduce the organization of the material.

Unlike most Dynamic Programming proofs, which utilize backward induction on the time-periods, we will have to proceed with a forward induction. Section 4.1 will present a test of the first step of the induction, and will then introduce a detailed analysis of the consequences of the induction hypothesis.

We will then separate the completion of the induction step into two parts. In the first part, handled in Section 4.2, by exploiting the structure provided by the forward induction hypothesis, and making critical use of the properties of the optimal control law $u_{k}^{*}\left(x_{k}\right)$ and optimal value function $J_{k}^{*}\left(x_{k}\right)$ (the solutions to problem $(D P)$ in Section 2), we will be able to introduce a candidate affine policy $q_{k}\left(\boldsymbol{w}^{k}\right)$. In Section

[^2]4.2.1, we will then prove that this policy is robustly feasible, and preserves the min-max value of the overall problem, $J_{m M}$, when used in conjunction with the original, convex state costs, $h_{k}\left(x_{k+1}\right)$.

Similarly, for the second part of the inductive step, (Section 4.3), by re-analyzing the feasible sets of the optimization problems resulting after the use of the (newly computed) affine policy $q_{k}\left(\boldsymbol{w}^{k}\right)$, we will determine a candidate affine cost $z_{k}\left(\boldsymbol{w}^{k+1}\right)$, which we will prove (Section 4.3.1) to be always larger than the original convex state costs, $h_{k}\left(x_{k+1}\right)$. However, despite this fact, in Section 4.3 .1 we will also show that when this affine cost is incurred, the overall min-max value $J_{m M}$ remains unchanged, which will complete the proof of the inductive step.

Section 4.3.2 will then conclude the proof of Theorem 3.1, and will outline several counterexamples that prevent an immediate extension of the result to more general cases.

### 4.1 Induction Hypothesis.

As mentioned before, the proof of the theorem will utilize a forward induction on the time-step $k$. Let us begin by verifying the induction at $k=1$.

Using the same notation as in Section 2, by taking the affine control to be $q_{1} \stackrel{\text { def }}{=} u_{1}^{*}\left(x_{1}\right)$, we immediately get that $q_{1}$, which is simply a constant, is robustly feasible, so (12) is obeyed.

Furthermore, since $q_{1}=u_{1}^{*}\left(x_{1}\right)$ is optimal, we can write the overall optimal objective value, $J_{m M}$, as:

$$
\begin{align*}
J_{m M}=J_{1}^{*}\left(x_{1}\right) & =\min _{u_{1} \in\left[L_{1}, U_{1}\right]}\left[c_{1} \cdot u_{1}+g_{1}\left(x_{1}+u_{1}\right)\right]=c_{1} \cdot q_{1}+g_{1}\left(x_{1}+q_{1}\right) \stackrel{(61)}{=} \\
& \stackrel{(61)}{=} c_{1} \cdot q_{1}+\max \left\{\left(h_{1}+J_{2}^{*}\right)\left(x_{1}+q_{1}+\underline{w}_{1}\right),\left(h_{1}+J_{2}^{*}\right)\left(x_{1}+q_{1}+\bar{w}_{1}\right)\right\} . \tag{17}
\end{align*}
$$

Next, we introduce the affine cost $z_{1}\left(w_{1}\right) \stackrel{\text { def }}{=} z_{1,0}+z_{1,1} \cdot w_{1}$, where we constrain the coefficients $z_{1, i}$ to satisfy the following system of linear equations:

$$
\left\{\begin{array}{l}
z_{1,0}+z_{1,1} \cdot \underline{w}_{1}=h_{1}\left(x_{1}+q_{1}+\underline{w}_{1}\right) \\
z_{1,0}+z_{1,1} \cdot \bar{w}_{1}=h_{1}\left(x_{1}+q_{1}+\bar{w}_{1}\right) .
\end{array}\right.
$$

Note that for fixed $x_{1}$ and $q_{1}$, the function $z_{1}\left(w_{1}\right)$ is nothing but a linear interpolation of the mapping $w_{1} \mapsto h_{1}\left(x_{1}+q_{1}+w_{1}\right)$, matching the value at points $\left\{\underline{w}_{1}, \bar{w}_{1}\right\}$. Since $h_{1}$ is convex, the linear interpolation defined above clearly dominates it, so condition (13) is readily satisfied. Furthermore, by (17), $J_{m M}$ is achieved for $w_{1} \in\left\{\underline{w}_{1}, \bar{w}_{1}\right\}$, so condition (14) is also obeyed.

Having checked the induction at time $k=1$, let us now assume that the statements of Theorem 3.1 are true for times $t=1, \ldots, k$. Equation (14) written for stage $k$ then yields:

$$
\begin{align*}
& \qquad \begin{aligned}
& J_{m M}=\max _{w_{1}, \ldots, w_{k}}\left[\sum_{t=1}^{k}\left(c_{t} \cdot q_{t}\left(\boldsymbol{w}^{t}\right)+z_{t}\left(\boldsymbol{w}^{t+1}\right)\right)+J_{k+1}^{*}\left(x_{1}+\sum_{t=1}^{k}\left(q_{t}\left(\boldsymbol{w}^{t}\right)+w_{t}\right)\right)\right]= \\
&=\max _{\left(\theta_{1}, \theta_{2}\right) \in \Theta}\left[\theta_{1}+J_{k+1}^{*}\left(\theta_{2}\right)\right], \\
& \text { where } \quad \Theta \stackrel{\text { def }}{=}\left\{\left(\theta_{1}, \theta_{2}\right): \theta_{1} \stackrel{\text { def }}{=} \sum_{t=1}^{k}\left(c_{t} \cdot q_{t}\left(\boldsymbol{w}^{t}\right)+z_{t}\left(\boldsymbol{w}^{t+1}\right)\right), \quad \theta_{2} \stackrel{\text { def }}{=} x_{1}+\sum_{t=1}^{k}\left(q_{t}\left(\boldsymbol{w}^{t}\right)+w_{t}\right)\right\} .
\end{aligned} .
\end{align*}
$$

Since $\left\{q_{t}\right\}_{1 \leq t \leq k}$ and $\left\{z_{t}\right\}_{1 \leq t \leq k}$ are affine functions, this implies that, although the uncertainties $w_{1}, \ldots, w_{k}$ lie in a set with $2^{\bar{k}}$ vertices (the hyper-rectangle $\mathcal{W}_{1} \times \cdots \times \mathcal{W}_{k}$ ), they are only able to affect the objective $J_{m M}$ through affine combinations taking values in the set $\Theta$. Such a polyhedron, arising as a 2-dimensional affine projection of a $k$-dimensional hyper-rectangle, is called a zonogon. It belongs to a larger class of polytopes, known as zonotopes, whose combinatorial structure and properties are well documented in the discrete and computational geometry literature. The interested reader is referred to Chapter 7 of Ziegler [2003] for a very nice and accessible introduction.

The main properties of a zonogon that we are interested in are summarized in Lemma 7.2, found in the Appendix. In particular, the set $\Theta$ is centrally symmetric, and has at most $2 k$ vertices (see Figure 2 for an


Figure 2: Zonogon obtained from projecting a hypercube in $\mathbf{R}^{6}$.
example). Furthermore, by numbering the vertices of $\Theta$ in counter-clockwise fashion, starting at:

$$
\begin{equation*}
\boldsymbol{v}_{0} \equiv \boldsymbol{v}^{-} \stackrel{\text { def }}{=} \underset{\theta_{1}}{\arg \max }\left\{\underset{\theta_{2}}{\arg \min }\{\boldsymbol{\theta} \in \Theta\}\right\} \tag{20}
\end{equation*}
$$

we can establish the following result concerning the points of $\Theta$ that are relevant in our problem:
Lemma 4.1. The maximum value in optimization problem (18) is achieved for $\left(\theta_{1}, \theta_{2}\right) \in\left\{\boldsymbol{v}_{0}, \boldsymbol{v}_{1}, \ldots, \boldsymbol{v}_{k}\right\}$.
Proof. The optimization problem described in (18) and (19) is a maximization of a convex function over a convex set. Therefore (see Section 32 of Rockafellar [1970]), the maximum is achieved at the extreme points of the set $\Theta$, namely on the set $\left\{\boldsymbol{v}_{0}, \boldsymbol{v}_{1}, \ldots, \boldsymbol{v}_{2 p-1}, \boldsymbol{v}_{2 p} \equiv \boldsymbol{v}_{0}\right\}$, where $2 p$ is the number of vertices of $\Theta$. Letting $\boldsymbol{O}$ denote the center of $\Theta$, by part (iii) of Lemma 7.2 in the Appendix, we have that the vertex symmetrically opposed to $\boldsymbol{v}^{-}$, namely $\boldsymbol{v}^{+} \stackrel{\text { def }}{=} 2 \boldsymbol{O}-\boldsymbol{v}^{-}$, satisfies $\boldsymbol{v}^{+}=\boldsymbol{v}_{p}$.

Consider any vertex $\boldsymbol{v}_{j}$ with $j \in\{p+1, \ldots, 2 p-1\}$. From the definition of $\boldsymbol{v}^{-}, \boldsymbol{v}^{+}$, for any such vertex, there exists a point $\boldsymbol{v}_{j}^{\#} \in\left[\boldsymbol{v}^{-}, \boldsymbol{v}^{+}\right]$, with the same $\theta_{2}$-coordinate as $\boldsymbol{v}_{j}$, but with a $\theta_{1}$-coordinate larger than $\boldsymbol{v}_{j}$ (refer to Figure 2). Since such a point will have an objective in problem (18) at least as large as $\boldsymbol{v}_{j}$, and $\boldsymbol{v}_{j}^{\#} \in\left[\boldsymbol{v}_{0}, \boldsymbol{v}_{p}\right]$, we can immediately conclude that the maximum of problem (18) is achieved on the set $\left\{\boldsymbol{v}_{0}, \ldots, \boldsymbol{v}_{p}\right\}$. Since $2 p \leq 2 k$ (see part (ii) of Lemma 7.2), we can immediately arrive at the conclusion of the lemma.

Since the argument presented in the lemma will be recurring throughout several of our proofs and constructions, we will end this subsection by introducing two useful definitions, and generalizing the previous result.

Consider the system of coordinates $\left(\theta_{1}, \theta_{2}\right)$ in $\mathbf{R}^{2}$, and let $\mathcal{S} \subset \mathbf{R}^{2}$ denote an arbitrary, finite set of points and $\mathcal{P}$ denote any (possibly non-convex) polygon such that its set of vertices is exactly $\mathcal{S}$. With $\boldsymbol{y}^{-} \stackrel{\text { def }}{=} \arg \max _{\theta_{1}}\left\{\arg \min _{\theta_{2}}\{\boldsymbol{\theta} \in \mathcal{S}\}\right\}, \boldsymbol{y}^{+} \stackrel{\text { def }}{=} \arg \max _{\theta_{1}}\left\{\arg \max _{\theta_{2}}\{\boldsymbol{\theta} \in \mathcal{S}\}\right\}$, by numbering the vertices of the convex hull of $\mathcal{S}$ in a counter-clockwise fashion, starting at $\boldsymbol{y}_{0} \stackrel{\text { def }}{=} \boldsymbol{y}^{-}$, and with $\boldsymbol{y}_{m}=\boldsymbol{y}^{+}$, we define the right side of $\mathcal{P}$ and the zonogon hull of $\mathcal{S}$ as follows:

Definition 4.1. The right side of an arbitrary polygon $\mathcal{P}$ is:

$$
\begin{equation*}
\text { r-side }(\mathcal{P}) \stackrel{\text { def }}{=}\left\{\boldsymbol{y}_{0}, \boldsymbol{y}_{1}, \ldots, \boldsymbol{y}_{m}\right\} \tag{21}
\end{equation*}
$$

Definition 4.2. The zonogon hull of a set of points $\mathcal{S}$ is:

$$
\begin{equation*}
\mathrm{z} \text {-hull }(\mathcal{S}) \stackrel{\text { def }}{=}\left\{\boldsymbol{y} \in \mathbf{R}^{2}: \boldsymbol{y}=\boldsymbol{y}_{0}+\sum_{i=1}^{m} w_{i} \cdot\left(\boldsymbol{y}_{i}-\boldsymbol{y}_{i-1}\right), 0 \leq w_{i} \leq 1\right\} \tag{22}
\end{equation*}
$$



Figure 3: Examples of zonogon hulls for different sets $\mathcal{S} \in \mathbf{R}^{2}$.

Intuitively, r-side $(\mathcal{P})$ represents exactly what the names hints at, i.e. the vertices found on the right side of $\mathcal{P}$. An equivalent definition using more familiar operators could be the following:

$$
\mathrm{r}-\operatorname{side}(\mathcal{P}) \equiv \operatorname{ext}\left(\operatorname{cone}\left(\left[\begin{array}{c}
0 \\
-1
\end{array}\right]\right)+\operatorname{conv}(\mathcal{P})\right)
$$

where cone $(\cdot)$ and $\operatorname{conv}(\cdot)$ represent the conic and convex hull, respectively, and $\operatorname{ext}(\cdot)$ denotes the set of extreme points.

Using Definition 7.1 in Section 7.2 of the Appendix, one can see that the zonogon hull of a set $\mathcal{S}$ is simply a zonogon that has exactly the same vertices on the right side as the convex hull of $\mathcal{S}$, i.e. r-side $(z-h u l l(\mathcal{S}))=r$-side $(\operatorname{conv}(\mathcal{S}))$. Some examples of zonogon hulls are shown in Figure 3 (note that the initial points in $\mathcal{S}$ do not necessarily fall inside the zonogon hull, and, as such, there is no general inclusion relation between the zonogon hull and the convex hull). The reason for introducing this object is that it allows for the following immediate generalization of Lemma 4.1:
Corollary 4.1. If $\mathcal{P}$ is any polygon in $\mathbf{R}^{2} \equiv\left(\theta_{1}, \theta_{2}\right)$ with a finite set $\mathcal{S}$ of vertices, and $f: \mathbf{R} \rightarrow \mathbf{R}$ is any convex function, then the following string of equalities holds:

$$
\begin{aligned}
\max _{\left(\theta_{1}, \theta_{2}\right) \in \mathcal{P}}\left[\theta_{1}+f\left(\theta_{2}\right)\right] & =\max _{\left(\theta_{1}, \theta_{2}\right) \in \operatorname{conv}(\mathcal{P})}\left[\theta_{1}+f\left(\theta_{2}\right)\right]=\max _{\left(\theta_{1}, \theta_{2}\right) \in \mathcal{S}}\left[\theta_{1}+f\left(\theta_{2}\right)\right]=\max _{\left(\theta_{1}, \theta_{2}\right) \in \mathrm{r}-\operatorname{side}(\mathcal{P})}\left[\theta_{1}+f\left(\theta_{2}\right)\right] \\
& =\max _{\left(\theta_{1}, \theta_{2}\right) \in \mathrm{z-hull}(\mathcal{S})}\left[\theta_{1}+f\left(\theta_{2}\right)\right]=\max _{\left(\theta_{1}, \theta_{2}\right) \in \mathrm{r}-\operatorname{side}(\mathrm{z}-\operatorname{hull}(\mathcal{S}))}\left[\theta_{1}+f\left(\theta_{2}\right)\right]
\end{aligned}
$$

Proof. The proof is identical to that of Lemma 4.1, and is omitted for brevity.
Using this result, whenever we will be faced with a maximization of a convex function $\theta_{1}+f\left(\theta_{2}\right)$, we will be able to switch between different feasible sets, without affecting the overall optimal value of the optimization problem.

In the context of Lemma 4.1, the above result allows us to restrict attention from a potentially large set of relevant points (the $2^{k}$ vertices of the hyper-rectangle $\mathcal{W}_{1} \times \cdots \times \mathcal{W}_{k}$ ), to the $k+1$ vertices found on the right side of the zonogon $\Theta$, which also gives insight into why the construction of an affine controller $q_{k+1}\left(\boldsymbol{w}^{k+1}\right)$ with $k+1$ degrees of freedom, yielding the same overall objective function value $J_{m M}$, might actually be possible.

In the remaining part of Section 4.1, we would like to further narrow down this set of relevant points, by using the structure and properties of the optimal control law $u_{k+1}^{*}\left(x_{k+1}\right)$ and optimal value function $J_{k+1}^{*}\left(x_{k+1}\right)$, derived in Section 2. Before proceeding, however, we will first reduce the notational clutter by introducing several simplifications and assumptions.

### 4.1.1 Simplified Notation and Assumptions.

To start, we will omit writing the time subscripts ( $k$ or $k+1$ ) whenever possible, and will take:

$$
\begin{equation*}
\theta_{1}(\boldsymbol{w}) \stackrel{\text { def }}{=} a_{0}+\sum_{i=1}^{k} a_{i} \cdot w_{i} ; \quad \theta_{2}(\boldsymbol{w}) \stackrel{\text { def }}{=} b_{0}+\sum_{i=1}^{k} b_{i} \cdot w_{i} ; \quad q_{k+1}(\boldsymbol{w}) \equiv q(\boldsymbol{w}) \stackrel{\text { def }}{=} q_{0}+\sum_{i=1}^{k} q_{i} \cdot w_{i} \tag{23}
\end{equation*}
$$

where $\boldsymbol{a}=\left(a_{1}, \ldots, a_{k}\right)$ and $\boldsymbol{b}=\left(b_{1}, \ldots, b_{k}\right)$ are the generators of the zonogon $\Theta$. We will use the same counter-clockwise numbering of the vertices of $\Theta$ as introduced earlier in Section 4.1:

$$
\begin{equation*}
\boldsymbol{v}_{0} \stackrel{\text { def }}{=} \boldsymbol{v}^{-}, \ldots, \boldsymbol{v}_{p} \stackrel{\text { def }}{=} \boldsymbol{v}^{+}, \ldots, \boldsymbol{v}_{2 p}=\boldsymbol{v}^{-} \tag{24}
\end{equation*}
$$

where $2 p$ is the number of vertices of $\Theta$, and we will use the following overloaded notation for $\theta_{1}$ and $\theta_{2}$ :

- $\theta_{i}(\boldsymbol{w})$, for $\boldsymbol{w} \in \mathbf{R}^{k}$, will denote the $\theta_{i}$ value assigned by the affine projection to $\boldsymbol{w}$, via (23).
- $\theta_{i}[\boldsymbol{v}]$, applied to $\boldsymbol{v} \in \mathbf{R}^{2}$, will be used to denote the $\theta_{i}$-coordinate of the point $\boldsymbol{v}$.

Also, since $\theta_{2} \equiv x_{k+1}$, instead of referring to $J_{k+1}^{*}\left(x_{k+1}\right)$ and $u_{k+1}^{*}\left(x_{k+1}\right)$, we will use $J^{*}\left(\theta_{2}\right)$ and $u^{*}\left(\theta_{2}\right)$, and we will occasionally use the short-hand notations $u^{*}\left(\boldsymbol{v}_{\boldsymbol{i}}\right), J^{*}\left(\boldsymbol{v}_{\boldsymbol{i}}\right)$ and $g\left(\boldsymbol{v}_{i}\right)$, instead of $u^{*}\left(\theta_{2}\left[\boldsymbol{v}_{i}\right]\right), J^{*}\left(\theta_{2}\left[\boldsymbol{v}_{i}\right]\right)$ and $g\left(\theta_{2}\left[\boldsymbol{v}_{\boldsymbol{i}}\right]+u^{*}\left(\theta_{2}\left[\boldsymbol{v}_{i}\right]\right)\right)$, respectively.

Since many of the sets of interest will lie in $\mathbf{R}^{2}$, for a system with coordinates $(x, y)$, we define the following convenient notation for the cotangent of the angle formed by an oriented line segment $[\boldsymbol{M}, \boldsymbol{N}]$ with the $x$-axis:

$$
\begin{equation*}
\operatorname{cotan}(\boldsymbol{M}, \boldsymbol{N}) \stackrel{\text { def }}{=} \frac{x_{N}-x_{M}}{y_{N}-y_{M}}, \quad \text { where } \boldsymbol{M}=\left(x_{M}, y_{M}\right) \in \mathbf{R}^{2}, \quad \boldsymbol{N}=\left(x_{N}, y_{N}\right) \in \mathbf{R}^{2} . \tag{25}
\end{equation*}
$$

We will also make the following simplifying assumptions:
ASSUMPTION 1. The uncertainty vector at time $k$, $\boldsymbol{w}^{k}=\left(w_{1}, \ldots, w_{k}\right)$, belongs to the unit hypercube $\mathcal{H}_{k}$ of $\mathbf{R}^{k}$, i.e. $\underline{w}_{i}=0, \bar{w}_{i}=1, \forall i=1, \ldots, k$.

