# Linearly Parameterized Bandits 

Paat Rusmevichientong<br>Cornell University<br>paatrus@cornell.edu

John N. Tsitsiklis<br>MIT<br>jnt@mit.edu

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#### Abstract

We consider bandit problems involving a large (possibly infinite) collection of arms, in which the expected reward of each arm is a linear function of an $r$-dimensional random vector $\mathbf{Z} \in \mathbb{R}^{r}$, where $r \geq 2$. The objective is to minimize the cumulative regret and Bayes risk. When the set of arms corresponds to the unit sphere, we prove that the regret and Bayes risk is of order $\Theta(r \sqrt{T})$, by establishing a lower bound for an arbitrary policy, and showing that a matching upper bound is obtained through a policy that alternates between exploration and exploitation phases. The phasebased policy is also shown to be effective if the set of arms satisfies a strong convexity condition. For the case of a general set of arms, we describe a near-optimal policy whose regret and Bayes risk admit upper bounds of the form $O\left(r \sqrt{T} \log ^{3 / 2} T\right)$.


## 1. Introduction

Since its introduction by Thompson (1933), the multiarmed bandit problem has served as an important model for decision making under uncertainty. Given a set of arms with unknown reward profiles, the decision maker must choose a sequence of arms to maximize the expected total payoff, where the decision in each period may depend on the previously observed rewards. The multiarmed bandit problem elegantly captures the tradeoff between the need to exploit arms with high payoff and the incentive to explore previously untried arms for information gathering purposes.

Much of the previous work on the multiarmed bandit problem assumes that the rewards of the arms are statistically independent (see, for example, Lai and Robbins (1985) and Lai (1987)). This assumption enables us to consider each arm separately, but it leads to policies whose regret scales linearly with the number of arms. Most policies that assume independence require each arm to be
tried at least once, and are impractical in settings involving many arms. In such settings, we want a policy whose regret is independent of the number of arms.

When the mean rewards of the arms are assumed to be independent random variables, Lai and Robbins (1985) show that the regret under an arbitrary policy must increase linearly with the number of arms. However, the assumption of independence is quite strong in practice. In many applications, the information obtained from pulling one arm can change our understanding of other arms. For instance, in marketing applications, we expect a priori that similar products should have similar sales. By exploiting the correlation among products/arms, we should be able to obtain a policy whose regret scales more favorably than traditional bandit algorithms that ignore correlation and assume independence.

Mersereau et al. (2009) propose a simple model that demonstrates the benefits of exploiting the underlying structure of the rewards. They consider a bandit problem where the expected reward of each arm is a linear function of an unknown scalar, with a known prior distribution. Since the reward of each arm depends on a single random variable, the mean rewards are perfectly correlated. They prove that, under certain assumptions, the cumulative Bayes risk over $T$ periods (defined below) under a greedy policy admits an $O(\log T)$ upper bound, independent of the number of arms.

In this paper, we extend the model of Mersereau et al. (2009) to the setting where the expected reward of each arm depends linearly on a multivariate random vector $\mathbf{Z} \in \mathbb{R}^{r}$. We concentrate on the case where $r \geq 2$, which is fundamentally different from the previous model because the mean rewards now depend on more than one random variable, and thus, they are no longer perfectly correlated. The bounds on the regret and Bayes risk and the policies found in Mersereau et al. (2009) no longer apply. To give a flavor for the differences, we will show that, in our model, the cumulative Bayes risk under an arbitrary policy is at least $\Omega(r \sqrt{T})$, which is significantly higher than the upper bound of $O(\log T)$ attainable when $r=1$.

The linearly parameterized bandit is an important model that has been studied by many researchers, including Ginebra and Clayton (1995), Abe and Long (1999), and Auer (2002). The results in this paper complement and extend the earlier and independent work of Dani et al. (2008a) in a number of directions. We provide a detailed comparison between our work and the existing literature in Sections 1.3 and 1.4 .

### 1.1 The Model

We have a compact set $\mathcal{U}_{r} \subset \mathbb{R}^{r}$ that corresponds to the set of arms, where $r \geq 2$. The reward $X_{t}^{u}$ of playing arm $\mathbf{u} \in \mathcal{U}_{r}$ in period $t$ is given by

$$
\begin{equation*}
X_{t}^{\mathbf{u}}=\mathbf{u}^{\prime} \mathbf{Z}+W_{t}^{\mathbf{u}} \tag{1}
\end{equation*}
$$

where $\mathbf{u}^{\prime} \mathbf{Z}$ is the inner product between the vector $\mathbf{u} \in \mathcal{U}_{r}$ and the random vector $\mathbf{Z} \in \mathbb{R}^{r}$. We assume that the random variables $W_{t}^{\mathbf{u}}$ are independent of each other and of $\mathbf{Z}$. Moreover, for each $\mathbf{u} \in \mathcal{U}_{r}$, the random variables $\left\{W_{t}^{\mathbf{u}}: t \geq 1\right\}$ are identically distributed, with $\mathrm{E}\left[W_{t}^{\mathbf{u}}\right]=0$ for all $t$ and $\mathbf{u}$. We allow the error random variables $W_{t}^{\mathbf{u}}$ to have unbounded support, provided that their moment generating functions satisfy certain conditions (given in Assumption (1). Each vector $\mathbf{u} \in \mathcal{U}_{r}$ simultaneously represents an arm and determines the expected reward of that arm. So, when it is clear from the context, we will interchangeably refer to a $\mathbf{u} \in \mathcal{U}_{r}$ as either a vector or an arm.

Let us introduce the following conventions and notation that will be used throughout the paper. We denote vectors and matrices in bold. All vectors are column vectors. For any vector $\mathbf{v} \in \mathbb{R}^{r}$, its transpose is denoted by $\mathbf{v}^{\prime}$, and is always a row vector. Let $\mathbf{0}$ denote the zero vector, and for $k=1, \ldots, r$, let $\mathbf{e}_{k}=(0, \ldots, 1, \ldots, 0)$ denote the standard unit vector in $\mathbb{R}^{r}$, with a 1 in the $k^{\text {th }}$ component and a 0 elsewhere. Also, let $\mathbf{I}_{k}$ denote the $k \times k$ identity matrix. We let $\mathbf{A}^{\prime}$ and $\operatorname{det}(\mathbf{A})$ denote the transpose and determinant of $\mathbf{A}$, respectively. If $\mathbf{A}$ is a symmetric positive semidefinite matrix, then $\lambda_{\min }(\mathbf{A})$ and $\lambda_{\max }(\mathbf{A})$ denote the smallest and the largest eigenvalues of $\mathbf{A}$, respectively. We use $\mathbf{A}^{1 / 2}$ to denote its symmetric nonnegative definite square root, so that $\mathbf{A}^{1 / 2} \mathbf{A}^{1 / 2}=\mathbf{A}$. If $\mathbf{A}$ is also positive definite, we let $\mathbf{A}^{-1 / 2}=\left(\mathbf{A}^{-1}\right)^{1 / 2}$. For any vector $\mathbf{v},\|\mathbf{v}\|=$ $\sqrt{\mathbf{v}^{\prime} \mathbf{v}}$ denotes the standard Euclidean norm, and for any positive definite matrix $\mathbf{A},\|\mathbf{v}\|_{\mathbf{A}}=\sqrt{\mathbf{v}^{\prime} \mathbf{A v}}$ denotes a corresponding weighted norm. For any two symmetric positive semidefinite matrices $\mathbf{A}$ and $\mathbf{B}$, we write $\mathbf{A} \leq \mathbf{B}$ if the matrix $\mathbf{B}-\mathbf{A}$ is positive semidefinite. Also, all logarithms $\log (\cdot)$ in the paper denote the natural log, with base $e$. A random variable is denoted by an uppercase letter while its realized values are denoted in lowercase.

For any $t \geq 1$, let $\mathcal{H}_{t-1}$ denote the set of possible histories until the end of period $t-1$. A policy $\psi=\left(\psi_{1}, \psi_{2}, \ldots\right)$ is a sequence of functions such that $\psi_{t}: \mathcal{H}_{t-1} \rightarrow \mathcal{U}_{r}$ selects an arm in period $t$ based on the history until the end of period $t-1$. For any policy $\psi$ and $\mathbf{z} \in \mathbb{R}^{r}$, the $T$-period cumulative regret under $\psi$ given $\mathbf{Z}=\mathbf{z}$, denoted by $\operatorname{Regret}(\mathbf{z}, T, \psi)$, is defined by

$$
\operatorname{Regret}(\mathbf{z}, T, \psi)=\sum_{t=1}^{T} \mathrm{E}\left[\max _{\mathbf{v} \in \mathcal{U}_{r}} \mathbf{v}^{\prime} \mathbf{z}-\mathbf{U}_{t}^{\prime} \mathbf{z} \mid \mathbf{Z}=\mathbf{z}\right]
$$

where for any $t \geq 1, \mathbf{U}_{t} \in \mathcal{U}_{r}$ is the arm chosen under $\psi$ in period $t$. Since $\mathcal{U}_{r}$ is compact, $\max _{\mathbf{v} \in \mathcal{U}_{r}} \mathbf{v}^{\prime} \mathbf{z}$ is well defined for all $\mathbf{z}$. The T-period cumulative Bayes risk under $\psi$ is defined by

$$
\operatorname{Risk}(T, \psi)=\mathrm{E}[\operatorname{Regret}(\mathbf{Z}, T, \psi)]
$$

where the expectation is taken with respect to the prior distribution of $\mathbf{Z}$. We aim to develop a policy that minimizes the cumulative regret and Bayes risk. We note that minimizing the $T$-period cumulative Bayes risk is equivalent to maximizing the expected total reward over $T$ periods.

To facilitate exposition, when we discuss a particular policy, we will drop the superscript and write $X_{t}$ and $W_{t}$ to denote $X_{t}^{\mathbf{U}_{t}}$ and $W_{t}^{\mathbf{U}_{t}}$, respectively, where $\mathbf{U}_{t}$ is the arm chosen by the policy in period $t$. With this convention, the reward obtained in period $t$ under a particular policy is simply $X_{t}=\mathbf{U}_{t}^{\prime} \mathbf{Z}+W_{t}$.

### 1.2 Potential Applications

Although our paper focuses on a theoretical analysis, we mention briefly potential applications to problems in marketing and revenue management. Suppose we have $m$ arms indexed by $\mathcal{U}_{r}=$ $\left\{\mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{m}\right\} \subset \mathbb{R}^{r}$. For $k=1, \ldots, r$, let $\phi_{k}=\left(u_{1, k}, u_{2, k}, \ldots, u_{m, k}\right) \in \mathbb{R}^{m}$ denote an mdimensional column vector consisting of the $k^{\text {th }}$ coordinates of the different vectors $\mathbf{u}_{\ell}$. Let $\boldsymbol{\mu}=\left(\mu_{1}, \ldots, \mu_{m}\right)$ be the column vector consisting of expected rewards, where $\mu_{\ell}$ denotes the expected reward of $\operatorname{arm} \mathbf{u}_{\ell}$. Under our formulation, the vector $\boldsymbol{\mu}$ lies in an $r$-dimensional subspace spanned by the vectors $\phi_{1}, \ldots, \phi_{r}$, that is, $\boldsymbol{\mu}=\sum_{k=1}^{r} Z_{k} \phi_{k}$, where $\mathbf{Z}=\left(Z_{1}, \ldots, Z_{r}\right)$. If each arm corresponds to a product to be offered to a customer, we can then interpret the vector $\phi_{k}$ as a feature vector or basis function, representing a particular characteristic of the products such as price or popularity. We can then interpret the random variables $Z_{1}, \ldots, Z_{r}$ as regression coefficients, obtained from approximating the vector of expected rewards using the basis functions $\boldsymbol{\phi}_{1}, \ldots, \boldsymbol{\phi}_{r}$, or more intuitively, as the weights associated with the different characteristics. Given a prior on the coefficients $Z_{k}$, our goal is to choose a sequence of products that gives the maximum expected total reward. This representation suggests that our model might be applicable to problems where we can approximate high-dimensional vectors using a linear combination of a few basis functions, an approach that has been successfully applied to high-dimensional dynamic programming problems (see Bertsekas and Tsitsiklis (1996) for an overview).

### 1.3 Related Literature

The multiarmed bandit literature can be divided into two streams, depending on the objective function criteria: maximizing the total discounted reward over an infinite horizon or minimizing the cumulative regret and Bayes risk over a finite horizon. Our paper focuses exclusively on the
second criterion. Much of the work in this area focuses on understanding the rate with which the regret and risk under various policies increase over time. In their pioneering work, Lai and Robbins (1985) establish an asymptotic lower bound of $\Omega(m \log T)$ for the $T$-period cumulative regret for bandit problems with $m$ independent arms whose mean rewards are "well-separated," where the difference between the expected reward of the best and second best arms is fixed and bounded away from zero. They further demonstrate a policy whose regret asymptotically matches the lower bound. In contrast, our paper focuses on problems where the number of arms is large (possibly infinite), and where the gap between the maximum expected reward and the expected reward of the second best arm can be arbitrarily small. Lail (1987) extends these results to a Bayesian setting, with a prior distribution on the reward characteristics of each arm. He shows that when we have $m$ arms, the $T$-period cumulative Bayes risk is of order $\Theta\left(m \log ^{2} T\right)$, when the prior distribution has a continuous density function satisfying certain properties (see Theorem 3 in Lai, 1987). Subsequent papers along this line include Agrawal et al. (1989), Agrawal (1995), and Auer et al. (2002).

There has been relatively little research, however, on policies that exploit the dependence among the arms. Thompson (1933) allows for correlation among arms in his initial formulation, though he only analyzes a special case involving independent arms. Robbins (1952) formulates a continuum-armed bandit regression problem, but does not provide an analysis of the regret or risk. Berry and Fristedt (1985) allow for dependence among arms in their formulation in Chapter 2, but mostly focus on the case of independent arms. Feldman (1962) and Keener (1985) consider two-armed bandit problems with two hidden states, where the rewards of each arm depend on the underlying state that prevails. Pressman and Sonin (1990) formulate a general multiarmed bandit problem with an arbitrary number of hidden states, and provide a detailed analysis for the case of two hidden states. Pandey et al. (2007) study bandit problems where the dependence of the arm rewards is represented by a hierarchical model.

A somewhat related literature on bandits with dependent arms is the recent work by Wang et al. (2005a,b) and Goldenshluger and Zeevi (2008, 2009) who consider bandit problems with two arms, where the expected reward of each arm depends on an exogenous variable that represents side information. These models, however, differ from ours because they assume that the side information variables are independent and identically distributed over time, and moreover, these variables are perfectly observed before we choose which arm to play. In contrast, we assume that the underlying random vector $\mathbf{Z}$ is unknown and fixed over time, to be estimated based on past rewards and decisions.

Our formulation can be viewed as a sequential method for maximizing a linear function based on noisy observations of the function values, and it is thus closely related to the field of stochas-
tic approximation, which was developed by Robbins and Monro (1951) and Kiefer and Wolfowitz (1952). We do not provide a comprehensive review of the literature here; interested readers are referred to an excellent survey by Lai (2003). In stochastic approximation, we wish to find an adaptive sequence $\left\{\mathbf{U}_{t} \in \mathbb{R}^{r}: t \geq 1\right\}$ that converges to a maximizer $\mathbf{u}^{*}$ of a target function, and the focus is on establishing the rate at which the mean squared error $\mathrm{E}\left[\left\|\mathbf{U}_{t}-\mathbf{u}^{*}\right\|^{2}\right]$ converges to zero (see, for example, Blum, 1954 and Cicek et al., 2009). In contrast, our cumulative regret and Bayes risk criteria take into account the cost associated with each observation. The different performance measures used in our formulation lead to entirely different policies and performance characteristics.

Our model generalizes the "response surface bandits" introduced by Ginebra and Clayton (1995), who assume a normal prior on $\mathbf{Z}$ and provide a simple tunable heuristic, without any analysis on the regret or risk. Abe and Long (1999), Auer (2002), and Dani et al. (2008a) all consider a special case of our model where the random vector $\mathbf{Z}$ and the error random variables $W_{t}^{\mathbf{u}}$ are bounded almost surely, and with the exception of the last paper, focus on the regret criterion. Abe and Long (1999) demonstrate a class of bandits where the dimension $r$ is at least $\Omega(\sqrt{T})$, and show that the $T$-period regret under an arbitrary policy must be at least $\Omega\left(T^{3 / 4}\right)$. Auer (2002) describes an algorithm based on least squares estimation and confidence bounds, and establishes an $O\left(\sqrt{r} \sqrt{T} \log ^{3 / 2}\left(T\left|\mathcal{U}_{r}\right|\right)\right)$ upper bound on the regret, for the case of finitely many arms. Dani et al. (2008a) show that the policy of Auer (2002) can be extended to problems having an arbitrary compact set of arms, and also make use of a barycentric spanner. They establish an $O\left(r \sqrt{T} \log ^{3 / 2} T\right)$ upper bound on the regret, and discuss a variation of the policy that is more computationally tractable (at the expense of higher regret). Dani et al. (2008a) also establish an $\Omega(r \sqrt{T})$ lower bound on the Bayes risk when the set of arms is the Cartesian product of circles $]$. However, this leaves a $O\left(\log ^{3 / 2} T\right)$ gap from the upper bound, leaving open the question of the exact order of regret and risk.

### 1.4 Contributions and Organizations

One of our contributions is proving that the regret and Bayes risk for a broad class of linearly parameterized bandits is of order $\Theta(r \sqrt{T})$. In Section 2, we establish an $\Omega(r \sqrt{T})$ lower bound for an arbitrary policy, when the set of arms is the unit sphere in $\mathbb{R}^{r}$. Then, in Section 3, we show that a matching $O(r \sqrt{T})$ upper bound can be achieved through a phase-based policy that alternates between exploration and exploitation phases. To the best of our knowledge, this is the first result that establishes matching upper and lower bounds for a class of linearly parameterized bandits.

[^0]Table 1 summarizes our results and provides a comparison with the results in Mersereau et al. (2009) for the case $r=1$. In the ensuing discussion of the bounds, we focus on the main parameters, $r$ and $T$, with more precise statements given in the theorems.

Although we obtain the same lower bound of $\Omega(r \sqrt{T})$, our example and proof techniques are very different from Dani et al. (2008a). We consider the unit sphere, with a multivariate normal prior on $\mathbf{Z}$, and standard normal errors. The analysis in Section 2 also illuminates the behavior of the least mean squares estimator in this setting, and we believe that it provides an approach that can be used to address more general classes of linear estimation and adaptive control problems.

We also prove that the phase-based policy remains effective (that is, admits an $O(r \sqrt{T})$ upper bound) for a broad class of bandit problems in which the set of arms is strongly convex 2 (defined in Section 3). To our knowledge, this is the first result that establishes the connection between a geometrical property (strong convexity) of the underlying set of arms and the effectiveness of separating exploration from exploitation. We suspect that strong convexity may have similar implications for other types of bandit and learning problems.

When the set of arms is an arbitrary compact set, the separation of exploration and exploitation may not be effective, and we consider in Section 4 an active exploration policy based on least squares estimation and confidence regions. We prove that the regret and risk under this policy are bounded above by $O\left(r \sqrt{T} \log ^{3 / 2} T\right)$, which is within a logarithmic factor of the lower bound. Our policy is closely related to the one considered in Auer (2002) and further analyzed in Dani et al. (2008a), with differences in a number of respects. First, our model allows the random vector $\mathbf{Z}$ and the errors $W_{t}^{\mathbf{u}}$ to have unbounded support, which requires a somewhat more complicated analysis. Second, our policy is an "anytime" policy, in the sense that the policy does not depend on the time horizon $T$ of interest. In contrast, the policies of Auer (2002) and Dani et al. (2008a) involve a certain parameter $\delta$ whose value must be set in advance as a function of the time horizon $T$ in order to obtain the $O\left(r \sqrt{T} \log ^{3 / 2} T\right)$ regret bound.

We finally comment on the case where the set of arms is finite and fixed. We show that the regret and risk under our active exploration policy increase gracefully with time, as $\log T$ and $\log ^{2} T$, respectively. These results show that our policy is within a constant factor of the asymptotic lower bounds established by Lai and Robbins (1985) and Lai (1987). In contrast, for the policies of Auer (2002) and Dani et al. (2008a), the available regret upper bounds grow over time as $\sqrt{T} \log ^{3 / 2} T$ and $\log ^{3} T$, respectively.

[^1]| Dimension <br> (r) | $\begin{gathered} \text { Set of } \\ \text { Arms }\left(\mathcal{U}_{r}\right) \end{gathered}$ | $T$-period Cumulative Regret |  | $T$-period Cumulative Bayes Risk |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | Lower Bound | Upper Bound | Lower Bound | Upper Bound |
| $r=1$ | Any Compact Set <br> (Mersereau et al., 2008) | $\Omega(\sqrt{T})$ | $O(\sqrt{T})$ | $\Omega(\log T)$ | $O(\log T)$ |
| $\begin{gathered} r \geq 2 \\ \text { (This Paper) } \end{gathered}$ | Unit Sphere <br> (Sections 2 and 3) | $\Omega(r \sqrt{T})$ | $O(r \sqrt{T})$ | $\Omega(r \sqrt{T})$ | $O(r \sqrt{T})$ |
|  | Any Compact Set (Section 4) | $\Omega(r \sqrt{T})$ | $O\left(r \sqrt{T} \log ^{3 / 2} T\right)$ | $\Omega(r \sqrt{T})$ | $O\left(r \sqrt{T} \log ^{3 / 2} T\right)$ |

Table 1: Regret and risk bounds for various values of $r$ and for different collections of arms.

