# Maximal lattice-free polyhedra: finiteness and an explicit description in dimension three 

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#### Abstract

A convex set with nonempty interior is maximal lattice-free if it is inclusion-maximal with respect to the property of not containing integer points in its interior. Maximal lattice-free convex sets are known to be polyhedra. The precision of a rational polyhedron $P$ in $\mathbb{R}^{d}$ is the smallest natural number $s$ such that $s P$ is an integral polyhedron. In this paper we show that, up to affine mappings preserving $\mathbb{Z}^{d}$, the number of maximal lattice-free rational polyhedra of a given precision $s$ is finite. Furthermore, we present the complete list of all maximal lattice-free integral polyhedra in dimension three. Our results are motivated by recent research on cutting plane theory in mixed-integer linear optimization.


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## 1 Introduction

A convex set $K \subseteq \mathbb{R}^{d}$ with nonempty interior is called lattice-free if the interior of $K$ does not contain a point of $\mathbb{Z}^{d}$ and maximal lattice-free if $K$ is inclusion-maximal in the class of lattice-free convex sets (for different definitions of lattice-freeness see, for instance, Rez86, Sca85, Seb99]). Every maximal lattice-free set $K$ is a polyhedron with an integer point in the relative interior of each facet of $K$ (see, for instance, Lov89, Proposition 3.3]).

The study of maximal lattice-free polyhedra is motivated by recent research in mixed-integer linear optimization. Cutting planes for mixed-integer linear programs can be obtained from a simultaneous consideration of several rows of a simplex tableau (see, for instance, ALW09, ALWW07, AWW09b, Bal71, BBCM09, BC07, CM08, DR08, DW08, Esp08, Zam09). Such cutting planes are deducible from lattice-free convex sets. Furthermore, the strongest cutting planes are derived from maximal lattice-free polyhedra. It is therefore natural to ask for a characterization of maximal lattice-free polyhedra. Since we aim at algorithmic applications, we restrict considerations to the class of maximal lattice-free rational polyhedra. In this paper we answer the following two questions:
I. Given the dimension $d \in \mathbb{N}$ and the precision of a maximal lattice-free rational polyhedron $P$, how many different shapes are possible for $P$ ?
II. How do maximal lattice-free integral polyhedra in dimension three look like?

The answer to the first question is that $P$ can only have finitely many shapes (a precise formulation will be given in the following section). In particular, we prove the following result.

Theorem 1.1. Let $\mathcal{I}^{d}$ denote the set of all lattice-free integral polyhedra $P \subseteq \mathbb{R}^{d}$ such that $P$ is not properly contained in another lattice-free integral polyhedron. Then there exists a constant $N$ depending only on $d$, and polyhedra $P_{1}, \ldots, P_{N} \in \mathcal{I}^{d}$ such that for every $P \in \mathcal{I}^{d}$ one has $P=U P_{j}+v$ for some unimodular matrix $U \in \mathbb{Z}^{d \times d}$, integral vector $v \in \mathbb{Z}^{d}$, and $j \in\{1, \ldots, N\}$.

The proof of Theorem 1.1 suggests that the constant $N$ grows rapidly in $d$. Moreover, the proof does not imply any quick constructive procedure for enumeration of the polyhedra $P_{1}, \ldots, P_{N}$. Having applications in mixed-integer cutting plane theory in mind, it is thus desirable to provide a precise classification for small dimensions. Notice that finite termination of a cutting plane algorithm only
requires cutting planes associated with lattice-free integral polyhedra as derived in Theorem 1.1 (see (BCM10, DL09, DPW10). The explicit description of $\mathcal{I}^{d}$ in Theorem 1.1 for $d=1,2$ is folklore. However, already the class $\mathcal{I}^{3}$ is rather complex. Thus, the complete enumeration of $\mathcal{I}^{d}$ for an arbitrary $d \geq 3$ is challenging. We provide a classification of an important subclass of $\mathcal{I}^{3}$.

Theorem 1.1 follows directly from Theorem 2.1 by setting $s=1$. The classification of a subclass of $\mathcal{I}^{3}$ is stated in Theorem 2.2 ,

## 2 Main results and notation

Let us first introduce the notation used in the formulations of our main results. (Introduction of the standard notation is postponed to the end of this section.) For the relevant background information in convex geometry, in particular with respect to polyhedra and lattices, we refer to the books Bar02, Gru07, GL87, Roc72.

The intersection of finitely many closed halfspaces is said to be a polyhedron. By $\mathcal{P}^{d}$ we denote the set of all polyhedra in $\mathbb{R}^{d}$ (where the elements of $\mathcal{P}^{d}$ do not have to be full-dimensional). A bounded polyhedron is called a polytope. A polyhedron $P \in \mathcal{P}^{d}$ is said to be integral if $P=\operatorname{conv}\left(P \cap \mathbb{Z}^{d}\right)$; and $P$ is said to be rational if $s P:=\left\{s x \in \mathbb{R}^{d}: x \in P\right\}$ is an integral polyhedron for some finite integer $s \geq 1$. The precision of a rational polyhedron $P$ is the smallest integer $s \geq 1$ such that $s P$ is an integral polyhedron.

If $\Lambda$ is a lattice in $\mathbb{R}^{d}$, then a polyhedron $P \in \mathcal{P}^{d}$ is said to be $\Lambda$-free if $\operatorname{int}(P) \cap \Lambda=\emptyset$. For $\Lambda=\mathbb{Z}^{d}$ we say "lattice-free" rather than " $\Lambda$-free". In this paper, we restrict $\Lambda$ to be $s \mathbb{Z}^{d}$ for some $s \in \mathbb{N}$.