Assumption 2. The zonogon $\Theta$ has a maximal number of vertices, i.e. $p=k$.
Assumption 3. The vertex of the hypercube projecting to $\boldsymbol{v}_{i}, i \in\{0, \ldots, k\}$, is exactly $[1,1, \ldots, 1,0, \ldots, 0]$, i.e. 1 in the first $k$ components and 0 thereafter (see Figure 2).

These assumptions are made only to facilitate the exposition, and result in no loss of generality. To see this, note that the conditions of Assumption 1 can always be achieved by adequate translation and scaling of the generators $\boldsymbol{a}$ and $\boldsymbol{b}$ (refer to Section 7.2 of the Appendix for more details), and Assumption 3 can be satisfied by renumbering the coordinates of the hyper-rectangle, i.e. renumbering the disturbances $w_{1}, \ldots, w_{k}$. As for Assumption 2, we argue that an extension of our construction to the degenerate case $p<k$ is immediate (one could also remove the degeneracy by applying an infinitesimal perturbation to the generators $\boldsymbol{a}$ or $\boldsymbol{b}$, with infinitesimal cost implications).

### 4.1.2 Further Analysis of the Induction Hypothesis.

With this simplified notation, by using (9) to express $J^{*}(\cdot)$ as a function of $u^{*}(\cdot)$ and $g(\cdot)$, we rewrite equation (18) as follows:

$$
\begin{align*}
& (O P T) \quad J_{m M}=\max _{\left(\tilde{\gamma}_{1}, \tilde{\gamma}_{2}\right) \in \tilde{\Gamma}}\left[\tilde{\gamma}_{1}+g\left(\tilde{\gamma}_{2}\right)\right], \\
& \text { where } \tilde{\Gamma} \stackrel{\text { def }}{=}\left\{\left(\tilde{\gamma}_{1}, \tilde{\gamma}_{2}\right): \tilde{\gamma}_{1} \stackrel{\text { def }}{=} \theta_{1}+c \cdot u^{*}\left(\theta_{2}\right), \quad \tilde{\gamma}_{2} \stackrel{\text { def }}{=} \theta_{2}+u^{*}\left(\theta_{2}\right), \quad\left(\theta_{1}, \theta_{2}\right) \in \Theta\right\} . \tag{26}
\end{align*}
$$

A characterization for the set $\tilde{\Gamma}=\left(\tilde{\gamma}_{1}, \tilde{\gamma}_{2}\right)$ can be obtained by replacing the optimal, piecewise affine control law $u^{*}\left(\theta_{2}\right)$, given by ${ }^{4}(8)$ in equation (26):

$$
\tilde{\Gamma}:\left(\tilde{\gamma}_{1}, \tilde{\gamma}_{2}\right)= \begin{cases}\left(\theta_{1}+c \cdot U, \theta_{2}+U\right), & \text { if } \theta_{2}<y^{*}-U  \tag{27}\\ \left(\theta_{1}-c \cdot \theta_{2}+c \cdot y^{*}, y^{*}\right), & \text { otherwise } \\ \left(\theta_{1}+c \cdot L, \theta_{2}+L\right), & \text { if } \theta_{2}>y^{*}-L\end{cases}
$$

Using the same overloaded notation ${ }^{5}$ for $\tilde{\gamma}_{1}, \tilde{\gamma}_{2}$ :

[^3]- $\tilde{\gamma}_{i}(\boldsymbol{v})$, applied to $\boldsymbol{v} \in \Theta \subset \mathbf{R}^{2}$, will refer to the $\tilde{\gamma}_{i}$ value assigned by the mapping (27) to $\boldsymbol{v}$
- $\tilde{\gamma}_{i}[\tilde{\boldsymbol{v}}]$, applied to $\tilde{\boldsymbol{v}} \in \Gamma \subset \mathbf{R}^{2}$, will denote the $\tilde{\gamma}_{i}$ coordinate of the point $\tilde{\boldsymbol{v}}$,
we can now provide a compact characterization for the maximizers in problem (OPT) from (26):
LEMMA 4.2. The maximum in problem $(O P T)$ over $\tilde{\Gamma}$ is reached on the right side of:

$$
\begin{equation*}
\Delta_{\Gamma} \stackrel{\text { def }}{=} \operatorname{conv}\left(\left\{\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}\right) \tag{28}
\end{equation*}
$$

where:

$$
\begin{equation*}
\tilde{\boldsymbol{v}}_{i} \stackrel{\text { def }}{=}\left(\tilde{\gamma}_{1}\left(\boldsymbol{v}_{i}\right), \tilde{\gamma}_{2}\left(\boldsymbol{v}_{i}\right)\right)=\left(\theta_{1}\left[\boldsymbol{v}_{i}\right]+c \cdot u^{*}\left(\boldsymbol{v}_{i}\right), \theta_{2}\left[\boldsymbol{v}_{i}\right]+u^{*}\left(\boldsymbol{v}_{i}\right)\right), \quad i \in\{0, \ldots, k\} . \tag{29}
\end{equation*}
$$

Proof. By Lemma 4.1, the maximum in (18) is reached at one of the vertices $\boldsymbol{v}_{0}, \boldsymbol{v}_{1}, \ldots, \boldsymbol{v}_{k}$ of the zonogon $\Theta$. Since this problem is equivalent to problem $(O P T)$ in $(26)$, written over $\tilde{\Gamma}$, we can immediately conclude that the maximum of the latter problem is reached at the points $\left\{\tilde{\boldsymbol{v}}_{i}\right\}_{1 \leq i \leq k}$ given by (29). Furthermore, since $g(\cdot)$ is convex (see Property P2 of the optimal DP solution, in Section 2), we can apply Corollary 4.1, and replace the points $\tilde{\boldsymbol{v}}_{i}$ with the right side of their convex hull, r-side $\left(\Delta_{\Gamma}\right)$, without changing the result of the optimization problem, which completes the proof.

Since this result will be central to our future construction and proof, we will spend the remaining part of the subsection discussing some of the properties of the main object of interest, the set of points on the right side of $\Delta_{\Gamma}$, r-side $\left(\Delta_{\Gamma}\right)$. To understand the geometry of the set $\Delta_{\Gamma}$ and its connection with the optimal control law, note that the mapping (27) from $\Theta$ to $\tilde{\Gamma}$ will discriminate points $\boldsymbol{\theta}=\left(\theta_{1}, \theta_{2}\right) \in \Theta$ depending on their position relative to the horizontal band:

$$
\begin{equation*}
\mathcal{B}_{L U} \stackrel{\text { def }}{=}\left\{\left(\theta_{1}, \theta_{2}\right) \in \mathbf{R}^{2}: \theta_{2} \in\left[y^{*}-U, y^{*}-L\right]\right\} \tag{30}
\end{equation*}
$$

In particular, from (27) and the definition of $\boldsymbol{v}_{0}, \ldots, \boldsymbol{v}_{k}$ in (24) and (20), we can distinguish a total of four distinct cases. The first three, shown in Figure 4, are very easy to analyze:


Figure 4: Trivial cases, when zonogon $\Theta$ lies entirely [C1] below, [C2] inside, or [C3] above the band $\mathcal{B}_{L U}$.
[C1] If the entire zonogon $\Theta$ falls below the band $\mathcal{B}_{L U}$, i.e. $\theta_{2}\left[\boldsymbol{v}_{k}\right]<y^{*}-U$, then $\tilde{\Gamma}$ is simply a translation of $\Theta$, by $(c \cdot U, U)$, so that r-side $\left(\Delta_{\Gamma}\right)=\left\{\tilde{\boldsymbol{v}}_{0}, \tilde{\boldsymbol{v}}_{1}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}$.
[C2] If $\Theta$ lies inside the band $\mathcal{B}_{L U}$, i.e. $y^{*}-U \leq \theta_{2}\left[\boldsymbol{v}_{0}\right] \leq \theta_{2}\left[\boldsymbol{v}_{k}\right] \leq y^{*}-L$, then all the points in $\tilde{\Gamma}$ will have $\tilde{\gamma}_{2}=y^{*}$, so $\tilde{\Gamma}$ will be a line segment, and $\mid \mathrm{r}$-side $\left(\Delta_{\Gamma}\right) \mid=1$.
[C3] If the entire zonogon $\Theta$ falls above the band $\mathcal{B}_{L U}$, i.e. $\theta_{2}\left[\boldsymbol{v}_{0}\right]>y^{*}-L$, then $\tilde{\Gamma}$ is again a translation of $\Theta$, by $(c \cdot L, L)$, so again r -side $\left(\Delta_{\Gamma}\right)=\left\{\tilde{\boldsymbol{v}}_{0}, \tilde{\boldsymbol{v}}_{1}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}$.

The remaining case, $[\mathbf{C} 4]$, is when $\Theta$ intersects the horizontal band $\mathcal{B}_{L U}$ in a nontrivial fashion. We can separate this situation in the three sub-cases shown in Figure 5, depending on the position of the vertex


Figure 5: Case [C4]. Original zonogon $\Theta$ (first row) and the set $\tilde{\Gamma}$ (second row) when $\boldsymbol{v}_{t}$ falls (a) under, (b) inside or (c) above the band $\mathcal{B}_{L U}$.
$\boldsymbol{v}_{t} \in \mathrm{r}$-side $(\Theta)$, where the index $t$ relates the per-unit control cost, $c$, with the geometrical properties of the zonogon:

$$
t \stackrel{\text { def }}{=} \begin{cases}0, & \text { if } \frac{a_{1}}{b_{1}} \leq c  \tag{31}\\ \max \left\{i \in\{1, \ldots, k\}: \frac{a_{i}}{b_{i}}>c\right\}, & \text { otherwise }\end{cases}
$$

We remark that the definition of $t$ is consistent, since, by the simplifying Assumption 3, the generators $\boldsymbol{a}, \boldsymbol{b}$ of the zonogon $\Theta$ always satisfy:

$$
\left\{\begin{array}{l}
\frac{a_{1}}{b_{1}}>\frac{a_{2}}{b_{2}}>\cdots>\frac{a_{k}}{b_{k}}  \tag{32}\\
b_{1}, b_{2}, \ldots, b_{k} \geq 0
\end{array}\right.
$$

An equivalent characterization of $\boldsymbol{v}_{t}$ can be obtained as the result of an optimization problem:

$$
\begin{equation*}
\boldsymbol{v}_{t} \equiv \underset{\theta_{2}}{\arg \min }\left\{\underset{\left(\theta_{1}, \theta_{2}\right) \in \Theta}{\arg \max }\left\{\theta_{1}-c \cdot \theta_{2}\right\}\right\} \tag{33}
\end{equation*}
$$

The following lemma summarizes all the relevant geometrical properties corresponding to this case:
Lemma 4.3. When the zonogon $\Theta$ has a non-trivial intersection with the band $\mathcal{B}_{L U}$ (case [ $\mathbf{C 4} 4$ ), the convex polygon $\Delta_{\Gamma}$ and the set of points on its right side, r-side $\left(\Delta_{\Gamma}\right)$, satisfy the following properties:

1. r-side $\left(\Delta_{\Gamma}\right)$ is the union of two sequences of consecutive vertices (one starting at $\tilde{\boldsymbol{v}}_{0}$, and one ending at $\tilde{\boldsymbol{v}}_{k}$ ), and possibly an additional vertex, $\tilde{\boldsymbol{v}}_{t}$ :

$$
\text { r-side }\left(\Delta_{\Gamma}\right)=\left\{\tilde{\boldsymbol{v}}_{0}, \tilde{\boldsymbol{v}}_{1}, \ldots, \tilde{\boldsymbol{v}}_{s}\right\} \cup\left\{\tilde{\boldsymbol{v}}_{t}\right\} \cup\left\{\tilde{\boldsymbol{v}}_{r}, \tilde{\boldsymbol{v}}_{r+1} \ldots, \tilde{\boldsymbol{v}}_{k}\right\}, \text { for some } s \leq r \in\{0, \ldots, k\} .
$$

2. With $\operatorname{cotan}(\cdot, \cdot)$ given by $(25)$ applied to the $\left(\tilde{\gamma}_{1}, \tilde{\gamma}_{2}\right)$ coordinates, we have that:

$$
\begin{cases}\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{s}, \tilde{\boldsymbol{v}}_{\min (t, r)}\right) \geq \frac{a_{s+1}}{b_{s+1}}, & \text { whenever } t>s  \tag{34}\\ \operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{\max (t, s)}, \tilde{\boldsymbol{v}}_{r}\right) \leq \frac{a_{r}}{b_{r}}, & \text { whenever } t<r\end{cases}
$$

While the proof of the lemma is slightly technical, which is why we have decided to leave it for Section 7.3 of the Appendix, its implications are more straightforward. In conjuction with Lemma 4.2, it provides a compact characterization of the points $\tilde{\boldsymbol{v}}_{i} \in \tilde{\Gamma}$ which are potential maximizers of problem $(O P T)$ in (26), which immediately narrows the set of relevant points $\boldsymbol{v}_{i} \in \Theta$ in optimization problem (18), and, implicitly, the set of disturbances $\boldsymbol{w} \in \mathcal{H}_{k}$ that can achieve the min-max cost $J_{m M}$.

### 4.2 Construction of the Affine Control Law.

Having analyzed the consequences that result from using the induction hypothesis of Theorem 3.1, we now return to the task of completing the inductive proof, which amounts to constructing an affine control law $q_{k+1}\left(\boldsymbol{w}^{k+1}\right)$ and an affine cost $z_{k+1}\left(\boldsymbol{w}^{k+2}\right)$ that verify conditions (12), (13) and (14) in Theorem 3.1. We will separate this task into two parts. In the current section, we will exhibit an affine control law $q_{k+1}\left(\boldsymbol{w}^{k+1}\right)$ that is robustly feasible, i.e. satisfies constraint (12), and that leaves the overall min-max cost $J_{m M}$ unchanged, when used at time $k+1$ in conjunction with the original convex state cost, $h_{k+1}\left(x_{k+2}\right)$. The second part of the induction, i.e. the construction of the affine costs $z_{k+1}\left(\boldsymbol{w}^{k+2}\right)$ satisfying (13) and (14), will be left for Section 4.3.

In the simplified notation introduced earlier, the problem we would like to solve is to find an affine control law $q(\boldsymbol{w})$ such that:

$$
\begin{aligned}
J_{m M} & =\max _{\boldsymbol{w} \in[0,1]^{k}}\left[\theta_{1}+c \cdot q(\boldsymbol{w})+g\left(\theta_{2}+q(\boldsymbol{w})\right)\right] \\
L & \leq q(\boldsymbol{w}) \leq U, \quad \forall \boldsymbol{w} \in \mathcal{H}_{k}
\end{aligned}
$$

The maximization represents the problem solved by the disturbances, when the affine controller, $q(\boldsymbol{w})$, is used instead of the optimal controller, $u^{*}\left(\theta_{2}\right)$. As such, the first equation amounts to ensuring that the overall objective function $J_{m M}$ remains unchanged, and the inequalities are a restatement of the robust feasibility condition. The system can be immediately rewritten as:

$$
\begin{align*}
(A F F) \quad J_{m M} & =\max _{\left(\gamma_{1}, \gamma_{2}\right) \in \Gamma}\left[\gamma_{1}+g\left(\gamma_{2}\right)\right]  \tag{35}\\
L & \leq q(\boldsymbol{w}) \leq U
\end{align*}
$$

where:

$$
\begin{equation*}
\Gamma \stackrel{\text { def }}{=}\left\{\left(\gamma_{1}, \gamma_{2}\right): \gamma_{1} \stackrel{\text { def }}{=} \theta_{1}+c \cdot q(\boldsymbol{w}), \quad \gamma_{2} \stackrel{\text { def }}{=} \theta_{2}+q(\boldsymbol{w}), \quad\left(\theta_{1}, \theta_{2}\right) \in \Theta\right\} . \tag{36}
\end{equation*}
$$

With this reformulation, all our decision variables, i.e. the affine coefficients of $q(\boldsymbol{w})$, have been moved to the feasible set $\Gamma$ of the maximization problem $(A F F)$ in (35). Note that, with an affine controller $q(\boldsymbol{w})=q_{0}+\boldsymbol{q}^{T} \boldsymbol{w}$, and $\theta_{1}, \theta_{2}$ affine in $\boldsymbol{w}(23)$, the feasible set $\Gamma$ will represent a new zonogon in $\mathbf{R}^{2}$, with generators given by $\boldsymbol{a}+\boldsymbol{c} \cdot \boldsymbol{q}$ and $\boldsymbol{b}+\boldsymbol{q}$. Furthermore, since the function $g$ is convex, the optimization problem $(A F F)$ over $\Gamma$ is of the exact same nature as that in (18), defined over the zonogon $\Theta$. Thus, in perfect analogy with our discussion in Section 4.1 (Lemma 4.1 and Corollary 4.1), we can conclude that the maximum in $(A F F)$ will occur at a vertex of $\Gamma$ found in $r$-side $(\Gamma)$.

In a different sense, note that optimization problem $(A F F)$ is also very similar to problem $(O P T)$ in (26), which was the problem solved by the uncertainties $\boldsymbol{w}$ when the optimal control law, $u^{*}\left(\theta_{2}\right)$, was used at time $k+1$. Since the optimal value of the latter problem is exactly equal to the overall min-max value, $J_{m M}$, we will interpret the equation in (35) as comparing the optimal values in the two optimization problems, $(A F F)$ and $(O P T)$.

As such, note that the same convex objective function, $\xi_{1}+g\left(\xi_{2}\right)$, is maximized in both problems, but over different feasible sets, $\tilde{\Gamma}$ for $(O P T)$ and $\Gamma$ for $(A F F)$, respectively. From Lemma 4.2 in Section 4.1.2, the maximum of problem $(O P T)$ is reached on the set r-side $\left(\Delta_{\Gamma}\right)$, where $\Delta_{\Gamma}=\operatorname{conv}\left(\left\{\tilde{\boldsymbol{v}}_{0}, \tilde{\boldsymbol{v}}_{1}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}\right)$. From the discussion in the previous paragraph, the maximum in problem ( $A F F$ ) occurs on r-side $(\Gamma)$. Therefore, in order to compare the two results of the maximization problems, we must relate the sets r-side $\left(\Delta_{\Gamma}\right)$ and r-side( $\Gamma$ ).

In this context, we introduce the central idea behind the construction of the affine control law, $q(\boldsymbol{w})$. Recalling the concept of a zonogon hull introduced in Definition 4.2, we argue that, if the affine coefficients of
the controller, $q_{0}, \boldsymbol{q}$, were computed in such a way that the zonogon $\Gamma$ actually corresponded to the zonogon hull of the set $\left\{\tilde{\boldsymbol{v}}_{0}, \tilde{\boldsymbol{v}}_{1}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}$, then, by using the result in Corollary 4.1, we could immediately conclude that the optimal values in $(O P T)$ and $(A F F)$ are the same.