We note that the bounds on the cumulative Bayes risk given in Table 1 hold under certain assumptions on the prior distribution of the random vector $\mathbf{Z}$. For $r=1, \mathbf{Z}$ is assumed to be a continuous random variable with a bounded density function (Theorem 3.2 in Mersereau et al., 2009). When the collection of arms is a unit sphere with $r \geq 2$, we require that both $\mathbf{E}[\|\mathbf{Z}\|]$ and $\mathrm{E}[1 /\|\mathbf{Z}\|]$ are bounded (see Theorems 2.1] and 3.1, and Lemma 3.2). For general compact sets of arms where our risk bound is not tight, we only require that $\|\mathbf{Z}\|$ has a bounded expectation.

## 2. Lower Bounds

In this section, we establish $\Omega(r \sqrt{T})$ lower bounds on the regret and risk under an arbitrary policy when the set of arms is the unit sphere. This result is stated in the following theorem 3

Theorem 2.1 (Lower Bounds). Consider a bandit problem where the set of arms is the unit sphere in $\mathbb{R}^{r}$, and $W_{t}^{\mathbf{u}}$ has a standard normal distribution with mean zero and variance one for all $t$ and $\mathbf{u}$. If $\mathbf{Z}$ has a multivariate normal distribution with mean $\mathbf{0}$ and covariance matrix $\mathbf{I}_{r} / r$, then for all policies $\psi$ and every $T \geq r^{2}$,

$$
\operatorname{Risk}(T, \psi) \geq 0.006 r \sqrt{T}
$$

Consequently, for any policy $\psi$ and $T \geq r^{2}$, there exists $\mathbf{z} \in \mathbb{R}^{r}$ such that

$$
\operatorname{Regret}(\mathbf{z}, T, \psi) \geq 0.006 r \sqrt{T}
$$

[^2]It suffices to establish the lower bound on the Bayes risk because the regret bound follows immediately. Throughout this section, we assume that $\mathcal{U}_{r}=\left\{\mathbf{u} \in \mathbb{R}^{r}:\|\mathbf{u}\|=1\right\}$. We fix an arbitrary policy $\psi$ and for any $t \geq 1$, we let $\mathbf{H}_{t}=\left(\mathbf{U}_{1}, X_{1}, \mathbf{U}_{2}, X_{2}, \ldots, \mathbf{U}_{t}, X_{t}\right)$ be the history up to time $t$. We also let $\widehat{\mathbf{Z}}_{t}$ denote the least mean squares estimator of $\mathbf{Z}$ given the history $\mathbf{H}_{t}$, that is,

$$
\widehat{\mathbf{Z}}_{t}=\mathrm{E}\left[\mathbf{Z} \mid \mathbf{H}_{t}\right] .
$$

Let $\mathbf{S}_{t}^{1}, \ldots, \mathbf{S}_{t}^{r-1}$ denote a collection of orthogonal unit vectors that are also orthogonal to $\widehat{\mathbf{Z}}_{t}$. Note that $\widehat{\mathbf{Z}}_{t}$ and $\mathbf{S}_{t}^{1}, \ldots, \mathbf{S}_{t}^{r-1}$ are functions of $\mathbf{H}_{t}$.

Since $\mathcal{U}_{r}$ is the unit sphere, $\max _{\mathbf{u} \in \mathcal{U}_{r}} \mathbf{u}^{\prime} \mathbf{z}=\left(\mathbf{z}^{\prime} \mathbf{z}\right) /\|\mathbf{z}\|=\|\mathbf{z}\|$, for all $\mathbf{z} \in \mathbb{R}^{r}$. Thus, the risk in period $t$ is given by $\mathrm{E}\left[\|\mathbf{Z}\|-\mathbf{U}_{t}^{\prime} \mathbf{Z}\right]$. The following lemma establishes a lower bound on the cumulative risk in terms of the estimator error variance and the total amount of exploration along the directions $\mathbf{S}_{T}^{1}, \ldots, \mathbf{S}_{T}^{r-1}$.

Lemma 2.2 (Risk Decomposition). For any $T \geq 1$,

$$
\operatorname{Risk}(T, \psi) \geq \frac{1}{2} \sum_{k=1}^{r-1} \mathrm{E}\left[\|\mathbf{Z}\| \sum_{t=1}^{T}\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right)^{2}+\frac{T}{\|\mathbf{Z}\|}\left\{\left(\mathbf{Z}-\widehat{\mathbf{Z}}_{T}\right)^{\prime} \mathbf{S}_{T}^{k}\right\}^{2}\right]
$$

Proof. Using the fact that for any two unit vectors $\mathbf{w}$ and $\mathbf{v}, 1-\mathbf{w}^{\prime} \mathbf{v}=\|\mathbf{w}-\mathbf{v}\|^{2} / 2$, the instantaneous regret in period $t$ satisfies

$$
\|\mathbf{Z}\|-\mathbf{U}_{t}^{\prime} \mathbf{Z}=\|\mathbf{Z}\|\left(1-\mathbf{U}_{t}^{\prime} \frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right)=\frac{\|\mathbf{Z}\|}{2}\left\|\mathbf{U}_{t}-\frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right\|^{2} \geq \frac{\|\mathbf{Z}\|}{2} \sum_{k=1}^{r-1}\left\{\left(\mathbf{U}_{t}-\frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right)^{\prime} \mathbf{S}_{T}^{k}\right\}^{2}
$$

where the inequality follows from the fact that $\mathbf{S}_{T}^{1}, \ldots, \mathbf{S}_{T}^{r-1}$ are orthogonal unit vectors. Therefore, the cumulative conditional risk satisfies

$$
\begin{aligned}
2 \sum_{t=1}^{T} \mathrm{E}\left[\|\mathbf{Z}\|-\mathbf{U}_{t}^{\prime} \mathbf{Z} \mid \mathbf{H}_{T}\right] & \geq \sum_{t=1}^{T} \mathrm{E}\left[\left.\|\mathbf{Z}\| \sum_{k=1}^{r-1}\left\{\left(\mathbf{U}_{t}-\frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right)^{\prime} \mathbf{S}_{T}^{k}\right\}^{2} \right\rvert\, \mathbf{H}_{T}\right] \\
& =\sum_{t=1}^{T} \sum_{k=1}^{r-1} \mathrm{E}\left[\left.\|\mathbf{Z}\|\left\{\left(\mathbf{U}_{t}-\frac{\mathbf{Z}}{\|\mathbf{Z}\|}\right)^{\prime} \mathbf{S}_{T}^{k}\right\}^{2} \right\rvert\, \mathbf{H}_{T}\right] \\
& =\sum_{t=1}^{T} \sum_{k=1}^{r-1} \mathrm{E}\left[\left.\|\mathbf{Z}\|\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right)^{2}-2\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right)\left(\mathbf{Z}^{\prime} \mathbf{S}_{T}^{k}\right)+\frac{\left(\mathbf{Z}^{\prime} \mathbf{S}_{T}^{k}\right)^{2}}{\|\mathbf{Z}\|} \right\rvert\, \mathbf{H}_{T}\right]
\end{aligned}
$$

with probability one. From the definition of $\mathbf{S}_{T}^{k}$, we have $\widehat{\mathbf{Z}}_{T}^{\prime} \mathbf{S}_{T}^{k}=0$ for $k=1 \ldots, r-1$. Therefore, for $t \leq T$,

$$
\mathrm{E}\left[\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right)\left(\mathbf{Z}^{\prime} \mathbf{S}_{T}^{k}\right) \mid \mathbf{H}_{T}\right]=\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right) \mathrm{E}\left[\mathbf{Z}^{\prime} \mid \mathbf{H}_{T}\right] \mathbf{S}_{T}^{k}=\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right) \widehat{\mathbf{Z}}_{T}^{\prime} \mathbf{S}_{T}^{k}=0
$$

which eliminates the middle term in the summand above. Furthermore, we see that $\mathbf{Z}^{\prime} \mathbf{S}_{T}^{k}=$ $\left(\mathbf{Z}-\widehat{\mathbf{Z}}_{T}\right)^{\prime} \mathbf{S}_{T}^{k}$ for all $k$. Thus, with probability one,

$$
\sum_{t=1}^{T} \mathrm{E}\left[\|\mathbf{Z}\|-\mathbf{U}_{t}^{\prime} \mathbf{Z} \mid \mathbf{H}_{T}\right] \geq \frac{1}{2} \sum_{k=1}^{r-1} \mathrm{E}\left[\left.\|\mathbf{Z}\| \sum_{t=1}^{T}\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right)^{2}+\frac{T}{\|\mathbf{Z}\|}\left\{\left(\mathbf{Z}-\widehat{\mathbf{Z}}_{T}\right)^{\prime} \mathbf{S}_{T}^{k}\right\}^{2} \right\rvert\, \mathbf{H}_{T}\right]
$$

and the desired result follows by taking the expectation of both sides.
Since $\mathbf{S}_{T}^{k}$ is orthogonal to $\widehat{\mathbf{Z}}_{T}$, we can interpret $\sum_{t=1}^{T}\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right)^{2}$ and $\left\{\left(\mathbf{Z}-\widehat{\mathbf{Z}}_{T}\right)^{\prime} \mathbf{S}_{T}^{k}\right\}^{2}$ as the total amount of exploration over $T$ periods and the squared estimation error, respectively, in the direction $\mathbf{S}_{T}^{k}$. Thus, Lemma 2.2 tells us that the cumulative risk is bounded below by the sum of the squared estimation error and the total amount of exploration in the past $T$ periods. This result suggests an approach for establishing a lower bound on the risk. If the amount of exploration $\sum_{t=1}^{T}\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right)^{2}$ is large, then the risk will be large. On the other hand, if the amount of exploration is small, we expect significant estimation errors, which in turn imply large risk. This intuition is made precise in Lemma [2.3, which relates the squared estimation error and the amount of exploration.

Lemma 2.3 (Little Exploration Implies Large Estimation Errors). For any $k$ and $T \geq 1$,

$$
\mathrm{E}\left[\left\{\left(\mathbf{Z}-\widehat{\mathbf{Z}}_{T}\right)^{\prime} \mathbf{S}_{T}^{k}\right\}^{2} \mid \mathbf{H}_{T}\right] \geq \frac{1}{r+\sum_{t=1}^{T}\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right)^{2}}
$$

with probability one.
Proof. Let $\mathbf{Q}_{T}=\widehat{\mathbf{Z}}_{T} /\left\|\widehat{\mathbf{Z}}_{T}\right\|$. For any $t$, we have that $\mathbf{U}_{t}=\sum_{k=1}^{r-1}\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right) \mathbf{S}_{T}^{k}+\left(\mathbf{U}_{t}^{\prime} \mathbf{Q}_{T}\right) \mathbf{Q}_{T}$. Let

$$
\mathbf{V}=\left[\begin{array}{llll}
\mathbf{S}_{T}^{1} & \mathbf{S}_{T}^{2} & \cdots & \mathbf{S}_{T}^{r-1}
\end{array} \mathbf{Q}_{T}\right]
$$

be an $r \times r$ orthonormal matrix whose columns are the vectors $\mathbf{S}_{T}^{1}, \ldots, \mathbf{S}_{T}^{r-1}$, and $\mathbf{Q}_{T}$, respectively. Then, it is easy to verify that

$$
\sum_{t=1}^{T} \mathbf{U}_{t} \mathbf{U}_{t}^{\prime}=\mathbf{V A} \mathbf{V}^{\prime}
$$

where $\mathbf{A}=\left(\begin{array}{ll}\boldsymbol{\Sigma} & \mathbf{c} \\ \mathbf{c}^{\prime} & a\end{array}\right)$, is an $r \times r$ matrix, with $a=\mathbf{Q}_{T}^{\prime}\left(\sum_{t=1}^{T} \mathbf{U}_{t} \mathbf{U}_{t}^{\prime}\right) \mathbf{Q}_{T}, \mathbf{c}$ is an $(r-1)$ dimensional column vector, and where $\boldsymbol{\Sigma}$ is an $(r-1) \times(r-1)$ matrix with $\boldsymbol{\Sigma}_{k \ell}=\left(\mathbf{S}_{T}^{k}\right)^{\prime}\left(\sum_{t=1}^{T} \mathbf{U}_{t} \mathbf{U}_{t}^{\prime}\right) \mathbf{S}_{T}^{\ell}=$ $\sum_{t=1}^{T}\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right)\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{\ell}\right)$ for $k, \ell=1, \ldots, r-1$,

Since $\mathbf{Z}$ has a multivariate normal prior distribution with covariance matrix $\mathbf{I}_{r} / r$, it is a standard result (use, for example, Corollary E.3.5 in Appendix E in Bertsekas, 1995) that

$$
\mathrm{E}\left[\left(\mathbf{Z}-\widehat{\mathbf{Z}}_{T}\right)\left(\mathbf{Z}-\widehat{\mathbf{Z}}_{T}\right)^{\prime} \mid \mathbf{H}_{T}\right]=\left(r \mathbf{I}_{r}+\sum_{t=1}^{T} \mathbf{U}_{t} \mathbf{U}_{t}^{\prime}\right)^{-1}=\mathbf{V}\left(r \mathbf{I}_{r}+\mathbf{A}\right)^{-1} \mathbf{V}^{\prime}
$$

Since $\mathbf{S}_{T}^{k}$ is a function of $\mathbf{H}_{T}$ and $\mathbf{V}^{\prime} \mathbf{S}_{T}^{k}=\mathbf{e}_{k}$, we have, for $k \leq r-1$, that

$$
\begin{aligned}
\mathrm{E}\left[\left\{\left(\mathbf{Z}-\widehat{\mathbf{Z}}_{T}\right)^{\prime} \mathbf{S}_{T}^{k}\right\}^{2} \mid \mathbf{H}_{T}\right] & =\left(\mathbf{V}^{\prime} \mathbf{S}_{T}^{k}\right)^{\prime}\left(r \mathbf{I}_{r}+\mathbf{A}\right)^{-1}\left(\mathbf{V}^{\prime} \mathbf{S}_{T}^{k}\right)=\left[\left(r \mathbf{I}_{r}+\mathbf{A}\right)^{-1}\right]_{k k} \\
& \geq \frac{1}{\left(r \mathbf{I}_{r}+\mathbf{A}\right)_{k k}}=\frac{1}{r+\sum_{t=1}^{T}\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right)^{2}}
\end{aligned}
$$

where the inequality follows from Fiedler's Inequality (see, for example, Theorem 2.1 in Fiedler and Pták, 1997), and the final equality follows from the definition of $\mathbf{A}$.

The next lemma gives a lower bound on the probability that $\mathbf{Z}$ is bounded away from the origin. The proof follows from simple calculations involving normal densities, and the details are given in Appendix A. 1 .

Lemma 2.4. For any $\theta \leq 1 / 2$ and $\beta>0, \operatorname{Pr}\{\theta \leq\|\mathbf{Z}\| \leq \beta\} \geq 1-4 \theta^{2}-\frac{1}{\beta^{2}}$.
The last lemma establishes a lower bound on the sum of the total amount of exploration and the squared estimation error, which is also the minimum cumulative Bayes risk along the direction $\mathbf{S}_{T}^{k}$ by Lemma 2.2.

Lemma 2.5 (Minimum Directional Risk). For $k=1, \ldots, r-1$, and $T \geq r^{2}$,

$$
\mathrm{E}\left[\|\mathbf{Z}\| \sum_{t=1}^{T}\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right)^{2}+\frac{T}{\|\mathbf{Z}\|}\left\{\left(\mathbf{Z}-\widehat{\mathbf{Z}}_{T}\right)^{\prime} \mathbf{S}_{T}^{k}\right\}^{2}\right] \geq 0.027 \sqrt{T} .
$$

We note that if $\|\mathbf{Z}\|$ were a constant, rather than a random variable, Lemma 2.5 would follow immediately. Hence, most of the work in the proof below involves constraining $\|\mathbf{Z}\|$ to a certain range $[\theta, \beta]$.

Proof. Consider an arbitrary $k$, and let $\Xi=\sum_{t=1}^{T}\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right)^{2}, \Gamma=\left\{\left(\mathbf{Z}-\widehat{\mathbf{Z}}_{T}\right)^{\prime} \mathbf{S}_{T}^{k}\right\}^{2}$. Our proof will make use of positive constants $\theta, \beta$, and $\eta$, whose values will be chosen later. Note that

$$
\begin{aligned}
\mathrm{E}\left[\left.\|\mathbf{Z}\| \Xi+\frac{T \Gamma}{\|\mathbf{Z}\|} \right\rvert\, \mathbf{H}_{T}\right] \geq & \mathrm{E}\left[\left.\left(\|\mathbf{Z}\| \Xi+\frac{T \Gamma}{\|\mathbf{Z}\|}\right) \mathbb{1}_{\{\theta \leq\|\mathbf{Z}\| \leq \beta\}} \mathbb{1}_{\{\Xi \geq \sqrt{T}\}} \right\rvert\, \mathbf{H}_{T}\right] \\
& +\mathrm{E}\left[\left.\left(\|\mathbf{Z}\| \Xi+\frac{T \Gamma}{\|\mathbf{Z}\|}\right) \mathbb{1}_{\{\theta \leq\|\mathbf{Z}\| \leq \beta\}} \mathbb{1}_{\{\Xi<\sqrt{T}\}} \right\rvert\, \mathbf{H}_{T}\right] \\
\geq & \theta \sqrt{T} \mathbb{1}_{\{\Xi \geq \sqrt{T}\}} \mathrm{E}\left[\mathbb{1}_{\{\theta \leq\|\mathbf{Z}\| \leq \beta\}} \mid \mathbf{H}_{T}\right] \\
& +\frac{T}{\beta} \mathbb{1}_{\{\Xi<\sqrt{T}\}} \mathrm{E}\left[\Gamma \mathbb{1}_{\{\theta \leq\|\mathbf{Z}\| \leq \beta\}} \mid \mathbf{H}_{T}\right],
\end{aligned}
$$

where we use the fact that $\Xi$ is a function of $\mathbf{H}_{T}$ in the final inequality. We will now lower bound the last term on the right hand side of the above inequality. Let $\Theta=E\left[\Gamma \mid \mathbf{H}_{T}\right]$. Since $\Theta$ is a
function of $\mathbf{H}_{T}$,

$$
\begin{aligned}
\frac{T}{\beta} \mathbb{1}_{\{\Xi<\sqrt{T}\}} \mathrm{E}\left[\Gamma \mathbb{1}_{\{\theta \leq\|\mathbf{Z}\| \leq \beta\}} \mid \mathbf{H}_{T}\right] & \geq \frac{T}{\beta} \mathbb{1}_{\{\Xi<\sqrt{T}\}} \mathrm{E}\left[\Gamma \mathbb{1}_{\{\theta \leq\|\mathbf{Z}\| \leq \beta\}} \mathbb{1}_{\{\Gamma \geq \eta \Theta\}} \mid \mathbf{H}_{T}\right] \\
& \geq \frac{\eta T}{\beta} \Theta \mathbb{1}_{\{\Xi<\sqrt{T}\}} \mathrm{E}\left[\mathbb{1}_{\{\theta \leq\|\mathbf{Z}\| \leq \beta\}} \mathbb{1}_{\{\Gamma \geq \eta \Theta\}} \mid \mathbf{H}_{T}\right] \\
& \geq \frac{\eta \sqrt{T}}{2 \beta} \mathbb{1}_{\{\Xi<\sqrt{T}\}} \mathrm{E}\left[\mathbb{1}_{\{\theta \leq\|\mathbf{Z}\| \leq \beta\}} \mathbb{1}_{\{\Gamma \geq \eta \Theta\}} \mid \mathbf{H}_{T}\right],
\end{aligned}
$$

where the last inequality follows from Lemma 2.3 which implies that, with probability one,

$$
\frac{\eta T}{\beta} \Theta \mathbb{1}_{\{\Xi<\sqrt{T}\}} \geq \frac{\eta T}{\beta} \cdot \frac{1}{r+\Xi} \mathbb{1}_{\{\Xi<\sqrt{T}\}} \geq \frac{\eta T}{\beta} \cdot \frac{1}{r+\sqrt{T}} \mathbb{1}_{\{\Xi<\sqrt{T}\}} \geq \frac{\eta \sqrt{T}}{2 \beta} \mathbb{1}_{\{\Xi<\sqrt{T}\}}
$$

and where the last inequality follows from the fact that $T \geq r^{2}$, and thus, $1 /(r+\sqrt{T}) \geq 1 /(2 \sqrt{T})$.
Putting everything together, we obtain

$$
\begin{aligned}
\mathrm{E}\left[\left.\|\mathbf{Z}\| \Xi+\frac{T \Gamma}{\|\mathbf{Z}\|} \right\rvert\, \mathbf{H}_{T}\right] \geq & \theta \sqrt{T} \mathbb{1}_{\{\Xi \geq \sqrt{T}\}} \mathrm{E}\left[\mathbb{1}_{\{\theta \leq\|\mathbf{Z}\| \leq \beta\}} \mid \mathbf{H}_{T}\right] \\
& +\frac{\eta \sqrt{T}}{2 \beta} \mathbb{1}_{\{\Xi<\sqrt{T}\}} \mathrm{E}\left[\mathbb{1}_{\{\theta \leq\|\mathbf{Z}\| \leq \beta\}} \mathbb{1}_{\{\Gamma \geq \eta \Theta\}} \mid \mathbf{H}_{T}\right] \\
\geq & \min \left\{\theta, \frac{\eta}{2 \beta}\right\} \sqrt{T} \mathrm{E}\left[\mathbb{1}_{\{\theta \leq\|\mathbf{Z}\| \leq \beta\}} \mathbb{1}_{\{\Gamma \geq \eta \Theta\}} \mid \mathbf{H}_{T}\right]
\end{aligned}
$$

with probability one. By the Bonferroni Inequality, we have that

$$
\begin{aligned}
\mathrm{E}\left[\mathbb{1}_{\{\theta \leq\|\mathbf{Z}\| \leq \beta\}} \mathbb{1}_{\{\Gamma \geq \eta \Theta\}} \mid \mathbf{H}_{T}\right] & =\operatorname{Pr}\left\{\theta \leq\|\mathbf{Z}\| \leq \beta \text { and } \Gamma \geq \eta \Theta \mid \mathbf{H}_{T}\right\} \\
& \geq \operatorname{Pr}\left\{\theta \leq\|\mathbf{Z}\| \leq \beta \mid \mathbf{H}_{T}\right\}+\operatorname{Pr}\left\{\Gamma \geq \eta \Theta \mid \mathbf{H}_{T}\right\}-1
\end{aligned}
$$

with probability one. Conditioned on $\mathbf{H}_{T},\left(\mathbf{Z}-\widehat{\mathbf{Z}}_{T}\right)^{\prime} \mathbf{S}_{T}^{k}$ is normally distributed with mean zero and variance

$$
\mathrm{E}\left[\left\{\left(\mathbf{Z}-\widehat{\mathbf{Z}}_{T}\right)^{\prime} \mathbf{S}_{T}^{k}\right\}^{2} \mid \mathbf{H}_{T}\right]=\mathrm{E}\left[\Gamma \mid \mathbf{H}_{T}\right]=\Theta .
$$