Our results are concerned with the interplay of the following three properties of polyhedra: integrality (abbreviated with " i "), $\Lambda$-freeness (abbreviated with " f " and an additional " s " in brackets to indicate the dependency on $\Lambda=s \mathbb{Z}^{d}$ ), and inclusion-maximality in a given class (abbreviated with " m "). By $\mathcal{P}_{\mathrm{i}}^{d}$ we denote the set of integral polyhedra belonging to $\mathcal{P}^{d}$, by $\mathcal{P}_{\text {if }}^{d}(s)$ the set of $\Lambda$-free polyhedra belonging to $\mathcal{P}_{\mathrm{i}}^{d}$, and by $\mathcal{P}_{\mathrm{ifm}}^{d}(s)$ the set of elements of $\mathcal{P}_{\text {if }}^{d}(s)$ which are maximal within $\mathcal{P}_{\text {if }}^{d}(s)$ with respect to inclusion.

Let $\operatorname{Aff}(\Lambda)$ denote the group of all affine transformations $T$ in $\mathbb{R}^{d}$ with $T(\Lambda)=\Lambda$. It is not hard to see that $\operatorname{Aff}(\Lambda) \subseteq \operatorname{Aff}\left(\mathbb{Z}^{d}\right)$. Henceforth, the transformations in $\operatorname{Aff}(\Lambda)$ are called $\Lambda$-preserving, while the transformations in $\operatorname{Aff}\left(\mathbb{Z}^{d}\right)$ are called unimodular. If a set $P$ can be mapped to a set $Q$ by a $\Lambda$-preserving transformation we simply say that both sets are equivalent. The group Aff $(\Lambda)$ has a natural action on $\mathcal{P}_{\mathrm{i}}^{d}$. Typically, we are interested in polyhedra in $\mathcal{P}_{\mathrm{i}}^{d}$ identified modulo $\operatorname{Aff}(\Lambda)$, since this identification does not change affine properties of integral polyhedra relative to the lattice $\Lambda$. In particular, two polyhedra $P, Q \in \mathcal{P}_{\mathrm{i}}^{d}$ which coincide up to an affine transformation in $\operatorname{Aff}(\Lambda)$ contain the same number of lattice points in $\mathbb{Z}^{d}$ and $\Lambda$ on corresponding faces.

Let us assume that $P \in \mathcal{P}^{d}$ is a maximal lattice-free rational polyhedron with precision $s$. Thus, $s P$ is an integral polyhedron and the maximality and lattice-freeness of $P$ with respect to the standard lattice $\mathbb{Z}^{d}$ transfers one-to-one into a maximality and $\Lambda$-freeness of $s P$ with respect to the lattice $\Lambda=s \mathbb{Z}^{d}$. Thus, instead of analyzing "maximal lattice-free rational polyhedra" (which correspond to cutting planes when rational data is assumed) we can equivalently consider the more convenient set of "maximal $\Lambda$-free integral polyhedra". Indeed, from an analytical point of view, the latter set is easier to handle since results from the literature can be used which are stated in terms of integral polyhedra. We are now ready to present our first main result.
Theorem 2.1. Let $d, s \in \mathbb{N}$. Then $\mathcal{P}_{\mathrm{ifm}}^{d}(s) / \operatorname{Aff}(\Lambda)$ is a finite set.
We now relate maximal $\Lambda$-free integral polyhedra to the set $\mathcal{P}_{\mathrm{ifm}}^{d}(s)$. Let $\mathcal{C}_{\mathrm{fm}}^{d}(s)$ be the class of all $\Lambda$-free convex sets in $\mathbb{R}^{d}$ which are not properly contained in another $\Lambda$-free convex set. The elements of $\mathcal{C}_{\mathrm{fm}}^{d}(s)$ are polyhedra (see Lov89, Proposition 3.3]). Thus, $\mathcal{C}_{\mathrm{fm}}^{d}(s)$ is the class of all maximal $\Lambda$-free polyhedra in $\mathbb{R}^{d}$. Let $\mathcal{P}_{\mathrm{fmi}}^{d}(s):=\mathcal{P}_{\mathrm{i}}^{d} \cap \mathcal{C}_{\mathrm{fm}}^{d}(s)$ be the class of all maximal $\Lambda$-free integral polyhedra in $\mathbb{R}^{d}$. By definition we have $\mathcal{P}_{\mathrm{fmi}}^{d}(s) \subseteq \mathcal{P}_{\mathrm{ifm}}^{d}(s)$. Both classes, $\mathcal{P}_{\mathrm{fmi}}^{d}(s)$ and $\mathcal{P}_{\mathrm{ifm}}^{d}(s)$, are of interest in cutting plane theory (see DPW10). In particular, the complete characterization of pairs of $s$ and $d$ for which the equality $\mathcal{P}_{\mathrm{fmi}}^{d}(s)=\mathcal{P}_{\mathrm{ifm}}^{d}(s)$ holds is unknown. For $d=1, s \geq 1$ and $d=2, s=1$ equality can be verified in a straightforward way. On the other hand, for $d \geq 2, s \geq 3$ the inclusion is strict. For instance, consider the polyhedron $Q_{s}^{d}:=\operatorname{conv}\left(\left\{o,(2 s+1) e_{1},(2 s+1) e_{1}+e_{2},(2 s-1) e_{1}+(2 s-1) e_{2}\right\}\right)+\operatorname{lin}\left(\left\{e_{3}, \ldots, e_{d}\right\}\right)$. It is easy to verify that $Q_{s}^{d} \in \mathcal{P}_{\text {ifm }}^{d}(s) \backslash \mathcal{P}_{\text {fmi }}^{d}(s)$. The remaining cases (that is, $d=2, s=2$ and $d \geq 3,1 \leq s \leq 2$ ) are open.

The finiteness of $\mathcal{P}_{\mathrm{fmi}}^{d}(s) / \operatorname{Aff}(\Lambda)$ follows directly from Theorem 2.1. This has two consequences: First, if we choose $s=1$, then for every dimension $d$, up to unimodular transformations, there is only a finite
number of maximal lattice-free integral polyhedra. Second, if we fix some integer $s \geq 1$ and consider the set of polytopes with vertices in $\frac{1}{s} \mathbb{Z}^{d}$, then there is only a finite number of maximal lattice-free polytopes in this set up to an affine transformation preserving $\mathbb{Z}^{d}$.