To this end, we introduce the following procedure for computing the affine control law $q(\boldsymbol{w})$ :

```
Algorithm 1 Compute affine controller \(q(\boldsymbol{w})\)
Require: \(\theta_{1}(\boldsymbol{w}), \theta_{2}(\boldsymbol{w}), g(\cdot), u^{*}(\cdot)\)
    if \(\left(\Theta\right.\) falls below \(\left.\mathcal{B}_{L U}\right)\) or \(\left(\Theta \subseteq \mathcal{B}_{L U}\right)\) or \(\left(\Theta\right.\) falls above \(\left.\mathcal{B}_{L U}\right)\) then
        Return \(q(\boldsymbol{w})=u^{*}\left(\theta_{2}(\boldsymbol{w})\right)\).
    else
        Apply the mapping (27) to obtain the points \(\tilde{\boldsymbol{v}}_{i}, i \in\{0, \ldots, k\}\).
        Compute the set \(\Delta_{\Gamma}=\operatorname{conv}\left(\left\{\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}\right)\).
        Let r-side \(\left(\Delta_{\Gamma}\right)=\left\{\tilde{\boldsymbol{v}}_{0}, \tilde{\boldsymbol{v}}_{1}, \ldots, \tilde{\boldsymbol{v}}_{s}\right\} \cup\left\{\tilde{\boldsymbol{v}}_{t}\right\} \cup\left\{\tilde{\boldsymbol{v}}_{r}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}\) be the set of points on the right side of \(\Delta_{\Gamma}\).
        Solve the following system for \(q_{0}, \ldots, q_{k}\) and \(K_{U}, K_{L}\) :
\[
\left\{\begin{array}{rlrl}
q_{0}+\cdots+q_{i} & =u^{*}\left(\boldsymbol{v}_{i}\right), & & \forall \tilde{\boldsymbol{v}}_{i} \in \mathrm{r}-\operatorname{side}\left(\Delta_{\Gamma}\right)  \tag{S}\\
& & \text { (matching) } \\
\frac{a_{i}+c \cdot q_{i}}{b_{i}+q_{i}} & =K_{U}, & & \forall i \in\{s+1, \ldots, \min (t, r)\} \\
& & (\text { alignment below } t) \\
\frac{a_{i}+c \cdot q_{i}}{b_{i}+q_{i}} & =K_{L}, & & \forall i \in\{\max (t, s)+1, \ldots, r\}
\end{array}\right.
\]
8: \(\quad\) Return \(q(\boldsymbol{w})=q_{0}+\sum_{i=1}^{k} q_{i} w_{i}\).
    end if
```

Before proving that the construction is well-defined and does produce the expected result, we will give some intuition for the constraints in system (37). In order to have the zonogon $\Gamma$ be the same as the zonogon hull of $\left\{\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}$, we must ensure that the vertices on the right side of $\Gamma$ exactly correspond to the points on the right side of $\Delta_{\Gamma}=\operatorname{conv}\left(\left\{\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}\right)$. This is achieved in two stages. First, we ensure that vertices $\boldsymbol{w}_{i}$ of the hypercube $\mathcal{H}_{k}$ that are mapped by the optimal control law $u^{*}(\cdot)$ into points $\tilde{\boldsymbol{v}}_{i} \in$ r-side $\left(\Delta_{\Gamma}\right)$ (through the succession of mappings $\boldsymbol{w}_{i} \stackrel{(19)}{\mapsto} \boldsymbol{v}_{i} \in \mathrm{r}$-side $(\Theta) \stackrel{(29)}{\mapsto} \tilde{\boldsymbol{v}}_{i} \in \mathrm{r}$-side $\left(\Delta_{\Gamma}\right)$ ), will be mapped by the affine control law, $q\left(\boldsymbol{w}_{i}\right)$, into the same point $\tilde{\boldsymbol{v}}_{i}$ (through the mappings $\boldsymbol{w}_{i} \stackrel{(19)}{\mapsto} \boldsymbol{v}_{i} \in \mathrm{r}$-side $(\Theta) \stackrel{(36)}{\mapsto} \tilde{\boldsymbol{v}}_{i} \in$ r-side $\left(\Delta_{\Gamma}\right)$ ). This is done in the first set of constraints, by matching the value of the optimal control law at any such points. Second, we ensure that any such matched points $\tilde{\boldsymbol{v}}_{i}$ actually correspond to the vertices on the right side of the zonogon $\Gamma$. This is done in the second and third set of constraints in (37), by computing the affine coefficients $q_{j}$ in such a way that the resulting segments in the generators of the zonogon $\Gamma$, namely $\binom{a_{j}+c \cdot q_{j}}{b_{j}+q_{j}}$, are all aligned, i.e. have the same cotangent, given by the $K_{U}, K_{L}$ variables. Geometrically, this exactly corresponds to the situation shown in Figure 6 below.

We remark that the above algorithm does not explicitly require that the control $q(\boldsymbol{w})$ be robustly feasible, i.e. second condition in (35). However, we will soon prove that this is a direct result of the way matching and alignment are performed in Algorithm 1.

### 4.2.1 Affine Controller Preserves Overall Objective and Is Robust.

In this section, we prove that the affine control law $q(\boldsymbol{w})$ produced by Algorithm 1 satisfies the requirements of (35), i.e. it is robustly feasible, and it preserves the overall objective function $J_{m M}$ when used in conjunction with the original convex state costs, $h(\cdot)$. With the exception of Corollary 4.1, all the key results that we will be using are contained in Section 4.1.2 (Lemmas 4.2 and 4.3). Therefore, we will preserve the same notation and case discussion as initially introduced there.

First consider the condition on line 1 of Algorithm 1, and note that this corresponds to the three trivial cases [C1], [C2] and [C3] of Section 4.1.2. In particular, since $\theta_{2} \equiv x_{k+1}$, we can use (8) to conclude that in these cases, the optimal control law $u^{*}(\cdot)$ is actually affine:


Figure 6: Outcomes from the matching and alignment performed in Algorithm 1.
[C1] If $\Theta$ falls below the band $\mathcal{B}_{L U}$, then the upper bound constraint on the control at time $k$ is always active, i.e. the optimal control is $u^{*}\left(\theta_{2}(\boldsymbol{w})\right)=U, \forall \boldsymbol{w} \in \mathcal{H}_{k}$.
[C2] If $\Theta \subseteq \mathcal{B}_{L U}$, then the constraints on the control at time $k$ are never active, i.e. $u^{*}\left(\theta_{2}(\boldsymbol{w})\right)=y^{*}-\theta_{2}(\boldsymbol{w})$, hence affine in $\boldsymbol{w}$, since $\theta_{2}$ is affine in $\boldsymbol{w}$, by (23).
[C3] If $\Theta$ falls above the band $\mathcal{B}_{L U}$, then the lower bound constraint on the control is always active, i.e. $u^{*}\left(\theta_{2}(\boldsymbol{w})\right)=L, \forall \boldsymbol{w} \in \mathcal{H}_{k}$.

Therefore, with the assignment in line 2 of Algorithm 1, we obtain an affine control law that is always feasible and also optimal.

When none of the trivial cases holds, we are in case [C4] of Section 4.1.2. Therefore, we can invoke the results from Lemma 4.3 to argue that the right side of the set $\Delta_{\Gamma}$ is exactly the set on line 7 of the algorithm, r-side $\left(\Delta_{\Gamma}\right)=\left\{\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{s}\right\} \cup\left\{\tilde{\boldsymbol{v}}_{t}\right\} \cup\left\{\tilde{\boldsymbol{v}}_{r}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}$. In this setting, we can now formulate the first claim about system (37) and its solution:

LEmma 4.4. System (37) is always feasible, and the solution satisfies:

1. $-b_{i} \leq q_{i} \leq 0, \forall i \in\{1, \ldots, k\}$.
2. $L \leq q(\boldsymbol{w}) \leq U, \forall \boldsymbol{w} \in \mathcal{H}_{k}$.

Proof. Note first that system (37) has exactly $k+3$ unknowns, two for the cotangents $K_{U}, K_{L}$, and one for each coefficient $q_{i}, 0 \leq i \leq k$. Also, since $\left|\mathrm{r}-\operatorname{side}\left(\Delta_{\Gamma}\right)\right| \leq\left|\operatorname{ext}\left(\Delta_{\Gamma}\right)\right| \leq k+1$, and there are exactly $\mid \mathrm{r}-$ side $\left(\Delta_{\Gamma}\right) \mid$ matching constraints, and $k+3-\left|\mathrm{r}-\operatorname{side}\left(\Delta_{\Gamma}\right)\right|$ alignment constraints, it can be immediately seen that the system is always feasible.

Consider any $q_{i}$ with $i \in\{1, \ldots, s\} \cup\{r+1, \ldots, k\}$. From the matching conditions, we have that $q_{i}=u^{*}\left(\boldsymbol{v}_{i}\right)-u^{*}\left(\boldsymbol{v}_{i-1}\right)$. By property P3 from Section 2, the difference in the values of the optimal control law $u^{*}(\cdot)$ satisfies:

$$
\begin{aligned}
u^{*}\left(\boldsymbol{v}_{i}\right)-u^{*}\left(\boldsymbol{v}_{i-1}\right) & \stackrel{\text { def }}{=} u^{*}\left(\theta_{2}\left[\boldsymbol{v}_{i}\right]\right)-u^{*}\left(\theta_{2}\left[\boldsymbol{v}_{i-1}\right]\right) \\
(\text { by P3 }) & =-f \cdot\left(\theta_{2}\left[\boldsymbol{v}_{i}\right]-\theta_{2}\left[\boldsymbol{v}_{i-1}\right]\right) \\
& \stackrel{(23)}{=}-f \cdot b_{i}, \quad \text { where } f \in[0,1] .
\end{aligned}
$$

Since, by (32), $b_{j} \geq 0, \forall j \in\{1, \ldots, k\}$, we immediately obtain $-b_{i} \leq q_{i} \leq 0$, for $i \in\{1, \ldots, s\} \cup\{r+1, \ldots, k\}$.
Now consider any index $i \in\{s+1, \ldots, t \wedge r\}$, where $t \wedge r \equiv \min (t, r)$. From the conditions in system (37)
for alignment below $t$, we have $q_{i}=\frac{a_{i}-K_{U} \cdot b_{i}}{K_{U}-c}$. By summing up all such relations, we obtain:

$$
\begin{aligned}
\sum_{i=s+1}^{t \wedge r} q_{i} & =\frac{\sum_{i=s+1}^{t \wedge r} a_{i}-K_{U} \cdot \sum_{i=s+1}^{t \wedge r} b_{i}}{K_{U}-c} \Leftrightarrow \quad \text { (using the matching) } \\
u^{*}\left(\boldsymbol{v}_{t \wedge r}\right)-u^{*}\left(\boldsymbol{v}_{s}\right) & =\frac{\sum_{i=s+1}^{t \wedge r} a_{i}-K_{U} \cdot \sum_{i=s+1}^{t \wedge r} b_{i}}{K_{U}-c} \Leftrightarrow \\
K_{U} & =\frac{\sum_{i=s+1}^{t \wedge r} a_{i}+c \cdot\left(u^{*}\left(\boldsymbol{v}_{t \wedge r}\right)-u^{*}\left(\boldsymbol{v}_{s}\right)\right)}{\sum_{i=s+1}^{t \wedge r} b_{i}+u^{*}\left(\boldsymbol{v}_{t \wedge r}\right)-u^{*}\left(\boldsymbol{v}_{s}\right)} \\
& =\frac{\left[\sum_{i=0}^{t \wedge r} a_{i}+c \cdot u^{*}\left(\boldsymbol{v}_{t \wedge r}\right)\right]-\left[\sum_{i=0}^{s} a_{i}+c \cdot u^{*}\left(\boldsymbol{v}_{s}\right)\right]}{\left[\sum_{i=0}^{t \wedge r} b_{i}+u^{*}\left(\boldsymbol{v}_{t \wedge r}\right)\right]-\left[\sum_{i=0}^{s} b_{i}+u^{*}\left(\boldsymbol{v}_{s}\right)\right]} \\
& \stackrel{(29)}{=} \frac{\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{t \wedge r}\right]-\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{s}\right] \stackrel{(25)}{\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{t \wedge r}\right]-\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{s}\right]} \stackrel{\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{s}, \tilde{\boldsymbol{v}}_{t \wedge r}\right) .}{ }}{} .
\end{aligned}
$$

In the first step, we have used the fact that both $\tilde{\boldsymbol{v}}_{s}$ and $\tilde{\boldsymbol{v}}_{\min (t, r)}$ are matched, hence the intermediate coefficients $q_{i}$ must sum to exactly the difference of the values of $u^{*}(\cdot)$ at $\boldsymbol{v}_{\min (t, r)}$ and $\boldsymbol{v}_{s}$ respectively. In this context, we can see that $K_{U}$ is simply the cotangent of the angle formed by the segment $\left[\tilde{\boldsymbol{v}}_{s}, \tilde{\boldsymbol{v}}_{\min (t, r)}\right]$ with the horizontal (i.e. $\tilde{\gamma}_{1}$ ) axis. In this case, we can immediately recall result (34) from Lemma 4.3, to argue that $K_{U} \geq \frac{a_{s+1}}{b_{s+1}}$. Combining with (31) and (32), we obtain:

$$
K_{U} \geq \frac{a_{s+1}}{b_{s+1}} \geq \stackrel{(32)}{\cdots} \geq \frac{a_{\min (t, r)}}{b_{\min (t, r)}} \geq \frac{a_{t}}{b_{t}} \stackrel{(31)}{>} c
$$

Therefore, we immediately have that for any $i \in\{s+1, \ldots, \min (t, r)\}$,

$$
\left\{\begin{array}{r}
a_{i}-K_{U} \cdot b_{i} \leq 0 \\
K_{U}-c>0
\end{array} \Rightarrow q_{i}=\frac{a_{i}-K_{U} \cdot b_{i}}{K_{U}-c} \leq 0, \quad\left\{\begin{array}{r}
a_{i}-c \cdot b_{i}>0 \\
q_{i}+b_{i}=\frac{a_{i}-c \cdot b_{i}}{K_{U}-c} \Rightarrow q_{i}+b_{i} \geq 0
\end{array}\right.\right.
$$

The argument for indices $i \in\{\max (t, s)+1, \ldots, r\}$ proceeds in exactly the same fashion, by recognizing that $K_{L}$ defined in the algorithm is the same as $\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{\max (t, s)}, \tilde{\boldsymbol{v}}_{r}\right)$, and then applying (34) to argue that $K_{L}<\frac{a_{r}}{b_{r}} \leq \frac{a_{\max (t, s)+1}}{b_{\max (t, s)+1}} \leq \frac{a_{t+1}}{b_{t+1}} \leq c$. This will allow us to use the same reasoning as above, completing the proof of part ( $i$ ) of the claim.

To prove part (ii), consider any $\boldsymbol{w} \in \mathcal{H}_{k} \stackrel{\text { def }}{=}[0,1]^{k}$. Using part (i), we obtain:

$$
\begin{aligned}
& q(\boldsymbol{w}) \stackrel{\text { def }}{=} q_{0}+\sum_{i=1}^{k} q_{i} \cdot w_{i} \leq\left(\text { since } w_{i} \in[0,1], q_{i} \leq 0\right) \leq q_{0} \stackrel{(* *)}{=} u^{*}\left(\boldsymbol{v}_{0}\right) \leq U \\
& q(\boldsymbol{w}) \geq q_{0}+\sum_{i=1}^{k} q_{i} \cdot 1 \stackrel{(* *)}{=} u^{*}\left(\boldsymbol{v}_{k}\right) \geq L
\end{aligned}
$$

Note that in step $(* *)$, we have critically used the result from Lemma 4.3 that, when $\Theta \nsubseteq \mathcal{B}_{L U}$, the points $\tilde{\boldsymbol{v}}_{0}, \tilde{\boldsymbol{v}}_{k}$ are always among the points on the right side of $\Delta_{\Gamma}$, and, therefore, we will always have the equations $q_{0}=u^{*}\left(\boldsymbol{v}_{0}\right), q_{0}+\sum_{i=1}^{k} q_{i}=u^{*}\left(\boldsymbol{v}_{k}\right)$ among the matching equations of system (37). For the last arguments, we have simply used the fact that the optimal control law, $u^{*}(\cdot)$, is always feasible, hence $L \leq u^{*}(\cdot) \leq U$.

This completes our first goal, namely proving that the affine controller $q(\boldsymbol{w})$ is always robustly feasible. To complete the construction, we introduce the following final result:

Lemma 4.5. Using the affine control law $q(\boldsymbol{w})$ computed in Algorithm 1 satisfies the first equation in (35).

Proof. From (36), the affine controller $q(\boldsymbol{w})$ induces the generators $\boldsymbol{a}+c \cdot \boldsymbol{q}$ and $\boldsymbol{b}+\boldsymbol{q}$ for the zonogon $\Gamma$. This implies that $\Gamma$ will be the Minkowski sum of the following segments in $\mathbf{R}^{2}$ :

$$
\begin{gather*}
{\left[\begin{array}{c}
a_{1}+c \cdot q_{1} \\
b_{1}+q_{1}
\end{array}\right], \ldots,\left[\begin{array}{c}
a_{s}+c \cdot q_{s} \\
b_{s}+q_{s}
\end{array}\right],\left[\begin{array}{c}
K_{U} \cdot\left(b_{s+1}+q_{s+1}\right) \\
b_{s+1}+q_{s+1}
\end{array}\right], \ldots,\left[\begin{array}{c}
K_{U} \cdot\left(b_{\min (t, r)}+q_{\min (t, r)}\right) \\
b_{\min (t, r)}+q_{\min (t, r)}
\end{array}\right],} \\
{\left[\begin{array}{c}
K_{L} \cdot\left(b_{\max (t, s)+1}+q_{\max (t, s)+1}\right) \\
b_{\max (t, s)+1}+q_{\max (t, s)+1}
\end{array}\right], \ldots,\left[\begin{array}{c}
K_{L} \cdot\left(b_{r}+q_{r}\right) \\
b_{r}+q_{r}
\end{array}\right],\left[\begin{array}{c}
a_{r+1}+c \cdot q_{r+1} \\
b_{r+1}+q_{r+1}
\end{array}\right], \ldots,\left[\begin{array}{c}
a_{k}+c \cdot q_{k} \\
b_{k}+q_{k}
\end{array}\right] .} \tag{38}
\end{gather*}
$$

From Lemma 4.4, we have that $q_{i}+b_{i} \geq 0, \forall i \in\{1, \ldots, k\}$. Therefore, if we consider the points in $\mathbf{R}^{2}$ :

$$
\boldsymbol{y}_{i}=\left(\sum_{j=0}^{i}\left(a_{j}+c \cdot q_{j}\right), \quad \sum_{j=0}^{i}\left(b_{j}+q_{j}\right)\right), \quad \forall i \in\{0, \ldots, k\},
$$

we can make the following simple observations:

- For any vertex $\boldsymbol{v}_{i} \in \Theta, i \in\{0, \ldots, k\}$, that is matched, i.e. $\tilde{\boldsymbol{v}}_{i} \in \mathrm{r}$-side $\left(\Delta_{\Gamma}\right)$, if we let $\boldsymbol{w}_{i}$ represent the unique $^{6}$ vertex of the hypercube $\mathcal{H}_{k}$ projecting onto $\boldsymbol{v}_{i}$, i.e. $\boldsymbol{v}_{i}=\left(\theta_{1}\left(\boldsymbol{w}_{i}\right), \theta_{2}\left(\boldsymbol{w}_{i}\right)\right)$, then we have:

$$
\boldsymbol{y}_{i} \stackrel{(36)}{=}\left(\gamma_{1}\left(\boldsymbol{w}_{i}\right), \gamma_{2}\left(\boldsymbol{w}_{i}\right)\right) \stackrel{(37)}{=}\left(\tilde{\gamma}_{1}\left(\boldsymbol{v}_{i}\right), \tilde{\gamma}_{2}\left(\boldsymbol{v}_{i}\right)\right) \stackrel{(29)}{=} \tilde{\boldsymbol{v}}_{i} .
$$

The first equality follows from the definition of the mapping that characterizes the zonogon $\Gamma$. The second equality follows from the fact that for any matched vertex $\boldsymbol{v}_{i}$, the coordinates in $\Gamma$ and $\tilde{\Gamma}$ are exactly the same, and the last equality is simply the definition of the point $\tilde{\boldsymbol{v}}_{i}$.

- For any vertex $\boldsymbol{v}_{i} \in \Theta, i \in\{0, \ldots, k\}$, that is not matched, we have:

$$
\begin{array}{ll}
\boldsymbol{y}_{i} \in\left[\boldsymbol{y}_{s}, \boldsymbol{y}_{\min (t, r)}\right], & \forall i \in\{s+1, \ldots, \min (t, r)-1\} \\
\boldsymbol{y}_{i} \in\left[\boldsymbol{y}_{\max (t, s)}, \boldsymbol{y}_{r}\right], & \forall i \in\{\max (t, s)+1, \ldots, r-1\}
\end{array}
$$

This can be seen directly from (38), since the segments in $\mathbf{R}^{2}$ given by $\left[\boldsymbol{y}_{s}, \boldsymbol{y}_{s+1}\right], \ldots,\left[\boldsymbol{y}_{\min (t, r)-1}, \boldsymbol{y}_{\min (t, r)}\right]$ are always aligned (with common cotangent, given by $K_{U}$ ), and, similarly, the segments $\left[\boldsymbol{y}_{\max (t, s)}, \boldsymbol{y}_{\max (t, s)+1}\right]$, $\ldots,\left[\boldsymbol{y}_{r-1}, \boldsymbol{y}_{r}\right]$ are also aligned (with common cotangent $K_{L}$ ).