Let $\Phi(\cdot)$ be the cumulative distribution function of the standard normal random variable, that is, $\Phi(x)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{x} e^{-u^{2} / 2} d u$. Then,

$$
\operatorname{Pr}\left\{\Gamma \geq \eta \Theta \mid \mathbf{H}_{T}\right\}=\operatorname{Pr}\left\{\left|\left(\mathbf{Z}-\widehat{\mathbf{Z}}_{T}\right)^{\prime} \mathbf{S}_{T}^{k}\right| \geq \sqrt{\eta} \sqrt{\Theta} \mid \mathbf{H}_{T}\right\}=2(1-\Phi(\sqrt{\eta}))
$$

from which it follows that, with probability one,

$$
\mathrm{E}\left[\mathbb{1}_{\{\theta \leq\|\mathbf{Z}\| \leq \beta\}} \mathbb{1}_{\{\Gamma \geq \eta \Theta\}} \mid \mathbf{H}_{T}\right] \geq \operatorname{Pr}\left\{\theta \leq\|\mathbf{Z}\| \leq \beta \mid \mathbf{H}_{T}\right\}+2(1-\Phi(\sqrt{\eta}))-1 .
$$

Therefore,

$$
\mathrm{E}\left[\left.\|\mathbf{Z}\| \Xi+\frac{T \Gamma}{\|\mathbf{Z}\|} \right\rvert\, \mathbf{H}_{T}\right] \geq \min \left\{\theta, \frac{\eta}{2 \beta}\right\}\left[\operatorname{Pr}\left\{\theta \leq\|\mathbf{Z}\| \leq \beta \mid \mathbf{H}_{T}\right\}+2(1-\Phi(\sqrt{\eta}))-1\right] \sqrt{T}
$$

with probability one, which implies that

$$
\begin{aligned}
\mathrm{E}\left[\|\mathbf{Z}\| \Xi+\frac{T \Gamma}{\|\mathbf{Z}\|}\right] & \geq \min \left\{\theta, \frac{\eta}{2 \beta}\right\}[\operatorname{Pr}\{\theta \leq\|\mathbf{Z}\| \leq \beta\}+2(1-\Phi(\sqrt{\eta}))-1] \sqrt{T}, \\
& \geq \min \left\{\theta, \frac{\eta}{2 \beta}\right\}\left[2(1-\Phi(\sqrt{\eta}))-\frac{1}{\beta^{2}}-4 \theta^{2}\right] \sqrt{T},
\end{aligned}
$$

where the last inequality follows from Lemma [2.4. Set $\theta=0.09, \beta=3$, and $\eta=0.5$, to obtain $\mathrm{E}\left[\|\mathbf{Z}\| \Xi+\frac{T \Gamma}{\|\mathbf{Z}\|}\right] \geq 0.027 \sqrt{T}$, which is the desired result.

Finally, here is the proof of Theorem [2.1.
Proof. It follows from Lemmas 2.2 and 2.5 that

$$
\begin{aligned}
\operatorname{Risk}(T, \psi) & \geq \frac{1}{2} \sum_{k=1}^{r-1} \mathrm{E}\left[\|\mathbf{Z}\| \sum_{t=1}^{T}\left(\mathbf{U}_{t}^{\prime} \mathbf{S}_{T}^{k}\right)^{2}+\frac{T}{\|\mathbf{Z}\|}\left\{\left(\mathbf{Z}-\widehat{\mathbf{Z}}_{T}\right)^{\prime} \mathbf{S}_{T}^{k}\right\}^{2}\right] \\
& \geq \frac{r-1}{2} \cdot 0.027 \sqrt{T} \geq \frac{r}{4} \cdot 0.027 \sqrt{T} \geq 0.006 r \sqrt{T}
\end{aligned}
$$

where we have used the fact $r \geq 2$, which implies that $r-1 \geq r / 2$.

## 3. Matching Upper Bounds

We have established $\Omega(r \sqrt{T})$ lower bounds when the set of arms $\mathcal{U}_{r}$ is the unit sphere. We now prove that a policy that alternates between exploration and exploitation phases yields matching upper bounds on the regret and risk, and is therefore optimal for this problem. Surprisingly, we will see that the phase-based policy is effective for a large class of bandit problems, involving a strongly convex set of arms. We introduce the following assumption on the tails of the error random variables $W_{t}^{\mathbf{u}}$ and on the set of arms $\mathcal{U}_{r}$, which will remain in effect throughout the rest of paper.

## Assumption 1.

(a) There exists a positive constant $\sigma_{0}$ such that for any $r \geq 2, \mathbf{u} \in \mathcal{U}_{r}, t \geq 1$, and $x \in \mathbb{R}$, we have $\mathbf{E}\left[e^{x W_{t}^{u}}\right] \leq e^{x^{2} \sigma_{0}^{2} / 2}$.
(b) There exist positive constants $\bar{u}$ and $\lambda_{0}$ such that for any $r \geq 2$,

$$
\max _{\mathbf{u} \in \mathcal{U}_{r}}\|\mathbf{u}\| \leq \bar{u}
$$

and the set of arms $\mathcal{U}_{r} \subset \mathbb{R}^{r}$ has $r$ linearly independent elements $\mathbf{b}_{1}, \ldots, \mathbf{b}_{r}$ such that $\lambda_{\min }\left(\sum_{k=1}^{r} \mathbf{b}_{k} \mathbf{b}_{k}^{\prime}\right) \geq \lambda_{0}$.

Under Assumption (a), the tails of the distribution of the errors $W_{t}^{\mathbf{u}}$ decay at least as fast as for a normal random variable with variance $\sigma_{0}^{2}$. The first part of Assumption (b) ensures that
the expected reward of the arms remain bounded as the dimension $r$ increases, while the arms $\mathbf{b}_{1}, \ldots, \mathbf{b}_{r}$ given in the second part of Assumption $1(b)$ will be used during the exploration phase of our policy.

Our policy - which we refer to as the Phased Exploration and Greedy Exploitation (PEGE) - operates in cycles, and in each cycle, we alternate between exploration and exploitation phases. During the exploration phase of cycle $c$, we play the $r$ linearly independent arms from Assumption 1(b). Using the rewards observed during the exploration phases in the past cycles, we compute an ordinary least squares (OLS) estimate $\widehat{\mathbf{Z}}(c)$. In the exploitation phase of cycle $c$, we use $\widehat{\mathbf{Z}}(c)$ as a proxy for $\mathbf{Z}$ and compute a greedy decision $\mathbf{G}(c) \in \mathcal{U}_{r}$ defined by:

$$
\begin{equation*}
\mathbf{G}(c)=\arg \max _{\mathbf{v} \in \mathcal{U}_{r}} \mathbf{v}^{\prime} \widehat{\mathbf{Z}}(c) \tag{2}
\end{equation*}
$$

where we break ties arbitrarily. We then play the $\operatorname{arm} \mathbf{G}(c)$ for an additional $c$ periods to complete cycle $c$. Here is a formal description of the policy.

Phased Exploration and Greedy Exploitation (PEGE)
Description: For each cycle $c \geq 1$, complete the following two phases.

1. Exploration ( $r$ periods): For $k=1,2, \ldots, r$, play arm $\mathbf{b}_{k} \in \mathcal{U}_{r}$ given in Assumption 1(b), and observe the reward $X^{\mathbf{b}_{k}}(c)$. Compute the OLS estimate $\widehat{\mathbf{Z}}(c) \in \mathbb{R}^{r}$, given by

$$
\widehat{\mathbf{Z}}(c)=\frac{1}{c}\left(\sum_{k=1}^{r} \mathbf{b}_{k} \mathbf{b}_{k}^{\prime}\right)^{-1} \sum_{s=1}^{c} \sum_{k=1}^{r} \mathbf{b}_{k} X^{\mathbf{b}_{k}}(s)=\mathbf{Z}+\frac{1}{c}\left(\sum_{k=1}^{r} \mathbf{b}_{k} \mathbf{b}_{k}^{\prime}\right)^{-1} \sum_{s=1}^{c} \sum_{k=1}^{r} \mathbf{b}_{k} W^{\mathbf{b}_{k}}(s),
$$

where for any $k, X^{\mathbf{b}_{k}}(s)$ and $W^{\mathbf{b}_{k}}(s)$ denote the observed reward and the error random variable associated with playing arm $\mathbf{b}_{k}$ in cycle $s$. Note that the last equality follows from Equation (11) defining our model.
2. Exploitation ( $c$ periods): Play the greedy $\operatorname{arm} \mathbf{G}(c)=\arg \max _{\mathbf{v} \in \mathcal{U}_{r}} \mathbf{v}^{\prime} \widehat{\mathbf{Z}}(c)$ for $c$ periods.

Since $\mathcal{U}_{r}$ is compact, for each $\mathbf{z} \in \mathbb{R}^{r}$, there is an optimal arm that gives the maximum expected reward. When this best arm varies smoothly with $\mathbf{z}$, we will show that the $T$-period regret and risk under the PEGE policy is bounded above by $O(r \sqrt{T})$. More precisely, we say that a set of arms $\mathcal{U}_{r}$ satisfies the smooth best arm response with parameter $J(\mathrm{SBAR}(J)$, for short) condition if for any nonzero vector $\mathbf{z} \in \mathbb{R}^{r} \backslash\{\mathbf{0}\}$, there is a unique best arm $\mathbf{u}^{*}(\mathbf{z}) \in \mathcal{U}_{r}$ that gives the maximum expected reward, and for any two unit vectors $\mathbf{z} \in \mathbb{R}^{r}$ an $\mathbf{y} \in \mathbb{R}^{r}$ with $\|\mathbf{z}\|=\|\mathbf{y}\|=1$, we have

$$
\left\|\mathbf{u}^{*}(\mathbf{z})-\mathbf{u}^{*}(\mathbf{y})\right\| \leq J\|\mathbf{z}-\mathbf{y}\|
$$

Even though the SBAR condition appears to be an implicit one, it admits a simple interpretation. According to Corollary 4 of Polovinkin (1996), a compact set $\mathcal{U}_{r}$ satisfies condition $\operatorname{SBAR}(J)$ if and only if it is strongly convex with parameter $J$, in the sense that the set $\mathcal{U}_{r}$ can be represented as the intersection of closed balls of radius $J$. Intuitively, the SBAR condition requires the boundary of $\mathcal{U}_{r}$ to have a curvature that is bounded below by a positive constant. For some examples, the unit ball satisfies the $\operatorname{SBAR}(1)$ condition. Furthermore, according to Theorem 3 of Polovinkin (1996), an ellipsoid of the form $\left\{\mathbf{u} \in R^{r}: \mathbf{u}^{\prime} \mathbf{Q}^{-1} \mathbf{u} \leq 1\right\}$, where $\mathbf{Q}$ is a symmetric positive definite matrix, satisfies the condition $\operatorname{SBAR}\left(\lambda_{\max }(\mathbf{Q}) / \sqrt{\lambda_{\min }(\mathbf{Q})}\right)$.

The main result of this section is stated in the following theorem. The proof is given in Section 3.1 .

Theorem 3.1 (Regret and Risk Under the Greedy Policy). Suppose that Assumption $\mathbb{1}$ holds and that the sets $\mathcal{U}_{r}$ satisfy the $\operatorname{SBAR}(J)$ condition. Then, there exists a positive constant $a_{1}$ that depends only on $\sigma_{0}, \bar{u}, \lambda_{0}$, and $J$, such that for any $\mathbf{z} \in \mathbb{R}^{r} \backslash\{\mathbf{0}\}$ and $T \geq r$,

$$
\operatorname{Regret}(\mathbf{z}, T, \text { PEGE }) \leq a_{1}\left(\|\mathbf{z}\|+\frac{1}{\|\mathbf{z}\|}\right) r \sqrt{T} .
$$

Suppose in addition, that there exists a constant $M>0$ such that for every $r \geq 2$ we have $\mathrm{E}[\|\mathbf{Z}\|] \leq$ $M$ and $\mathrm{E}[1 /\|\mathbf{Z}\|] \leq M$. Then, there exists a positive constant $a_{2}$ that depends only on $\sigma_{0}, \bar{u}, \lambda_{0}$, $J$, and $M$, such that for any $T \geq r$,

$$
\operatorname{Risk}(T, \operatorname{PEGE}) \leq a_{2} r \sqrt{T} .
$$

Dependence on $\|\mathbf{z}\|$ in the regret bound: By Assumption $\mathbb{1}(b)$, for any $\mathbf{z} \in \mathbb{R}^{r}$, the instantaneous regret under $\operatorname{arm} \mathbf{v} \in \mathcal{U}$ is bounded by $\max _{\mathbf{u} \in \mathcal{U}} \mathbf{z}^{\prime}(\mathbf{u}-\mathbf{v}) \leq 2 \bar{u}\|\mathbf{z}\|$. Thus, $2 \bar{u}\|\mathbf{z}\| T$ provides a trivial upper bound on the $T$-period cumulative regret under the PEGE policy. Combining this with Theorem 3.1, we have that

$$
\operatorname{Regret}(\mathbf{z}, T, \text { PEGE }) \leq \max \left\{a_{1}, 2 \bar{u}\right\} \cdot \min \left\{\left(\|\mathbf{z}\|+\frac{1}{\|\mathbf{z}\|}\right) r \sqrt{T},\|\mathbf{z}\| T\right\}
$$

The above result shows that the performance of our policy does not deteriorate as the norm of $\mathbf{z}$ approaches zero.

Intuitively, the requirement $\mathrm{E}[\|\mathbf{Z}\|] \leq M$ in Theorem 3.1 implies that, as $r$ increases, the maximum expected reward (over all arms) remains bounded. Moreover, the assumption on the boundedness of $\mathrm{E}[1 /\|\mathbf{Z}\|]$ means that $\mathbf{Z}$ does not have too much mass near the origin. The following lemma provides conditions under which this assumption holds, and shows that the case of the multivariate normal distribution used in Theorem 2.1 is also covered. The proof is given in Appendix A. 2

Lemma 3.2 (Small Mass Near the Origin).
(a) Suppose that there exist constants $M_{0}$ and $\rho \in(0,1]$ such that for any $r \geq 2$, the random variable $\|\mathbf{Z}\|$ has a density function $g: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$such that $g(x) \leq M_{0} x^{\rho}$ for all $x \in[0, \rho]$. Then, $\mathrm{E}[1 /\|\mathbf{Z}\|] \leq M$, where $M$ depends only on $M_{0}$ and $\rho$.
(b) Suppose that for any $r \geq 2$, the random vector $\mathbf{Z}$ has a multivariate normal distribution with mean $\mathbf{0} \in \mathbb{R}^{r}$ and covariance matrix $\mathbf{I}_{r} / r$. Then, $\mathrm{E}[\|\mathbf{Z}\|] \leq 1$ and $\mathrm{E}[1 /\|\mathbf{Z}\|] \leq \sqrt{\pi}$.

The following corollary shows that the example in Section 2admits tight matching upper bounds on the regret and risk.

Corollary 3.3 (Matching Upper Bounds). Consider a bandit problem where the set of arms is the unit sphere in $\mathbb{R}^{r}$, and where $W_{t}^{\mathbf{u}}$ has a standard normal distribution with mean zero and variance one for all $t$ and $\mathbf{u}$. Then, there exists an absolute constant $a_{3}$ such that for any $\mathbf{z} \in \mathbb{R}^{r} \backslash\{\mathbf{0}\}$ and $T \geq r$,

$$
\operatorname{Regret}(\mathbf{z}, T, \text { PEGE }) \leq a_{3}\left(\|\mathbf{z}\|+\frac{1}{\|\mathbf{z}\|}\right) r \sqrt{T} .
$$

Moreover, if $\mathbf{Z}$ has a multivariate normal distribution with mean $\mathbf{0}$ and covariance matrix $\mathbf{I}_{r} / r$, then for all $T \geq r$,

$$
\operatorname{Risk}(T, \text { PEGE }) \leq a_{3} r \sqrt{T} .
$$

Proof. Since the set of arms is the unit sphere and the errors are standard normal, Assumption 1 is satisfied with $\sigma_{0}=\bar{u}=\lambda_{0}=1$. Moreover, as already discussed, the unit sphere satisfies the $\operatorname{SBAR}(1)$ condition. Finally, By Lemma 3.2, the random vector $\mathbf{Z}$ satisfies the hypotheses of Theorem 3.1. The regret and risk bounds then follow immediately.

### 3.1 Proof of Theorem 3.1

The proof of Theorem 3.1 relies on the following upper bound on the square of the norm difference between $\widehat{\mathbf{Z}}(c)$ and $\mathbf{Z}$.

Lemma 3.4 (Bound on Squared Norm Difference). Under Assumption 团, there exists a positive constant $h_{1}$ that depends only on $\sigma_{0}, \bar{u}$, and $\lambda_{0}$ such that for any $\mathbf{z} \in \mathbb{R}^{r}$ and $c \geq 1$,

$$
\mathrm{E}\left[\|\widehat{\mathbf{Z}}(c)-\mathbf{z}\|^{2} \mid \mathbf{Z}=\mathbf{z}\right] \leq \frac{h_{1} r}{c}
$$

Proof. Recall from the definition of the PEGE policy that the estimate $\widehat{\mathbf{Z}}(c)$ at the end of the exploration phase of cycle $c$ is given by

$$
\widehat{\mathbf{Z}}(c)=\mathbf{Z}+\frac{1}{c}\left(\sum_{k=1}^{r} \mathbf{b}_{k} \mathbf{b}_{k}^{\prime}\right)^{-1} \sum_{s=1}^{c} \sum_{k=1}^{r} \mathbf{b}_{k} W^{\mathbf{b}_{k}}(s)=\mathbf{Z}+\frac{1}{c} \sum_{s=1}^{c} \mathbf{B} \mathbf{V}(s),
$$

where $\mathbf{B}=\left(\sum_{k=1}^{r} \mathbf{b}_{k} \mathbf{b}_{k}^{\prime}\right)^{-1}$ and $\mathbf{V}(s)=\sum_{k=1}^{r} \mathbf{b}_{k} W^{\mathbf{b}_{k}}(s)$ ．Note that the mean－zero random vari－ ables $W^{\mathbf{b}_{k}}(s)$ are independent of each other and their variance is bounded by some constant $\gamma_{0}$ that depends only on $\sigma_{0}$ ．Then，it follows from Assumption $\square$ that

$$
\begin{aligned}
\mathrm{E}\left[\|\widehat{\mathbf{Z}}(c)-\mathbf{z}\|^{2} \mid \mathbf{Z}=\mathbf{z}\right] & =\frac{1}{c^{2}} \sum_{s=1}^{c} \mathrm{E}\left[\mathbf{V}(s)^{\prime} \mathbf{B}^{2} \mathbf{V}(s)\right]=\frac{1}{c^{2}} \sum_{s=1}^{c} \sum_{k=1}^{r} \mathrm{E}\left[\left(W^{\mathbf{b}_{k}}(s)\right)^{2}\right] \mathbf{b}_{k}^{\prime} \mathbf{B}^{2} \mathbf{b}_{k} \\
& \leq \frac{\gamma_{0}}{c} \sum_{k=1}^{r} \mathbf{b}_{k}^{\prime} \mathbf{B}^{2} \mathbf{b}_{k} \leq \frac{\gamma_{0}}{c} \sum_{k=1}^{r} \lambda_{\max }\left(\mathbf{B}^{2}\right)\left\|\mathbf{b}_{k}\right\|^{2} \leq \frac{\gamma_{0} \bar{u}^{2} r}{\lambda_{0}^{2} c}
\end{aligned}
$$

which is the desired result．
The next lemma gives an upper bound on the difference between two normalized vectors in terms of the difference of the original vectors．

Lemma 3.5 （Difference Between Normalized Vectors）．For any $\mathbf{z}, \mathbf{w} \in \mathbb{R}^{r}$ ，not both equal to zero，

$$
\left\|\frac{\mathbf{w}}{\|\mathbf{w}\|}-\frac{\mathbf{z}}{\|\mathbf{z}\|}\right\| \leq \frac{2\|\mathbf{w}-\mathbf{z}\|}{\max \{\|\mathbf{z}\|,\|\mathbf{w}\|\}}
$$

where we define $\mathbf{0} /\|\mathbf{0}\|$ to be some fixed unit vector．
Proof．The inequality is easily seen to hold if either $\mathbf{w}=\mathbf{0}$ or $\mathbf{z}=\mathbf{0}$ ．So，assume that both $\mathbf{w}$ and $\mathbf{z}$ are nonzero．Using the triangle inequality and the fact that $|\|\mathbf{w}\|-\|\mathbf{z}\|| \leq\|\mathbf{w}-\mathbf{z}\|$ ，we have that

$$
\left\|\frac{\mathrm{w}}{\|\mathrm{w}\|}-\frac{\mathbf{z}}{\|\mathrm{z}\|}\right\| \leq\left\|\frac{\mathrm{w}}{\|\mathrm{w}\|}-\frac{\mathbf{z}}{\|\mathrm{w}\|}\right\|+\left\|\frac{\mathrm{z}}{\|\mathrm{w}\|}-\frac{\mathbf{z}}{\|\mathrm{z}\|}\right\|=\frac{\|\mathbf{w}-\mathbf{z}\|}{\|\mathrm{w}\|}+\|\mathbf{z}\|\left|\frac{1}{\|\mathrm{w}\|}-\frac{1}{\|\mathrm{z}\|}\right| \leq \frac{2\|\mathbf{w}-\mathbf{z}\|}{\|\mathrm{w}\|} .
$$