The second part of the paper deals with the classification of the set $\mathcal{P}_{\mathrm{fmi}}^{3}(1)$. As we show later (in Proposition (3.1), we can restrict ourselves to polytopes within $\mathcal{P}_{\text {fmi }}^{3}(1)$. Let $\mathcal{M}^{d}$ be the set of all maximal lattice-free integral polytopes in $\mathbb{R}^{d}$. In dimension one, the set $\mathcal{M}^{1}$ consists of all intervals $[n, n+1]$ for an integer $n$. Thus, up to a unimodular transformation, $[0,1]$ is the only maximal lattice-free integral polytope. In dimension two, it is easy to see that every element of $\mathcal{M}^{2}$ is equivalent to $\operatorname{conv}\left(\left\{o, 2 e_{1}, 2 e_{2}\right\}\right)$. In AWW09a it has been shown that, up to a unimodular transformation, there are only seven different simplices in $\mathcal{M}^{3}$. In this paper we complete the classification of elements of $\mathcal{M}^{3}$ by proving the following theorem.

Theorem 2.2. Let $P \in \mathcal{M}^{3}$. Then, up to a unimodular transformation, $P$ is one of the following polytopes (see Figure 1):

- one of the seven simplices

$$
\begin{aligned}
& M_{1}=\operatorname{conv}\left(\left\{o, 2 e_{1}, 3 e_{2}, 6 e_{3}\right\}\right), \\
& M_{2}=\operatorname{conv}\left(\left\{o, 2 e_{1}, 4 e_{2}, 4 e_{3}\right\}\right), \\
& M_{3}=\operatorname{conv}\left(\left\{o, 3 e_{1}, 3 e_{2}, 3 e_{3}\right\}\right), \\
& M_{4}=\operatorname{conv}\left(\left\{o, e_{1}, 2 e_{1}+4 e_{2}, 3 e_{1}+4 e_{3}\right\}\right), \\
& M_{5}=\operatorname{conv}\left(\left\{o, e_{1}, 2 e_{1}+5 e_{2}, 3 e_{1}+5 e_{3}\right\}\right), \\
& M_{6}=\operatorname{conv}\left(\left\{o, 3 e_{1}, e_{1}+3 e_{2}, 2 e_{1}+3 e_{3}\right\}\right), \\
& M_{7}=\operatorname{conv}\left(\left\{o, 4 e_{1}, e_{1}+2 e_{2}, 2 e_{1}+4 e_{3}\right\}\right),
\end{aligned}
$$

- the pyramid $M_{8}=\operatorname{conv}(B \cup\{a\})$ with the base $B=\operatorname{conv}\left(\left\{ \pm 2 e_{1}, \pm 2 e_{2}\right\}\right)$ and the apex $a=(1,1,2)$,
- the pyramid $M_{9}=\operatorname{conv}(B \cup\{a\})$ with the base $B=\operatorname{conv}\left(\left\{-e_{1},-e_{2}, 2 e_{1}, 2 e_{2}\right\}\right)$ and the apex $a=$ $(1,1,3)$,
- the prism $M_{10}=\operatorname{conv}(B \cup(B+u))$ with the bases $B$ and $B+u$, where $B=\operatorname{conv}\left(\left\{e_{1}, e_{2},-\left(e_{1}+e_{2}\right)\right\}\right)$ and $u=(1,2,3)$,
- the prism $M_{11}=\operatorname{conv}(B \cup(B+u))$ with the bases $B$ and $B+u$, where $B=\operatorname{conv}\left(\left\{ \pm e_{1}, 2 e_{2}\right\}\right)$ and $u=(1,0,2)$,
- the parallelepiped $M_{12}=\operatorname{conv}\left(\left\{\sigma_{1} u_{1}+\sigma_{2} u_{2}+\sigma_{3} u_{3}: \sigma_{1}, \sigma_{2}, \sigma_{3} \in\{0,1\}\right\}\right)$ where $u_{1}=(-1,1,0)$, $u_{2}=(1,1,0)$, and $u_{3}=(1,1,2)$.

We now introduce some further notation. Throughout the paper, $d \in \mathbb{N}$ is the dimension of the underlying space. Elements of $\mathbb{R}^{d}$ are considered to be column vectors. Transposition is denoted by $(\cdot)^{\top}$ and the origin by $o$. We denote by $e_{j}$ the $j$ th unit vector. Its size will always be clear from the context. For $x, y \in \mathbb{R}^{d}$, we denote by $[x, y]$ the line segment with endpoints $x$ and $y$, and by $[x, y\rangle$ the ray (i.e., the half-line) emanating from $x$ and passing through $y$. An additive subgroup $\Lambda$ of $\mathbb{R}^{d}$ is said to be a lattice if the intersection of $\Lambda$ with every compact set of $\mathbb{R}^{d}$ is finite. In this paper, for the sake of simplicity, we fix our underlying lattice to be $\mathbb{Z}^{d}$, though, due to affine invariance, the obtained results are independent of the concrete choice of the lattice.

Given a set $K \subseteq \mathbb{R}^{d}$, we use the functionals $\operatorname{conv}(K)$ (convex hull of $K$ ), aff( $K$ ) (affine hull of $K$ ), $\operatorname{lin}(K)$ (linear hull of $K$ ), $\operatorname{int}(K)$ (interior of $K$ ), $\operatorname{relint}(K)$ (relative interior of $K$ ), relbd $(K)$ (relative boundary of $K$ ), $\operatorname{rec}(K)$ (recession cone of $K$ ), and vert $(K)$ (set of vertices of $K$ ). For $K \subseteq \mathbb{R}^{d}$, $\operatorname{vol}(K)$ denotes the volume of $K$ in $\operatorname{aff}(K)$.