This exactly corresponds to the situation shown earlier in Figure 6. By combining the two observations, it can be seen that the points $\left\{\boldsymbol{y}_{0}, \boldsymbol{y}_{1}, \ldots, \boldsymbol{y}_{s}, \boldsymbol{y}_{\max (t, s)}, \boldsymbol{y}_{\min (t, r)}, \boldsymbol{y}_{r}, \ldots, \boldsymbol{y}_{k}\right\}$ will satisfy the following properties:

$$
\begin{aligned}
\boldsymbol{y}_{i} & =\tilde{\boldsymbol{v}}_{i}, \forall \tilde{\boldsymbol{v}}_{i} \in \mathrm{r}-\operatorname{side}\left(\Delta_{\Gamma}\right) \\
\operatorname{cotan}\left(\boldsymbol{y}_{0}, \boldsymbol{y}_{1}\right) & \geq \operatorname{cotan}\left(\boldsymbol{y}_{1}, \boldsymbol{y}_{2}\right) \geq \cdots \geq \operatorname{cotan}\left(\boldsymbol{y}_{s-1}, \boldsymbol{y}_{s}\right) \geq \operatorname{cotan}\left(\boldsymbol{y}_{s}, \boldsymbol{y}_{\min (t, r)}\right) \geq \\
& \geq \operatorname{cotan}\left(\boldsymbol{y}_{\max (t, s)}, \boldsymbol{y}_{r}\right) \geq \operatorname{cotan}\left(\boldsymbol{y}_{r}, \boldsymbol{y}_{r+1}\right) \geq \cdots \geq \operatorname{cotan}\left(\boldsymbol{y}_{k-1}, \boldsymbol{y}_{k}\right)
\end{aligned}
$$

where the second relation follows simply because the points $\tilde{\boldsymbol{v}}_{i} \in \mathrm{r}$-side $\left(\Delta_{\Gamma}\right)$ are extreme points on the right side of a convex hull, and thus satisfy the same string of inequalities. This immediately implies that this set of $\boldsymbol{y}_{i}$ (corresponding to $\tilde{\boldsymbol{v}}_{i} \in$ r-side $\left(\Delta_{\Gamma}\right)$ ) exactly represent the right side of the zonogon $\Gamma$, which, in turn, implies that $\Gamma \equiv \mathrm{z}$-hull $\left(\left\{\tilde{\boldsymbol{v}}_{0}, \tilde{\boldsymbol{v}}_{1}, \ldots, \tilde{\boldsymbol{v}}_{s}, \tilde{\boldsymbol{v}}_{\max (t, s)}, \tilde{\boldsymbol{v}}_{\min (t, r)}, \tilde{\boldsymbol{v}}_{r}, \tilde{\boldsymbol{v}}_{r+1}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}\right)$. But then, by Corollary 4.1, the maximum value of problem $(O P T)$ in (26) is equal to the maximum value of problem ( $A F F$ ) in (35), and, since the former is always $J_{m M}$, so is that latter.

This concludes the construction of the affine control law $q(\boldsymbol{w})$. We have shown that the policy computed by Algorithm 1 satisfies all the conditions in (35), i.e. is robustly feasible (by Lemma 4.4) and, when used in conjunction with the original convex state costs, preserves the overall optimal min-max value $J_{m M}$ (Lemma 4.5).

[^4]
### 4.3 Construction of the Affine State Cost.

Note that we have essentially completed the first part of the induction step. For the second part, we would still need to show how an affine stage cost can be computed, such that constraints (13) and (14) are satisfied. We will return temporarily to the notation containing time indices, so as to put the current state of the proof into perspective.

In solving problem $(A F F)$ of (35), we have shown that there exists an affine $q_{k+1}\left(\boldsymbol{w}^{k+1}\right)$ such that:

$$
\begin{aligned}
J_{m M} & =\max _{\boldsymbol{w}^{k+1} \in \mathcal{H}_{k}}\left[\theta_{1}\left(\boldsymbol{w}^{k+1}\right)+c_{k+1} \cdot q_{k+1}\left(\boldsymbol{w}^{k+1}\right)+g_{k+1}\left(\theta_{2}\left(\boldsymbol{w}^{k+1}\right)+q_{k+1}\left(\boldsymbol{w}^{k+1}\right)\right)\right] \\
& \stackrel{(36)}{=} \max _{\boldsymbol{w}^{k+1} \in \mathcal{H}_{k}}\left[\gamma_{1}\left(\boldsymbol{w}^{k+1}\right)+g_{k+1}\left(\gamma_{2}\left(\boldsymbol{w}^{k+1}\right)\right)\right] .
\end{aligned}
$$

Using the definition of $g_{k+1}(\cdot)$ from (7), we can write the above (only retaining the second term) as:

$$
\begin{aligned}
J_{m M} & =\max _{\boldsymbol{w}^{k+1} \in \mathcal{H}_{k}}\left[\gamma_{1}\left(\boldsymbol{w}^{k+1}\right)+\max _{w_{k+1} \in \mathcal{W}_{k+1}}\left[h_{k+2}\left(\gamma_{2}\left(\boldsymbol{w}^{k+1}\right)+w_{k+2}\right)+J_{k+2}^{*}\left(\gamma_{2}\left(\boldsymbol{w}^{k+1}\right)+w_{k+2}\right)\right]\right] \\
& \stackrel{\text { def }}{=} \max _{\boldsymbol{w}^{k+2} \in \mathcal{H}_{k+1}}\left[\pi_{1}\left(\boldsymbol{w}^{k+2}\right)+h_{k+2}\left(\pi_{2}\left(\boldsymbol{w}^{k+2}\right)\right)+J_{k+2}^{*}\left(\pi_{2}\left(\boldsymbol{w}^{k+2}\right)\right)\right],
\end{aligned}
$$

where $\pi_{1}\left(\boldsymbol{w}^{k+2}\right) \stackrel{\text { def }}{=} \gamma_{1}\left(\boldsymbol{w}^{k+1}\right)$, and $\pi_{2}\left(\boldsymbol{w}^{k+2}\right) \stackrel{\text { def }}{=} \gamma_{2}\left(\boldsymbol{w}^{k+1}\right)+w_{k+2}$. Is is easy to note that:

$$
\begin{equation*}
\Pi \stackrel{\text { def }}{=}\left(\pi_{1}\left(\boldsymbol{w}^{k+2}\right), \pi_{2}\left(\boldsymbol{w}^{k+2}\right)\right) \tag{39}
\end{equation*}
$$

represents yet another zonogon, obtained by projecting a hyper-rectangle $\mathcal{H}_{k+1} \subset \mathbf{R}^{k+1}$ into $\mathbf{R}^{2}$. It has a particular shape relative to the zonogon $\Gamma=\left(\gamma_{1}, \gamma_{2}\right)$, since the generators of $\Pi$ are simply obtained by appending a 0 and a 1 , respectively, to the generators of $\Gamma$, which implies that $\Pi$ is the convex hull of two translated copies of $\Gamma$, where the translation is occuring on the $\pi_{2}$ axis. As it turns out, this fact will bear little importance for the discussion to follow, so we include it here only for completeness.

In this context, the problem we would like to solve is to replace the convex function $h_{k+2}\left(\pi_{2}\left(\boldsymbol{w}^{k+2}\right)\right)$ with an affine function $z_{k+2}\left(\boldsymbol{w}^{k+2}\right)$, such that the analogues of conditions (13) and (14) are obeyed:

$$
\begin{aligned}
z_{k+2}\left(\boldsymbol{w}^{k+2}\right) & \geq h_{k+2}\left(\pi_{2}\left(\boldsymbol{w}^{k+2}\right)\right), \quad \forall \boldsymbol{w}^{k+2} \in \mathcal{H}_{k+1} \\
J_{m M} & =\max _{\boldsymbol{w}^{k+2} \in \mathcal{H}_{k+1}}\left[\pi_{1}\left(\boldsymbol{w}^{k+2}\right)+z_{k+2}\left(\boldsymbol{w}^{k+2}\right)+J_{k+2}^{*}\left(\pi_{2}\left(\boldsymbol{w}^{k+2}\right)\right)\right] .
\end{aligned}
$$

We can now switch back to the simplified notation, where all the time subscripts or superscripts are removed. Furthermore, to preserve as much of the familiar notation from Section 4.1.1, we will denote the generators of zonogon $\pi$ by $\boldsymbol{a}, \boldsymbol{b}$, so that we have:

$$
\begin{equation*}
\pi_{1}(\boldsymbol{w})=a_{0}+\sum_{i=1}^{k+1} a_{i} \cdot w_{i} \quad, \quad \pi_{2}(\boldsymbol{w})=b_{0}+\sum_{i=1}^{k+1} b_{i} \cdot w_{i} \tag{40}
\end{equation*}
$$

In perfect analogy to our discussion in Section 4.1, we can introduce:

$$
\begin{equation*}
\boldsymbol{v}^{-} \stackrel{\text { def }}{=} \underset{\pi_{1}}{\arg \max }\left\{\underset{\pi_{2}}{\arg \min }\{\boldsymbol{\pi} \in \Pi\}\right\} ; \quad \boldsymbol{v}^{+} \stackrel{\text { def }}{=} 2 \boldsymbol{O}-\boldsymbol{v}^{-} \quad(\boldsymbol{O} \text { is the center of } \Pi) \tag{41}
\end{equation*}
$$

$$
\boldsymbol{v}_{0} \stackrel{\text { def }}{=} \boldsymbol{v}^{-}, \ldots, \boldsymbol{v}_{p_{1}} \stackrel{\text { def }}{=} \boldsymbol{v}^{+}, \ldots, \boldsymbol{v}_{2 p_{1}}=\boldsymbol{v}^{-} \quad(\text { counter-clockwise numbering of the vertices of } \Pi)
$$

Without loss of generality, we will, again, work under Assumptions 1, 2, and 3, i.e. we will analyze the case when $\mathcal{H}_{k+1}=[0,1]^{k+1}, p_{1}=k+1$ (the zonogon $\Pi$ has a maximal number of vertices), and $\boldsymbol{v}_{i}=$ $[1,1, \ldots, 1,0, \ldots, 0]$ (ones in the first $i$ positions). Furthermore, we will again use $\pi_{1,2}(\boldsymbol{w})$ for $\boldsymbol{w} \in \mathcal{H}_{k+1}$ to denote the mapping from $\mathcal{H}_{k+1} \mapsto \Pi \subset \mathbf{R}^{2}$, and $\pi_{1,2}[\boldsymbol{v}]$ to denote the coordinates of the point $\boldsymbol{v} \in \mathbf{R}^{2}$, and will use the shorthand notations $h\left(\boldsymbol{v}_{i}\right), J^{*}\left(\boldsymbol{v}_{i}\right)$ instead of $h\left(\pi_{2}\left[\boldsymbol{v}_{i}\right]\right)$ and $J^{*}\left(\pi_{2}\left[\boldsymbol{v}_{i}\right]\right)$, respectively.

With the simplified notation, the goal is to find $z(\boldsymbol{w})$ such that:

$$
\begin{align*}
z(\boldsymbol{w}) & \geq h\left(\pi_{2}(\boldsymbol{w})\right), \quad \forall \boldsymbol{w} \in \mathcal{H}_{k+1}  \tag{42}\\
\max _{\left(\pi_{1}, \pi_{2}\right) \in \Pi}\left[\pi_{1}+h\left(\pi_{2}\right)+J^{*}\left(\pi_{2}\right)\right] & =\max _{\boldsymbol{w} \in \mathcal{H}_{k+1}}\left[\pi_{1}(\boldsymbol{w})+z(\boldsymbol{w})+J^{*}\left(\pi_{2}(\boldsymbol{w})\right)\right] \tag{43}
\end{align*}
$$

In (43), the first maximization problem corresponds to the problem solved by the uncertainties, $\boldsymbol{w}$, when the original convex state cost, $h\left(\pi_{2}\right)$, is incurred. As such, the result of the maximization is always exactly equal to $J_{m M}$, the overall min-max value. The second maximization corresponds to the problem solved by the uncertainties when the affine cost, $z(\boldsymbol{w})$, is incurred instead of the convex cost. Requiring that the two optimal values be equal thus amounts to preserving the overall min-max value, $J_{m M}$, under the affine cost.

Since $h$ and $J^{*}$ are convex (see property $\mathbf{P 2}$ in Section 2 ), we can immediately use Lemma 4.1 to conclude that the optimal value in the first maximization problem in (43) is reached at one of the vertices $\boldsymbol{v}_{0}, \ldots, \boldsymbol{v}_{k+1}$ found in r-side(П). Therefore, by introducing the points:

$$
\begin{equation*}
\tilde{\boldsymbol{v}}_{i} \stackrel{\text { def }}{=}\left(\pi_{1}\left[\boldsymbol{v}_{i}\right]+h\left(\boldsymbol{v}_{i}\right), \pi_{2}\left[\boldsymbol{v}_{i}\right]\right), \forall i \in\{0, \ldots, k+1\}, \tag{44}
\end{equation*}
$$

we can immediately conclude the following result:

Lemma 4.6. The maximum in problem:

$$
\begin{gather*}
(O P T) \max _{\left(\tilde{\pi}_{1}, \tilde{\pi}_{2}\right)}\left[\tilde{\pi}_{1}+J^{*}\left(\tilde{\pi}_{2}\right)\right]  \tag{45}\\
\text { where } \quad \tilde{\pi}_{1} \stackrel{\text { def }}{=} \pi_{1}+h\left(\pi_{2}\right), \quad \tilde{\pi}_{2} \stackrel{\text { def }}{=} \pi_{2}, \quad\left(\pi_{1}, \pi_{2}\right) \in \Pi,
\end{gather*}
$$

is reached on the right side of:

$$
\begin{equation*}
\Delta_{\Pi} \stackrel{\text { def }}{=} \operatorname{conv}\left(\left\{\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{k+1}\right\}\right) \tag{46}
\end{equation*}
$$

Proof. The result is analogous to Lemma 4.2, and the proof is a rehashing of similar ideas. In particular, first note that problem $(O P T)$ is a rewriting of the first maximization problem in (43). Therefore, since the maximum of the latter problem is reached at the vertices $\boldsymbol{v}_{i}, i \in\{0, \ldots, k+1\}$, of zonogon $\Pi$, by using (44) we can conclude that the maximum in problem $(O P T)$ must be reached on the set $\left\{\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{k+1}\right\}$. Noting that the function maximized in $(O P T)$ is convex, this set of points can be replaced with its convex hull, $\Delta_{\Pi}$, without affecting the result [see Section 32 of Rockafellar, 1970]. Furthermore, since the function maximized is of the form $\xi_{1}+f\left(\xi_{2}\right)$, with $f$ convex, by applying the results in Corollary 4.1, and replacing the set by the right-side of its convex hull, r-side $\left(\Delta_{\Pi}\right)$, the result of the optimization would remain unchanged.

Continuing the analogy with the construction in Section 4.2, we rewrite the second optimization in (43) as:

$$
\begin{align*}
& (A F F) \max _{\left(\hat{\pi}_{1}, \hat{\pi}_{2}\right) \in \hat{\Pi}}\left[\hat{\pi}_{1}+J^{*}\left(\hat{\pi}_{2}\right)\right],  \tag{47}\\
& \text { where } \hat{\Pi} \stackrel{\text { def }}{=}\left\{\left(\hat{\pi}_{1}, \hat{\pi}_{2}\right): \hat{\pi}_{1}(\boldsymbol{w}) \stackrel{\text { def }}{=} \pi_{1}(\boldsymbol{w})+z(\boldsymbol{w}), \quad \hat{\pi}_{2}(\boldsymbol{w}) \stackrel{\text { def }}{=} \pi_{2}(\boldsymbol{w}), \boldsymbol{w} \in \mathcal{H}_{k+1}\right\} \text {. }
\end{align*}
$$

In order to examine the maximum in problem $(A F F)$, we remark that its feasible set, $\hat{\Pi} \subset \mathbf{R}^{2}$, also represents a zonogon, with generators given by $\boldsymbol{a}+\boldsymbol{z}$ and $\boldsymbol{b}$, respectively. Therefore, by Lemma 4.1, the maximum of problem $(A F F)$ is reached at one of the vertices on the right side of $\hat{\Pi}$.

Using the same key idea from the construction of the affine control law, we now argue that, if the coefficients of the affine cost, $z_{i}$, were computed in such a way that $\hat{\Pi}$ represented the zonogon hull of the set of points $\left\{\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{k+1}\right\}$, then (by Corollary 4.1), the maximum value of problem $(A F F)$ would be the same as the maximum value of problem $(O P T)$.

To this end, we introduce the following procedure for computing the affine cost $z(\boldsymbol{w})$ :

```
Algorithm 2 Compute affine stage cost \(z(\boldsymbol{w})\)
Require: \(\pi_{1}(\boldsymbol{w}), \pi_{2}(\boldsymbol{w}), h(\cdot), J^{*}(\cdot)\).
    Apply the mapping (44) to obtain \(\tilde{\boldsymbol{v}}_{i}, \forall i \in\{0, \ldots, k+1\}\).
    Compute the set \(\Delta_{\Pi}=\operatorname{conv}\left(\left\{\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{k+1}\right\}\right)\).
    Let r-side \(\left(\Delta_{\Pi}\right) \stackrel{\text { def }}{=}\left\{\tilde{\boldsymbol{v}}_{s(1)}, \ldots, \tilde{\boldsymbol{v}}_{s(n)}\right\}\), where \(s(1) \leq s(2) \leq \cdots \leq s(n) \in\{0, \ldots, k+1\}\) are the sorted
    indices of points on the right side of \(\Delta_{\Pi}\).
    4: Solve the following system for \(z_{j},(j \in\{0, \ldots, k+1\})\), and \(K_{s(i)},(i \in\{2, \ldots, n\})\) :
\[
\left\{\begin{align*}
z_{0}+z_{1}+\cdots+z_{s(i)} & =h\left(\boldsymbol{v}_{s(i)}\right), & & \forall \tilde{\boldsymbol{v}}_{s(i)} \in \operatorname{r-side}\left(\Delta_{\Pi}\right)  \tag{48}\\
\frac{z_{j}+a_{j}}{b_{j}} & =K_{s(i)}, & & \forall j \in\{s(i-1)+1, \ldots, s(i)\}, \forall i \in\{2, \ldots, n\},
\end{align*}\right. \text { (alignment) }
\]
```

5: Return $z(\boldsymbol{w})=z_{0}+\sum_{i=1}^{k+1} z_{i} \cdot w_{i}$.

To visualize how the algorithm is working, an extended example is included in Figure 7.


Figure 7: Matching and alignment performed in Algorithm 2.

The intuition behind the construction is the same as that presented in Section 4.2. In particular, the matching constraints in system (48) ensure that for any vertex $\boldsymbol{w}$ of the hypercube $\mathcal{H}_{k+1}$ that corresponds to a potential maximizer in problem $(O P T)$ (through $\boldsymbol{w} \in \mathcal{H}_{k+1} \stackrel{(40)}{\mapsto} \boldsymbol{v}_{i} \in \Pi \stackrel{(44)}{\mapsto} \tilde{\boldsymbol{v}}_{i} \in \mathrm{r}$-side $\left(\Delta_{\Pi}\right)$ ), the value of the affine cost $z(\boldsymbol{w})$ is equal to the value of the initial convex cost, $h\left(\boldsymbol{v}_{i}\right) \equiv h\left(\pi_{2}(\boldsymbol{w})\right)$, implying that the value in problem $(A F F)$ of $(47)$ at $\left(\hat{\pi}_{1}(\boldsymbol{w}), \hat{\pi}_{2}(\boldsymbol{w})\right)$ is equal to the value in problem (OPT) of (45) at $\tilde{\boldsymbol{v}}_{i}$. The alignment constraints in system (48) ensure that any such matched points, $\left(\hat{\pi}_{1}(\boldsymbol{w}), \hat{\pi}_{2}(\boldsymbol{w})\right.$ ), actually correspond to the vertices on the right side of the zonogon $\hat{\Pi}, r$-side $(\hat{\Pi})$, which implies that, as desired, $\hat{\Pi} \equiv \mathrm{z}$-hull $\left(\left\{\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{k+1}\right\}\right)$.