By symmetry，we also have $\left\|\frac{\mathbf{w}}{\|\mathbf{w}\|}-\frac{\mathbf{z}}{\|\mathbf{z}\|}\right\| \leq \frac{2\|\mathbf{w}-\mathbf{z}\|}{\|\mathbf{z}\|}$ ，which gives the desired result．
The following lemma gives an upper bound on the expected instantaneous regret under the greedy decision $\mathbf{G}(c)$ during the exploitation phase of cycle $c$ ．

Lemma 3.6 （Regret Under the Greedy Decision）．Suppose that Assumption $⿴ 囗 十 ⺝$ holds and the sets $\mathcal{U}_{r}$ satisfy the $S B A R(J)$ condition．Then，there exists a positive constant $h_{2}$ that depends only on $\sigma_{0}, \bar{u}, \lambda_{0}$ ，and $J$ ，such that for any $\mathbf{z} \in \mathbb{R}^{r}$ and $c \geq 1$ ，

$$
\mathrm{E}\left[\max _{\mathbf{u} \in \mathcal{U}^{r}} \mathbf{z}^{\prime}(\mathbf{u}-\mathbf{G}(c)) \mid \mathbf{Z}=\mathbf{z}\right] \leq \frac{r h_{2}}{c\|\mathbf{z}\|},
$$

Proof．The result is trivially true when $\mathbf{z}=\mathbf{0}$ ．So，let us fix some $\mathbf{z} \in \mathbb{R}^{r} \backslash\{\mathbf{0}\}$ ．By comparing the greedy decision $\mathbf{G}(c)$ with the best $\operatorname{arm} \mathbf{u}^{*}(\mathbf{z})$ ，we see that the instantaneous regret satisfies

$$
\begin{aligned}
\mathbf{z}^{\prime}\left(\mathbf{u}^{*}(\mathbf{z})-\mathbf{G}(c)\right) & =(\mathbf{z}-\widehat{\mathbf{Z}}(c))^{\prime} \mathbf{u}^{*}(\mathbf{z})+\left(\mathbf{u}^{*}(\mathbf{z})-\mathbf{G}(c)\right)^{\prime} \widehat{\mathbf{Z}}(c)+(\widehat{\mathbf{Z}}(c)-\mathbf{z})^{\prime} \mathbf{G}(c) \\
& \leq(\mathbf{z}-\widehat{\mathbf{Z}}(c))^{\prime} \mathbf{u}^{*}(\mathbf{z})+(\widehat{\mathbf{Z}}(c)-\mathbf{z})^{\prime} \mathbf{G}(c) \\
& =(\widehat{\mathbf{Z}}(c)-\mathbf{z})^{\prime}\left(\mathbf{G}(c)-\mathbf{u}^{*}(\mathbf{z})\right)=(\widehat{\mathbf{Z}}(c)-\mathbf{z})^{\prime}\left(\mathbf{u}^{*}(\widehat{\mathbf{Z}}(c))-\mathbf{u}^{*}(\mathbf{z})\right),
\end{aligned}
$$

where the inequality follows from the definition of the greedy decision in Equation (2), and the final equality follows from the fact that $\mathbf{G}(c)=\mathbf{u}^{*}(\widehat{\mathbf{Z}}(c))$. As a convention, we define $\mathbf{0} /\|\mathbf{0}\|$ to some fixed unit vector and set $\mathbf{u}^{*}(\mathbf{0})=\mathbf{u}^{*}(\mathbf{0} /\|\mathbf{0}\|)$.

It then follows from the Cauchy-Schwarz Inequality that, with probability one,

$$
\begin{aligned}
\mathbf{z}^{\prime}\left(\mathbf{u}^{*}(\mathbf{z})-\mathbf{G}(c)\right) & \leq\|\widehat{\mathbf{Z}}(c)-\mathbf{z}\|\left\|\mathbf{u}^{*}(\widehat{\mathbf{Z}}(c))-\mathbf{u}^{*}(\mathbf{z})\right\| \\
& =\|\widehat{\mathbf{Z}}(c)-\mathbf{z}\|\left\|\mathbf{u}^{*}\left(\frac{\widehat{\mathbf{Z}}(c)}{\|\widehat{\mathbf{Z}}(c)\|}\right)-\mathbf{u}^{*}\left(\frac{\mathbf{z}}{\|\mathbf{z}\|}\right)\right\| \\
& \leq J\|\widehat{\mathbf{Z}}(c)-\mathbf{z}\|\left\|\frac{\widehat{\mathbf{Z}}(c)}{\|\widehat{\mathbf{Z}}(c)\|}-\frac{\mathbf{z}}{\|\mathbf{z}\|}\right\| \leq \frac{2 J\|\widehat{\mathbf{Z}}(c)-\mathbf{z}\|^{2}}{\|\mathbf{z}\|},
\end{aligned}
$$

where the equality follows from the fact that $\mathbf{u}^{*}(\mathbf{z})=\mathbf{u}^{*}(\lambda \mathbf{z})$ for all $\lambda>0$. The second inequality follows from condition $\operatorname{SBAR}(J)$, and the final inequality follows from Lemma 3.5, The desired result follows by taking conditional expectations, given $\mathbf{Z}=\mathbf{z}$, and applying Lemma 3.4,

We can now complete the proof of Theorem [3.1, by adding the regret over the differnt times and cycles. By Assumption $⿴$ and the Cauchy-Schwarz Inequality, the instantaneous regret from playing any $\operatorname{arm} \mathbf{u} \in \mathcal{U}_{r}$ is bounded above by $\max _{\mathbf{v} \in \mathcal{U}_{r}} \mathbf{z}^{\prime}(\mathbf{v}-\mathbf{u}) \leq 2 \bar{u}\|\mathbf{z}\|$. Consider an arbitrary cycle $c$. Then, the total regret incurred during the exploration phase (with $r$ periods) in this cycle is bounded above by $2 \bar{u} r\|\mathbf{z}\|$. During the exploitation phase of cycle $c$, we always play the greedy $\operatorname{arm} \mathbf{G}(c)$. The expected instantaneous regret in each period during the exploitation phase is bounded above by $r h_{2} / c\|\mathbf{z}\|$. So, the total regret during cycle $c$ is bounded above by $2 \bar{u} r\|\mathbf{z}\|+h_{2} r /\|\mathbf{z}\|$. Summing over $K$ cycles, we obtain

$$
\operatorname{Regret}\left(\mathbf{z}, r K+\sum_{c=1}^{K} c, \text { PEGE }\right) \leq h_{3} r\|\mathbf{z}\| K+h_{4} \sum_{c=1}^{K} \frac{r}{\|\mathbf{z}\|} \text {, }
$$

for some positive constants $h_{3}$ and $h_{4}$ that depend only on $\sigma_{0}, \bar{u}, \lambda_{0}$, and $J$.
Consider an arbitrary time period $T \geq r$ and $\mathbf{z} \in \mathbb{R}^{r}$. Let $K_{0}=\lceil\sqrt{2 T}\rceil$. Note that the total time periods after $K_{0}$ cycles is at least $T$ because $r K_{0}+\sum_{c=1}^{K_{0}} c \geq \sum_{c=1}^{K_{0}} c=\frac{K_{0}\left(K_{0}+1\right)}{2} \geq \frac{K_{0}^{2}}{2} \geq T$. Since the cumulative regret is nondecreasing over time, it follows that

$$
\begin{aligned}
\operatorname{Regret}(\mathbf{z}, T, \text { PEGE }) & \leq \operatorname{Regret}\left(\mathbf{z}, r K_{0}+\sum_{c=1}^{K_{0}} c, \text { PEGE }\right) \\
& \leq h_{3} r\|\mathbf{z}\| K_{0}+h_{4} \frac{r K_{0}}{\|\mathbf{z}\|} \leq 3 \max \left\{h_{3}, h_{4}\right\}\left(\|\mathbf{z}\|+\frac{1}{\|\mathbf{z}\|}\right) r \sqrt{T},
\end{aligned}
$$

where the final inequality follows because $K_{0}=\lceil\sqrt{2 T}\rceil \leq 3 \sqrt{T}$. The risk bound follows by taking expectations and using the assumption on the boundedness of $\mathrm{E}[\|\mathbf{Z}\|]$ and $\mathrm{E}[1 /\|\mathbf{Z}\|]$.

## 4. A Policy for General Bandits

We have shown that when a bandit has a smooth best arm response, the PEGE policy achieves optimal $O(r \sqrt{T})$ regret and Bayes risk. The general idea is that when the estimation error is small, the instantaneous regret of the greedy decision based on our estimate $\widehat{\mathbf{Z}}(c)$ can be of the same order as $\|\mathbf{Z}-\widehat{\mathbf{Z}}(c)\|$. However, under the smoothness assumption, this upper bound on the instantaneous regret is improved to $O\left(\|\mathbf{Z}-\widehat{\mathbf{Z}}(c)\|^{2}\right)$, as shown in the proof of Lemma 3.6, and this enables us to separate exploration from exploitation.

However, if the number of arms is finite or if the collection of arms is an arbitrary compact set, then the PEGE policy may not be effective. This is because a small estimation error may have a disproportionately large effect on the arm chosen by a greedy policy, leading to a large instantaneous regret. In this section, we discuss a policy - which we refer to as the Uncertainty Ellipsoid (UE) policy - that can be applied to any bandit problem, at the price of slightly higher regret and Bayes risk. In contrast to the PEGE policy, the UE policy combines active exploration and exploitation in every period.

As discussed in the introduction, the UE policy is closely related to the algorithms described in Auer (2002) and Dani et al. (2008a), but also has the "anytime" property (the policy does not require prior knowledge of the time horizon $T$ ), and also allows the random vector $\mathbf{Z}$ and the errors $W_{t}^{\mathbf{u}}$ to be unbounded. For the sake of completeness, we give a detailed description of our policy and state the regret and risk bounds that we obtain. The reader can find the proofs of these bounds in Appendix B

To facilitate exposition, we introduce a constant that will appear in the description of the policy, namely,

$$
\begin{equation*}
\kappa_{0}=2 \sqrt{1+\log \left(1+\frac{36 \bar{u}^{2}}{\lambda_{0}}\right)} \tag{3}
\end{equation*}
$$

where the parameters $\bar{u}$ and $\lambda_{0}$ are given in Assumption The UE policy maintains, at each time period $t$, the following two pieces of information.

1. The ordinary least squares (OLS) estimate defined as follows: if $\mathbf{U}_{1}, \ldots, \mathbf{U}_{t}$ are the arms chosen during the first $t$ periods, then the OLS estimate $\widehat{\mathbf{Z}}_{t}$ is given byt:

$$
\begin{equation*}
\mathbf{C}_{t}=\left(\sum_{s=1}^{t} \mathbf{U}_{s} \mathbf{U}_{s}^{\prime}\right)^{-1}, \quad \mathbf{M}_{t}=\sum_{s=1}^{t} \mathbf{U}_{s} W_{s}, \quad \text { and } \quad \widehat{\mathbf{Z}}_{t}=\mathbf{C}_{t} \sum_{s=1}^{t} \mathbf{U}_{s} X_{s}=\mathbf{Z}+\mathbf{C}_{t} \mathbf{M}_{t} \tag{4}
\end{equation*}
$$

[^3]In contrast to the PEGE policy, whose estimates relied only on the rewards observed in the exploration phases, the estimate $\widehat{\mathbf{Z}}_{t}$ incorporates all available information up to time $t$. We initialize the policy by playing $r$ linearly independent arms, so that $\mathbf{C}_{t}$ is positive definite for $t \geq r$.
2. An uncertainty ellipsoid $\mathcal{E}_{t} \subseteq \mathbb{R}^{r}$ associated with the estimate $\widehat{\mathbf{Z}}_{t}$, defined by,

$$
\begin{equation*}
\mathcal{E}_{t}=\left\{\mathbf{w} \in \mathbb{R}^{r}: \mathbf{w}^{\prime} \mathbf{C}_{t}^{-1} \mathbf{w} \leq\left(\alpha \sqrt{\log t} \sqrt{\min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}\right)^{2}\right\} \quad \text { and } \quad \alpha=4 \sigma_{0} \kappa_{0}^{2} \tag{5}
\end{equation*}
$$

where the parameters $\sigma_{0}$ and $\kappa_{0}$ are given in Assumption (1) and Equation (3). The uncertainty ellipsoid $\mathcal{E}_{t}$ represents the set of likely "errors" associated with the estimate $\widehat{\mathbf{Z}}_{t}$. We define the uncertainty radius $R_{t}^{\mathbf{u}}$ associated with each arm $\mathbf{u}$ as follows:

$$
\begin{equation*}
R_{t}^{\mathbf{u}}=\max _{\mathbf{v} \in \mathcal{E}_{t}} \mathbf{v}^{\prime} \mathbf{u}=\alpha \sqrt{\log t} \sqrt{\min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}\|\mathbf{u}\|_{\mathbf{C}_{t}} . \tag{6}
\end{equation*}
$$

A formal description of the policy is given below.

## Uncertainty Ellipsoid (UE)

Initialization: During the first $r$ periods, play the $r$ linearly independent arms $\mathbf{b}_{1}, \mathbf{b}_{2}, \ldots, \mathbf{b}_{r}$ given in Assumption (b). Determine the OLS estimate $\widehat{\mathbf{Z}}_{r}$, the uncertainty ellipsoid $\mathcal{E}_{r}$, and the uncertainty radius associated with each arm.
Description: For $t \geq r+1$, do the following:
(i) Let $\mathbf{U}_{t} \in \mathcal{U}_{r}$ be an arm that gives the maximum estimated reward over the ellipsoid $\widehat{\mathbf{Z}}_{t-1}+\mathcal{E}_{t-1}$, that is,

$$
\begin{equation*}
\mathbf{U}_{t}=\arg \max _{\mathbf{v} \in \mathcal{U}_{r}}\left\{\mathbf{v}^{\prime} \widehat{\mathbf{Z}}_{t-1}+\max _{\mathbf{w} \in \mathcal{E}_{t-1}} \mathbf{w}^{\prime} \mathbf{v}\right\}=\arg \max _{\mathbf{v} \in \mathcal{U}_{r}}\left\{\mathbf{v}^{\prime} \widehat{\mathbf{Z}}_{t-1}+R_{t-1}^{\mathbf{v}}\right\} \tag{7}
\end{equation*}
$$

where the uncertainty radius $R_{t-1}^{\mathrm{v}}$ is defined in Equation (6); ties are broken arbitrarily.
(ii) Play arm $\mathbf{U}_{t}$ and observe the resulting reward $X_{t}$.
(iii) Update the OLS estimate $\widehat{\mathbf{Z}}_{t}$, the uncertainty ellipsoid $\mathcal{E}_{t}$, and the uncertainty radius $R_{t}^{\mathbf{u}}$ of each arm $\mathbf{u}$, using the formulas in Equations (4), (5), and (6).

By choosing an arm that maximizes the estimated reward over the ellipsoid $\widehat{\mathbf{Z}}_{t}+\mathcal{E}_{t}$, our policy involves simultaneous exploitation (via the term $\mathbf{v}^{\prime} \widehat{\mathbf{Z}}_{t}$ ) and exploration (via the term $R_{t}^{\mathbf{v}}=$ $\max _{\mathbf{w} \in \mathcal{E}_{t}} \mathbf{w}^{\prime} \mathbf{v}$ ) in every period. The ellipsoid $\mathcal{E}_{t}$ reflects the uncertainty in our OLS estimate $\widehat{\mathbf{Z}}_{t}$. It generalizes the classical upper confidence index introduced by Lai and Robbins (1985), to account for correlations among the arm rewards. In the special case of $r$ independent arms where $\mathcal{U}_{r}=\left\{\mathbf{e}_{1}, \ldots, \mathbf{e}_{r}\right\}$, it is easy to verify that for each $\operatorname{arm} \mathbf{e}_{\ell}$, the expression $\mathbf{e}_{\ell}^{\prime} \widehat{\mathbf{Z}}_{t}+R_{t}^{\mathbf{e}_{\ell}}$ coincides (up
to a scaling constant) with the upper confidence bound used by Auer et al. (2002). Our definition of the uncertainty radius involves an extra factor of $\sqrt{\min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}$, in order to handle the case where the arms are not standard unit vectors, and the rewards are correlated.

The main results of this section are given in the following two theorems. The first theorem establishes upper bounds on the regret and risk when the set of arms is an arbitrary compact set. This result shows that the UE policy is nearly optimal, admitting upper bounds that are within a logarithmic factor of the $\Omega(r \sqrt{T})$ lower bounds given in Theorem 2.1. Although the proof of this theorem makes use of somewhat different (and novel) large deviation inequalities for adaptive least squares estimators, the argument shares similarities with the proofs given in Dani et al. (2008a), and we omit the details. The reader can find a complete proof in Appendix B. 2 ,

Theorem 4.1 (Bounds for General Compact Sets of Arms). Under Assumption [1, there exist positive constants $a_{4}$ and $a_{5}$ that depend only on the parameters $\sigma_{0}, \bar{u}$, and $\lambda_{0}$, such that for all $T \geq r+1$ and $\mathbf{z} \in \mathbb{R}^{r}$,

$$
\operatorname{Regret}(\mathbf{z}, T, \mathrm{UE}) \leq a_{4} r\|\mathbf{z}\|+a_{5} r \sqrt{T} \log ^{3 / 2} T,
$$

and

$$
\operatorname{Risk}(T, \mathrm{UE}) \leq a_{4} r \mathrm{E}[\|\mathbf{Z}\|]+a_{5} r \sqrt{T} \log ^{3 / 2} T .
$$

For any $\operatorname{arm} \mathbf{u} \in \mathcal{U}_{r}$ and $\mathbf{z} \in \mathbb{R}^{r}$, let $\Delta^{\mathbf{u}}(\mathbf{z})$ denote the difference between the maximum expected reward and the expected reward of $\operatorname{arm} \mathbf{u}$ when $\mathbf{Z}=\mathbf{z}$, that is,

$$
\Delta^{\mathbf{u}}(\mathbf{z})=\max _{\mathbf{v} \in \mathcal{U}_{r}} \mathbf{v}^{\prime} \mathbf{z}-\mathbf{u}^{\prime} \mathbf{z}
$$

When the number of arms is finite, it turns out that we can obtain bounds on regret and risk that scale more gracefully over time, growing as $\log T$ and $\log ^{2} T$, respectively. This result is stated in Theorem 4.2, which shows that, for a fixed set of arms, the UE policy is asymptotically optimal as a function time, within a constant factor of the lower bounds established by Lai and Robbins (1985) and Lai (1987).

Theorem 4.2 (Bounds for Finitely Many Arms). Under Assumption [1, there exist positive constants $a_{6}$ and $a_{7}$ that depend only on the parameters $\sigma_{0}, \bar{u}$, and $\lambda_{0}$ such that for all $T \geq r+1$ and $\mathrm{z} \in \mathbb{R}^{r}$,

$$
\operatorname{Regret}(\mathbf{z}, T, \mathrm{UE}) \leq a_{6}\left|\mathcal{U}_{r}\right|\|\mathbf{z}\|+a_{7}\left|\mathcal{U}_{r}\right| \sum_{\mathbf{u} \in \mathcal{U}_{r}} \min \left\{\frac{\log T}{\Delta^{\mathbf{u}}(\mathbf{z})}, T \Delta^{\mathbf{u}}(\mathbf{z})\right\}
$$

Moreover, suppose that there exists a positive constant $M_{0}$ such that, for all arms $\mathbf{u}$, the distribution of the random variable $\Delta^{\mathbf{u}}(\mathbf{Z})$ is described by a point mass at 0 , and a density function that is
bounded above by $M_{0}$ on $\mathbb{R}_{+}$. Then, there exist positive constants $a_{8}$ and $a_{9}$ that depend only on the parameters $\sigma_{0}, \bar{u}, \lambda_{0}$, and $M_{0}$, such that for all $T \geq r+1$,

$$
\operatorname{Risk}(T, \mathrm{UE}) \leq a_{8}\left|\mathcal{U}_{r}\right| \mathrm{E}[\|\mathbf{Z}\|]+a_{9}\left|\mathcal{U}_{r}\right|^{2} \log ^{2} T
$$

Proof. For any arm $\mathbf{u} \in \mathcal{U}_{r}$ and $\mathbf{z} \in \mathbb{R}^{r}$, let the random variable $N^{\mathbf{u}}(\mathbf{z}, T)$ denote the total number of times that the arm $\mathbf{u}$ is chosen during periods 1 through $T$, given that $\mathbf{Z}=\mathbf{z}$. Using an argument similar to the one in Auer et al. (2002), we can show that

$$
\mathrm{E}\left[N^{\mathbf{u}}(\mathbf{z}, T) \mid \mathbf{Z}=\mathbf{z}\right] \leq 6+\frac{4 \alpha^{2}\left|\mathcal{U}_{r}\right| \log T}{\left(\Delta^{\mathbf{u}}(\mathbf{z})\right)^{2}}
$$

The reader can find a proof of this result in Appendix B.3.
The regret bound in Theorem 4.2 then follows immediately from the above upper bound and the fact that $N^{\mathbf{u}}(\mathbf{z}, T) \leq T$ with probability one, because

$$
\begin{aligned}
\operatorname{Regret}(\mathbf{z}, T, \mathrm{UE}) & =\sum_{\mathbf{u} \in \mathcal{U}_{r}} \Delta^{\mathbf{u}}(\mathbf{z}) \mathrm{E}\left[N^{\mathbf{u}}(\mathbf{z}, T) \mid \mathbf{Z}=\mathbf{z}\right] \leq \sum_{\mathbf{u} \in \mathcal{U}_{r}} \Delta^{\mathbf{u}}(\mathbf{z}) \min \left\{6+\frac{4 \alpha^{2}\left|\mathcal{U}_{r}\right| \log T}{\left(\Delta^{\mathbf{u}}(\mathbf{z})\right)^{2}}, T\right\} \\
& \leq 6 \sum_{\mathbf{u} \in \mathcal{U}_{r}} \Delta^{\mathbf{u}}(\mathbf{z})+\max \left\{4 \alpha^{2}, 1\right\}\left|\mathcal{U}_{r}\right| \sum_{\mathbf{u} \in \mathcal{U}_{r}} \min \left\{\frac{\log T}{\Delta^{\mathbf{u}}(\mathbf{z})}, T \Delta^{\mathbf{u}}(\mathbf{z})\right\},
\end{aligned}
$$

and the desired result follows from the fact that $\Delta^{\mathbf{u}}(\mathbf{z})=\max _{\mathbf{v} \in \mathcal{U}_{r}}(\mathbf{v}-\mathbf{u})^{\prime} \mathbf{z} \leq 2 \bar{u}\|\mathbf{z}\|$, by the Cauchy-Schwarz Inequality.