The dual lattice of $\Lambda=s \mathbb{Z}^{d}$ is $\Lambda^{*}=\frac{1}{s} \mathbb{Z}^{d}$. By $\pi$ we denote the projection onto the first $d-1$ coordinates, i.e., the mapping $\pi(x):=\left(x_{1}, \ldots, x_{d-1}\right)$, where $x:=\left(x_{1}, \ldots, x_{d}\right) \in \mathbb{R}^{d}$. This implies $\pi(\Lambda)=s \mathbb{Z}^{d-1}$. If $K \subseteq \mathbb{R}^{d}$ is a closed convex set with nonempty interior, then the lattice width of $K$ (with respect to the lattice $\Lambda$ ) is defined by

$$
w_{\Lambda}(K):=\min _{u \in \Lambda^{*} \backslash\{o\}} w(K, u),
$$

where $w(K, u)$, for $u \in \mathbb{R}^{d}$, is the width function given by

$$
w(K, u):=\max _{x \in K} u^{\top} x-\min _{x \in K} u^{\top} x
$$



Figure 1: All maximal lattice-free integral polytopes in dimension three

The lattice width of $K$ with respect to $\Lambda$ can be seen as the smallest number of "lattice slices" of $K$ along any nonzero vector in $\Lambda^{*}$.

Theorem [2.1] is proved in Section 3. In Section 4 we introduce the tools which we need for proving Theorem 2.2 and we explain the idea of the proof. The proof of Theorem [2.2 is given in Sections 5 . 8

## 3 The finiteness proof

In this section, we prove Theorem 2.1 Let us first highlight the main steps of the proof.

1. Reduction to polytopes. Every unbounded $P \in \mathcal{P}_{\mathrm{ifm}}^{d}(s)$ is the direct product of an affine space and a polytope in $\mathcal{P}_{\mathrm{ifm}}^{k}(s)$ for some $1 \leq k \leq d$ (see Proposition 3.1). Thus, it suffices to verify finiteness only for polytopes within $\mathcal{P}_{\text {ifm }}^{d}(s)$.
2. Bounding $|\boldsymbol{P} \cap \boldsymbol{\Lambda}|$. Consider a polytope $P \in \mathcal{P}_{\text {ifm }}^{d}(s)$. We construct an upper bound on the number of points of $\Lambda$ on the boundary of $P$. For that, we use the lattice diameter. The lattice diameter of $P$ with respect to $\Lambda$ is defined as the maximum of $|l \cap P \cap \Lambda|-1$ over all lines $l$. We show that the lattice diameter of $P$ is bounded from above in terms of $d$ and $s$ only. This is done as follows.

We assume by contradiction that, for some line $l,|l \cap P \cap \Lambda|-1$ is a large number $M$. By a $\Lambda$-preserving transformation, $l=\operatorname{lin}\left(\left\{e_{d}\right\}\right)$. Let $P^{\prime}$ be the projection of $P$ onto the first $d-1$ coordinates. Then $\pi(l)=o$ and from $P \in \mathcal{P}_{\text {ifm }}^{d}(s)$ it follows $\operatorname{int}\left(P^{\prime}\right) \cap \pi(\Lambda) \neq \emptyset$ (see Lemma 3.7). Choose $p \in \operatorname{int}\left(P^{\prime}\right) \cap \pi(\Lambda)$. Then we construct a $k$-dimensional simplex $S$ with vertices
$o=p_{0}, p_{1}, \ldots, p_{k}$ in $\mathbb{Z}^{d-1}$ such that $p$ is the only point of $\pi(\Lambda)$ in the relative interior of $S$. This construction is the key ingredient in our proof (see Lemma 3.6). Let $\lambda_{0}, \ldots, \lambda_{k}$ be the barycentric coordinates of $p$ with respect to $S$. By results of Hen83, LZ91 (see Theorem 3.4), the $\lambda_{i}$ 's are bounded from below in terms of $d$ and $s$ only. The length of $(p+l) \cap P$ is bounded from below in terms of $\lambda_{0}$ and $M$. On the other hand, since $P$ is $\Lambda$-free, the length of $(p+l) \cap P$ is at most $s$. So, if $M$ is too large, this leads to a contradiction.
The upper bound on the lattice diameter implies an upper bound on $|P \cap \Lambda|$ (see Lemma 3.8).
3. Conclusion of finiteness. The upper bound on $|P \cap \Lambda|$ together with results of Hen83, LZ91] (see Theorem 3.5) implies an upper bound on the volume of $P$ (see Theorem 3.3). All bounds only depend on $d$ and $s$. This, in turn, yields finiteness of $\mathcal{P}_{\mathrm{ifm}}^{d}(s)$ (see Theorem 3.2).

The fact that we can restrict to the study of polytopes in $\mathcal{P}_{\mathrm{ifm}}^{d}(s)$ is a consequence of the following proposition. We point out that a similar result is true for the set $\mathcal{C}_{\mathrm{fm}}^{d}(s)$ as well (see Lov89, Proposition 3.1]).

Proposition 3.1. Let $d, s \in \mathbb{N}$ and let $P \in \mathcal{P}_{\mathrm{ifm}}^{d}(s)$. Then there exists some $k \in\{1, \ldots, d\}$ and a polytope $P^{\prime} \in \mathcal{P}_{\mathrm{ifm}}^{k}(s)$ such that $P \equiv P^{\prime} \times \mathbb{R}^{d-k}(\bmod \operatorname{Aff}(\Lambda))$.

Proof. If $P$ is bounded, the assertion is trivial as we let $k=d$ and $P^{\prime}=P$. Let $P$ be unbounded. By an inductive argument, it suffices to show the existence of $P^{\prime} \in \mathcal{P}_{\text {ifm }}^{d-1}(s)$ such that $P \equiv P^{\prime} \times \mathbb{R}(\bmod \operatorname{Aff}(\Lambda))$.

Since $P$ is unbounded, the recession cone of $P$ contains nonzero vectors. Since $P$ is integral, the recession cone of $P$ is an integral polyhedron (see, for example, Sch86, §16.2]). Thus, the recession cone of $P$ contains a nonzero integer vector $u$. By scaling, we can assume that $u \in \Lambda$.