We conclude our preliminary remarks by noting that system (48) does not directly impose the robust domination constraint (42). However, as we will soon see, this result is a byproduct of the way the matching and alignment are performed in Algorithm 2.

### 4.3.1 Affine Cost $z(\cdot)$ Dominates Convex Cost $h(\cdot)$ and Preserves Overall Objective.

In this section, we will prove that the affine cost $z(\boldsymbol{w})$ computed in Algorithm 2 not only robustly dominates the original convex cost (42), but also preserves the overall min-max value (43).

The following lemma summarizes the first main result:
Lemma 4.7. System (48) is always feasible, and the solution $z(\boldsymbol{w})$ always satisfies equation (43).

Proof. We first note that $s(1)=0$ and $s(n)=k+1$, i.e. $\tilde{\boldsymbol{v}}_{0}, \tilde{\boldsymbol{v}}_{k+1} \in \mathrm{r}$-side $\left(\Delta_{\Pi}\right)$. To see why that is the case, note that, by (41), $\boldsymbol{v}_{0}$ will always have the smallest $\pi_{2}$ coordinate in the zonogon $\Pi$. Since the transformation (44) yielding $\tilde{\boldsymbol{v}}_{i}$ leaves the second coordinate unchanged, it will always be true that $\tilde{\boldsymbol{v}}_{0}=\arg \max _{\tilde{\pi}_{1}} \arg \min _{\tilde{\pi}_{2}}\left\{\tilde{\boldsymbol{v}}_{i}, i \in\{0, \ldots, k+1\}\right\}$, which will immediately imply that $\tilde{\boldsymbol{v}}_{0} \in \mathrm{r}-\operatorname{side}\left(\Delta_{\Pi}\right)$. The proof for $\tilde{\boldsymbol{v}}_{k+1}$ follows in an identical matter, since $\boldsymbol{v}_{k+1}$ has the largest $\pi_{2}$ coordinate in $\Pi$.

It can then be checked that the following choice of $z_{i}$ always satisfies system (48):

$$
\begin{aligned}
z_{0} & =h\left(\boldsymbol{v}_{0}\right) ; \quad z_{j}=K_{s(i)} \cdot b_{j}-a_{j}, \quad \forall j \in\{s(i-1)+1, \ldots, s(i)\}, \quad \forall i \in\{2, \ldots, n\}, \\
K_{s(i)} & =\frac{z_{s(i-1)+1}+\cdots+z_{s(i)}+a_{s(i-1)+1}+\cdots+a_{s(i)}}{b_{s(i-1)+1}+\cdots+b_{s(i)}}=\frac{h\left(v_{s(i)}\right)-h\left(v_{s(i-1)}\right)+a_{s(i-1)+1}+\cdots+a_{s(i)}}{b_{s(i-1)+1}+\cdots+b_{s(i)}} .
\end{aligned}
$$

The proof of the second part of the lemma will be analogous to that of Lemma 4.5. To start, consider the feasible set of problem $(A F F)$ in (47), namely the zonogon $\hat{\Pi}$, and note that, from (40), its generators are given by $\boldsymbol{a}+\boldsymbol{z}, \boldsymbol{b}$ :

$$
\left[\begin{array}{c}
\boldsymbol{a}+\boldsymbol{z}  \tag{49}\\
\boldsymbol{b}
\end{array}\right]=\left[\begin{array}{ccccc}
a_{1}+z_{1} & \ldots & a_{s(i)}+z_{s(i)} & a_{s(i)+1}+z_{s(i)+1} & \ldots \\
b_{k+1}+z_{k+1} \\
b_{1} & \ldots & b_{s(1)} & b_{s(1)+1} & \ldots
\end{array}\right] . b_{k+1} .
$$

By introducing the following points in $\mathbf{R}^{2}$ :

$$
\boldsymbol{y}_{i}=\left(\sum_{j=0}^{i}\left(a_{j}+z_{j}\right), \sum_{j=0}^{i} b_{j}\right)
$$

we have the following simple claims:

- For any $\boldsymbol{v}_{i} \in \Pi$ that is matched, i.e. $\tilde{\boldsymbol{v}}_{i} \in \mathrm{r}$-side $\left(\Delta_{\Pi}\right)$, with $\boldsymbol{w}_{i}=[1,1, \ldots, 1,0, \ldots, 0]$ denoting the unique $^{7}$ vertex of $\mathcal{H}_{k+1}$ satisfying $\left(\pi_{1}\left(\boldsymbol{w}_{i}\right), \pi_{2}\left(\boldsymbol{w}_{i}\right)\right)=\boldsymbol{v}_{i}$, we have:

$$
\boldsymbol{y}_{i} \stackrel{(47)}{=}\left(\pi_{1}\left(\boldsymbol{w}_{i}\right)+z\left(\boldsymbol{w}_{i}\right), \pi_{2}\left(\boldsymbol{w}_{i}\right)\right) \stackrel{(48)}{=}\left(\pi_{1}\left[\boldsymbol{v}_{i}\right]+h\left(\boldsymbol{v}_{i}\right), \pi_{2}\left[\boldsymbol{v}_{i}\right]\right) \stackrel{(44)}{=} \tilde{\boldsymbol{v}}_{i} .
$$

The first equality follows from the definition of the zonogon $\hat{\Pi}$, the second follows because any $\tilde{\boldsymbol{v}}_{i} \in$ r-side $\left(\Delta_{\Pi}\right)$ is matched in system (48), and the third equality represents the definition of the points $\tilde{\boldsymbol{v}}_{i}$.

- For any vertex $\boldsymbol{v}_{j} \in \Pi$, that is not matched, i.e. $\tilde{\boldsymbol{v}}_{j} \notin \mathrm{r}$-side $\left(\Delta_{\Pi}\right)$, and $s(i)<j<s(i+1)$ for some $i$, we have $\boldsymbol{y}_{j} \in\left[\boldsymbol{y}_{s(i)}, \boldsymbol{y}_{s(i+1)}\right]$. This can be seen by using the alignment conditions in system (48) in conjunction with (49), since the segments in $\mathbf{R}^{2}$ given by $\left[\boldsymbol{y}_{s(i)}, \boldsymbol{y}_{s(i)+1}\right],\left[\boldsymbol{y}_{s(i)+1}, \boldsymbol{y}_{s(i)+2}\right], \ldots,\left[\boldsymbol{y}_{s(i+1)-1}, \boldsymbol{y}_{s(i+1)}\right]$ will always be parallel, with common cotangent given by $K_{s(i+1)}$.

For a geometric interpretation, the reader is referred back to Figure 7. Corroborating these results with the fact that $\left\{\tilde{\boldsymbol{v}}_{s(1)}, \ldots, \tilde{\boldsymbol{v}}_{s(n)}\right\}=\mathrm{r}$-side $\left(\Delta_{\Pi}\right)$ must always satisfy:

$$
\begin{equation*}
\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{s(1)}, \tilde{\boldsymbol{v}}_{s(2)}\right) \geq \operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{s(2)}, \tilde{\boldsymbol{v}}_{s(3)}\right) \geq \cdots \geq \operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{s(n-1)}, \tilde{\boldsymbol{v}}_{s(n)}\right) \tag{50}
\end{equation*}
$$

we immediately obtain that the points $\left\{\boldsymbol{y}_{s(1)}, \boldsymbol{y}_{s(2)}, \ldots, \boldsymbol{y}_{s(n)}\right\}$ (corresponding to $\tilde{\boldsymbol{v}}_{i} \in$ r-side $\left.\left(\Delta_{\Pi}\right)\right)$ exactly represent the right side of the zonogon $\hat{\Pi}$, which, in turn, implies that $\hat{\Pi} \equiv \mathrm{z}$-hull $\left(\left\{\tilde{\boldsymbol{v}}_{0}, \tilde{\boldsymbol{v}}_{1}, \ldots, \tilde{\boldsymbol{v}}_{k+1}\right\}\right)$. But then, by Corollary 4.1, the maximum value of problem $(O P T)$ in (45) is equal to the maximum value of problem $(A F F)$ in (47), and, since the former is always $J_{m M}$, so is that latter.

In order to complete the second step of the induction, we must only show that the robust domination constraint (42) is also obeyed:

$$
z(\boldsymbol{w}) \geq h\left(\pi_{2}(\boldsymbol{w})\right) \Leftrightarrow z_{0}+z_{1} \cdot w_{1}+\cdots+z_{k+1} \cdot w_{k+1} \geq h\left(b_{0}+b_{1} \cdot w_{1}+\cdots+b_{k+1} \cdot w_{k+1}\right), \forall \boldsymbol{w} \in \mathcal{H}_{k+1} .
$$

The following lemma will take us very close to the desired result:

[^5]LEmma 4.8. The coefficients for the affine cost $z(\boldsymbol{w})$ computed in Algorithm 2 will always satisfy the following property:
$h\left(b_{0}+b_{j(1)}+\cdots+b_{j(m)}\right) \leq z_{0}+z_{j(1)}+\cdots+z_{j(m)}, \forall j(1), \ldots, j(m) \in\{1, \ldots, k+1\}, \forall m \in\{1, \ldots, k+1\}$.
Proof. Before proceeding with the proof, we will first list several properties related to the construction of the affine cost. We claim that, upon termination, Algorithm 2 will produce a solution to the following system:

$$
\begin{align*}
& \begin{cases}z_{0} & =h\left(\boldsymbol{v}_{s(1)}\right) \\
z_{0}+z_{1}+\cdots+z_{s(2)} & =h\left(\boldsymbol{v}_{s(2)}\right) \\
\quad \vdots & \vdots \\
z_{0}+z_{1}+\cdots+z_{s(n)} & =h\left(\boldsymbol{v}_{s(n)}\right) \\
\frac{z_{1}+a_{1}}{b_{1}} & =\cdots=\frac{z_{s(2)}+a_{s(2)}}{b_{s(2)}}=K_{s(2)} \\
\quad \vdots & \quad \vdots \\
\frac{z_{s(n-1)+1}+a_{s(n-1)+1}}{b_{s(n-1)+1}} & =\cdots=\frac{z_{s(n)}+a_{s(n)}}{b_{s(n)}}=K_{s(n)}\end{cases}  \tag{51}\\
& K_{s(2)} \geq \cdots \geq K_{s(n)}  \tag{52}\\
& \left\{\begin{array}{l}
\frac{h\left(\boldsymbol{v}_{j}\right)-h\left(\boldsymbol{v}_{0}\right)+a_{1}+\cdots+a_{j}}{b_{1}+\cdots+b_{j}} \leq K_{s(2)} \leq \frac{h\left(\boldsymbol{v}_{s(2)}\right)-h\left(\boldsymbol{v}_{j}\right)+a_{j+1}+\cdots+a_{s(1)}}{b_{j+1}+\cdots+b_{s(1)}}, \\
\forall j \in\{1, \ldots, s(2)-1\} \\
\vdots \quad \vdots \\
\frac{h\left(\boldsymbol{v}_{j}\right)-h\left(\boldsymbol{v}_{s(n-1)}\right)+a_{s(n-1)+1}+\cdots+a_{j}}{b_{s(n-1)+1}+\cdots+b_{j}} \leq K_{s(n)} \leq \frac{h\left(\boldsymbol{v}_{s(n)}\right)-h\left(\boldsymbol{v}_{j}\right)+a_{j+1}+\cdots+a_{s(n)}}{b_{j+1}+\cdots+b_{s(n)}}, \\
\forall j \in\{s(n-1)+1, \ldots, s(n)-1\} .
\end{array}\right. \tag{53}
\end{align*}
$$

Let us explain the significance of all the equations. (51) is simply a rewriting of the original system (48), which states that at any vertex $\boldsymbol{v}_{s(i)}$, the value of the affine function should exactly match the value assigned by the convex function $h(\cdot)$, and the coefficients $z_{i}$ between any two matched vertices should be such that the resulting segments, $\left[z_{j}+a_{j}, b_{j}\right]$, are aligned (i.e. the angle they form with the $\tilde{\pi}_{1}$ axis has the same cotangent, specified by $K_{(\cdot)}$ variables). We note that we have explicitly used the fact that $s(1)=0$, which we have shown in the first paragraph of the proof of Lemma 4.7.

Equation (52) is a simple restatement of (50), that the cotangents on the right side of a convex hull must be decreasing.

Equation (53) is a direct consequence of the fact that $\left\{\tilde{\boldsymbol{v}}_{s(1)}, \tilde{\boldsymbol{v}}_{s(2)}, \ldots, \tilde{\boldsymbol{v}}_{s(n)}\right\}$ represent r-side $\left(\Delta_{\Pi}\right)$. To see why that is, consider an arbitrary $j \in\{s(i)+1, \ldots, s(i+1)-1\}$. Since $\tilde{\boldsymbol{v}}_{j} \notin \mathrm{r}$-side $\left(\Delta_{\Pi}\right)$, we have:

$$
\begin{aligned}
\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{s(i)}, \tilde{\boldsymbol{v}}_{j}\right) & \leq \operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{j}, \tilde{\boldsymbol{v}}_{s(i+1)}\right){ }^{(40)_{\Leftrightarrow}(44)} \\
\frac{a_{s(i)+1}+\cdots+a_{j}+h\left(\boldsymbol{v}_{j}\right)-h\left(\boldsymbol{v}_{s(i)}\right)}{b_{s(i)+1}+\cdots+b_{j}} & \leq \frac{a_{j+1}+\cdots+a_{s(i+1)}+h\left(\boldsymbol{v}_{s(i+1)}\right)-h\left(\boldsymbol{v}_{j}\right)}{b_{j+1}+\cdots+b_{s(i+1)}} \Leftrightarrow \\
\frac{a_{s(i)+1}+\cdots+a_{j}+h\left(\boldsymbol{v}_{j}\right)-h\left(\boldsymbol{v}_{s(i)}\right)}{b_{s(i)+1}+\cdots+b_{j}} & \leq K_{s(2)} \leq \frac{a_{j+1}+\cdots+a_{s(i+1)}+h\left(\boldsymbol{v}_{s(i+1)}\right)-h\left(\boldsymbol{v}_{j}\right)}{b_{j+1}+\cdots+b_{s(i+1)}},
\end{aligned}
$$

where, in the last step, we have used the mediant inequality ${ }^{8}$ and the fact that, from $(51), K_{s(2)}=$ $\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{s(i)}, \tilde{\boldsymbol{v}}_{s(i+1)}\right)=\frac{a_{s(i)+1}+\cdots+a_{s(i+1)}+h\left(\boldsymbol{v}_{s(i+1)}\right)-h\left(\boldsymbol{v}_{s(i)}\right)}{b_{s(i)+1}+\cdots+b_{s(i+1)}}$ (refer back to Figure 7 for a geometrical interpretation).

With these observations, we will now prove the claim of the lemma. The strategy of the proof will be to use induction on the size of the subsets, $m$. First, we will show the property for any subset of indices $j(1), \ldots, j(m) \in\{s(1)=0, \ldots, s(2)\}$, and will then extend it to $j(1), \ldots, j(m) \in\{s(i)+1, \ldots, s(i+1)\}$ for any $i$, and then to any subset of $\{1, \ldots, k+1\}$.

[^6]The following implications of the conditions (51), (52) and (53), are stated here for convenience, since they will be used throughout the rest of the proof:

$$
\begin{align*}
& h\left(\boldsymbol{v}_{s(1)}\right)=h\left(\boldsymbol{v}_{0}\right)=z_{0} ; \quad h\left(\boldsymbol{v}_{s(2)}\right)=z_{0}+z_{1}+\cdots+z_{s(2)} .  \tag{54}\\
& h\left(\boldsymbol{v}_{j}\right)-h\left(\boldsymbol{v}_{0}\right) \leq z_{1}+\cdots+z_{j}, \quad \forall j \in\{1, \ldots, s(2)-1\} .  \tag{55}\\
& \frac{z_{1}}{b_{1}} \leq \cdots \leq \frac{z_{j}}{b_{j}} \leq \cdots \leq \frac{z_{s(2)}}{b_{s(2)}}, \quad \forall j \in\{1, \ldots, s(2)-1\} . \tag{56}
\end{align*}
$$

Their proofs are straightforward. (54) follows directly from system (51), and:

$$
\begin{aligned}
& \frac{h\left(\boldsymbol{v}_{j}\right)-h\left(\boldsymbol{v}_{0}\right)+a_{1}+\cdots+a_{j}}{b_{1}+\cdots+b_{j}} \stackrel{(53)}{\leq} K_{s(2)} \stackrel{(51)}{=} \frac{z_{1}+\cdots+z_{j}+a_{1}+\cdots+a_{j}}{b_{1}+\cdots+b_{j}} \Rightarrow \text { (55) true. } \\
& \left\{\begin{array}{l}
(51): \frac{a_{1}+z_{1}}{b_{1}}=\cdots=\frac{a_{j}+z_{j}}{b_{j}}=\cdots=\frac{a_{s(2)}+z_{s(2)}}{b_{s(2)}} \\
\Pi \text { zonogon } \Rightarrow \frac{a_{1}}{b_{1}}>\cdots>\frac{a_{j}}{b_{j}}>\cdots>\frac{a_{s(2)}}{b_{s(2)}} \quad \Rightarrow \text { (56) true. }
\end{array}\right.
\end{aligned}
$$

We can now proceed with the proof, by checking the induction for $m=1$. We would like to show that:

$$
h\left(b_{0}+b_{j}\right) \leq z_{0}+z_{j}, \quad \forall j \in\{1, \ldots, s(2)\}
$$

Writing $b_{0}+b_{j}$ as:

$$
\begin{aligned}
b_{0}+b_{j} & =(1-\lambda) \cdot b_{0}+\lambda \cdot\left(b_{0}+\cdots+b_{j}\right) \\
\lambda & =\frac{b_{j}}{b_{1}+\cdots+b_{j}},
\end{aligned}
$$

we obtain:

$$
\begin{aligned}
h\left(b_{0}+b_{j}\right) & \leq(1-\lambda) \cdot h\left(b_{0}\right)+\lambda \cdot \underbrace{h\left(b_{0}+\cdots+b_{j}\right)}_{\equiv h\left(\boldsymbol{v}_{j}\right)} \\
& =h\left(\boldsymbol{v}_{0}\right)+\frac{b_{j}}{b_{1}+\cdots+b_{j}}\left[h\left(\boldsymbol{v}_{j}\right)-h\left(\boldsymbol{v}_{0}\right)\right] \leq(\text { by }(54) \text { if } j=s(2) \text { or (55) otherwise }) \\
& \leq z_{0}+\frac{b_{j}}{b_{1}+\cdots+b_{j}}\left(z_{1}+\cdots+z_{j}\right) \leq(\text { by }(56) \text { and the mediant inequality }) \\
& \leq z_{0}+z_{j}
\end{aligned}
$$

Assume the property is true for any subsets of size $m$. Consider a subset $j(1), \ldots, j(m), j(m+1)$, and, without loss of generality, let $j(m+1)$ be the largest index. With the convex combination:

$$
\begin{aligned}
& b^{*} \stackrel{\text { def }}{=} b_{0}+b_{j(1)}+\cdots+b_{j(m)}+b_{j(m+1)} \\
& \quad=(1-\lambda) \cdot\left(b_{0}+b_{j(1)}+\cdots+b_{j(m)}\right)+\lambda \cdot\left(b_{0}+b_{1}+\cdots+b_{j(m+1)-1}+b_{j(m+1)}\right)
\end{aligned}
$$

$$
\text { where } \lambda=\frac{b_{j(m+1)}}{\left(b_{1}+b_{2}+\cdots+b_{j(m+1)}\right)-\left(b_{j(1)}+b_{j(2)}+\cdots+b_{j(m)}\right)} \text {, }
$$

we obtain:

$$
\begin{aligned}
h\left(b^{*}\right) \leq & (1-\lambda) \cdot h\left(b_{0}+b_{j(1)}+\cdots+b_{j(m)}\right)+\lambda \cdot h\left(\boldsymbol{v}_{i(m+1)}\right) \leq(\text { by induction hypothesis and }(54),(55)) \\
\leq & (1-\lambda) \cdot\left(z_{0}+z_{j(1)}+\cdots+z_{j(m)}\right)+\lambda \cdot\left(z_{0}+z_{1}+\cdots+z_{i(m+1)}\right) \\
= & z_{0}+z_{j(1)}+\cdots+z_{j(m)}+\frac{b_{j(m+1)}}{\left(b_{1}+b_{2}+\cdots+b_{j(m+1)}\right)-\left(b_{j(1)}+b_{j(2)}+\cdots+b_{j(m)}\right)} . \\
& \quad \cdot\left[\left(z_{1}+z_{2}+\cdots+z_{j(m+1)}\right)-\left(z_{j(1)}+z_{j(2)}+\cdots+z_{j(m)}\right)\right] \leq(\text { by }(56) \text { and mediant inequality }) \\
\leq & z_{0}+z_{j(1)}+\cdots+z_{j(m)}+z_{j(m+1)} .
\end{aligned}
$$