We will now establish an upper bound on the Bayes risk. From the regret bound, it suffices to show that for any $\mathbf{u} \in \mathcal{U}_{r}$,

$$
\mathrm{E}\left[\min \left\{\frac{\log T}{\Delta^{\mathbf{u}}(\mathbf{Z})}, T \Delta^{\mathbf{u}}(\mathbf{Z})\right\}\right] \leq\left(M_{0}+1\right) \log T+M_{0} \log ^{2} T
$$

Let $q^{\mathbf{u}}(\cdot)$ denote the density function associated with the random variable $\Delta^{\mathbf{u}}(\mathbf{Z})$. Then,

$$
\begin{aligned}
\mathrm{E}\left[\min \left\{\frac{\log T}{\Delta^{\mathbf{u}}(\mathbf{Z})}, T \Delta^{\mathbf{u}}(\mathbf{Z})\right\}\right]= & \int_{0}^{\sqrt{\frac{\log T}{T}}} \min \left\{\frac{\log T}{x}, T x\right\} q^{\mathbf{u}}(x) d x \\
& +\int_{\sqrt{\frac{\log T}{T}}}^{1} \min \left\{\frac{\log T}{x}, T x\right\} q^{\mathbf{u}}(x) d x+\int_{1}^{\infty} \min \left\{\frac{\log T}{x}, T x\right\} q^{\mathbf{u}}(x) d x
\end{aligned}
$$

We will now proceed to bound each of the three terms on the right hand side of the above equality. Having assumed that $q^{\mathbf{u}}(\cdot) \leq M_{0}$, the first term satisfies

$$
\int_{0}^{\sqrt{(\log T) / T}} \min \left\{\frac{\log T}{x}, T x\right\} q^{\mathbf{u}}(x) d x \leq M_{0} \int_{0}^{\sqrt{(\log T) / T}} T x d x=\left.M_{0} T \frac{x^{2}}{2}\right|_{0} ^{\sqrt{(\log T) / T}} \leq M_{0} \log T
$$

For the second term, note that

$$
\begin{aligned}
\int_{\sqrt{(\log T) / T}}^{1} \min \left\{\frac{\log T}{x}, T x\right\} q^{\mathbf{u}}(x) d x & \leq M_{0} \int_{\sqrt{(\log T) / T}}^{1} \frac{\log T}{x} d x=M_{0} \log T \cdot\left(\left.\log x\right|_{\sqrt{(\log T) / T}} ^{1}\right) \\
& =M_{0}(\log T) \cdot \frac{\log T-\log \log T}{2} \leq M_{0} \log ^{2} T,
\end{aligned}
$$

where the last inequality follows from the fact that $\log T-\log \log T \leq 2 \log T$ for all $T \geq 2$. To evaluate the last term, note that $\frac{\log T}{x} \leq \log T$ for all $x \geq 1$, and thus, $\int_{1}^{\infty} \min \left\{\frac{\log T}{x}, T x\right\} q^{\mathbf{u}}(x) d x \leq$ $\log T \int_{1}^{\infty} q^{\mathbf{u}}(x) \leq \log T$. Putting everything together, we have that $\mathrm{E}\left[\min \left\{\frac{\log T}{\Delta^{\mathbf{u}}(\mathbf{Z})}, T \Delta^{\mathbf{u}}(\mathbf{Z})\right\}\right] \leq$ $\left(M_{0}+1\right) \log T+M_{0} \log ^{2} T$, which is the desired result.

We conclude this section by giving an example of a random vector $\mathbf{Z}$ that satisfies the condition in Theorem 4.2, A similar example also appears in Example 2 of Lai (1987).

Example 4.3 (IID Random Variables). Suppose $\mathcal{U}_{r}=\left\{\mathbf{e}_{1}, \ldots, \mathbf{e}_{r}\right\}$ and $\mathbf{Z}=\left(Z_{1}, \ldots, Z_{r}\right)$, where the random variables $Z_{k}$ are independent and identically distributed with a common cumulative distribution function $F$ and a density function $f: \mathbb{R} \rightarrow \mathbb{R}$ which is bounded above by $M$. Then, for each $k$, the random variable $\Delta^{\mathbf{e}_{k}}(\mathbf{Z})$ is given by $\Delta^{\mathbf{e}_{k}}(\mathbf{Z})=\left(\max _{j=1, \ldots, r} Z_{j}\right)-Z_{k}=$ $\max \left\{0, \max _{j \neq k}\left\{Z_{j}-Z_{k}\right\}\right\}$. It is easy to verify that $\Delta^{\mathbf{e}_{k}}(\mathbf{Z})$ has a point mass at 0 and a continuous density function $q_{k}(\cdot)$ on $\mathbb{R}_{+}$given by: for any $x>0$,

$$
q_{k}(x)=(r-1) \int\left\{F\left(z_{k}+x\right)\right\}^{r-2} f\left(z_{k}+x\right) f\left(z_{k}\right) d z_{k} \leq(r-1) M
$$

### 4.1 Regret Bounds for Polyhedral Sets of Arms

In this section, we focus on the regret profiles when the set of arms $\mathcal{U}_{r}$ is a polyhedral set. Let $\mathcal{E}\left(\mathcal{U}_{r}\right)$ denote the set of extreme points of $\mathcal{U}_{r}$. From a standard result in linear programming, for all $\mathbf{z} \in \mathbb{R}^{r}$,

$$
\max _{\mathbf{u} \in \mathcal{U}_{r}} \mathbf{u}^{\prime} \mathbf{z}=\max _{\mathbf{u} \in \mathcal{E}\left(\mathcal{U}_{r}\right)} \mathbf{u}^{\prime} \mathbf{z}
$$

Since a polyhedral set has a finite number of extreme points $\left(\left|\mathcal{E}\left(\mathcal{U}_{r}\right)\right|<\infty\right)$, the parameterized bandit problem can be reduced to the standard multi-armed bandit problem, where each arm corresponds to an extreme point of $\mathcal{U}_{r}$. We can thus apply the algorithm of Lai and Robbins (1985) and obtain the following upper bound on the $T$-period cumulative regret for polyhedra

$$
\begin{equation*}
\operatorname{Regret}(\mathbf{z}, T, \text { Lai's Algorithm })=O\left(\frac{\left|\mathcal{E}\left(\mathcal{U}_{r}\right)\right| \cdot \log T}{\min \left\{\Delta^{\mathbf{u}}(\mathbf{z}): \Delta^{\mathbf{u}}(\mathbf{z})>0\right\}}\right) \tag{8}
\end{equation*}
$$

where the denominator corresponds to the difference between the expected reward of the optimal and the second best extreme points. The algorithm of Lai and Robbins (1985) is effective only when the polyhedral set $\mathcal{U}_{r}$ has a small number of extreme points, as shown by the following examples.

Example 4.4 (Simplex). Suppose $\mathcal{U}_{r}=\left\{\mathbf{u} \in \mathbb{R}^{r}: \sum_{i=1}^{r}\left|u_{i}\right| \leq 1\right\}$ is an $r$-dimensional unit simplex. Then, $\mathcal{U}_{r}$ has $2 r$ extreme points, and Equation (8) gives an $O(r \log T)$ upper bound on the regret.

Example 4.5 (Linear Constraints). Suppose that $\mathcal{U}_{r}=\left\{\mathbf{u} \in \mathbb{R}^{r}: \mathbf{A u} \leq \mathbf{b}\right.$ and $\left.\mathbf{u} \geq \mathbf{0}\right\}$, where $\mathbf{A}$ is a $p \times r$ matrix with $p \leq r$. It follows from the standard linear programming theory that every extreme point is a basic feasible solution, which has at most $p$ nonzero coordinates (see, for example, Bertsimas and Tsitsiklis, 1997). Thus, the number of extreme points is bounded above by $\binom{r+p}{p}=O\left((2 r)^{p}\right)$, and Equation (8) gives an $O\left((2 r)^{p} \log T\right)$ upper bound on the regret.

In general, the number of extreme points of a polyhedron can be very large, rendering the bandit algorithm of Lai and Robbins (1985) ineffective; consider, for example, the $r$-dimensional cube $\mathcal{U}_{r}=\left\{\mathbf{u} \in \mathbb{R}^{r}:\left|u_{i}\right| \leq 1\right.$ for all $\left.i\right\}$, which has $2^{r}$ extreme points. Moreover, we cannot apply the results and algorithms from Section 3 to the convex hull of $\mathcal{U}_{r}$. This is because the convex hull of a polyhedron is not strongly convex (it cannot be written as an intersection of Euclidean balls), and thus, it does not satisfy the required $\operatorname{SBAR}(\cdot)$ condition in Theorem 3.1. The UE policy in the previous section gives $O\left(r \sqrt{T} \log ^{3 / 2} T\right)$ regret and risk upper bounds. However, finding an algorithm specifically for polyhedral sets that yields an $O(r \sqrt{T})$ regret upper bound (without an additional logarithmic factor) remains an open question.

## 5. Conclusion

We analyzed a class of multiarmed bandit problems where the expected reward of each arm depends linearly on an unobserved random vector $\mathbf{Z} \in \mathbb{R}^{r}$, with $r \geq 2$. Our model allows for correlations among the rewards of different arms. When we have a smooth best arm response, we showed that a policy that alternates between exploration and exploitation is optimal. For a general bandit, we proposed a near-optimal policy that performs active exploration in every period. For finitely many arms, our policy achieves asymptotically optimal regret and risk as a function of time, but scales with the square of the number of arms. Improving the dependence on the number of arms remains an open question. It would also be interesting to study more general correlation structures. Our formulation assumes that the vector of expected rewards lies in an $r$-dimensional subspace spanned by a known set of basis functions that describe the characteristics of the arms. Extending our work to a setting where the basis functions are unknown has the potential to broaden the applicability of our model.

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## A. Properties of Normal Vectors

In this section, we prove that if $\mathbf{Z}$ has a multivariate normal distribution with mean $\mathbf{0} \in \mathbb{R}^{r}$ and covariance matrix $\mathbf{I}_{r} / r$, then $\mathbf{Z}$ has the properties described in Lemmas 2.4 and 3.2.

## A. 1 Proof of Lemma 2.4

We want to establish a lower bound on $\operatorname{Pr}\{\theta \leq\|\mathbf{Z}\| \leq \beta\}$. Let $\mathbf{Y}=\left(Y_{1}, \ldots, Y_{r}\right)$ denote the standard multivariate normal random vector with mean $\mathbf{0}$ and identity covariance matrix $\mathbf{I}_{r}$. By our hypothesis, $\mathbf{Z}$ has the same distribution as $\mathbf{Y} / \sqrt{r}$, which implies that

$$
\operatorname{Pr}\{\theta \leq\|\mathbf{Z}\| \leq \beta\}=\operatorname{Pr}\{\theta \sqrt{r} \leq\|\mathbf{Y}\| \leq \beta \sqrt{r}\}=1-\operatorname{Pr}\left\{\|\mathbf{Y}\|^{2}<\theta^{2} r\right\}-\operatorname{Pr}\left\{\|\mathbf{Y}\|^{2}>\beta^{2} r\right\}
$$

By definition, $\|\mathbf{Y}\|^{2}=Y_{1}^{2}+\cdots+Y_{r}^{2}$ has a chi-square distribution with $r$ degrees of freedom. By the Markov Inequality, $\operatorname{Pr}\left\{\|\mathbf{Y}\|^{2}>\beta^{2} r\right\} \leq \mathrm{E}\left[\|\mathbf{Y}\|^{2}\right] /\left(\beta^{2} r\right)=1 / \beta^{2}$. We will now establish an upper bound on $\operatorname{Pr}\left\{\|\mathbf{Y}\|^{2}<\theta^{2} r\right\}$. Note that, for any $\lambda>0$,

$$
\operatorname{Pr}\left\{\|\mathbf{Y}\|^{2}<\theta^{2} r\right\}=\operatorname{Pr}\left\{e^{-\lambda \sum_{k=1}^{r} Y_{k}^{2}}>e^{-\lambda \theta^{2} r}\right\} \leq e^{\lambda \theta^{2} r} \cdot \mathbf{E}\left[\prod_{k=1}^{r} e^{-\lambda Y_{k}^{2}}\right]=\left(\frac{e^{\lambda \theta^{2}}}{\sqrt{1+2 \lambda}}\right)^{r},
$$

where last equality follows from the fact that $Y_{1}, \ldots, Y_{r}$ are independent standard normal random variables and thus, $\mathrm{E}\left[e^{-\lambda Y_{k}^{2}}\right]=1 / \sqrt{1+2 \lambda}$ for $\lambda>0$. Set $\lambda=1 / \theta^{2}$, and use the facts $\theta \leq 1 / 2 \leq$ $\sqrt{2} / e$ and $r \geq 2$, to obtain

$$
\operatorname{Pr}\left\{\|\mathbf{Y}\|^{2}<\theta^{2} r\right\} \leq\left(\frac{e \theta}{\sqrt{2+\theta^{2}}}\right)^{r} \leq\left(\frac{e \theta}{\sqrt{2}}\right)^{r} \leq\left(\frac{e \theta}{\sqrt{2}}\right)^{2}=\frac{e^{2} \theta^{2}}{2} \leq 4 \theta^{2}
$$

which implies that $\operatorname{Pr}\{\theta \leq\|\mathbf{Z}\| \leq \beta\} \geq 1-\frac{1}{\beta^{2}}-4 \theta^{2}$, which is the desired result.

## A. 2 Proof of Lemma 3.2

For part (a) of the lemma, we have

$$
\mathrm{E}[1 /\|\mathbf{Z}\|]=\int_{0}^{\infty} \frac{1}{x} g(x) d x \leq M_{0} \int_{0}^{\rho} x^{\rho-1} d x+\frac{1}{\rho} \int_{\rho}^{\infty} g(x) d x \leq M_{0} \frac{\rho^{\rho}}{\rho}+\frac{1}{\rho} .
$$

For the proof of part (b), let $\mathbf{Y}=\left(Y_{1}, \ldots, Y_{r}\right)$ be a standard multivariate normal random vector with mean $\mathbf{0}$ and identity covariance matrix, $\mathbf{I}_{r}$. Then, $\mathbf{Z}$ has the same distribution as $\mathbf{Y} / \sqrt{r}$. Note that $\|\mathbf{Y}\|^{2}$ has a chi-square distribution with $r$ degrees of freedom. Thus,

$$
\mathrm{E}[\|\mathbf{Z}\|]=\frac{1}{\sqrt{r}} \mathrm{E}[\|\mathbf{Y}\|] \leq \frac{1}{\sqrt{r}} \sqrt{\mathrm{E}\left[\|\mathbf{Y}\|^{2}\right]}=\frac{1}{\sqrt{r}} \sqrt{r}=1 .
$$

We will now establish an upper bound on $\mathrm{E}[1 /\|\mathbf{Z}\|]=\sqrt{r} \mathrm{E}[1 /\|\mathbf{Y}\|]$. For $r=2$, since $\|\mathbf{Y}\|$ has a chi distribution with two degrees of freedom, we have that

$$
\mathrm{E}[1 /\|\mathbf{Z}\|]=\sqrt{2} \int_{0}^{\infty} \frac{1}{x} \cdot x e^{-x^{2} / 2} d x=\sqrt{2} \int_{0}^{\infty} e^{-x^{2} / 2} d x=\sqrt{\pi} .
$$

Consider the case where $r \geq 3$. Then,

$$
\mathrm{E}[1 /\|\mathbf{Z}\|]=\sqrt{r} \mathrm{E}[1 /\|\mathbf{Y}\|] \leq \sqrt{r} \sqrt{\mathrm{E}\left[1 /\|\mathbf{Y}\|^{2}\right]} .
$$

Using the formula for the density of the chi-square distribution, we have

$$
\begin{aligned}
\mathrm{E}\left[1 /\|\mathbf{Y}\|^{2}\right] & =\int_{0}^{\infty} \frac{1}{x} \cdot \frac{1}{2^{r / 2} \Gamma(r / 2)} x^{(r / 2)-1} e^{-x / 2} d x \\
& =\frac{2^{(r / 2)-1}}{2^{r / 2}} \cdot \frac{\Gamma((r / 2)-1)}{\Gamma(r / 2)} \cdot \int_{0}^{\infty} \frac{1}{2^{(r-2) / 2} \Gamma((r-2) / 2)} x^{((r-2) / 2)-1} e^{-x / 2} d x \\
& =\frac{1}{2((r / 2)-1)}=\frac{1}{r-2} \leq \frac{3}{r},
\end{aligned}
$$

where the third equality follows from the fact that $\Gamma(r / 2)=((r / 2)-1) \cdot \Gamma((r / 2)-1)$ for $r \geq 3$ and the integrand is the density function of the chi-square distribution with $r-2$ degrees of freedom and evaluates to 1 . The last inequality follows because $r \geq 3$. Thus, we have $\mathrm{E}[1 /\|\mathbf{Z}\|] \leq \sqrt{3} \leq \sqrt{\pi}$, which is the desired result.

## B. Proof of Theorems 4.1 and 4.2

In the next section, we establish large deviation inequalities for adaptive least squares estimators (with unbounded error random variables), which will be used in the proof of Theorems 4.1 and 4.2, given in Sections B. 2 and B.3, respectively.

## B. 1 Large Deviation Inequalities

The first result extend the standard Chernoff Inequality to our setting involving uncertainty ellipsoids when we have finitely many arms.

Theorem B. 1 (Chernoff Inequality for Uncertainty Ellipsoids with Finitely Many Arms). Under Assumption 11, for any $t \geq r, \mathbf{x} \in \mathbb{R}^{r}, \mathbf{z} \in \mathbb{R}^{r}$, and $\zeta>0$,

$$
\operatorname{Pr}\left\{\mathbf{x}^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>\zeta \sigma_{0}\|\mathbf{x}\|_{\mathbf{C}_{t}} \mid \mathbf{Z}=\mathbf{z}\right\} \leq t^{5\left|\mathcal{U}_{r}\right|} e^{-\zeta^{2} / 2}
$$

and

$$
\operatorname{Pr}\left\{\left(\mathbf{U}_{t+1}-\mathbf{x}\right)^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>\zeta \sigma_{0}\left\|\mathbf{U}_{t+1}-\mathbf{x}\right\|_{\mathbf{C}_{t}} \mid \mathbf{Z}=\mathbf{z}\right\} \leq t^{5\left|\mathcal{U}_{r}\right|} e^{-\zeta^{2} / 2}
$$

Proof. We will only prove the second inequality because the proof of the first one follows the same argument. If the sequence of arms $\mathbf{U}_{1}, \mathbf{U}_{2}, \ldots$ is deterministic (and thus, the matrix $\mathbf{C}_{t}$ is also deterministic), then

$$
\frac{\left(\mathbf{U}_{t+1}-\mathbf{x}\right)^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)}{\left\|\mathbf{U}_{t+1}-\mathbf{x}\right\|_{\mathbf{C}_{t}}}=\sum_{s=1}^{t} \frac{\left(\mathbf{U}_{t+1}-\mathbf{x}\right)^{\prime} \mathbf{C}_{t} \mathbf{U}_{s}}{\left\|\mathbf{U}_{t+1}-\mathbf{x}\right\|_{\mathbf{C}_{t}}} W_{s}
$$

and

$$
\sum_{s=1}^{t}\left(\frac{\left(\mathbf{U}_{t+1}-\mathbf{x}\right)^{\prime} \mathbf{C}_{t} \mathbf{U}_{s}}{\left\|\mathbf{U}_{t+1}-\mathbf{x}\right\|_{\mathbf{C}_{t}}}\right)^{2}=\frac{\left(\mathbf{U}_{t+1}-\mathbf{x}\right)^{\prime} \mathbf{C}_{t}\left(\sum_{s=1}^{t} \mathbf{U}_{s} \mathbf{U}_{s}^{\prime}\right) \mathbf{C}_{t}\left(\mathbf{U}_{t+1}-\mathbf{x}\right)}{\left(\mathbf{U}_{t+1}-\mathbf{x}\right)^{\prime} \mathbf{C}_{t}\left(\mathbf{U}_{t+1}-\mathbf{x}\right)}=1
$$

The classical Chernoff Inequality for the sum of independent random variables (see, for example, Chapter 1 in Dudley, 1999) then yields

$$
\begin{aligned}
& \operatorname{Pr}\left\{\left(\mathbf{U}_{t+1}-\mathbf{x}\right)^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>\zeta \sigma_{0}\left\|\mathbf{U}_{t+1}-\mathbf{x}\right\|_{\mathbf{C}_{t}} \mid \mathbf{Z}=\mathbf{z}\right\} \\
& \quad \leq \exp \left\{-\frac{\zeta^{2} \sigma_{0}^{2}}{2 \sigma_{0}^{2} \sum_{s=1}^{t}\left(\left(\mathbf{U}_{t+1}-\mathbf{x}\right)^{\prime} \mathbf{C}_{t} \mathbf{U}_{s} /\left\|\mathbf{U}_{t+1}-\mathbf{x}\right\|_{\mathbf{C}_{t}}\right)^{2}}\right\}=e^{-\zeta^{2} / 2}
\end{aligned}
$$

In our setting, however, the arms $\mathbf{U}_{t}$ are random variables that depend on the accumulated history, and we cannot apply the standard Chernoff inequality directly. If $N^{\mathbf{u}}(\mathbf{z}, t)$ denotes the total number of times that arm $\mathbf{u}$ has been chosen during the first $t$ periods given that $\mathbf{Z}=\mathbf{z}$, then

$$
\mathbf{C}_{t}=\left(\sum_{s=1}^{t} \mathbf{U}_{s} \mathbf{U}_{s}^{\prime}\right)^{-1}=\left(\sum_{\mathbf{u} \in \mathcal{U}_{r}} N^{\mathbf{u}}(\mathbf{z}, t) \mathbf{u} \mathbf{u}^{\prime}\right)^{-1}
$$

which shows that the matrix $\mathbf{C}_{t}$ is completely determined by the nonnegative integer random variables $N^{\mathbf{u}}(\mathbf{z}, t)$. Since $0 \leq N^{\mathbf{u}}(\mathbf{z}, t) \leq t$, the number of possible values of the vector $\left(N^{\mathbf{u}}(\mathbf{z}, t): \mathbf{u} \in \mathcal{U}_{r}\right)$ is at most $t^{\left|\mathcal{U}_{r}\right|}$. It then follows easily that the number of different values of the ordered pair $\left(\mathbf{U}_{t+1}, \mathbf{C}_{t}\right)$ is at most $\left|\mathcal{U}_{r}\right| t^{\left|\mathcal{U}_{r}\right|} \leq t^{5\left|\mathcal{U}_{r}\right|}$. To get the desired result, we can then use the union bound and apply the classical Chernoff Inequality to each ordered pair.