Applying a $\Lambda$-preserving transformation we assume that $u=s e_{d}$. The polyhedron $P^{\prime}:=\pi(P) \subseteq \mathbb{R}^{d-1}$ is $\pi(\Lambda)$-free. In fact, assume there exists a point $p^{\prime} \in \operatorname{int}\left(P^{\prime}\right) \cap \pi(\Lambda)$. Then $\operatorname{int}(P) \cap \pi^{-1}\left(p^{\prime}\right)$ is nonempty and contains infinitely many points of $\Lambda$, a contradiction to the choice of $P$.

Since $P^{\prime}$ is $\pi(\Lambda)$-free, $\pi^{-1}\left(P^{\prime}\right)$ is $\Lambda$-free. By construction, $P \subseteq \pi^{-1}\left(P^{\prime}\right)$, and since $P$ is maximal in $\mathcal{P}_{\text {if }}^{d}(s)$ we even have $P=\pi^{-1}\left(P^{\prime}\right)$. Furthermore, $P^{\prime} \in \mathcal{P}_{\mathrm{ifm}}^{d-1}(s)$. In fact, if $P^{\prime}$ were not maximal in $\mathcal{P}_{\text {if }}^{d-1}(s)$ we could find $P^{\prime \prime} \in \mathcal{P}_{\text {if }}^{d-1}(s)$ such that $P^{\prime} \varsubsetneqq P^{\prime \prime}$. Then $P$ is properly contained in the $\Lambda$-free integral polyhedron $\pi^{-1}\left(P^{\prime \prime}\right)$, a contradiction to the assumptions on $P$. By construction, $P \equiv$ $P^{\prime} \times \mathbb{R}(\bmod \operatorname{Aff}(\Lambda))$.

The following is well-known (see, for instance, [LZ91, Theorem 2]).
Theorem 3.2. Let $d, s \in \mathbb{N}$ and let $\mathcal{X} \subseteq \mathcal{P}_{i}^{d}$ be a set of polytopes. Then the set $\mathcal{X} / \operatorname{Aff}(\Lambda)$ (where $\Lambda=s \mathbb{Z}^{d}$ ) is finite if and only if the volume of all elements of $\mathcal{X}$ is bounded from above by a constant depending only on $d$ and $s$.

In the remainder of this section we prepare the proof of the following theorem.
Theorem 3.3. Let $d, s \in \mathbb{N}$. Then there exists a constant $V(d, s)>0$ such that for every polytope $P$ in $\mathcal{P}_{\mathrm{ifm}}^{d}(s)$ the inequality $\operatorname{vol}(P) \leq V(d, s)$ is fulfilled.

Once, Theorem 3.3 is proven, Theorem 2.1 is a direct consequence of Proposition 3.1. Theorem 3.2 and Theorem 3.3

The proof of Theorem [3.3 relies on results of Hensley Hen83] and Lagarias and Ziegler [LZ91. Hensley [Hen83] showed that the volume and the total number of integer points of a $d$-dimensional integral polyhedron with precisely $k>0$ interior integer points can be bounded in terms of $d$ and $k$ only. Lagarias and Ziegler LZ91 improved these bounds and generalized parts of Hensley's results. In this paper, we shall use the main results as well as some intermediate assertions from Hen83 and [Z91.

A polytope $S$ is said to be a simplex if $S$ is the convex hull of finitely many affinely independent points. If $S$ is a simplex in $\mathbb{R}^{d}$ with vertices $p_{0}, \ldots, p_{k}(0 \leq k \leq d)$ and $p$ is a point in $S$, then $p$ can be uniquely represented by $p=\sum_{j=0}^{k} \lambda_{j} p_{j}$, where $\lambda_{0}, \ldots, \lambda_{k} \geq 0$ and $\lambda_{0}+\cdots+\lambda_{k}=1$. The values $\lambda_{0}, \ldots, \lambda_{k}$ are called the barycentric coordinates of $p$ with respect to the simplex $S$. The point $p$ lies in the relative interior of $S$ if and only if $\lambda_{0}, \ldots, \lambda_{k}>0$.

Theorem 3.4. (Hen83, Theorem 3.1] and LZ91, Lemma 2.2].) Let $d, s \in \mathbb{N}$. Then there exists a constant $\lambda^{*}(d, s)>0$ such that, for every d-dimensional integral simplex $S$ in $\mathbb{R}^{d}$ with precisely one interior point $p$ in $s \mathbb{Z}^{d}$, all barycentric coordinates $\lambda_{i}(i=0, \ldots, d)$ of $p$ with respect to $S$ satisfy $\lambda_{i} \geq \lambda^{*}(d, s)$.

Note that, in the formulation of Theorem 3.4 $\lambda^{*}(d, s)$ is not necessarily best possible. Once some $\lambda^{*}(d, s)$ is known, then any smaller positive constant works as well. Thus, it is always possible and will be convenient later to choose the values $\lambda^{*}(d, s)$ to be nonincreasing in $d \in \mathbb{N}$. In fact, the best known concrete values for $\lambda^{*}(d, s)$, given in LZ91, Lemma 2.2], are nonincreasing in $d$.

Theorem 3.5. (Hen83, Theorem 3.6] and [Z91, Theorem 1].) Let $d, s, k \in \mathbb{N}$ and let $\Lambda=s \mathbb{Z}^{d}$. Let $\mathcal{X}$ be the class of all d-dimensional polytopes $P \in \mathcal{P}_{\mathrm{i}}^{d}$ with $1 \leq|\operatorname{int}(P) \cap \Lambda| \leq k$. Then there exists a constant $V(d, s, k)>0$ such that for every $P \in \mathcal{X}$ one has $\operatorname{vol}(P) \leq V(d, s, k)$. In particular, $\mathcal{X} / \operatorname{Aff}(\Lambda)$ is a finite set.

We have mentioned all results from the literature that are needed to prove Theorem 3.3 Let us now show our assertion. We point out that in the remainder of this section, for all statements and proofs, we always assume $\Lambda=s \mathbb{Z}^{d}$.