We claim that the exact same procedure can be repeated for a subset of indices from $\{s(i)+1, \ldots, s(i+1)\}$, for any index $i \in\{1, \ldots, n-1\}$. We would simply be using the adequate inequality from (53), and the statements equivalent to (54), (55) and (56). The following results would be immediate:

$$
\begin{gather*}
h\left(\left(b_{0}+b_{1}+\cdots+b_{s(i)}\right)+b_{j(1)}+\cdots+b_{j(m)}\right) \leq\left(z_{0}+z_{1}+\cdots+z_{s(i)}\right)+z_{j(1)}+\cdots+z_{j(m)}  \tag{57}\\
\forall i \in\{1, \ldots, n\}, \forall j(1), \ldots, j(m) \in\{s(i)+1, \ldots, s(i+1)\}
\end{gather*}
$$

Note that instead of the term $b_{0}$ for the argument of $h(\cdot)$, we would use the complete sum $b_{0}+b_{1}+\cdots+b_{s(i)}$, and, similarly, instead of $z_{0}$ we would have the complete sum $z_{0}+z_{1}+\cdots+z_{s(i)}$. With these results, we can make use of the increasing increments property of convex functions:

$$
\frac{h\left(x_{1}+\Delta\right)-h\left(x_{1}\right)}{\Delta} \leq \frac{h\left(x_{2}+\Delta\right)-h\left(x_{2}\right)}{\Delta}, \quad \forall \Delta>0, x_{1} \leq x_{2}
$$

to obtain the following result:

$$
\begin{aligned}
& h(b_{0}+\underbrace{b_{j(1)}+\cdots+b_{j(m)}}_{j(\cdot) \in\{1, \ldots, s(2)\}}+\underbrace{b_{i(1)}+\cdots+b_{i(l)}}_{i(\cdot) \in\{s(2)+1, \ldots, s(3)\}})-h\left(b_{0}+b_{j(1)}+\cdots+b_{j(m)}\right) \leq \\
& \leq h(b_{0}+\underbrace{b_{1}+\cdots+b_{s(2)}}_{\text {all indices in }}+b_{i(1)}+\cdots+b_{i(l)})-\underbrace{=}_{\text {def }} h\left(\boldsymbol{v}_{s(2)}\right) \\
& \leq\left(z_{0}+z_{1}+\cdots+z_{s(2)}\right)+z_{i(1)}+\cdots+z_{i(l)}-\left(z_{0}+z_{1}+\cdots+z_{s(2)}\right) \\
& =z_{i(1)}+\cdots+z_{i(l)} \Rightarrow \\
& h\left(b_{0}+b_{j(1)}+\cdots+b_{j(m)}+b_{i(1)}+\cdots+b_{i(l)}\right) \leq h\left(b_{0}+b_{j(1)}+\cdots+b_{j(m)}\right)+z_{i(1)}+\cdots+z_{i(l)}^{(54),(57)} \leq \\
& \quad \leq z_{0}+z_{j(1)}+\cdots+z_{j(m)}+z_{i(1)}+\cdots+z_{i(l)} .
\end{aligned}
$$

We showed the property for indices drawn only from the first two intervals, $\{s(1)+1, \ldots, s(2)\}$ and $\{s(2)+$ $1, \ldots, s(3)\}$, but it should be clear how the argument can be immediately extended to any collection of indices, drawn from any intervals. We omit the details for brevity, and conclude that the claim of the lemma is true.

We are now ready for the last major result:
Lemma 4.9. The affine cost $z(\boldsymbol{w})$ computed by Algorithm 2 always dominates the convex cost $h\left(\pi_{2}(\boldsymbol{w})\right)$ :

$$
h\left(b_{0}+\sum_{i=1}^{k+1} b_{i} \cdot w_{i}\right) \leq z_{0}+\sum_{i=1}^{k+1} z_{i} \cdot w_{i}, \quad \forall \boldsymbol{w} \in \mathcal{H}_{k+1}=[0,1]^{k+1}
$$

Proof. Note first that the function $f(\boldsymbol{w}) \stackrel{\text { def }}{=} h\left(b_{0}+\sum_{i=1}^{k+1} b_{i} \cdot w_{i}\right)-\left(z_{0}+\sum_{i=1}^{k+1} z_{i} \cdot w_{i}\right)$ is a convex function of $\boldsymbol{w}$. Furthermore, the result of Lemma 4.8 can be immediately rewritten as:

$$
h\left(b_{0}+\sum_{i=1}^{k+1} b_{i} \cdot w_{i}\right) \leq z_{0}+\sum_{i=1}^{k+1} z_{i} \cdot w_{i}, \forall \boldsymbol{w} \in\{0,1\}^{k+1} \Leftrightarrow f(\boldsymbol{w}) \leq 0, \forall \boldsymbol{w} \in\{0,1\}^{k+1}
$$

Since the maximum of a convex function on a polytope occurs on the extreme points of the polytope, and $\operatorname{ext}\left(\mathcal{H}_{k+1}\right)=\{0,1\}^{k+1}$, we immediately have that: $\max \boldsymbol{w} \in \mathcal{H}_{k+1} f(\boldsymbol{w})=\max _{\boldsymbol{w} \in\{0,1\}^{k+1}} f(\boldsymbol{w}) \leq 0$, which completes the proof of the lemma.

We can now conclude the proof of correctness in the construction of the affine stage cost, $z(\boldsymbol{w})$. With Lemma 4.9, we have that the affine cost always dominates the convex cost $h(\cdot)$, thus condition (42) is obeyed. Furthermore, from Lemma 4.7, the overall min-max cost remains unchanged even when incurring the affine stage cost, $z(\boldsymbol{w})$, hence condition (43) is also true. This completes the construction of the affine cost, and hence also the full step of the induction hypothesis.

### 4.3.2 Proof of Main Theorem.

To finalize the current section, we summarize the steps that have lead us to the result, thereby proving the main Theorem 3.1.

Proof of Theorem 3.1. In Section 4.1, we have verified the induction hypothesis at time $k=1$. With the induction hypothesis assumed true for times $t=1, \ldots, k$, we have listed the initial consequences in Lemma 4.1 and Corollary 4.1 of Section 4.1.1. By exploring the structure of the optimal control law, $u_{k+1}^{*}\left(x_{k+1}\right)$, and the optimal value function, $J_{k+1}^{*}\left(x_{k+1}\right)$, in Section 4.1.2, we have finalized the analysis of the induction hypothesis, and summarized our findings in Lemmas 4.2 and 4.3.

Section 4.2 then introduced the main construction of the affine control law, $q_{k+1}\left(\boldsymbol{w}^{k+1}\right)$, which was shown to be robustly feasible (Lemma 4.4). Furthermore, in Lemma 4.5, we have shown that, when used in conjuction with the original convex state costs, $h_{k+1}\left(x_{k+2}\right)$, this affine control preserves the min-max value of the overall problem.

In Section 4.3, we have also introduced an affine stage cost, $z_{k+2}\left(\boldsymbol{w}^{k+2}\right)$, which, if incurred at time $k+1$, will always preserve the overall min-max value (Lemma 4.7), despite being always larger than the original convex cost, $h_{k+1}\left(x_{k+2}\right)$ (Lemma 4.9).

### 4.3.3 Counterexamples for potential extensions.

On first sight, one might be tempted to believe that the results in Theorem 3.1 could be immediately extended to more general problems. In particular, one could be tempted to ask one of the following natural questions:

1. Would both results of Theorem 3.1 (i.e. existence of affine control laws and existence of affine stage costs) hold for a problem which also included linear constraints coupling the controls $u_{t}$ across different time-steps? (see Ben-Tal et al. [2005b] for a situation when this might be of interest)
2. Would both results of Theorem 3.1 hold for multi-dimensional linear systems? (i.e. problems where $x_{k} \in \mathbf{R}^{d}, \forall k$, with $d \geq 2$ )
3. Are affine policies in the disturbances optimal for the two problems above?

In the rest of the current section, we would like to show how these questions can all be answered negatively, using the following simple counterexample:

$$
\begin{align*}
& T=4, c_{k}=1, h_{k}\left(x_{k+1}\right)=\max \left\{18.5 \cdot x_{k+1},-24 \cdot x_{k+1}\right\}, L_{k}=0, U_{k}=\infty, \forall k \in\{1, \ldots, 4\}, \\
& w_{1} \in[-7,0], w_{2} \in[-11,0], w_{3} \in[-8,0], w_{4} \in[-44,0]  \tag{CEx}\\
& \sum_{i=1}^{k} u_{i} \leq 10 \cdot k, \forall k \in\{1, \ldots, 4\} .
\end{align*}
$$

The first two rows describe a one-dimensional problem that fits the conditions of Problem 1.1 in Section 1. The third row corresponds to a coupling constraint for controls at different times, so that the problem fits question (i) above. Furthermore, since the state in such a problem consists of two variables (one for $x_{k}$ and one for $\sum_{i=1}^{k} u_{k}$ ), the example also fits question (ii) above.

The optimal min-max value for the counterexample ( $C E x$ ) above can be found by solving a stochastic optimization problem (see Ben-Tal et al. [2005b]), in which non-anticipatory decisions are computed at all the extreme points of the uncertainty set, i.e. for $\left\{\underline{w}_{1}, \bar{w}_{1}\right\} \times\left\{\underline{w}_{2}, \bar{w}_{2}\right\} \times\left\{\underline{w}_{3}, \bar{w}_{3}\right\} \times\left\{\underline{w}_{4}, \bar{w}_{4}\right\}$. The resulting model, which is a large linear program, can be solved to optimality, resulting in a corresponding value of approximately 838.493 for problem ( $C E x$ ).

To compute the optimal min-max objective obtained by using affine policies $q_{k}\left(\boldsymbol{w}^{k}\right)$ and incurring affine costs $z_{k}\left(\boldsymbol{w}^{k+1}\right)$, one can ammend the model $(A A R C)$ from Section 3 by including constraints for the cumulative controls (see Ben-Tal et al. [2005b] for details), and then using (15) to rewrite the resulting model as a linear program. The optimal value of this program for counterexample ( $C E x$ ) was approximately 876.057, resulting in a gap of $4.4 \%$, and thus providing a negative answer to questions (i) and (ii).

To investigate question (iii), we remark that the smallest objective achievable by using affine policies of the type $q_{k}\left(\boldsymbol{w}^{k}\right)$ can be found by solving another stochastic optimization problem, having as decision
variables the affine coefficients $\left\{q_{k, t}\right\}_{0 \leq t<k \leq T}$, as well as (non-anticipatory) stage cost variables $z_{k}^{\boldsymbol{w}}$ for every time step $k \in\{1, \ldots, T\}$ and every extreme point $\boldsymbol{w}$ of the uncertainty set. Solving the resulting linear program for instance ( $C E x$ ) gave an optimal value of 873.248 , so strictly larger than the (true) optimum (838.493), and strictly smaller than the optimal value of the model utilizing both affine control policies and affine stage costs (876.057).

Thus, with question (iii) also answered negatively, we conclude that policies that are affine in the disturbances, $q_{k}\left(\boldsymbol{w}^{k}\right)$, are in general suboptimal for problems with cumulative control constraints or multiple dimensions, and that replacing the convex state costs $h_{k}\left(x_{k+1}\right)$ by (larger) affine costs $z_{k}\left(\boldsymbol{w}^{k+1}\right)$ would, in general, result in even further deterioration of the objective.

## 5 An application in inventory management.

In this section, we would like to explore our results in connection with a classical inventory problem. This idea was originally introduced by Ben-Tal et al. [2005b], in the context of a more general model: the retailersupplier with flexible commitment contracts problem (RSFC). We will first describe a simplified version of the problem, and then draw a very interesting connection with our results.

The setting is the following: consider a single-product, single-echelon, multi-period supply chain, in which inventories are managed periodically over a planning horizon of $T$ periods. The unknown demands $w_{t}$ from customers arrive at the (unique) echelon, henceforth referred to as the retailer, and are satisfied from the on-hand inventory, denoted by $x_{t}$ at the beginning of period $t$. The retailer can replenish the inventory by placing orders $u_{t}$, at the beginning of each period $t$, for a cost of $c_{t}$ per unit of product. These orders are immediately available, i.e. there is no lead-time in the system, but there are capacities on how much the retailer can order: $L_{t} \leq u_{t} \leq U_{t}$. After the demand $w_{t}$ is realized, the retailer incurs holding costs $H_{t} \cdot \max \left\{0, x_{t}+u_{t}-w_{t}\right\}$ for all the amounts of supply stored on her premises, as well as penalties $B_{t} \cdot \max \left\{w_{t}-x_{t}-u_{t}, 0\right\}$, for any demand that is backlogged.

In the spirit of robust optimization, we will assume that the only information available about the demand at time $t$ is that it resides within a certain inverval centered around a nominal (or mean) demand $\bar{d}_{t}$, which results in the uncertainty set $\mathcal{W}_{t}=\left\{\left|w_{t}-\bar{d}_{t}\right| \leq \rho \cdot \bar{d}_{t}\right\}$, where $\rho \in[0,1]$ can be interpreted as an uncertainty level. As such, if we take the objective function to be minimized as the cost resulting in the worst-case scenario, we immediately obtain an instance of our original Problem 1.1, with $\alpha_{t}=\beta_{t}=1, \gamma_{t}=-1$, and the convex state costs $h_{t}(\cdot)$ denoting the Newsvendor cost, $h_{t}\left(x_{t+1}\right)=H_{t} \cdot \max \left\{x_{t}+u_{t}-w_{t}, 0\right\}+B_{t} \cdot \max \left\{w_{t}-\right.$ $\left.x_{t}-u_{t}, 0\right\}$.

Therefore, the results in Theorem 3.1 are immediately applicable to conclude that no loss of optimality is incurred when we restrict attention to affine order quantities $q_{t}$ that depend on the history of available demands at time $t, q_{t}\left(\boldsymbol{w}^{t}\right)=q_{t, 0}+\sum_{\tau=1}^{t-1} q_{t, \tau} \cdot w_{\tau}$, and when we replace the Newsvendor costs $h_{t}\left(x_{t+1}\right)$ by some (potentially larger) affine costs $z_{t}\left(\boldsymbol{w}^{t+1}\right)$. The main advantage is that, with these substitutions, the problem of finding the optimal affine policies becomes an LP (see the discussion in Section 3 and the paper by Ben-Tal et al. [2005b] for more details).

The more interesting connection with our results comes if we recall the construction in Algorithm 1. In particular, we have the following simple claim:
Proposition 5.1. If the affine orders $q_{t}\left(\boldsymbol{w}^{t}\right)$ computed in Algorithm 1 are implemented at every time step $t$, and we let: $x_{k}\left(\boldsymbol{w}^{k}\right)=x_{1}+\sum_{t=1}^{k-1}\left(q_{t}\left(\boldsymbol{w}^{t}\right)-w_{t}\right) \stackrel{\text { def }}{=} x_{t, 0}+\sum_{t=1}^{k-1} x_{k, t} \cdot w_{t}$ denote the affine dependency of the inventory $x_{k}$ on the history of demands, $\boldsymbol{w}^{k}$, then:

1. If a certain demand $w_{t}$ is fully satisfied by time $k \geq t+1$, i.e. $x_{k, t}=0$, then all the (affine) orders $q_{\tau}$ placed after time $k$ will not depend on $w_{t}$.
2. Every demand $w_{t}$ is at most satisfied by the future orders $q_{k}, k \geq t+1$, and the coefficient $q_{k, t}$ represents what fraction of the demand $w_{t}$ is satisfied by the order $q_{k}$.
Proof. To prove the first claim, recall that, in our notation from Section 4.1.1, $x_{k} \equiv \theta_{2}=b_{0}+\sum_{t=1}^{k-1} b_{t} \cdot w_{t}$. Applying part $(i)$ of Lemma 4.4 in the current setting ${ }^{9}$, we have that $0 \leq q_{k, t} \leq-x_{k, t}$. Therefore, if

[^7]$x_{k, t}=0$, then $q_{k, t}=0$, which implies that $x_{k+1, t}=0$. By induction, we immediately get that $q_{\tau, t}=0, \forall \tau \in$ $\{k, \ldots, T\}$.

To prove the second part, note that any given demand, $w_{t}$, initially has an affine coefficient of -1 in the state $x_{t+1}$, i.e. $x_{t+1, t}=-1$. By part (i) of Lemma 4.4, $0 \leq q_{t+1, t} \leq-x_{t+1, t}=1$, so that $q_{t+1, t}$ represents a fraction of the demand $w_{t}$ satisfied by the order $q_{t+1}$. Furthermore, $x_{t+2, t}=x_{t+1, t}+q_{t+1, t} \in[-1,0]$, so, by induction, we immediately have that $q_{k, t} \in[0,1], \forall k \geq t+1$, and $\sum_{k=t+1}^{T} q_{k, t} \leq 1$.

In view of this result, if we think of $\left\{q_{k}\right\}_{k \geq t+1}$ as future orders that are partially satisfying the demand $w_{t}$, then every future order quantity $q_{k}\left(\boldsymbol{w}^{k}\right)$ will satisfy exactly a fraction of the demand $w_{t}$ (since the coefficient for $w_{t}$ in $q_{k}$ will always be in $[0,1]$ ), and every demand will be at most satisfied by the sequence of orders following after it appears. This interpretation bears some similarity with the unit decomposition approach of Muharremoglu and Tsitsiklis [2007], where every unit of supply can be interpreted as satisfying a particular unit of the demand. Here, we are accounting for fractions of the total demand, as being satisfied by future order quantities.

## 6 Conclusions. Future Directions.

We have presented a novel approach for theoretically handling robust, multi-stage decision problems. The method strongly utilized the connections between the geometrical properties of the feasible sets (zonogons), and the objective functions being optimized, in order to prune the set of relevant points and derive properties about the optimal policies for the problem. We have also shown an interesting implication of our theoretical results in the context of a classical problem in inventory management.

On a theoretical level, one immediate direction of future research would be to study systems with mixed (polyhedral) constraints, on both state and control at time $t$. Furthermore, we would like to explore the possibility of utilizing the same proof technique in the context of multi-dimensional problems, as well as for more complicated uncertainty sets $\mathcal{W}$.

Second, we would like to better understand the connections between the matching performed in Algorithm 2 and the properties of convex (and supermodular) functions, and explore extensions of the approach to handle cost functions that are not necessarily convex, as well as non-linear cost structures for the control $u_{t}$. Another potential area of interest would be to use our analysis tools to quantify the performance of affine policies even in problems where they are known to be suboptimal (such as the one suggested in Section 4.3.3). This could potentially lead to fast approximation algorithms, with solid theoretical foundations.

On a practical level, we would like to explore potential applications arising in robust portfolio optimization, as well as operations management. Also, we would like to construct a procedure that mimics the behavior of our algorithms, but does not require knowledge of the optimal value functions $J^{*}(\cdot)$ or optimal controllers $u^{*}(\cdot)$. One potential idea would be to explore which types of cuts could be added to the linear program $(A A R C)$, to ensure that it computes a solution as "close" to the affine controller $q(\boldsymbol{w})$ as possible.