When the number of arms is infinite, the bounds in Theorem B. 1 are vacuous. The following theorem provides an extension of the Chernoff inequality to the case of infinitely many arms.

Theorem B. 2 (Chernoff Inequality for Uncertainty Ellipsoids with Infinitely Many Arms). Under Assumption $\mathbf{1}$, for any $t \geq r, \mathbf{x} \in \mathbb{R}^{r}, \mathbf{z} \in \mathbb{R}^{r}$, and $\zeta \geq 2$,

$$
\operatorname{Pr}\left\{\mathbf{x}^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>\zeta \kappa_{0} \sigma_{0} \sqrt{\log t}\|\mathbf{x}\|_{\mathbf{C}_{t}} \mid \mathbf{Z}=\mathbf{z}\right\} \leq t^{r \kappa_{0}^{2}} e^{-\zeta^{2} / 4}
$$

and

$$
\operatorname{Pr}\left\{\left(\mathbf{U}_{t+1}-\mathbf{x}\right)^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>\zeta \kappa_{0} \sigma_{0} \sqrt{\log t}\left\|\mathbf{U}_{t+1}-\mathbf{x}\right\|_{\mathbf{C}_{t}} \mid \mathbf{Z}=\mathbf{z}\right\} \leq t^{r} \kappa_{0}^{2} e^{-\zeta^{2} / 4}
$$

The proof of Theorem B. 2 makes use of the following series of lemmas. The first lemma establishes a tail inequality for a ratio of two random variables. De la Peña et al. (2004) gave a proof of this result in Corollary 2.2 (page 1908) of their paper .

Lemma B. 3 (Exponential Inequality for Ratios, De la Peña et al., 2004). Let $A$ and $B$ be two random variables such that $B \geq 0$ with probability one and $\mathrm{E}\left[e^{\gamma A-\left(\gamma^{2} B^{2} / 2\right)}\right] \leq 1$ for all $\gamma \in \mathbb{R}$. Then, for all $\zeta \geq \sqrt{2}$ and $y>0$,

$$
\operatorname{Pr}\left\{|A| \geq \zeta \sqrt{\left(B^{2}+y\right)\left(1+\frac{1}{2} \log \left(1+\frac{B^{2}}{y}\right)\right)}\right\} \leq e^{-\zeta^{2} / 2}
$$

Recall from Equation (4) that $\mathbf{M}_{t}=\sum_{s=1}^{t} \mathbf{U}_{s} W_{s}$ is the martingale associated with the least squares estimate $\widehat{\mathbf{Z}}_{t}$. The next lemma establishes a martingale inequality associated with the inner product $\mathbf{x}^{\prime} \mathbf{M}_{t}$ for an arbitrary vector $\mathbf{x} \in \mathbb{R}^{r}$. This result is based on Lemma B. 3 with $A=\frac{\mathbf{x}^{\prime} \mathbf{M}_{t}}{\sigma_{0}}=\sum_{s=1}^{t} \frac{\left(\mathbf{x}^{\prime} \mathbf{U}_{s}\right)}{\sigma_{0}} W_{s}$ and $B=\|\mathbf{x}\|_{\mathbf{C}_{t}^{-1}}=\sqrt{\mathbf{x}^{\prime} \mathbf{C}_{t}^{-1} \mathbf{x}}=\sqrt{\sum_{s=1}^{t}\left(\mathbf{x}^{\prime} \mathbf{U}_{s}\right)^{2}}$. We then use upper and lower bounds on $B^{2}$ to establish bounds on the term $\log \left(1+\frac{B^{2}}{y}\right)$, for a suitable choice of $y$, giving us the desired result.

Lemma B. 4 (Martingale Inequality). Under Assumption for fony $\mathrm{x} \in \mathbb{R}^{r}, t \geq 1$, and $\zeta \geq \sqrt{2}$,

$$
\operatorname{Pr}\left\{\left|\mathbf{x}^{\prime} \mathbf{M}_{t}\right|>\zeta \kappa_{0} \sigma_{0} \sqrt{\log t}\|\mathbf{x}\|_{\mathbf{C}_{t}^{-1}}\right\}=\operatorname{Pr}\left\{\mathbf{x}^{\prime} \mathbf{M}_{t} \mathbf{M}_{t}^{\prime} \mathbf{x}>\zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2}(\log t)\left(\mathbf{x}^{\prime} \mathbf{C}_{t}^{-1} \mathbf{x}\right)\right\} \leq e^{-\zeta^{2} / 2} .
$$

Proof. Let $\mathbf{x} \in \mathbb{R}^{r}$ and $t \geq 1$ be given. Without loss of generality, we can assume that $\|\mathbf{x}\|=1$. Let the random variables $A$ and $B$ be defined by

$$
A=\frac{\mathbf{x}^{\prime} \mathbf{M}_{t}}{\sigma_{0}}=\sum_{s=1}^{t} \frac{\left(\mathbf{x}^{\prime} \mathbf{U}_{s}\right)}{\sigma_{0}} W_{s} \quad \text { and } \quad B=\|\mathbf{x}\|_{\mathbf{C}_{t}^{-1}}=\sqrt{\mathbf{x}^{\prime} \mathbf{C}_{t}^{-1} \mathbf{x}}=\sqrt{\sum_{s=1}^{t}\left(\mathbf{x}^{\prime} \mathbf{U}_{s}\right)^{2}}
$$

For any $s$, let $\mathbf{H}_{s}=\left(\mathbf{U}_{1}, X_{1}, W_{1}, \ldots, \mathbf{U}_{s}, X_{s}, W_{s}\right)$ the history until the end of period $s$. By definition, $\mathbf{U}_{s}$ is a function of $\mathbf{H}_{s-1}$, and it follows from Assumption $\mathbb{1}(\mathrm{a})$ that for any $\gamma \in \mathbb{R}$,

$$
\mathrm{E}\left[\left.e^{\frac{\gamma}{\sigma_{0}}\left(\mathbf{x}^{\prime} \mathbf{U}_{s}\right) W_{s}-\frac{\gamma^{2}\left(\mathbf{x}^{\prime} \mathbf{U}_{s}\right)^{2}}{2}} \right\rvert\, \mathbf{H}_{s-1}\right]=e^{-\frac{\gamma^{2}\left(\mathbf{x}^{\prime} \mathbf{U}_{s}\right)^{2}}{2}} \mathrm{E}\left[\left.e^{\frac{\gamma}{\sigma_{0}}\left(\mathbf{x}^{\prime} \mathbf{U}_{s}\right) W_{s}} \right\rvert\, \mathbf{H}_{s-1}\right] \leq 1
$$

Using a standard argument involving iterated expectations, we obtain

$$
\mathrm{E}\left[e^{\gamma A-\left(\gamma^{2} B^{2} / 2\right)}\right]=\mathrm{E}\left[e^{\sum_{s=1}^{t}\left(\frac{\gamma}{\sigma_{0}}\left(\mathbf{x}^{\prime} \mathbf{U}_{s}\right) W_{s}-\frac{\gamma^{2}\left(\mathbf{x}^{\prime} \mathbf{U}_{s}\right)^{2}}{2}\right)}\right]=\mathrm{E}\left[\prod_{s=1}^{t} e^{\left(\frac{\gamma}{\sigma_{0}}\left(\mathbf{x}^{\prime} \mathbf{U}_{s}\right) W_{s}-\frac{\gamma^{2}\left(\mathrm{x}^{\prime} \mathbf{U}_{s}\right)^{2}}{2}\right)}\right] \leq 1
$$

We can thus apply Lemma B.3 to the random variables $A$ and $B$. Moreover, it follows from the definition of $\bar{u}$ and $\lambda_{0}$ in Assumption (b) that, with probability one,

$$
\lambda_{0} \leq \lambda_{\min }\left(\sum_{s=1}^{t} \mathbf{U}_{s} \mathbf{U}_{s}^{\prime}\right) \leq \mathbf{x}^{\prime}\left(\sum_{s=1}^{t} \mathbf{U}_{s} \mathbf{U}_{s}^{\prime}\right) \mathbf{x}=B^{2}=\sum_{s=1}^{t}\left(\mathbf{x}^{\prime} \mathbf{U}_{s}\right)^{2} \leq t \bar{u}^{2}
$$

Therefore, $B^{2}+\lambda_{0} \leq 2 B^{2}$, and

$$
\begin{aligned}
1+\frac{1}{2} \log \left(1+\frac{B^{2}}{\lambda_{0}}\right) & \leq 1+\frac{1}{2} \log \left(1+\frac{t \bar{u}^{2}}{\lambda_{0}}\right) \leq \frac{1}{2}\left(\log t+2+\log \left(1+\frac{\bar{u}^{2}}{\lambda_{0}}\right)\right) \\
& \leq \frac{\log t}{2}\left(1+\frac{2+\log \left(1+\left(\bar{u}^{2} / \lambda_{0}\right)\right)}{\log t}\right) \leq \frac{\kappa_{0}^{2} \log t}{2},
\end{aligned}
$$

where the last inequality follows from the definition of $\kappa_{0}$ and the fact that $t \geq r \geq 2$. These two upper bounds imply that $\sqrt{\left(B^{2}+\lambda_{0}\right)\left(1+\frac{1}{2} \log \left(1+\frac{B^{2}}{\lambda_{0}}\right)\right)} \leq \kappa_{0} \sqrt{\log t} B$. Therefore,

$$
\operatorname{Pr}\left\{\left|\mathbf{x}^{\prime} \mathbf{M}_{t}\right|>\zeta \kappa_{0} \sigma_{0} \sqrt{\log t}\|\mathbf{x}\|_{\mathbf{C}_{t}^{-1}}\right\} \leq \operatorname{Pr}\left\{|A|>\zeta \sqrt{\left(B^{2}+\lambda_{0}\right)\left(1+\frac{1}{2} \log \left(1+\frac{B^{2}}{\lambda_{0}}\right)\right)}\right\}
$$

and the desired result then follows immediately from Lemma B.3.
The next and final lemma extends the previous result to show that the matrix $\zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2}(\log t) \mathbf{C}_{t}^{-1}-$ $\mathbf{M}_{t} \mathbf{M}_{t}^{\prime}$ is positive semidefinite with a high probability. The proof of this result makes use of the fact that for the matrix $\zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2}(\log t) \mathbf{C}_{t}^{-1}-\mathbf{M}_{t} \mathbf{M}_{t}^{\prime}$ to be positive semidefinite, it suffices for the inequality $\mathbf{x}^{\prime} \mathbf{M}_{t} \mathbf{M}_{t}^{\prime} \mathbf{x} \leq \zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2}(\log t) \mathbf{x}^{\prime} \mathbf{C}_{t}^{-1} \mathbf{x}$ to hold for vectors $\mathbf{x}$ in a sufficiently dense subset. We can then apply Lemma B. 4 for each such vector $\mathbf{x}$ and use the union bound.

Lemma B.5. Under Assumption 1, for any $t \geq r$ and $\zeta \geq 2$,

$$
\operatorname{Pr}\left\{\mathbf{M}_{t} \mathbf{M}_{t}^{\prime} \leq \zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2}(\log t) \mathbf{C}_{t}^{-1}\right\} \geq 1-t^{r \kappa_{0}^{2}} e^{-\zeta^{2} / 4},
$$

Proof. Let $\mathcal{S}^{r}=\left\{\mathrm{x} \in \mathbb{R}^{r}:\|\mathrm{x}\|=1\right\}$ denote the unit sphere in $\mathbb{R}^{r}$. Let $\delta>0$ be defined by:

$$
\delta=\frac{\lambda_{0}}{9 \bar{u}^{2} t},
$$

where the constants $\lambda_{0}$ and $\bar{u}$ are given in Assumption (b). Without loss of generality, we can assume that $\delta \leq 1 / 2$ and that $1 / \delta$ is an integer. Let $\mathcal{X}^{r}$ be a covering of $\mathcal{S}^{r}$, that is, for any $\mathrm{x} \in \mathcal{S}^{r}$, there exists $\mathbf{y} \in \mathcal{X}^{r}$ such that $\|\mathbf{x}-\mathbf{y}\| \leq \delta$. It is easy to verify that $\mathcal{X}^{r}$ can be chosen to have a cardinality of at most $(2 \sqrt{r} / \delta)^{r}$ because we can consider a rectangular grid on $[-1,1]^{r}$ with a grid spacing of $\delta / \sqrt{r}$. Then, for any point $\mathbf{x} \in \mathcal{S}^{r}$, there is a point $\mathbf{y}$ on the rectangular grid such that the magnitude of each component of $\mathbf{x}-\mathbf{y}$ is at most $\delta / \sqrt{r}$, which implies that $\|\mathbf{x}-\mathbf{y}\| \leq \delta$.

Let $t \geq r$ and $\zeta \geq 2$ be given. To facilitate our exposition, let $\beta=\zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2} \log t$. Let $\mathcal{G}$ denote the event that the following inequalities hold:

$$
\mathbf{e}_{i}^{\prime} \mathbf{M}_{t} \mathbf{M}_{t}^{\prime} \mathbf{e}_{i} \leq \beta \mathbf{e}_{i} \mathbf{C}_{t}^{-1} \mathbf{e}_{i}, \quad i=1,2, \ldots, r, \quad \text { and } \quad \mathbf{y}^{\prime} \mathbf{M}_{t} \mathbf{M}_{t}^{\prime} \mathbf{y} \leq \frac{\beta}{2} \mathbf{y}^{\prime} \mathbf{C}_{t}^{-1} \mathbf{y}, \quad \forall \mathbf{y} \in \mathcal{X}^{r}
$$

Using the union bound, it follows from Lemma B. 4 that the event $\mathcal{G}$ happens with a probability at least

$$
\begin{aligned}
1-\left|\mathcal{X}^{r}\right| e^{-\zeta^{2} / 4}-r e^{-\zeta^{2} / 2} & \geq 1-\left(\frac{2 \sqrt{r}}{\delta}\right)^{r} e^{-\zeta^{2} / 4}-r e^{-\zeta^{2} / 2} \geq 1-\left(\left(\frac{2 \sqrt{r}}{\delta}\right)^{r}+r\right) e^{-\zeta^{2} / 4} \\
& \geq 1-\left(\frac{4 \sqrt{r}}{\delta}\right)^{r} e^{-\zeta^{2} / 4} \geq 1-\left(\frac{36 \bar{u}^{2} t^{2}}{\lambda_{0}}\right)^{r} e^{-\zeta^{2} / 4} \geq 1-t^{r} \kappa_{0}^{2} e^{-\zeta^{2} / 4},
\end{aligned}
$$

where we have used the fact that $t \geq \sqrt{r} \geq 2$ in the penultimate inequality. The final inequality follows from the definition of $\kappa_{0}$ in Equation (3), which implies that $\kappa_{0}^{2} \geq 4\left(1+\log \left(36 \bar{u}^{2} / \lambda_{0}\right)\right) \geq 4$, and thus, $\frac{36 \bar{u}^{2} t^{2}}{\lambda_{0}} \leq t^{2} e^{\kappa_{0}^{2} / 4} \leq t^{\kappa_{0}^{2} / 2}\left(t^{2}\right)^{\kappa_{0}^{2} / 4}=t^{\kappa_{0}^{2}}$.

To complete the proof, it suffices to show that when the event $\mathcal{G}$ occurs, we have that $\mathbf{x}^{\prime} \mathbf{M}_{t} \mathbf{M}_{t}^{\prime} \mathbf{x} \leq$ $\beta \mathbf{x}^{\prime} \mathbf{C}_{t}^{-1} \mathbf{x}$ for all $\mathbf{x} \in \mathcal{S}^{r}$. Consider an arbitrary $\mathbf{x} \in \mathcal{S}^{r}$, and let $\mathbf{y} \in \mathcal{X}^{r}$ be such that $\|\mathbf{x}-\mathbf{y}\| \leq \delta$. This implies that $\|\mathbf{x}+\mathbf{y}\| \leq\|2 \mathbf{x}\|+\|\mathbf{y}-\mathbf{x}\| \leq 2+\delta \leq 3$. Moreover, $\mathbf{x}^{\prime} \mathbf{M}_{t} \mathbf{M}_{t}^{\prime} \mathbf{x}-\mathbf{y}^{\prime} \mathbf{M}_{t} \mathbf{M}_{t}^{\prime} \mathbf{y}=$ $(\mathbf{x}-\mathbf{y})^{\prime} \mathbf{M}_{t} \mathbf{M}_{t}^{\prime}(\mathbf{x}+\mathbf{y}) \leq 3 \delta\left\|\mathbf{M}_{t}\right\|^{2}$ where we use the Cauchy-Schwarz for the last inequality.

Similarly, we can show that for all $s, \mathbf{y}^{\prime} \mathbf{U}_{s} \mathbf{U}_{s}^{\prime} \mathbf{y} \leq \mathbf{x}^{\prime} \mathbf{U}_{s} \mathbf{U}_{s}^{\prime} \mathbf{x}+3 \delta \bar{u}^{2}$. Summing over all $s$, we obtain that $\mathbf{y}^{\prime} \mathbf{C}_{t}^{-1} \mathbf{y} \leq \mathbf{x} \mathbf{C}_{t}^{-1} \mathbf{x}+3 \delta t \bar{u}^{2}$. Putting everything together, we have that

$$
\begin{aligned}
\mathbf{x}^{\prime} \mathbf{M}_{t} \mathbf{M}_{t}^{\prime} \mathbf{x} & \leq \mathbf{y}^{\prime} \mathbf{M}_{t} \mathbf{M}_{t}^{\prime} \mathbf{y}+3 \delta\left\|\mathbf{M}_{t}\right\|^{2} \leq \frac{\beta}{2} \mathbf{y}^{\prime} \mathbf{C}_{t}^{-1} \mathbf{y}+3 \delta\left\|\mathbf{M}_{t}\right\|^{2} \\
& \leq \frac{\beta}{2} \mathbf{x}^{\prime} \mathbf{C}_{t}^{-1} \mathbf{x}+\frac{3 \beta}{2} \delta t \bar{u}^{2}+3 \delta\left\|\mathbf{M}_{t}\right\|^{2} \leq \beta \mathbf{x}^{\prime} \mathbf{C}_{t}^{-1} \mathbf{x}-\frac{\beta}{2} \lambda_{0}+\frac{3 \beta}{2} \delta t \bar{u}^{2}+3 \delta\left\|\mathbf{M}_{t}\right\|^{2},
\end{aligned}
$$

where the last inequality follows from the fact that $\mathbf{C}_{t}^{-1}=\sum_{s=1}^{t} \mathbf{U}_{s} \mathbf{U}_{s}^{\prime} \geq \lambda_{0} \mathbf{I}_{r}$ from our definition of $\lambda_{0}$. Finally, note that under the event $\mathcal{G}$,

$$
\begin{aligned}
\left\|\mathbf{M}_{t}\right\|^{2} & =\sum_{i=1}^{r} \mathbf{e}_{i}^{\prime} \mathbf{M}_{t} \mathbf{M}_{t}^{\prime} \mathbf{e}_{i} \leq \beta \sum_{i=1}^{r} \mathbf{e}_{i}^{\prime} \mathbf{C}_{t}^{-1} \mathbf{e}_{i}=\beta \sum_{s=1}^{t} \sum_{i=1}^{r} \mathbf{e}_{i}^{\prime} \mathbf{U}_{s} \mathbf{U}_{s}^{\prime} \mathbf{e}_{i} \\
& =\beta \sum_{s=1}^{t} \sum_{i=1}^{r}\left|\mathbf{e}_{i}^{\prime} \mathbf{U}_{s}\right|^{2}=\beta \sum_{s=1}^{t}\left\|\mathbf{U}_{s}\right\|^{2} \leq \beta t \bar{u}^{2},
\end{aligned}
$$

which implies that

$$
-\frac{\beta}{2} \lambda_{0}+\frac{3 \beta}{2} \delta t \bar{u}^{2}+3 \delta\left\|\mathbf{M}_{t}\right\|^{2} \leq-\frac{\beta}{2} \lambda_{0}+\frac{9 \beta}{2} \delta t \bar{u}^{2}=\frac{\beta}{2}\left(9 \delta t \bar{u}^{2}-\lambda_{0}\right)=0,
$$

where the last inequality follows from the definition of $\delta$. Thus, we have that $\mathbf{x}^{\prime} \mathbf{M}_{t} \mathbf{M}_{t}^{\prime} \mathbf{x} \leq \beta \mathbf{x}^{\prime} \mathbf{C}_{t}^{-1} \mathbf{x}$, which is the desired result.