Let $a \in \Lambda$ and let $\mathcal{X}^{d}(a)$ be the class of all polyhedra $P \in \mathcal{P}_{\mathrm{i}}^{d}$ such that $a \in \operatorname{relbd}(P)$ and $\operatorname{relint}(P) \cap \Lambda \neq$ $\emptyset$. On $\mathcal{X}^{d}(a)$ we introduce the partial order $\preceq$ as follows: for $P, Q \in \mathcal{X}^{d}(a)$ we define $P \preceq Q$ if and only if $\operatorname{relint}(P) \subseteq \operatorname{relint}(Q)$. Let us verify that the binary relation $\preceq$ is indeed a partial order. The property $P \preceq P$ is obvious. If $P \preceq Q$ and $Q \preceq P$, then $\operatorname{relint}(P)=\operatorname{relint}(Q)$. Since $P$ and $Q$ are closed it follows $P=Q$. If $P \preceq Q$ and $Q \preceq R$, then $\operatorname{relint}(P) \subseteq \operatorname{relint}(Q) \subseteq \operatorname{relint}(R)$. Thus $P \preceq R$.

By $\mathcal{R}^{d}(a)$ we denote the set of the minimal elements of the poset $\left(\mathcal{X}^{d}(a), \preceq\right)$, i.e., the set of the elements $Q \in \mathcal{X}^{d}(a)$ such that there exists no $P \in \mathcal{X}^{d}(a)$ with $P \preceq Q$ and $P \neq Q$. We emphasize that elements of $\mathcal{X}^{d}(a)$ and $\mathcal{R}^{d}(a)$ do not have to be full-dimensional. It is not hard to verify that for every $P \in \mathcal{X}^{d}(a)$ there exists $Q \in \mathcal{R}^{d}(a)$ such that $Q \preceq P$. If $P$ is bounded, this follows from the fact that the set of all $Q \in \mathcal{X}^{d}(a)$ satisfying $Q \preceq P$ is finite as $\left|P \cap \mathbb{Z}^{d}\right|<+\infty$. If $P$ is unbounded we replace $P$ by $\bar{P}=\operatorname{conv}\left(P \cap B \cap \mathbb{Z}^{d}\right)$, where $B$ is a sufficiently large box centered at $a$ and such that $\operatorname{relint}(\bar{P}) \cap \Lambda \neq \emptyset$. Then we apply the argument for the bounded case to $\bar{P}$. We remark that for $P, Q \in \mathcal{X}^{d}(a)$ the condition $\operatorname{relint}(P) \subseteq \operatorname{relint}(Q)$ holds if and only if one has $P \subseteq Q$ and $\operatorname{relint}(P) \cap \operatorname{relint}(Q) \neq \emptyset$. This follows from standard results in convexity (see, for example, [Roc72, Theorem 6.5]).

It turns out that the elements of $\mathcal{R}^{d}(a)$ have a very specific shape which is described as follows.
Lemma 3.6. Let $a \in \Lambda$ and $P \in \mathcal{R}^{d}(a)$. Then $P$ has the following properties.
I. $P$ is a simplex of dimension $k \in\{1, \ldots, d\}$.
II. $a \in \operatorname{vert}(P)$.
III. relint $(P) \cap \Lambda$ consists of precisely one point.
$I V$. The facet $F$ of $P$ opposite to the vertex a satisfies $F \cap \mathbb{Z}^{d}=\operatorname{vert}(F)$.
Proof. Let $P \in \mathcal{R}^{d}(a)$ and $q \in \operatorname{relint}(P) \cap \Lambda$ be arbitrary. We consider $2 q-a$ (the reflection of $a$ with respect to $q$ ). First assume that $2 q-a \in P$. Then $q \in \operatorname{relint}(P) \cap \operatorname{relint}([a, 2 q-a])$ and $[a, 2 q-a] \subseteq P$. Thus, since $P \in \mathcal{R}^{d}(a)$ we have $P=[a, 2 q-a]$. Again, since $P \in \mathcal{R}^{d}(a), q$ is the only point of $\Lambda$ in relint $(P)$. For such a $P$, Parts IV follow immediately. In the remainder of the proof let $2 q-a \notin P$.

Parts § and [I]. Let $b$ be the intersection point of $[a, q\rangle$ and $\operatorname{relbd}(P)$. Since $q \in \operatorname{relint}([a, b])$ we have $q=(1-\lambda) a+\lambda b$ for some $0<\lambda<1$. Consider a facet $F$ of $P$ which contains $b$. Since $P$ is integral, also $F$ is integral, i.e., $F=\operatorname{conv}\left(F \cap \mathbb{Z}^{d}\right)$. By Carathéodory's theorem, there exist affinely independent points $q_{1}, \ldots, q_{k} \in F \cap \mathbb{Z}^{d}$ such that $b=\lambda_{1} q_{1}+\cdots+\lambda_{k} q_{k}$ for some $\lambda_{1}, \ldots, \lambda_{k}>0$ with $\lambda_{1}+\cdots+\lambda_{k}=1$. Thus, $q=(1-\lambda) q_{0}+\lambda \lambda_{1} q_{1} \cdots+\lambda \lambda_{k} q_{k}$, where $q_{0}:=a$. The point $a=q_{0}$ does not belong to aff $(F)$. In fact, otherwise $a \in P \cap \operatorname{aff}(F)=F$ and since $b \in F$ we get $q \in F$, a contradiction to $q \in \operatorname{relint}(P)$. Hence $q_{0}, \ldots, q_{k}$ are affinely independent. Since $P \in \mathcal{R}^{d}(a)$, we have $P=\operatorname{conv}\left(\left\{q_{0}, \ldots, q_{k}\right\}\right)$. Hence $P$ is a simplex and $a$ is a vertex of $P$.