## 7 Appendix.

### 7.1 Dynamic Programming Solution.

This section contains a detailed proof for the solution of the Dynamic Programming formulation, initially introduced in Section 2. Recall that the problem we would like to solve is the following:

$$
\begin{aligned}
& J_{m M} \stackrel{\text { def }}{=} \min _{u_{1}}\left[c_{1} \cdot u_{1}+\max _{w_{1}}\left[h_{1}\left(x_{2}\right)+\cdots+\min _{u_{k}}\left[c_{k} \cdot u_{k}+\max _{w_{k}}\left[h_{k}\left(x_{k+1}\right)+\ldots\right.\right.\right.\right. \\
& \left.\left.+\min _{u_{T}}\left(c_{T} \cdot u_{T}+\max _{w_{T}} h_{T}\left(x_{T+1}\right)\right) \cdots\right] \cdots\right] \\
& \text { s.t. } \quad x_{k+1}=x_{k}+u_{k}+w_{k} \\
& L_{k} \leq u_{k} \leq U_{k} \quad \forall k \in\{1,2, \ldots, T\}, \\
& w_{k} \in \mathcal{W}_{k}=\left[\underline{w}_{k}, \bar{w}_{k}\right]
\end{aligned}
$$

which gives rise to the corresponding Bellman recursion:

$$
J_{k}^{*}\left(x_{k}\right) \stackrel{\text { def }}{=} \min _{L_{k} \leq u_{k} \leq U_{k}}\left[c_{k} \cdot u_{k}+\max _{w_{k} \in \mathcal{W}_{k}}\left[h_{k}\left(x_{k}+u_{k}+w_{k}\right)+J_{k+1}^{*}\left(x_{k}+u_{k}+w_{k}\right)\right]\right]
$$

According to our definition of running cost and cost-to-go, the cost at $T+1$ is $J_{T+1}^{*}=0$, which yields the following Bellman recursion at time $T$ :

$$
J_{T}^{*}\left(x_{T}\right) \stackrel{\text { def }}{=} \min _{L_{T} \leq u_{T} \leq U_{T}}\left[c_{T} \cdot u_{T}+\max _{w_{T} \in \mathcal{W}_{T}} h_{T}\left(x_{T}+u_{T}+w_{T}\right)\right]
$$

First consider the inner (maximization) problem. Letting $y_{T} \stackrel{\text { def }}{=} x_{T}+u_{T}$, we obtain:

$$
\begin{align*}
g_{T}\left(y_{T}\right) & \stackrel{\text { def }}{=} \max _{w_{T} \in\left[\underline{w}_{T}, \bar{w}_{T}\right]} h_{T}\left(x_{T}+u_{T}+w_{T}\right) \\
\left(\text { since } h_{T}(\cdot) \text { convex }\right) & =\max \left\{h_{T}\left(y_{T}+\underline{w}_{T}\right), h_{T}\left(y_{T}+\bar{w}_{T}\right)\right\} \tag{58}
\end{align*}
$$

Note that $g_{T}$ is the maximum of two convex, coercive functions of $y_{T}$, hence it is also convex and coercive (see Theorem 5.5 in Rockafellar [1970] or Proposition 1.2.4 in Bertsekas [2003] for details). The outer (minimization) problem at time $T$ becomes:

$$
\begin{aligned}
J_{T}^{*}\left(x_{T}\right) & =\min _{L_{T} \leq u_{T}(\cdot) \leq U_{T}} c_{T} \cdot u_{T}+g_{T}\left(x_{T}+u_{T}\right) \\
& =-c_{T} \cdot x_{T}+\min _{L_{T} \leq u_{T}(\cdot) \leq U_{T}}\left[c_{T} \cdot\left(x_{T}+u_{T}\right)+g_{T}\left(x_{T}+u_{T}\right)\right]
\end{aligned}
$$

For any $x_{T}, c_{T} \cdot\left(x_{T}+u_{T}\right)+g_{T}\left(x_{T}+u_{T}\right)$ is a convex function of its argument $y_{T}=x_{T}+u_{T}$. As such, by defining $\left[\underline{y}_{t}, \bar{y}_{t}\right]$ to be the compact set of minimizers of the convex and coercive function $c_{T} \cdot y+g_{T}(y)$, we obtain that the optimal controller and optimal value function at time $T$ will be:

$$
\begin{align*}
& u_{T}^{*}\left(x_{T}\right)= \begin{cases}U_{T}, & \text { if } x_{T}<\underline{y}_{T}-U_{T} \\
-x_{T}+y^{*}, & \text { otherwise } \\
L_{T}, & \text { if } x_{T}>\bar{y}_{T}-L_{T}\end{cases}  \tag{59}\\
& J_{T}^{*}\left(x_{T}\right)= \begin{cases}c_{T} \cdot U_{T}+g_{T}\left(x_{T}+U_{T}\right), & \text { if } x_{T}<\underline{y}_{T}-U_{T} \\
c_{T} \cdot\left(y^{*}-x_{T}\right)+g_{T}\left(y^{*}\right), & \text { otherwise } \\
c_{T} \cdot L_{T}+g_{T}\left(x_{T}+L_{T}\right), & \text { if } x_{T}>\bar{y}_{T}-L_{T}\end{cases} \tag{60}
\end{align*}
$$

where $y^{*} \in\left[\underline{y}_{T}, \bar{y}_{T}\right]$.
The following properties are immediately obvious:

1. $u_{T}^{*}\left(x_{T}\right)$ is piecewise affine (with at most 3 pieces), continuous, monotonically decreasing in $x_{T}$.
2. $J_{T}^{*}\left(x_{T}\right)$ is convex, since it represents a partial minimization of a convex function with respect to one of the variables (see Proposition 2.3.6 in Bertsekas [2003]).

The results can be immediately extended by induction on $k$ :
LEMMA 7.1. The optimal control policy $u_{k}^{*}\left(x_{k}\right)$ is piecewise affine, with at most 3 pieces, continuous, and monotonically decreasing in $x_{k}$. The optimal objective function $J_{k}^{*}\left(x_{k}\right)$ is convex in $x_{t}$.

Proof. The induction is checked at $k=T$. Assume the property is true at $k+1$. Letting $y_{k} \xlongequal{\text { def }} x_{k}+u_{k}$, the Bellman recursion at $k$ becomes:

$$
\begin{aligned}
& J_{k}^{*}\left(x_{k}\right) \stackrel{\text { def }}{=} \min _{L_{k} \leq u_{k} \leq U_{k}}\left[c_{k} \cdot u_{k}+g_{k}\left(x_{k}+u_{k}\right)\right] \\
& g_{k}\left(y_{k}\right) \stackrel{\text { def }}{=} \max _{w_{k} \in \mathcal{W}_{k}}\left[h_{k}\left(y_{k}+w_{k}\right)+J_{k+1}^{*}\left(y_{k}+w_{k}\right)\right]
\end{aligned}
$$

Consider first the maximization problem. Since $h_{k}$ is convex, and (by the induction hypothesis) $J_{k+1}^{*}$ is also convex, the maximum will be reached on the boundary of $\mathcal{W}_{k}=\left[\underline{w}_{k}, \bar{w}_{k}\right]$ :

$$
\begin{equation*}
g_{k}\left(y_{k}\right)=\max \left\{h_{k}\left(y_{k}+\underline{w}_{k}\right)+J_{k+1}^{*}\left(y_{k}+\underline{w}_{k}\right), \quad h_{k}\left(y_{k}+\bar{w}_{k}\right)+J_{k+1}^{*}\left(y_{k}+\bar{w}_{k}\right)\right\} \tag{61}
\end{equation*}
$$

and $g_{k}\left(y_{k}\right)$ will be also be convex. The minimization problem becomes:

$$
\begin{align*}
J_{k}^{*}\left(x_{k}\right) & =\min _{L_{k} \leq u_{k} \leq U_{k}}\left[c_{k} \cdot u_{k}+g_{k}\left(x_{k}+u_{k}\right)\right] \\
& =-c_{k} \cdot x_{k}+\min _{L_{k} \leq u_{k} \leq U_{k}}\left[c_{k} \cdot\left(x_{k}+u_{k}\right)+g_{k}\left(x_{k}+u_{k}\right)\right] \tag{62}
\end{align*}
$$

Defining, as before, $\left[\underline{y}_{k}, \bar{y}_{k}\right]$ as the set of minimizers of $c_{k} \cdot y+g_{k}(y)$, we get:

$$
\begin{align*}
& u_{k}^{*}\left(x_{k}\right)= \begin{cases}U_{k}, & \text { if } x_{k}<\underline{y}_{k}-U_{k} \\
-x_{k}+y^{*}, & \text { otherwise } \\
L_{k}, & \text { if } x_{k}>\bar{y}_{k}-L_{k}\end{cases}  \tag{63}\\
& J_{k}^{*}\left(x_{k}\right)= \begin{cases}c_{k} \cdot U_{k}+g_{k}\left(x_{k}+U_{k}\right), & \text { if } x_{k}<\underline{y}_{k}-U_{k} \\
c_{k} \cdot\left(y^{*}-x_{k}\right)+g_{k}\left(y^{*}\right), & \text { otherwise } \\
c_{k} \cdot L_{k}+g_{k}\left(x_{k}+L_{k}\right), & \text { if } x_{k}>\bar{y}_{k}-L_{k}\end{cases} \tag{64}
\end{align*}
$$

where $y^{*} \in\left[\underline{y}_{k}, \bar{y}_{k}\right]$.
In particular, $u_{k}^{*}$ will be piecewise affine with 3 pieces, continuous, monotonically decreasing, and $J_{k}^{*}$ will be convex (as the partial minimization of a convex function with respect to one of the variables). A typical example of the optimal control law and the optimal value function is shown in Figure 1 of Section 2.

### 7.2 Zonotopes and Zonogons.

In this section of the Appendix, we would like to outline several useful properties of the main geometrical objects of interest in our exposition, namely zonotopes. The presentation here parallels that in Chapter 7 of Ziegler [2003], to which the interested reader is referred to for a much more comprehensive treatment.

Zonotopes are special polytopes that can be viewed in various ways: as projections of cubes, as Minkowski sums of line segments, and as sets of bounded linear combinations of vector configurations. Each description gives a different insight into the combinatorics of zonotopes, and there exist some very interesting results that unify the different descriptions under a common theory. For our purposes, it will be sufficient to understand zonotopes under the first two descriptions. In particular, letting $\mathcal{H}_{k}$ denote the $k$-dimensional hypercube, $\mathcal{H}_{k}=\left\{\boldsymbol{w} \in \mathbf{R}^{k}: 0 \leq w_{i} \leq 1, \forall i\right\}$, we can introduce the following definition:

Definition 7.1 (7.13 in Ziegler [2003]). A zonotope is the image of a cube under an affine projection, that is, a d-polytope $Z \subseteq \mathbf{R}^{d}$ of the form

$$
\begin{aligned}
Z=Z(V) & :=V \cdot \mathcal{H}_{k}+\boldsymbol{z}=\left\{V \boldsymbol{w}+\boldsymbol{z}: \boldsymbol{w} \in \mathcal{H}_{k}\right\} \\
& =\left\{\boldsymbol{x} \in \mathbf{R}^{d}: \boldsymbol{x}=\boldsymbol{z}+\sum_{i=1}^{k} w_{i} \boldsymbol{v}_{i}, 0 \leq y_{i} \leq 1\right\}
\end{aligned}
$$

for some matrix (vector configuration) $V=\left(\boldsymbol{v}_{1}, \ldots, \boldsymbol{v}_{k}\right) \in \mathbf{R}^{d \times k}$.
The rows of the matrix $V$ are often referred to as the generators defining the zonotope. An equivalent description of the zonotope can be obtained by recalling that every $k$-cube $\mathcal{H}_{k}$ is a product of line segments $\mathcal{H}_{k}=\mathcal{H}_{1} \times \cdots \times \mathcal{H}_{1}$. Since for a linear operator $\pi$ we always have: $\pi\left(\mathcal{H}_{1} \times \cdots \times \mathcal{H}_{1}\right)=\pi\left(\mathcal{H}_{1}\right)+\cdots+\pi\left(\mathcal{H}_{1}\right)$, by considering an affine map given by $\pi(\boldsymbol{w})=V \boldsymbol{w}+\boldsymbol{z}$, it is easy to see that every zonotope is the Minkowski sum of a set of line segments:

$$
Z(V)=\left[0, \boldsymbol{v}_{1}\right]+\cdots+\left[0, \boldsymbol{v}_{p}\right]+\boldsymbol{z}
$$

For completeness, we remark that there is no loss of generality in regarding a zonotope as a projection from the unit hypercube $\mathcal{H}_{k}$, since any projection from an arbitrary hyper-rectangle in $\mathbf{R}^{k}$ can always be seen as a projection from the unit hypercube in $\mathbf{R}^{k}$. To see this, consider an arbitrary hyper-rectangle in $\mathbf{R}^{k}$ :

$$
\mathcal{W}_{k}=\left[\underline{w}_{1}, \bar{w}_{1}\right] \times\left[\underline{w}_{2}, \bar{w}_{2}\right] \times \cdots \times\left[\underline{w}_{k}, \bar{w}_{k}\right],
$$

and note that, with $V \in \mathbf{R}^{d \times k}$, and $\boldsymbol{a}^{T} \in \mathbf{R}^{k}$ denoting the $j$-th row of $V$, the $j$-th component of $Z(V) \stackrel{\text { def }}{=} V$. $\mathcal{W}_{k}+\boldsymbol{z}$ can be written:

$$
Z(V)_{j} \stackrel{\text { def }}{=} z_{j}+\sum_{i=1}^{k}\left(a_{i} \cdot w_{i}\right)=\left(z_{j}+\sum_{i=1}^{k} a_{i} \cdot \underline{w}_{i}\right)+\sum_{i=1}^{k} a_{i} \cdot\left(\bar{w}_{i}-\underline{w}_{i}\right) \cdot y_{i}, \text { where } y_{i} \in[0,1], \forall 1 \leq i \leq k
$$

An example of a subclass of zonotopes are the zonogons, which are all centrally symmetric, 2-dimensional $2 p$-gons, arising as the projection of $p$-cubes to the plane. An example is shown in Figure 2 of Section 4.1. These are the main objects of interest in our treatment, and the following lemma summarizes their most important properties:

Lemma 7.2. Let $\mathcal{H}_{k}=[0,1]^{k}$ be a $k$-dimensional hypercube, $k \geq 2$. For fixed $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{R}^{k}$ and $a_{0}, b_{0} \in \mathbf{R}$, consider the affine transformation $\pi: \mathbf{R}^{k} \rightarrow \mathbf{R}^{2}, \pi(\boldsymbol{w})=\left[\begin{array}{c}\boldsymbol{a}^{T} \\ \boldsymbol{b}^{T}\end{array}\right] \cdot \boldsymbol{w}+\left[\begin{array}{c}a_{0} \\ b_{0}\end{array}\right]$ and the zonogon $\Theta \subset \mathbf{R}^{2}$ :

$$
\Theta=\pi\left(\mathcal{H}_{k}\right) \stackrel{\text { def }}{=}\left\{\boldsymbol{\theta} \in \mathbf{R}^{2}: \exists \boldsymbol{w} \in \mathcal{H}_{k} \text { s.t. } \boldsymbol{\theta}=\pi(\boldsymbol{w})\right\} .
$$

If we let $\mathcal{V}_{\Theta}$ denote the set of vertices of $\Theta$, then the following properties are true:

1. $\exists \boldsymbol{O} \in \Theta$ such that $\Theta$ is symmetric around $\boldsymbol{O}: \forall \boldsymbol{x} \in \Theta \Rightarrow 2 \boldsymbol{O}-\boldsymbol{x} \in \Theta$.
2. $\left|\mathcal{V}_{\Theta}\right|=2 p \leq 2 k$ vertices. Also, $p<k$ if and only if $\exists i \neq j \in\{1, \ldots, k\}$ such that $\operatorname{rank}\left(\left[\begin{array}{ll}a_{i} & a_{j} \\ b_{i} & b_{j}\end{array}\right]\right)<$ 2.
3. If we number the vertices of $\mathcal{V}_{\Theta}$ in cyclic order:

$$
\mathcal{V}_{\Theta}=\left(\boldsymbol{v}_{0}, \ldots, \boldsymbol{v}_{i}, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_{2 p-1}\right) \quad\left(\boldsymbol{v}_{2 p+i} \stackrel{\text { def }}{=} \boldsymbol{v}_{(2 p+i)} \bmod (2 p)\right)
$$

then $2 \boldsymbol{O}-\boldsymbol{v}_{i}=\boldsymbol{v}_{i+p}$, and we have the following representation for $\Theta$ as a Minkowski sum of line segments:

$$
\begin{aligned}
& \Theta=\boldsymbol{O}+\left[-\frac{\boldsymbol{v}_{1}-\boldsymbol{v}_{0}}{2}, \frac{\boldsymbol{v}_{1}-\boldsymbol{v}_{0}}{2}\right]+\cdots+\left[-\frac{\boldsymbol{v}_{p}-\boldsymbol{v}_{p-1}}{2}, \frac{\boldsymbol{v}_{p}-\boldsymbol{v}_{p-1}}{2}\right] \\
& \stackrel{\text { def }}{=} \boldsymbol{O}+\sum_{i=1}^{p} \lambda_{i} \cdot \frac{\boldsymbol{v}_{i}-\boldsymbol{v}_{i-1}}{2}, \quad-1 \leq \lambda_{i} \leq 1
\end{aligned}
$$

4. If $\exists \boldsymbol{w}_{1}, \boldsymbol{w}_{2} \in \mathcal{H}_{k}$ such that $\boldsymbol{v}_{1} \stackrel{\text { def }}{=} \pi\left(\boldsymbol{w}_{1}\right)=\boldsymbol{v}_{2} \stackrel{\text { def }}{=} \pi\left(\boldsymbol{w}_{2}\right)$ and $\boldsymbol{v}_{1,2} \in \mathcal{V}_{\Theta}$, then $\exists j \in\{1, \ldots, k\}$ such that $a_{j}=b_{j}=0$.
5. With the same numbering from (iii) and $k=p$, for any $i \in\{0, \ldots, 2 p-1\}$, the vertices of the hypercube that are projecting to $\boldsymbol{v}_{i}$ and $\boldsymbol{v}_{i+1}$, respectively, are adjacent, i.e. they only differ in exactly one component.

Proof. We will omit a complete proof of the lemma, and will instead simply suggest the main ideas needed for checking the validity of the statements.

For part $(i)$, it is easy to argue that the center of the hypercube, $\boldsymbol{O}_{\mathcal{H}}=[1 / 2,1 / 2, \ldots, 1 / 2]^{T}$, will always project into the center of the zonogon, i.e. $\boldsymbol{O}=\pi\left(\boldsymbol{O}_{\mathcal{H}}\right)$. This implies that any zonogon will be centrally symmetric, and will therefore have an even number of vertices.

Part ( (ii) can be shown by induction on the dimension $k$ of the hypercube, $\mathcal{H}_{k}$. For instance, to prove the first claim, note that the projection of a polytope is simply the convex hull of the projections of the vertices, and therefore projecting a hypercube of dimension $k$ simply amounts to projecting two hypercubes of dimension $k-1$, one for $w_{k}=0$ and another for $w_{k}=1$, and then taking the convex hull of the two resulting polytopes. It is easy to see that these two polytopes in $\mathbf{R}^{2}$ are themselves zonogons, and are translated copies of each other (by an amount $\left[a_{k}, b_{k}\right]^{T}$ ). Therefore, by the induction hypothesis, they have at most $2(k-1)$ vertices, and taking their convex hull introduces at most two new vertices, for a total of at most $2(k-1)+2=2 k$ vertices. The second claim can be proved in a similar fashion.

One way to prove part (iii) is also by induction on $p$, by taking any pair of opposite (i.e. parallel, of the same length) edges and showing that they correspond to a Minkowski summand of the zonogon.

Part (iv) also follows by induction. Using the same argument as for part (ii), note that the only ways to have two distinct vertices of the hypercube $\mathcal{H}_{k}$ (of dimension $k$ ) project onto the same vertex of the zonogon $\Theta$ is to either have this situation happen for one of the two $k-1$ dimensional hypercubes (in which case the induction hypothesis would complete the proof), or to have zero translation between the two zonogons, which could only happen if $a_{k}=b_{k}=0$.

Part $(v)$ follows by using parts (iii) and (iv) and the definition of a zonogon as the Minkowski sum of line segments. In particular, since the difference between two consecutive vertices of the zonogon, $\boldsymbol{v}_{i}, \boldsymbol{v}_{i+1}$, for the case $k=p$, is always given by a single column of the projection matrix (i.e. $\left[a_{j}, b_{j}\right]^{T}$, for some $j$ ), then the unique vertices of $\mathcal{H}_{k}$ that were projecting onto $\boldsymbol{v}_{i}$ and $\boldsymbol{v}_{i+1}$, respectively, must be incidence vectors that differ in exactly one component, i.e. are adjacent on the hypercube $\mathcal{H}_{k}$.