We are now ready to give a proof of Theorem B.2.
Proof. It suffices to establish the second inequality in Theorem B. 2 because the proof for the first inequality follows the same argument. It follows from the Cauchy-Schwarz inequality that

$$
\frac{\left(\mathbf{U}_{t+1}-\mathbf{x}\right)^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)}{\left\|\mathbf{U}_{t+1}-\mathbf{x}\right\|_{\mathbf{C}_{t}}}=\frac{\left(\mathbf{U}_{t+1}-\mathbf{x}\right)^{\prime} \mathbf{C}_{t}^{1 / 2} \mathbf{C}_{t}^{-1 / 2}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)}{\left\|\mathbf{C}_{t}^{1 / 2}\left(\mathbf{U}_{t+1}-\mathbf{x}\right)\right\|} \leq\left\|\mathbf{C}_{t}^{-1 / 2}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)\right\|
$$

with probability one. Therefore,

$$
\begin{aligned}
\operatorname{Pr} & \left\{\left(\mathbf{U}_{t+1}-\mathbf{x}\right)^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>\zeta \kappa_{0} \sigma_{0} \sqrt{\log t}\left\|\mathbf{U}_{t+1}-\mathbf{x}\right\|_{\mathbf{C}_{t}} \mid \mathbf{Z}=\mathbf{z}\right\} \\
& \leq \operatorname{Pr}\left\{\left\|\mathbf{C}_{t}^{-1 / 2}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)\right\|>\zeta \kappa_{0} \sigma_{0} \sqrt{\log t} \mid \mathbf{Z}=\mathbf{z}\right\} \\
& =\operatorname{Pr}\left\{\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)^{\prime} \mathbf{C}_{t}^{-1}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>\zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2} \log t \mid \mathbf{Z}=\mathbf{z}\right\} \\
& =\operatorname{Pr}\left\{\mathbf{M}_{t}^{\prime} \mathbf{C}_{t} \mathbf{M}_{t}>\zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2} \log t \mid \mathbf{Z}=\mathbf{z}\right\},
\end{aligned}
$$

where the last equality follows from the definition of the least squares estimate $\widehat{\mathbf{Z}}_{t}$.
It is a well-known result in linear algebra (see, for example, Theorem 1.3.3 in Bhatia, 2007) that if $\mathbf{A}$ and $\mathbf{B}$ are two symmetric positive definite matrices, then the block matrix $\left(\begin{array}{cc}\mathbf{A} & \mathbf{X} \\ \mathbf{X}^{\prime} & \mathbf{B}\end{array}\right)$ is positive semidefinite if and only if $\mathbf{X B}^{-1} \mathbf{X}^{\prime} \leq \mathbf{A}$. Applying this result to the two "equivalent" $(r+1) \times(r+1)$ matrices $\left(\begin{array}{cc}\zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2} \log t & \mathbf{M}_{t}^{\prime} \\ \mathbf{M}_{t} & \mathbf{C}_{t}^{-1}\end{array}\right) \quad$ and $\left(\begin{array}{cc}\mathbf{C}_{t}^{-1} & \mathbf{M}_{t} \\ \mathbf{M}_{t}^{\prime} & \zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2} \log t\end{array}\right)$, we conclude that $\mathbf{M}_{t}^{\prime} \mathbf{C}_{t} \mathbf{M}_{t} \leq \zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2} \log t$ if and only if $\mathbf{M}_{t} \mathbf{M}_{t}^{\prime} \leq \zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2}(\log t) \mathbf{C}_{t}^{-1}$. The desired result then
follows from the fact that

$$
\begin{aligned}
\operatorname{Pr}\left\{\mathbf{M}_{t}^{\prime} \mathbf{C}_{t} \mathbf{M}_{t}>\zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2} \log t \mid \mathbf{Z}=\mathbf{z}\right\} & =1-\operatorname{Pr}\left\{\mathbf{M}_{t}^{\prime} \mathbf{C}_{t} \mathbf{M}_{t} \leq \zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2} \log t \mid \mathbf{Z}=\mathbf{z}\right\} \\
& =1-\operatorname{Pr}\left\{\mathbf{M}_{t} \mathbf{M}_{t}^{\prime} \leq \zeta^{2} \kappa_{0}^{2} \sigma_{0}^{2}(\log t) \mathbf{C}_{t}^{-1} \mid \mathbf{Z}=\mathbf{z}\right\} \\
& \leq t^{r \kappa_{0}^{2}} e^{-\zeta^{2} / 4}
\end{aligned}
$$

where the last inequality follows from Lemma B.5.

## B. 2 Bounds for General Compact Sets of Arms: Proof of Theorem 4.1

The proof of Theorem 4.1 makes use of a number of auxiliary results. The first result provides a motivation for the choice of the parameter $\alpha$ in Equation (5) and our definition of the uncertainty radius $R_{t}^{\mathbf{u}}$ in Equation (6). They are chosen to keep the probability of overestimating the reward of an arm by more than $R_{t}^{\mathbf{u}}$ bounded by $1 / t^{2}$. This will limit the growth rate of the cumulative regret due to such overestimation.

Lemma B. 6 (Large Deviation Inequalities for the Uncertainty Radius). Under Assumption 1, for any $\operatorname{arm} \mathbf{u} \in \mathcal{U}_{r}$ and $t \geq r$,

$$
\operatorname{Pr}\left\{\mathbf{u}^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>R_{t}^{\mathbf{u}} \mid \mathbf{Z}=\mathbf{z}\right\} \leq \frac{1}{t^{2}}
$$

and for any $\mathbf{x} \in \mathbb{R}^{r}$,

$$
\operatorname{Pr}\left\{\left(\mathbf{U}_{t+1}-\mathbf{x}\right)^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>\alpha \sqrt{\log t} \sqrt{\min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}\left\|\mathbf{U}_{t+1}-\mathbf{x}\right\|_{\mathbf{C}_{t}} \mid \mathbf{Z}=\mathbf{z}\right\} \quad \leq \frac{1}{t^{2}}
$$

where $\alpha=4 \sigma_{0} \kappa_{0}^{2}$.
Proof. It suffices to establish the first inequality because the proof of the second one is exactly the same. Let $\beta_{t}=4 \sigma_{0} \kappa_{0}^{2} \sqrt{\log t} \sqrt{\min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}$. Recall from Equations (5) and (6) that $R_{t}^{\mathbf{u}}=\beta_{t}\|\mathbf{u}\|_{\mathbf{C}_{t}}$. By applying Theorem B.1 (with $\zeta=4 \kappa_{0}^{2} \sqrt{\log t} \sqrt{\min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}$ ) and Theorem B. 2 (with $\zeta=4 \kappa_{0} \sqrt{\min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}$ ), we obtain
$\operatorname{Pr}\left\{\mathbf{u}^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>R_{t}^{\mathbf{u}} \mid \mathbf{Z}=\mathbf{z}\right\} \leq \min \left\{t^{5\left|\mathcal{U}_{r}\right|} e^{-8 \kappa_{0}^{4}(\log t) \min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}, t^{r \kappa_{0}^{2}} e^{-4 \kappa_{0}^{2} \min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}\right\}$.
There are two cases to consider: $r \log t>\left|\mathcal{U}_{r}\right|$ and $r \log t \leq\left|\mathcal{U}_{r}\right|$. Suppose that $r \log t>\left|\mathcal{U}_{r}\right|$. Then, $\operatorname{Pr}\left\{\mathbf{u}^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>R_{t}^{\mathbf{u}} \mid \mathbf{Z}=\mathbf{z}\right\} \leq t^{5\left|\mathcal{U}_{r}\right|} e^{-8 \kappa_{0}^{4}(\log t) \min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}=t^{5\left|\mathcal{U}_{r}\right|} e^{-8 \kappa_{0}^{4}(\log t)\left|\mathcal{U}_{r}\right|}=\frac{t^{5\left|\mathcal{U}_{r}\right|}}{t^{8 \kappa_{0}^{4}\left|\mathcal{U}_{r}\right|}} \leq \frac{1}{t^{2}}$, where the last inequality follows from the fact that $\left(8 \kappa_{0}^{4}-5\right)\left|\mathcal{U}_{r}\right| \geq 2$. In the second case where $r \log t \leq\left|\mathcal{U}_{r}\right|$, we have that
$\operatorname{Pr}\left\{\mathbf{u}^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>R_{t}^{\mathbf{u}} \mid \mathbf{Z}=\mathbf{z}\right\} \leq t^{r \kappa_{0}^{2}} e^{-4 \kappa_{0}^{2} \min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}=t^{r \kappa_{0}^{2}} e^{-4 \kappa_{0}^{2} r \log t}=\frac{t^{r \kappa_{0}^{2}}}{t^{4 r \kappa_{0}^{2}}}=\frac{1}{t^{3 r \kappa_{0}^{2}}} \leq \frac{1}{t^{2}}$,
where the last inequality follows from the fact that $3 r \kappa_{0}^{2} \geq 2$. Since the probability is bounded by $1 / t^{2}$ in both cases, this gives the desired result.

For any $t \geq 1$, let the random variable $Q_{t}(\mathbf{z})$ denote the instantaneous regret in period $t$ given that $\mathbf{Z}=\mathbf{z}$, that is,

$$
\begin{equation*}
Q_{t}(\mathbf{z})=\max _{\mathbf{v} \in \mathcal{U}_{r}} \mathbf{v}^{\prime} \mathbf{z}-\mathbf{U}_{t}^{\prime} \mathbf{z} \tag{9}
\end{equation*}
$$

Lemma B.6 shows that the probability of a large estimation error in period $t$ is at most $O\left(1 / t^{2}\right)$. Consequently, as shown in the following lemma, the probability of having a large instantaneous regret in period $t$ is also small.

Lemma B. 7 (Instantaneous Regret Bound). Under Assumption $\mathbb{1}$, for all $t \geq r$ and $\mathbf{z} \in \mathbb{R}^{r}$,

$$
\operatorname{Pr}\left\{Q_{t+1}(\mathbf{z})>2 \alpha \sqrt{\log t} \sqrt{\min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}} \mid \mathbf{Z}=\mathbf{z}\right\} \leq \frac{1}{t^{2}}
$$

Proof. Let $\mathbf{z} \in \mathbb{R}^{r}$ be given and let $\mathbf{w}$ denote an optimal arm, that is, $\max _{\mathbf{v} \in \mathcal{U}_{r}} \mathbf{v}^{\prime} \mathbf{z}=\mathbf{w}^{\prime} \mathbf{z}$. To facilitate our discussion, let $\beta_{t}=\alpha \sqrt{\log t} \sqrt{\min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}$. Then, it follows from the definition of the uncertainty radius in Equation (6) and the definition of $\mathbf{U}_{t+1}$ in Equation (17) that

$$
\mathbf{U}_{t+1}^{\prime} \widehat{\mathbf{Z}}_{t}+\beta_{t}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}} \geq \mathbf{w}^{\prime} \widehat{\mathbf{Z}}_{t}+\beta_{t}\|\mathbf{w}\|_{\mathbf{C}_{t}}
$$

which implies that

$$
\begin{aligned}
\beta_{t}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}} & \geq\left(\mathbf{w}-\mathbf{U}_{t+1}\right)^{\prime} \widehat{\mathbf{Z}}_{t}+\beta_{t}\|\mathbf{w}\|_{\mathbf{C}_{t}} \\
& =\left(\mathbf{w}-\mathbf{U}_{t+1}\right)^{\prime} \mathbf{z}+\left(\mathbf{w}-\mathbf{U}_{t+1}\right)^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)+\beta_{t}\|\mathbf{w}\|_{\mathbf{C}_{t}} \\
& =Q_{t+1}(\mathbf{z})+\left(\mathbf{w}-\mathbf{U}_{t+1}\right)^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)+\beta_{t}\|\mathbf{w}\|_{\mathbf{C}_{t}} .
\end{aligned}
$$

Suppose that the event $Q_{t+1}(\mathbf{z})>2 \beta_{t}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}$ occurs. Then, it follows that

$$
\beta_{t}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}>2 \beta_{t}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}+\left(\mathbf{w}-\mathbf{U}_{t+1}\right)^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)+\beta_{t}\|\mathbf{w}\|_{\mathbf{C}_{t}}
$$

which implies that $\left(\mathbf{U}_{t+1}-\mathbf{w}\right)^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>\beta_{t}\left(\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}+\|\mathbf{w}\|_{\mathbf{C}_{t}}\right) \geq \beta_{t}\left\|\mathbf{U}_{t+1}-\mathbf{w}\right\|_{\mathbf{C}_{t}}$. Thus,

$$
\begin{aligned}
& \operatorname{Pr}\left\{Q_{t+1}(\mathbf{z})>2 \beta_{t}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}} \mid \mathbf{Z}=\mathbf{z}\right\} \\
& \quad \leq \operatorname{Pr}\left\{\left(\mathbf{U}_{t+1}-\mathbf{w}\right)^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>\beta_{t}\left\|\mathbf{U}_{t+1}-\mathbf{w}\right\|_{\mathbf{C}_{t}} \mid \mathbf{Z}=\mathbf{z}\right\} \leq \frac{1}{t^{2}},
\end{aligned}
$$

the last inequality follows from Lemma B. 6
Lemma B. 7 suggests the following approach for bounding the cumulative regret over $T$ periods. In the first $r$ periods (during the initialization), we incur a regret of $O(r)$. For each time period between $r+1$ and $T$, we consider the two cases: 1) where the instantaneous regret is large with $Q_{t+1}(\mathbf{z})>2 \alpha \sqrt{\log t} \sqrt{\min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}$; and, 2) the instantaneous regret is small.

By the above lemma, the contribution to the cumulative regret from the first case is bounded above by $O\left(\sum_{t} 1 / t^{2}\right)$, which is finite. In the second case, we have a simple upper bound of $2 \alpha \sqrt{r}(\log t)\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}$ for the instantaneous regret. This argument leads to the following bound on the cumulative regret over $T$ periods.

Lemma B. 8 (Regret Decomposition). Under Assumption $\mathbb{1}$, for all $T \geq r+1$ and $\mathbf{z} \in \mathbb{R}^{r}$,

$$
\operatorname{Regret}(\mathbf{z}, T, \mathrm{UE}) \leq 2 \bar{u}(r+2)\|\mathbf{z}\|+2 \alpha \sqrt{r}(\log T) \sqrt{T} \mathrm{E}\left[\sqrt{\sum_{t=r}^{T-1}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2}} \mid \mathbf{Z}=\mathbf{z}\right]
$$

Proof. Let $\mathbf{z} \in \mathbb{R}^{r}$ be given. By the Cauchy-Schwarz Inequality and Assumption $\mathbb{T}(\mathrm{b})$, we have the following upper bound on the instantaneous regret for all $t$ and $\mathbf{z} \in \mathbb{R}^{r}: Q_{t}(\mathbf{z})=\max _{\mathbf{v} \in \mathcal{U}_{r}}\left(\mathbf{v}-\mathbf{U}_{t}\right)^{\prime} \mathbf{z} \leq$ $2 \bar{u}\|\mathbf{z}\|$. Therefore,

$$
\operatorname{Regret}(\mathbf{z}, T, \mathrm{UE}) \leq 2 \bar{u} r\|\mathbf{z}\|+\mathrm{E}\left[\sum_{t=r}^{T-1} Q_{t+1}(\mathbf{z}) \mid \mathbf{Z}=\mathbf{z}\right] .
$$

For any $t \geq r$, let the indicator random variable $G_{t+1}(\mathbf{z})$ be defined by:

$$
G_{t+1}(\mathbf{z})=\mathbb{1}\left[Q_{t+1}(\mathbf{z}) \leq 2 \alpha \sqrt{\log t} \sqrt{\min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}\right]
$$

The contribution to the expected instantaneous regret $\mathrm{E}\left[Q_{t+1}(\mathbf{z}) \mid \mathbf{Z}=\mathbf{z}\right]$ comes from two cases: 1) when $G_{t+1}(\mathbf{z})=0$ and 2 ) when $G_{t+1}(\mathbf{z})=1$. We will upper bound each of these two contributions separately. In the first case, we know from Lemma B. 7 that $\operatorname{Pr}\left\{G_{t+1}(\mathbf{z})=0 \mid \mathbf{Z}=\mathbf{z}\right\}=$ $\operatorname{Pr}\left\{Q_{t+1}(\mathbf{z})>2 \alpha \sqrt{\log t} \sqrt{\min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}} \mid \mathbf{Z}=\mathbf{z}\right\} \leq 1 / t^{2}$. Since $\sum_{t=1}^{\infty} 1 / t^{2} \leq 2$, we have that

$$
\mathrm{E}\left[\sum_{t=r}^{T-1}\left(1-G_{t+1}(\mathbf{z})\right) Q_{t+1}(\mathbf{z}) \mid \mathbf{Z}=\mathbf{z}\right] \leq 2 \bar{u}\|\mathbf{z}\| \sum_{t=r}^{T-1} \operatorname{Pr}\left\{G_{t+1}(\mathbf{z})=0 \mid \mathbf{Z}=\mathbf{z}\right\} \leq 4 \bar{u}\|\mathbf{z}\|
$$

On the other hand, when $G_{t+1}(\mathbf{z})=1$, we have that $Q_{t+1}(\mathbf{z}) \leq 2 \alpha \sqrt{r}(\log t)\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}$. This implies that, with probability one,

$$
\begin{aligned}
\sum_{t=r}^{T-1} G_{t+1}(\mathbf{z}) Q_{t+1}(\mathbf{z}) & \leq 2 \alpha \sqrt{r} \sum_{t=r}^{T-1}(\log t)\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}} \leq 2 \alpha \sqrt{r} \sqrt{\sum_{t=r}^{T-1} \log ^{2} t} \times \sqrt{\sum_{t=r}^{T-1}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2}} \\
& \leq 2 \alpha \sqrt{r}(\log T) \sqrt{T} \sqrt{\sum_{t=r}^{T-1}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2}}
\end{aligned}
$$

where we use the Cauchy-Schwarz Inequality in the second inequality and the final inequality follows from the fact that $\sqrt{\sum_{t=r}^{T-1} \log ^{2} t} \leq \sqrt{\sum_{t=1}^{T-1} \log ^{2} T} \leq(\log T) \sqrt{T}$. Putting the two cases together gives the desired upper bound because

$$
\mathrm{E}\left[\sum_{t=r}^{T-1} Q_{t+1}(\mathbf{z}) \mid \mathbf{Z}=\mathbf{z}\right] \leq 4 \bar{u}\|\mathbf{z}\|+2 \alpha \sqrt{r}(\log T) \sqrt{T} \mathrm{E}\left[\sqrt{\sum_{t=r}^{T-1}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2}} \mid \mathbf{Z}=\mathbf{z}\right]
$$

The eigenvectors of the matrix $\mathbf{C}_{t}=\left(\sum_{s=1}^{t} \mathbf{U}_{s} \mathbf{U}_{s}^{\prime}\right)^{-1}$ reflect the directions of the arms that are chosen during the first $t$ periods. The corresponding eigenvalues then measure the frequency with which these directions are explored. Frequently explored directions will have small eigenvalues, while the eigenvalues for unexplored directions will be large. Thus, the weighted norm $\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}$ has two interpretations. First, it measures the size of the regret in period $t+1$. In addition, since $\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2}$ is a linear combination of the eigenvalues of $\mathbf{C}_{t}$, it also reflects the amount of exploration in period $t+1$ in the unexplored directions.

The above interpretation suggests that if we incur large regrets in the past (equivalently, we have done a lot of exploration), then the current regret should be small. Our intuition is confirmed in the following lemma that establishes a recursive relationship between the weighted norm $\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}$ in period $t+1$ and the norms in the preceding periods.