In the remainder of the proof let $P=\operatorname{conv}\left(\left\{q_{0}, \ldots, q_{k}\right\}\right)$ with $q_{0}:=a$ and $q_{1}, \ldots, q_{k}$ defined as above.
Part III, For $j=0, \ldots, k$ let $P_{j}$ be the simplex with vertices $\left\{q, q_{0}, \ldots, q_{k}\right\} \backslash\left\{q_{j}\right\}$. It can be verified with straightforward arguments that $P=P_{0} \cup \cdots \cup P_{k}$ and the relative interiors of the simplices $P_{j}$ are pairwise disjoint. For proving Part III we argue by contradiction. We assume that $\operatorname{relint}(P) \cap \Lambda$ contains $q^{\prime}$ with $q^{\prime} \neq q$. First we show that $q^{\prime} \in P_{0}$. Assume the contrary. Then $q^{\prime} \in P_{j}$ for some $j \in\{1, \ldots, k\}$. Let $F$ be the face of $P_{j}$ with $q^{\prime} \in \operatorname{relint}(F)$. Since $q^{\prime} \notin P_{0}, a$ is a vertex of $F$. The existence of $F \nsubseteq P$ with $q^{\prime} \in \operatorname{relint}(F)$ and $a \in \operatorname{vert}(F)$ contradicts the fact that $P \in \mathcal{R}^{d}(a)$. Hence $q^{\prime} \in P_{0}$. We define $Q:=\operatorname{conv}\left(\left(P_{0} \cap \mathbb{Z}^{d}\right) \backslash\{q\}\right)$. Since $q^{\prime} \in \operatorname{relint}(P)$, and $q^{\prime}, q_{1}, \ldots, q_{k} \in Q$, the polytope $Q$ has the same dimension as $P$. We have $[a, q\rangle \cap Q=\left[b, b^{\prime}\right]$, where $b \in \operatorname{relint}(P)$ and $b^{\prime} \in \operatorname{relbd}(P)$.

Since $q \in \operatorname{relint}([a, b])$ one has $q=(1-\lambda) a+\lambda b$ for some $0<\lambda<1$. Let now $G$ be the facet of $Q$ containing $b$. The point $a=q_{0}$ does not belong to aff $(G)$. In fact, otherwise aff $(G)$ would contain $\left[b, b^{\prime}\right]$, which implies aff $(G) \cap \operatorname{relint} Q \neq \emptyset$, a contradiction. Using Carathéodory's theorem, let $p_{1}, \ldots, p_{m}$ be affinely independent vertices of $G$ such that $b=\lambda_{1} p_{1}+\cdots+\lambda_{m} p_{m}$ for some $\lambda_{1}, \ldots, \lambda_{m}>0$ with $\lambda_{1}+\cdots+\lambda_{m}=1$. Then $q=(1-\lambda) p_{0}+\lambda \lambda_{1} p_{1}+\cdots+\lambda \lambda_{m} p_{m}$ with $p_{0}:=a$. Since $p_{0} \notin \operatorname{aff}(G)$ and since $p_{1}, \ldots, p_{m} \in G$ are affinely independent, we see that $p_{0}, \ldots, p_{m}$ are affinely independent. The simplex $S=\operatorname{conv}\left(\left\{p_{0}, \ldots, p_{m}\right\}\right)$ is properly contained in $P$, contains $a$ on its relative boundary and satisfies $q \in \operatorname{relint}(S) \cap \operatorname{relint}(P)$, a contradiction to the fact that $P \in \mathcal{R}^{d}(a)$. This shows Part 【II,

PartIV. We argue by contradiction. Let $F$ be the facet of $P$ opposite to $a$ and assume that $\operatorname{vert}(F) \nsubseteq$ $F \cap \mathbb{Z}^{d}$. Let $S_{1}, \ldots, S_{m}$ be a triangulation constructed on the points $F \cap \mathbb{Z}^{d}$. Then, $S_{1}, \ldots, S_{m}$ are simplices with pairwise disjoint interiors having the same dimension as $F$ and such that $F \cap \mathbb{Z}^{d}=\bigcup_{i=1}^{m} \operatorname{vert}\left(S_{i}\right)$, $F=\bigcup_{i=1}^{m} S_{i}$, and for every $S_{i}, \operatorname{vert}\left(S_{i}\right)$ are the only integer points in $S_{i}$. By assumption, we have $S_{i} \neq F$ for every $i$.

Then there exists a simplex $S_{j}$ such that $[a, q\rangle \cap S_{j}$ is nonempty. Let $b$ be the point $[a, q\rangle \cap S_{j}$. Further on, let $G$ be the face of $S_{j}$ with $b \in \operatorname{relint}(G)$. By construction, $q \in \operatorname{relint}(\bar{P})$ where $\bar{P}:=\operatorname{conv}(\{a\} \cup G)$ and $\bar{P} \nsubseteq P$. This contradicts the fact that $P \in \mathcal{R}^{d}(a)$.

Lemma 3.6 and the following Lemma 3.7 are used in the proof of Lemma 3.8.
Lemma 3.7. Let $P \in \mathcal{P}_{\mathrm{ifm}}^{d}(s)$ be a polytope. Then $\operatorname{int}(\pi(P)) \cap \pi(\Lambda) \neq \emptyset$.
Proof. If $P^{\prime}:=\pi(P)$ satisfies $\operatorname{int}\left(P^{\prime}\right) \cap \pi(\Lambda)=\emptyset$, then $\pi^{-1}\left(P^{\prime}\right)$ is $\Lambda$-free and integral, and then in view of the maximality of $P$, one has $\pi^{-1}\left(P^{\prime}\right) \subseteq P$ which contradicts the boundedness of $P$.

In the following lemma we prove that the number of points of $\Lambda$ on the boundary of a polytope $P \in \mathcal{P}_{\mathrm{ifm}}^{d}(s)$ is bounded by a constant which is dependent only on $d$ and $s$.
Lemma 3.8. Let $d, s \in \mathbb{N}$. Then there exists a constant $N(d, s)>0$ such that every polytope $P \in \mathcal{P}_{\mathrm{ifm}}^{d}(s)$ contains at most $N(d, s)$ points in $\Lambda$.