### 7.3 Technical Lemmas.

This section of the Appendix contains a detailed proof for the technical Lemma 4.3 introduced in Section 4.1.2, which we include below, for convenience.

LEMMA 4.3. When the zonogon $\Theta$ has a non-trivial intersection with the band $\mathcal{B}_{L U}$ (case [C4]), the convex polygon $\Delta_{\Gamma}$ and the set of points on its right side, r-side $\left(\Delta_{\Gamma}\right)$, satisfy the following properties:

1. r-side $\left(\Delta_{\Gamma}\right)$ is the union of two sequences of consecutive vertices (one starting at $\tilde{\boldsymbol{v}}_{0}$, and one ending at $\tilde{\boldsymbol{v}}_{k}$ ), and possibly an additional vertex, $\tilde{\boldsymbol{v}}_{t}$ :

$$
\mathrm{r}-\operatorname{side}\left(\Delta_{\Gamma}\right)=\left\{\tilde{\boldsymbol{v}}_{0}, \tilde{\boldsymbol{v}}_{1}, \ldots, \tilde{\boldsymbol{v}}_{s}\right\} \cup\left\{\tilde{\boldsymbol{v}}_{t}\right\} \cup\left\{\tilde{\boldsymbol{v}}_{r}, \tilde{\boldsymbol{v}}_{r+1} \ldots, \tilde{\boldsymbol{v}}_{k}\right\}, \text { for some } s \leq r \in\{0, \ldots, k\} .
$$

2. With $\operatorname{cotan}(\cdot, \cdot)$ given by $(25)$ applied to the $\left(\tilde{\gamma}_{1}, \tilde{\gamma}_{2}\right)$ coordinates, we have that:

$$
\begin{cases}\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{s}, \tilde{\boldsymbol{v}}_{\min (t, r)}\right) \geq \frac{a_{s+1}}{b_{s+1}}, & \text { whenever } t>s \\ \operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{\max (t, s)}, \tilde{\boldsymbol{v}}_{r}\right) \leq \frac{a_{r}}{b_{r}}, & \text { whenever } t<r\end{cases}
$$

Proof of Lemma 4.3. In the following exposition, we will use the same notation as introduced in Section 4.1.2. Recall that case [C4] on which the lemma is focused corresponds to a nontrivial intersection of the zonotope $\Theta$ with the horizontal band $\mathcal{B}_{L U}$ defined in (30). As suggested in Figure 5 of Section 4.1.2, this case can be separated into three subcases, depending on the position of the vertex $\boldsymbol{v}_{t}$ relative to the band $\mathcal{B}_{L U}$, where the index $t$ is defined in (31). Since the proof of all three cases is essentially identical, we will focus on the more "complicated" situation, namely when $\boldsymbol{v}_{t} \in \mathcal{B}_{L U}$. The corresponding arguments for the other two cases should be straightforward.

First, recall that $\Delta_{\Gamma}$ is given by (28), i.e. $\Delta_{\Gamma}=\operatorname{conv}\left(\left\{\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}\right)$, where the points $\tilde{\boldsymbol{v}}_{i}$ are given by (29), which results from applying mapping (27) to $\boldsymbol{v}_{i} \in \Theta$. From Definition 4.1 of the right side, it can be seen that the points of interest to us, namely r-side $\left(\Delta_{\Gamma}\right)$, will be a maximal subset $\left\{\tilde{\boldsymbol{v}}_{i(1)}, \tilde{\boldsymbol{v}}_{i(2)}, \ldots, \tilde{\boldsymbol{v}}_{i(m)}\right\} \subseteq$ $\left\{\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}$, satisfying:

$$
\left\{\begin{array}{l}
\tilde{\boldsymbol{v}}_{i(1)}=\underset{\tilde{\gamma}_{1}}{\arg \max } \underset{\tilde{\gamma}_{2}}{\arg \min }\left[\tilde{\gamma}: \tilde{\boldsymbol{\gamma}}=\left(\tilde{\gamma}_{1}, \tilde{\gamma}_{2}\right) \in\left\{\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}\right]  \tag{65}\\
\tilde{\boldsymbol{v}}_{i(m)}=\underset{\tilde{\gamma}_{1}}{\arg \max } \underset{\tilde{\gamma}_{2}}{\arg \max }\left[\tilde{\boldsymbol{\gamma}}: \tilde{\gamma}=\left(\tilde{\gamma}_{1}, \tilde{\gamma}_{2}\right) \in\left\{\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{k}\right\}\right] \\
\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{i(1)}, \tilde{\boldsymbol{v}}_{i(2)}\right)>\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{i(2)}, \tilde{\boldsymbol{v}}_{i(3)}\right)>\cdots>\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{i(m-1)}, \tilde{\boldsymbol{v}}_{i(m)}\right)
\end{array}\right.
$$

For the analysis, we will find it useful to define the following two indices:

$$
\begin{equation*}
\hat{s} \stackrel{\text { def }}{=} \min \left\{i \in\{0, \ldots, k\}: \theta_{2}\left(\boldsymbol{v}_{i}\right) \geq y^{*}-U\right\}, \quad \hat{r} \stackrel{\text { def }}{=} \max \left\{i \in\{0, \ldots, k\}: \theta_{2}\left(\boldsymbol{v}_{i}\right) \leq y^{*}-L\right\} \tag{66}
\end{equation*}
$$

In particular, $\hat{s}$ is the index of the first vertex of r-side $(\Theta)$ falling inside $\mathcal{B}_{L U}$, and $\hat{r}$ is the index of the last vertex of r-side $(\Theta)$ falling inside $\mathcal{B}_{L U}$. Since we are in the situation when $\boldsymbol{v}_{t} \in \mathcal{B}_{L U}$, it can be seen that $0 \leq \hat{s} \leq t \leq \hat{r} \leq k$, and thus, from (31) (the definition of $t$ ) and (32) (typical conditions for the right side of a zonogon):

$$
\begin{equation*}
\frac{a_{1}}{b_{1}}>\cdots>\frac{a_{\hat{s}}}{b_{\hat{s}}}>\cdots>\frac{a_{t}}{b_{t}}>c \geq \frac{a_{t+1}}{b_{t+1}}>\cdots>\frac{a_{\hat{r}}}{b_{\hat{r}}}>\cdots>\frac{a_{k}}{b_{k}} \tag{67}
\end{equation*}
$$

With this new notation, we proceed to prove the first result in the claim. First, consider all the vertices $\boldsymbol{v}_{i} \in$ r-side $(\Theta)$ falling strictly below the band $\mathcal{B}_{L U}$, i.e. satisfying $\theta_{2}\left[\boldsymbol{v}_{i}\right]<y^{*}-U$. From the definition of $\hat{s}$, (66), these are exactly $\boldsymbol{v}_{0}, \ldots, \boldsymbol{v}_{\hat{s}-1}$, and mapping (27) applied to them will yield: $\tilde{\boldsymbol{v}}_{i}=$ $\left(\theta_{1}\left[\boldsymbol{v}_{i}\right]+c \cdot U, \theta_{2}\left[\boldsymbol{v}_{i}\right]+U\right)$. In other words, any such points will simply be translated by $(c \cdot U, U)$. Similarly, any points $\boldsymbol{v}_{i} \in$ r-side $(\Theta)$ falling strictly above the band $\mathcal{B}_{L U}$, i.e. $\theta_{2}\left[\boldsymbol{v}_{i}\right]>y^{*}-L$, will be translated by $(c \cdot L, L)$, so that we have:

$$
\begin{array}{ll}
\tilde{\boldsymbol{v}}_{i}=\boldsymbol{v}_{i}+(c \cdot U, U), & i \in\{0, \ldots, \hat{s}-1\} \\
\tilde{\boldsymbol{v}}_{i}=\boldsymbol{v}_{i}+(c \cdot L, L), & i \in\{\hat{r}+1, \ldots, k\} \tag{68}
\end{array}
$$

which immediately implies, since $\boldsymbol{v}_{i} \in \mathrm{r}$-side $(\Theta)$, that:

$$
\left\{\begin{array}{l}
\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{0}, \tilde{\boldsymbol{v}}_{1}\right)>\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{1}, \tilde{\boldsymbol{v}}_{2}\right)>\cdots>\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{\hat{s}-2}, \tilde{\boldsymbol{v}}_{\hat{s}-1}\right)  \tag{69}\\
\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{\hat{r}+1}, \tilde{\boldsymbol{v}}_{\hat{r}+2}\right)>\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{\hat{r}+2}, \tilde{\boldsymbol{v}}_{\hat{r}+3}\right)>\cdots>\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{k-1}, \tilde{\boldsymbol{v}}_{k}\right)
\end{array}\right.
$$

For any vertices inside $\mathcal{B}_{L U}$, i.e. $\boldsymbol{v}_{i} \in$ r-side $(\Theta) \cap \mathcal{B}_{L U}$, mapping (27) will yield:

$$
\begin{equation*}
\tilde{\boldsymbol{v}}_{i}=\left(\theta_{1}\left[\boldsymbol{v}_{i}\right]-c \cdot \theta_{2}\left[\boldsymbol{v}_{i}\right]+c \cdot y^{*}, y^{*}\right), \quad i \in\{\hat{s}, \ldots, t, \ldots, \hat{r}\}, \tag{70}
\end{equation*}
$$

that is, they will be mapped into points with the same $\tilde{\gamma}_{2}$ coordinates. Furthermore, using (23), it can be seen that $\tilde{\boldsymbol{v}}_{t}$ will have the largest $\tilde{\gamma}_{1}$ coordinate among all such $\tilde{\boldsymbol{v}}_{i}$ :

$$
\begin{align*}
& \tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{t}\right]-\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{i}\right] \stackrel{\text { def }}{=} \theta_{1}\left[\boldsymbol{v}_{t}\right]-\theta_{1}\left[\boldsymbol{v}_{i}\right]-c \cdot\left(\theta_{2}\left[\boldsymbol{v}_{t}\right]-\theta_{2}\left[\boldsymbol{v}_{i}\right]\right) \\
& \stackrel{(23)}{=} \begin{cases}\sum_{j=i+1}^{t} a_{j}-c \cdot \sum_{j=i+1}^{t} b_{j} \stackrel{(67)}{\geq} 0, & \text { if } \hat{s} \leq i<t \\
-\sum_{j=t+1}^{i} a_{j}+c \cdot \sum_{j=t+1}^{i} b_{j} \stackrel{(67)}{\geq} 0, & \text { if } t<i \leq \hat{r}\end{cases} \tag{71}
\end{align*}
$$

Furthermore, since the mapping (27) yielding $\tilde{\gamma}_{2}$ is only a function of $\theta_{2}$, and is monotonic non-decreasing (strictly monotonic increasing outside the band $\mathcal{B}_{L U}$ ), vertices $\boldsymbol{v}_{0}, \ldots, \boldsymbol{v}_{k} \in \mathrm{r}$-side $(\Theta)$ will be mapped into points $\tilde{\boldsymbol{v}}_{0}, \ldots, \tilde{\boldsymbol{v}}_{k} \in \tilde{\Gamma}$ with non-decreasing $\tilde{\gamma}_{2}$ coordinates:
$\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{0}\right]<\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{1}\right]<\cdots<\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{\hat{s}-1}\right]<y^{*}=\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{\hat{s}}\right]=\cdots=\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{t}\right]=\cdots=\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{\hat{r}}\right]<\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{\hat{r}+1}\right]<\cdots<\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{k}\right]$.
Therefore, combining this fact with (69) and (71), we can conclude that the points $\tilde{\boldsymbol{v}}_{i}$ satisfying conditions (65) are none other than:

$$
\mathrm{r}-\operatorname{side}\left(\Delta_{\Gamma}\right)=\left\{\tilde{\boldsymbol{v}}_{0}, \tilde{\boldsymbol{v}}_{1}, \ldots, \tilde{\boldsymbol{v}}_{s}, \tilde{\boldsymbol{v}}_{t}, \tilde{\boldsymbol{v}}_{r}, \tilde{\boldsymbol{v}}_{r+1}, \tilde{\boldsymbol{v}}_{k}\right\}
$$

where the indices $s$ and $r$ are given as:

$$
\begin{align*}
& s \stackrel{\text { def }}{=}\left\{\begin{array}{l}
\max \left\{i \in\{1, \ldots, \hat{s}-1\}: \operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{i-1}, \tilde{\boldsymbol{v}}_{i}\right)>\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{i}, \tilde{\boldsymbol{v}}_{t}\right)\right\} \\
0, \text { if the above condition is never true },
\end{array}\right. \\
& r \stackrel{\text { def }}{=}\left\{\begin{array}{l}
\min \left\{i \in\{\hat{r}+1, \ldots, k-1\}: \operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{t}, \tilde{\boldsymbol{v}}_{i}\right)>\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{i}, \tilde{\boldsymbol{v}}_{i+1}\right)\right\} \\
k, \text { if the above condition is never true. }
\end{array}\right. \tag{72}
\end{align*}
$$

This completes the proof of part $(i)$ of the Lemma. We remark that, for the cases when $\boldsymbol{v}_{t}$ falls strictly below $\mathcal{B}_{L U}$ or strictly above $\mathcal{B}_{L U}$, one can repeat the exact same reasoning, and immediately argue that the same result would hold.

In order to prove the first claim in part (ii), we first recall that, from (72), if $s<\hat{s}-1$, we must have:

$$
\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{s}, \tilde{\boldsymbol{v}}_{s+1}\right) \leq \operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{s+1}, \tilde{\boldsymbol{v}}_{t}\right)
$$

since otherwise, we would have taken $s+1$ instead of $s$ in (72). But this immediately implies that:

$$
\begin{aligned}
& \operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{s}, \tilde{\boldsymbol{v}}_{s+1}\right) \leq \operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{s+1}, \tilde{\boldsymbol{v}}_{t}\right) \stackrel{(25)}{\Leftrightarrow} \\
& \frac{\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{s+1}\right]-\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{s}\right]}{\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{s+1}\right]-\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{s}\right]} \leq \frac{\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{t}\right]-\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{s+1}\right]}{\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{t}\right]-\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{s+1}\right]} \Rightarrow \text { (mediant inequality) } \\
& \frac{\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{s+1}\right]-\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{s}\right]}{\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{s+1}\right]-\tilde{\gamma}_{2}\left[\boldsymbol{v}_{s}\right]} \leq \frac{\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{t}\right]-\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{s}\right]}{\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{t}\right]-\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{s}\right]} \stackrel{(68)}{\Leftrightarrow}
\end{aligned}
$$

which is exactly the first claim in part (ii). Thus, the only case to discuss is $s=\hat{s}-1$. Since $s \geq 0$, it must be that, in this case, there are vertices $\boldsymbol{v}_{i} \in$ r-side $(\Theta)$ falling strictly below the band $\mathcal{B}_{L U}$. Therefore, we can introduce the following point in $\Theta$ :

$$
\begin{equation*}
M \stackrel{\text { def }}{=} \underset{\theta_{1}}{\arg \max }\left\{\left(\theta_{1}, \theta_{2}\right) \in \Theta: \theta_{2}=y^{*}-U\right\} \tag{73}
\end{equation*}
$$

Referring back to Figure 6 in Section 4.2, it can be seen that $M$ represents the point with smallest $\theta_{2}$ coordinate in $\mathcal{B}_{L U} \cap \mathrm{r}$-side $(\Theta)$, and $M \in\left[\boldsymbol{v}_{\hat{s}-1}, \boldsymbol{v}_{\hat{s}}\right]$. If we let $\left(\theta_{1}[M], \theta_{2}[M]\right)$ denote the coordinates of $M$, then by applying mapping (27) to $M$, the coordinates of the point $\tilde{M} \in \tilde{\Gamma}$ are:

$$
\begin{equation*}
\tilde{M}=\left(\theta_{1}[M]+c \cdot U, \theta_{2}[M]+U\right)=\left(\theta_{1}[M]+c \cdot U, y^{*}\right) \tag{74}
\end{equation*}
$$

Furthermore, a similar argument with (71) can be invoked to show that $\tilde{\gamma}_{1}[\tilde{M}] \leq \tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{t}\right]$. With $s=\hat{s}-1$, we then have:

$$
\begin{aligned}
\operatorname{cotan}\left(\tilde{\boldsymbol{v}}_{s}, \tilde{\boldsymbol{v}}_{t}\right) & \stackrel{(25)}{=} \frac{\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{t}\right]-\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{\hat{s}-1}\right]}{\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}^{\prime}\right]-\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{\hat{s}-1}\right]} \geq \quad\left(\text { since } \tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{t}\right]=\tilde{\gamma}_{2}[\tilde{M}]=y^{*}>\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{\hat{s}-1}\right]\right) \\
& \geq \frac{\tilde{\gamma}_{1}[\tilde{M}]-\tilde{\gamma}_{1}\left[\tilde{\boldsymbol{v}}_{\hat{s}-1}\right]}{\tilde{\gamma}_{2}[\tilde{M}]-\tilde{\gamma}_{2}\left[\tilde{\boldsymbol{v}}_{\hat{s}-1}\right]} \\
& \stackrel{(68),(74)}{=} \frac{\theta_{1}[M]-\theta_{1}\left[\boldsymbol{v}_{\hat{s}-1}\right]}{\theta_{2}[M]-\theta_{2}\left[\boldsymbol{v}_{\hat{s}-1}\right]}=\quad\left(\text { since } M \in\left[\boldsymbol{v}_{\hat{s}-1}, \boldsymbol{v}_{\hat{s}}\right]\right) \\
& =\frac{a_{s+1}}{b_{s+1}},
\end{aligned}
$$

which completes the proof of the first claim in part (ii).
The proof of the second claim in (ii) proceeds in an analogous fashion, by first examining the trivial case $r>\hat{r}+1$ in (72), and then introducing $N \stackrel{\text { def }}{=} \arg \max _{\theta_{1}}\left\{\left(\theta_{1}, \theta_{2}\right) \in \Theta: \theta_{2}=y^{*}-L\right\}$ for the case $r=\hat{r}+1$.

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[^1]:    ${ }^{1}$ Such a system can always be obtained by the linear change of variables $\tilde{x}_{k}=\frac{x_{k}}{\prod_{i=1}^{k-1} \alpha_{i}}$, and by suitably scaling the bounds $L_{k}, U_{k}, \underline{w}_{k}, \bar{w}_{k}$.

[^2]:    ${ }^{2}$ Such constraints may lead to infeasible problems. For example: $T=1, x_{1}=0, u_{1} \in[0,1], w_{1} \in[0,1], x_{2} \in[5,10]$.
    ${ }^{3}$ The functions $\tilde{h}_{k}(\cdot)$ are convex in $x_{k+1}$, since $\mathbf{1}\left(\left[L_{k+1}, U_{k+1}\right]\right)$, the indicator function of a convex set, is convex (see pages 28, 33 in Rockafellar [1970] or Example 7.1.2 in Bertsekas [2003]), and the sum of convex functions remains convex.

[^3]:    ${ }^{4}$ For simplicity, we focus on the case when $g(\cdot)$ has a unique minimizer, such that $\underline{y}=\bar{y}=y^{*}$ in (8), (9). The analysis could be immediately extended to the set of minimizers, $[\underline{y}, \bar{y}]$.
    ${ }^{5}$ To avoid confusion, in addition to different types of parentheses, we will consistenly use tilded arguments to denote points $\tilde{\boldsymbol{v}}_{i} \in \tilde{\Gamma}$, and non-tilded arguments for $\boldsymbol{v} \in \Theta$.

[^4]:    ${ }^{6}$ This vertex is unique due to our standing Assumption 2 that the number of vertices in $\Theta$ is $2 k$ (also see part (iv) of Lemma 7.2 in the Appendix).

[^5]:    ${ }^{7}$ Recall that we are, again, working under Assumption 2, which implies uniqueness by part (iv) of Lemma 7.2 in the Appendix)

[^6]:    ${ }^{8}$ If $b, d>0$ and $\frac{a}{b} \leq \frac{c}{d}$, then $\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$.

[^7]:    ${ }^{9}$ The signs of the inequalities are changed because every disturbance, $w_{t}$, is entering the system dynamics with a coefficient -1 , instead of +1 , as was the case in the discussion from Section 4.1.1.