Lemma B. 9 (Large Past Regrets Imply Small Current Regret). Under Assumption 1, for all $t \geq r$ and $\mathbf{z} \in \mathbb{R}^{r}$, with probability one,

$$
0 \leq\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2} \leq \frac{\bar{u}^{2}}{\lambda_{0}} \quad \text { and } \quad\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2} \leq \frac{\left\{\left(\bar{u}^{2} / \lambda_{0}\right) \cdot(t+1)\right\}^{r}}{\prod_{s=r}^{t-1}\left(1+\left\|\mathbf{U}_{s+1}\right\|_{\mathbf{C}_{s}}^{2}\right)}
$$

Proof. For any $t \geq r$, let $\mathbf{\Upsilon}_{t}=\left(\mathbf{C}_{t}\right)^{-1}=\sum_{s=1}^{t} \mathbf{U}_{s} \mathbf{U}_{s}^{\prime}$. By the Rayleigh Principle,

$$
\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2}=\mathbf{U}_{t+1}^{\prime} \mathbf{C}_{t} \mathbf{U}_{t+1} \leq \lambda_{\max }\left(\mathbf{C}_{t}\right)\left\|\mathbf{U}_{t+1}\right\|^{2}=\frac{\left\|\mathbf{U}_{t+1}\right\|^{2}}{\lambda_{\min }\left(\mathbf{\Upsilon}_{t}\right)} \leq \frac{\bar{u}^{2}}{\lambda_{0}}
$$

where the last inequality follows from the definition of $\bar{u}$ and the fact that

$$
\lambda_{\min }\left(\mathbf{\Upsilon}_{t}\right)=\lambda_{\min }\left(\mathbf{\Upsilon}_{r}+\sum_{s=r+1}^{t} \mathbf{U}_{s} \mathbf{U}_{s}^{\prime}\right) \geq \lambda_{\min }\left(\mathbf{\Upsilon}_{r}\right) \geq \lambda_{0}
$$

where the last equality follows from the fact that $\mathbf{\Upsilon}_{r}=\sum_{k=1}^{r} \mathbf{b}_{k} \mathbf{b}_{k}^{\prime}$ where the vectors $\mathbf{b}_{1}, \ldots, \mathbf{b}_{r}$ are given in Assumption $1(\mathrm{~b})$. This proves the claimed upper bound on $\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2}$.

We will now establish the inequality that relates $\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2}$ to $\left\|\mathbf{U}_{s+1}\right\|_{\mathbf{C}_{s}}^{2}$ for $s<t$. Note that

$$
\begin{align*}
\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2} & =\mathbf{U}_{t+1}^{\prime} \mathbf{C}_{t} \mathbf{U}_{t+1} \leq 1+\mathbf{U}_{t+1}^{\prime} \mathbf{C}_{t} \mathbf{U}_{t+1} \\
& =\frac{\operatorname{det}\left(\mathbf{\Upsilon}_{t}\right) \cdot\left(1+\mathbf{U}_{t+1}^{\prime} \mathbf{C}_{t} \mathbf{U}_{t+1}\right)}{\operatorname{det}\left(\boldsymbol{\Upsilon}_{t}\right)}=\frac{\operatorname{det}\left(\mathbf{\Upsilon}_{t}+\mathbf{U}_{t+1} \mathbf{U}_{t+1}^{\prime}\right)}{\operatorname{det}\left(\boldsymbol{\Upsilon}_{t}\right)}=\frac{\operatorname{det}\left(\mathbf{\Upsilon}_{t+1}\right)}{\operatorname{det}\left(\boldsymbol{\Upsilon}_{t}\right)}, \tag{10}
\end{align*}
$$

where the second to last equality follows the matrix determinant lemma.
We will now establish bounds on the determinants $\operatorname{det}\left(\boldsymbol{\Upsilon}_{t+1}\right)$ and $\operatorname{det}\left(\mathbf{\Upsilon}_{t}\right)$. Note that

$$
\lambda_{\max }\left(\mathbf{\Upsilon}_{t+1}\right) \leq \operatorname{tr}\left(\mathbf{\Upsilon}_{t+1}\right)=\sum_{s=1}^{t+1} \operatorname{tr}\left(\mathbf{U}_{s} \mathbf{U}_{s}^{\prime}\right)=\sum_{s=1}^{t+1}\left\|\mathbf{U}_{s}\right\|^{2} \leq(t+1) \bar{u}^{2}
$$

where the last inequality follows from the definition of $\bar{u}$. Therefore, $\operatorname{det}\left(\mathbf{\Upsilon}_{t+1}\right) \leq\left[\lambda_{\max }\left(\mathbf{\Upsilon}_{t+1}\right)\right]^{r} \leq$ $(t+1)^{r} \bar{u}^{2 r}$. Moreover, using Equation (10) repeatedly, we obtain

$$
\operatorname{det}\left(\mathbf{\Upsilon}_{t}\right)=\operatorname{det}\left(\mathbf{\Upsilon}_{r}\right) \prod_{s=r}^{t-1}\left(1+\left\|\mathbf{U}_{s+1}\right\|_{\mathbf{C}_{s}}^{2}\right) \geq \lambda_{0}^{r} \prod_{s=r}^{t-1}\left(1+\left\|\mathbf{U}_{s+1}\right\|_{\mathbf{C}_{s}}^{2}\right),
$$

where the last inequality follows from the fact that $\mathbf{\Upsilon}_{r}=\sum_{k=1}^{r} \mathbf{b}_{k} \mathbf{b}_{k}^{\prime}$ and $\operatorname{det}\left(\mathbf{\Upsilon}_{r}\right) \geq\left[\lambda_{\text {min }}\left(\mathbf{\Upsilon}_{r}\right)\right]^{r} \geq$ $\lambda_{0}^{r}$, where the vectors $\mathbf{b}_{1}, \ldots, \mathbf{b}_{r}$ and the parameter $\lambda_{0}$ are defined in Assumption (b).

Putting everything together, we have that

$$
\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2} \leq \frac{\operatorname{det}\left(\mathbf{\Upsilon}_{t+1}\right)}{\operatorname{det}\left(\mathbf{\Upsilon}_{t}\right)} \leq \frac{(t+1)^{r} \bar{u}^{2 r}}{\lambda_{0}^{r} \prod_{s=r}^{t-1}\left(1+\left\|\mathbf{U}_{s+1}\right\|_{\mathbf{C}_{s}}^{2}\right)}=\frac{\left\{\left(\bar{u}^{2} / \lambda_{0}\right) \cdot(t+1)\right\}^{r}}{\prod_{s=r}^{t-1}\left(1+\left\|\mathbf{U}_{s+1}\right\|_{\mathbf{C}_{s}}^{2}\right)}
$$

which is the desired result.
The above result shows that if the weighted norms in the preceding periods, as measured by $\prod_{s=r}^{t-1}\left(1+\left\|\mathbf{U}_{s+1}\right\|_{\mathbf{C}_{s}}^{2}\right)$, are large, then the weighted norm in the current period $t+1$ will be small. Moreover, since the weighted norm in the current period depends on the product of the norms in the past, we hope that the growth rate of the sum $\sum_{t=r}^{T-1}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2}$ should be small. To formalize our conjecture, we introduce a related optimization problem. For any $c \geq 0$ and $t \geq 1$, let $V^{*}(c, t)$ be defined by:

$$
\begin{aligned}
& V^{*}(c, t)= \max \\
& \sum_{s=1}^{t} y_{s} \\
& \text { s.t. } \quad 0 \leq y_{s} \leq c \quad \text { and } \quad y_{s} \leq \frac{\{c \cdot(r+s)\}^{r}}{\prod_{q=1}^{s-1}\left(1+y_{q}\right)}, \quad s=1,2, \ldots, t,
\end{aligned}
$$

where we define $\prod_{q=1}^{0}\left(1+y_{q}\right)=1$. The following lemma gives an upper bound in terms of the function $V^{*}$.

Lemma B. 10 (Bounds on the Growth Rate of $\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2}$ ). Under Assumption $\mathbb{1}$, for any $T \geq r+1$ and $\mathbf{z} \in \mathbb{R}^{r}$, with probability one, $\sum_{t=r}^{T-1}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2} \leq V^{*}\left(\bar{u}^{2} / \lambda_{0}, T-r\right)$.

Proof. For all $s \geq 1$, let $y_{s}=\left\|\mathbf{U}_{r+s}\right\|_{\mathbf{C}_{r+s-1}}^{2}$. Then, $\sum_{t=r}^{T-1}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2}=\sum_{s=1}^{T-r} y_{s}$. Let $c_{0}=$ $\bar{u}^{2} / \lambda_{0}$. It follows from Lemma B. 9 that for all $s$, with probability one, $0 \leq y_{s} \leq c_{0}$ and $y_{s} \leq$ $\left\{c_{0}(r+s)\right\}^{r} / \prod_{q=1}^{s-1}\left(1+y_{q}\right)$. Therefore, we have $\sum_{t=r}^{T-1}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2} \leq V^{*}\left(c_{0}, T-r\right)$.

It follows from Lemma B. 10 that it suffices to develop an upper bound on $V^{*}(c, t)$. This result is given in the following lemma.

Lemma B. 11 (Optimization Bound). For all $c \geq 0$, and $t \geq 1$,

$$
V^{*}(c, t) \leq 2 c_{0}\left(r \log c_{0}+(r+1) \log (r+t+1)\right),
$$

where $c_{0}=\max \{1, c\}$.

Proof. Any feasible solution $\left\{y_{s}: s=1, \ldots, t\right\}$ for the problem defining $V^{*}(c, t)$, also satisfies the constraints

$$
0 \leq \frac{y_{s}}{2 c_{0}} \leq \frac{y_{s}}{c_{0}} \leq 1 \quad \text { and } \quad \frac{y_{s}}{2 c_{0}} \leq \frac{\left\{c_{0} \cdot(r+s)\right\}^{r}}{\prod_{q=1}^{s-1}\left(1+\left(y_{q} / c_{0}\right)\right)} \leq\left\{c_{0} \cdot(r+s)\right\}^{r} e^{-\sum_{q=1}^{s-1} y_{q} /\left(2 c_{0}\right)},
$$

where the last inequality follows from the fact that for any $a \in[0,1]$, we have $1+a \geq e^{a / 2}$. Thus, by letting $a_{s}=y_{s} / 2 c_{0}$, we obtain $V^{*}(c, t) \leq 2 c_{0} W^{*}\left(c_{0}, t\right)$, where $W^{*}\left(c_{0}, t\right)$ is the maximum possible value of $\sum_{s=1}^{t} a_{s}$, subject to

$$
0 \leq a_{s} \leq 1 \quad \text { and } \quad a_{s} \leq\left\{c_{0} \cdot(r+s)\right\}^{r} e^{-\sum_{q=1}^{s-1} a_{s}}
$$

Let us introduce a continuous-time variable $\tau$, and define $a(\tau)=a_{s}$, for $\tau \in[s-1, s)$. Let $b(\tau)=\int_{0}^{\tau} a\left(\tau^{\prime}\right) d \tau^{\prime}$, and note that $b(s)=\sum_{q=1}^{s} a_{q}$. For any $\tau \in[s-1, s)$, we have $\dot{b}(\tau)=a_{s} \leq\left\{c_{0} \cdot(r+s)\right\}^{r} e^{-\sum_{q=1}^{s-1} a_{s}} \leq\left\{c_{0} \cdot(r+\tau+1)\right\}^{r} e^{a_{s}-\sum_{q=1}^{s} a_{q}} \leq\left\{c_{0} \cdot(r+\tau+1)\right\}^{r} e^{-b(\tau)+1}$.

Let $d(\tau)=e^{b(\tau)}$. Then, for any $\tau \geq 0$,

$$
\dot{d}(\tau)=d(\tau) \dot{b}(\tau) \leq e^{b(\tau)}\left\{c_{0} \cdot(r+\tau+1)\right\}^{r} e^{-b(\tau)+1}=\left\{c_{0} \cdot(r+\tau+1)\right\}^{r} e .
$$

By integrating both sides, we obtain $d(t) \leq \frac{e c_{0}^{r}(r+t+1)^{r+1}}{r+1}$ for all $t \geq 0$. Since $e /(r+1) \leq 1$ because $r \geq 2$, taking logarithms, we obtain

$$
\sum_{q=1}^{t} a_{s}=b(t)=\log d(t) \leq r \log c_{0}+(r+1) \log (r+t+1)
$$

The right-hand side above is therefore an upper bound on $W^{*}\left(c_{0}, t\right)$, which leads to the upper bound on $V^{*}(c, t)$, which gives the desired result.

Finally, here is the proof of Theorem 4.1.
Proof. It suffices to establish the regret bound because the risk bound follows immediately from taking the expectation. Let $A_{0}=\max \left\{1, \bar{u}^{2} / \lambda_{0}\right\}$. It follows from Lemmas B.8, B.10, and B.11 that

$$
\begin{aligned}
\operatorname{Regret}(\mathbf{z}, T, \mathrm{UE}) & \leq 2 \bar{u}(r+2)\|\mathbf{z}\|+2 \alpha \sqrt{r}(\log T) \sqrt{T} \mathrm{E}\left[\sqrt{\sum_{t=r}^{T-1}\left\|\mathbf{U}_{t+1}\right\|_{\mathbf{C}_{t}}^{2}} \mid \mathbf{Z}=\mathbf{z}\right] \\
& \leq 2 \bar{u}(r+2)\|\mathbf{z}\|+2 \alpha \sqrt{r}(\log T) \sqrt{T} \sqrt{V^{*}\left(\bar{u}^{2} / \lambda_{0}, T-r\right)} \\
& \leq 2 \bar{u}(r+2)\|\mathbf{z}\|+2 \alpha \sqrt{r}(\log T) \sqrt{T} \sqrt{2 A_{0}\left\{r \log A_{0}+(r+1) \log (T+1)\right\}} \\
& \leq a_{4} r\|\mathbf{z}\|+a_{5} r \sqrt{T} \log ^{3 / 2} T,
\end{aligned}
$$

for some positive constants $a_{4}$ and $a_{5}$ that depend only on $\sigma_{0}, \bar{u}$, and $\lambda_{0}$.

## B. 3 Bounds for Finitely Many Arms: Proof of Theorem 4.2

Recall that for any $\mathbf{z} \in \mathbb{R}^{r}$ and $\mathbf{u} \in \mathcal{U}_{r}, N^{\mathbf{u}}(\mathbf{z}, T)$ is the number of times that arm $\mathbf{u}$ has been chosen during the first $T$ periods. To complete the proof of Theorem 4.2, it suffices to show that

$$
\mathrm{E}\left[N^{\mathbf{u}}(\mathbf{z}, T) \mid \mathbf{Z}=\mathbf{z}\right] \leq 6+\frac{4 \alpha^{2}\left|\mathcal{U}_{r}\right| \log T}{\left(\Delta^{\mathbf{u}}(\mathbf{z})\right)^{2}}
$$

Let us fix some $\mathbf{z} \in \mathbb{R}^{r}$ and $\mathbf{u} \in \mathcal{U}_{r}$. Since $N^{\mathbf{u}}(\mathbf{z}, t)$ is nondecreasing in $t$, we can show that for any positive integer $\theta, N^{\mathbf{u}}(\mathbf{z}, T) \leq \theta+\sum_{t=r}^{T-1} \mathbb{1}_{\left\{\mathbf{U}_{t+1}=\mathbf{u} \text { and } N^{\mathbf{u}}(\mathbf{z}, t) \geq \theta\right\}}$. Suppose that $\mathbf{w}$ is the optimal arm, that is, $\max _{\mathbf{v} \in \mathcal{U}_{r}} \mathbf{v}^{\prime} \mathbf{z}=\mathbf{w}^{\prime} \mathbf{z}$. Then, we have that

$$
\mathbb{1}_{\left\{\mathbf{U}_{t+1}=\mathbf{u}\right\}} \leq \mathbb{1}_{\left\{\mathbf{u}^{\prime} \widehat{\mathbf{Z}}_{t}+R_{t}^{\mathbf{u}} \geq \mathbf{w}^{\prime} \widehat{\mathbf{z}}_{t}+R_{t}^{\mathbf{w}}\right\}} \leq \mathbb{1}_{\left\{\mathbf{u}^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>R_{t}^{\mathbf{u}}\right\}}+\mathbb{1}_{\left\{\mathbf{w}^{\prime}\left(\widehat{\mathbf{z}}_{t}-\mathbf{z}\right)<-R_{t}^{\mathbf{w}}\right\}}+\mathbb{1}_{\left\{(\mathbf{w}-\mathbf{u})^{\prime} \mathbf{z} \leq 2 R_{t}^{\mathbf{u}}\right\}} .
$$

Since $\operatorname{Pr}\left\{\mathbf{u}^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)>R_{t}^{\mathbf{u}} \mid \mathbf{Z}=\mathbf{z}\right\}$ and $\operatorname{Pr}\left\{\mathbf{w}^{\prime}\left(\widehat{\mathbf{Z}}_{t}-\mathbf{z}\right)<-R_{t}^{\mathbf{w}} \mid \mathbf{Z}=\mathbf{z}\right\}$ are bounded above by $1 / t^{2}$ by Lemma B.6, we can show that

$$
\begin{aligned}
\mathrm{E}\left[N^{\mathbf{u}}(\mathbf{z}, T) \mid \mathbf{Z}=\mathbf{z}\right] & \leq \theta+2 \sum_{t=1}^{\infty} \frac{1}{t^{2}}+\sum_{t=r}^{T-1} \operatorname{Pr}\left\{(\mathbf{w}-\mathbf{u})^{\prime} \mathbf{z} \leq 2 R_{t}^{\mathbf{u}} \text { and } N^{\mathbf{u}}(\mathbf{z}, t) \geq \theta \mid \mathbf{Z}=\mathbf{z}\right\}, \\
& \leq 4+\theta+\sum_{t=r}^{T-1} \operatorname{Pr}\left\{(\mathbf{w}-\mathbf{u})^{\prime} \mathbf{z} \leq 2 R_{t}^{\mathbf{u}} \text { and } N^{\mathbf{u}}(\mathbf{z}, t) \geq \theta \mid \mathbf{Z}=\mathbf{z}\right\}
\end{aligned}
$$

Let $\mathbf{H}=\sum_{\mathbf{v} \subseteq \mathcal{U}_{1}: \mathbf{v} \neq \mathbf{u}} N^{\mathbf{v}}(\mathbf{z}, t) \mathbf{v v}^{\prime}$. It follows from Equation (4) and the Sherman-Morrison Formula (see Sherman and Morrison, 1950) that

$$
\mathbf{C}_{t}=\left(\mathbf{H}+N^{\mathbf{u}}(\mathbf{z}, t) \mathbf{u u}^{\prime}\right)^{-1}=\mathbf{H}^{-1}-\frac{N^{\mathbf{u}}(\mathbf{z}, t) \mathbf{H}^{-1} \mathbf{u u}^{\prime} \mathbf{H}^{-1}}{1+N \mathbf{u}(\mathbf{z}, t) \mathbf{u}^{\prime} \mathbf{H}^{-1} \mathbf{u}}
$$

which implies that

$$
\|\mathbf{u}\|_{\mathbf{C}_{t}}^{2}=\mathbf{u}^{\prime} \mathbf{C}_{t} \mathbf{u}=\mathbf{u}^{\prime} \mathbf{H}^{-1} \mathbf{u}-\frac{N^{\mathbf{u}}(\mathbf{z}, t)\left(\mathbf{u}^{\prime} \mathbf{H}^{-1} \mathbf{u}\right)^{2}}{1+N^{\mathbf{u}}(\mathbf{z}, t) \mathbf{u}^{\prime} \mathbf{H}^{-1} \mathbf{u}}=\frac{\mathbf{u}^{\prime} \mathbf{H}^{-1} \mathbf{u}}{1+N^{\mathbf{u}}(\mathbf{z}, t) \mathbf{u}^{\prime} \mathbf{H}^{-1} \mathbf{u}} \leq \frac{1}{N^{\mathbf{u}}(\mathbf{z}, t)}
$$

and therefore, $2 R_{t}^{\mathbf{u}}=2 \alpha \sqrt{\log t} \sqrt{\min \left\{r \log t,\left|\mathcal{U}_{r}\right|\right\}}\|\mathbf{u}\|_{\mathbf{C}_{t}} \leq\left(2 \alpha \sqrt{\left|\mathcal{U}_{r}\right| \log t}\right) / \sqrt{N^{\mathbf{u}}(\mathbf{z}, t)}$.
By setting $\theta=1+\left\lceil\frac{4 \alpha^{2}\left|\mathcal{U}_{r}\right| \log T}{\left(\Delta^{\mathbf{u}}(\mathbf{z})\right)^{2}}\right\rceil$, we conclude that $2 R_{t}^{\mathbf{u}}<\Delta^{\mathbf{u}}(\mathbf{z})=(\mathbf{w}-\mathbf{u})^{\prime} \mathbf{z}$ whenever $N^{\mathbf{u}}(\mathbf{z}, t) \geq \theta$. This implies that $\mathbb{1}_{\left\{(\mathbf{w}-\mathbf{u})^{\prime} \mathbf{z} \leq 2 R_{t}^{\mathbf{u}} \text { and } N^{\mathbf{u}}(\mathbf{z}, t) \geq \theta\right\}}=0$, and we have that

$$
\mathrm{E}\left[N^{\mathbf{u}}(\mathbf{z}, T) \mid \mathbf{Z}=\mathbf{z}\right] \leq 4+1+\left\lceil\frac{4 \alpha^{2}\left|\mathcal{U}_{r}\right| \log T}{\left(\Delta^{\mathbf{u}}(\mathbf{z})\right)^{2}}\right\rceil \leq 6+\frac{4 \alpha^{2}\left|\mathcal{U}_{r}\right| \log T}{\left(\Delta^{\mathbf{u}}(\mathbf{z})\right)^{2}}
$$

which is the desired result.


[^0]:    ${ }^{1}$ The original lower bound (Theorem 3 on page 360 of Dani et al., 2008a) was not entirely correct; a correct version was provided later, in Dani et al. (2008b).

[^1]:    ${ }^{2}$ One can show that the Cartesian product of circles is not strongly convex, and thus, our phase-based policy cannot be applied to give the matching upper bound for the example used in Dani et al. (2008a).

[^2]:    ${ }^{3}$ The result of Theorem 2.1 easily extends to the case where the covariance matrix is $\mathbf{I}_{r}$, rather than $\mathbf{I}_{r} / r$. The proof is essentially the same.

[^3]:    ${ }^{4}$ Let us note that we are abusing notation here. Throughout this section $\widehat{\mathbf{Z}}_{t}$ stands for the OLS estimate, which is different from the least mean squares estimator $\mathrm{E}\left[\mathbf{Z} \mid \mathbf{H}_{t}\right]$ introduced in Section 2,