Proof. Let $P \in \mathcal{P}_{\text {ifm }}^{d}(s)$ be a polytope. For the purpose of deriving a contradiction assume that $|P \cap \Lambda| \geq$ $M^{d}+1$, where $M=\left\lceil\frac{1}{\lambda^{*}(d, s)}+1\right\rceil$ with $\lambda^{*}(d, s)$ defined as in the formulation of Theorem 3.4. Thus, there exist two distinct points $v, w \in P \cap \Lambda$ such that $\frac{1}{s} v \equiv \frac{1}{s} w(\bmod M)$. Then we can choose pairwise distinct $z_{0}, \ldots, z_{M}$ in $P \cap \Lambda \cap \operatorname{aff}(\{v, w\})$ such that $\operatorname{conv}\left(\left\{z_{0}, \ldots, z_{M}\right\}\right) \cap \Lambda=\left\{z_{0}, \ldots, z_{M}\right\}$. Performing a $\Lambda$-preserving transformation we assume that $z_{j}=j \cdot s e_{d}$ for $j=0, \ldots, M$. One has $\pi\left(z_{j}\right)=o$ for every $j=0, \ldots, M$. Since $M \geq 2$ (which follows from $\lambda^{*}(d, s)>0$ ), $o$ is a boundary point of $P^{\prime}:=\pi(P)$, otherwise $P$ would not be $\Lambda$-free. By Lemma 3.7 $\operatorname{int}\left(P^{\prime}\right) \cap \pi(\Lambda) \neq \emptyset$.

By construction, $P^{\prime}$ is integral and belongs to $\mathcal{X}^{d-1}(o)$. Thus, there exists a polytope $Q \in \mathcal{R}^{d-1}(o)$ with $\operatorname{relint}(Q) \subseteq \operatorname{int}\left(P^{\prime}\right)$. By Lemma [3.6, $Q$ is a simplex with precisely one point of $\pi(\Lambda)$, say $p$, in the relative interior. Let $k$ be the dimension of $Q$ and let $p_{0}, \ldots, p_{k}$ be the vertices of $Q$ with $p_{0}=o$. By Theorem 3.4, if $p=\sum_{j=0}^{k} \lambda_{j} p_{j}$ with $\lambda_{0}, \ldots, \lambda_{k}>0$ and $\lambda_{0}+\cdots+\lambda_{k}=1$, then one has $\lambda_{j} \geq \lambda^{*}(d, s)$ for every $j=0, \ldots, k$, where $\lambda^{*}(d, s)$ is a constant as in the formulation of Theorem 3.4 For a point $x \in P^{\prime}$, let $f(x)$ denote the length of the line segment $\pi^{-1}(x) \cap P$ (and thus, $f$ represents an "X-Ray picture" of $P)$. Employing the convexity of $P$ we see that $f(\cdot)$ is concave on $P^{\prime}$. Consequently,

$$
f(p)=f\left(\sum_{j=0}^{k} \lambda_{j} p_{j}\right) \geq \sum_{j=0}^{k} \lambda_{j} f\left(p_{j}\right) \geq \lambda_{0} f\left(p_{0}\right) \geq \lambda^{*}(d, s) s M>s
$$

On the other hand, since $p \in \operatorname{int}\left(P^{\prime}\right) \cap \pi(\Lambda)$, we have $f(p) \leq s$ as otherwise $P$ would not be $\Lambda$-free. Thus, this gives a contradiction to our assumption on $|P \cap \Lambda|$. It follows that $P$ contains at most $M^{d}$ points in $\Lambda$ and we can choose $N(d, s):=M^{d}$.

Proof of Theorem 3.3. Let $P \in \mathcal{P}_{\text {ifm }}^{d}(s)$ be a polytope. In the following, we enlarge $P$ to a polytope $Q \in \mathcal{P}_{\mathrm{i}}^{d}$ such that $P \subseteq Q$ and $\emptyset \neq \operatorname{int}(Q) \cap \Lambda \subseteq P \cap \Lambda$. By Lemma 3.8 this implies $1 \leq|\operatorname{int}(Q) \cap \Lambda| \leq$ $|P \cap \Lambda| \leq N(d, s)$, with $N(d, s)$ defined in the formulation of Lemma 3.8 Then, by Theorem 3.5 $\operatorname{vol}(P) \leq \operatorname{vol}(Q) \leq V(d, s, N(d, s))$ with $V(d, s, N(d, s))$ defined according to Theorem3.5. Consequently, $\operatorname{vol}(P) \leq V(d, s):=V(d, s, N(d, s))$.

Let us now construct the polytope $Q$. For that, we consider a sequence of polytopes $P^{i}$ which we define iteratively. Choose an arbitrary $p_{1} \in \Lambda$ such that $p_{1} \notin P$ and let $P^{1}:=\operatorname{conv}\left(P \cup\left\{p_{1}\right\}\right)$. For $i \geq 1$,
we proceed as follows. If $\operatorname{int}\left(P^{i}\right) \cap \Lambda \subseteq P \cap \Lambda$, then we stop and define $Q:=P^{i}$. Otherwise, we select $p_{i+1} \in\left(\operatorname{int}\left(P^{i}\right) \cap \Lambda\right) \backslash(P \cap \Lambda)$ and set $P^{i+1}:=\operatorname{conv}\left(P \cup\left\{p_{i+1}\right\}\right)$. Note that $P^{i+1} \nsubseteq P^{i}$ for all $i \geq 1$ and that the sequence is finite since $P$ is a polytope. Eventually, we construct a polytope $Q \in \mathcal{P}_{\mathrm{i}}^{d}$ such that $P \subseteq Q$ and $\operatorname{int}(Q) \cap \Lambda \subseteq P \cap \Lambda$. Furthermore, $\operatorname{int}(Q) \cap \Lambda \neq \emptyset$ since $P$ is properly contained in $Q$ and $P$ is maximal $\Lambda$-free.

Proof of Theorem [2.1] The theorem follows directly from Proposition 3.1, Theorem 3.2, and Theorem 3.3.

Remark 3.9. (The role of $\mathcal{R}^{d}(a)$.) In our proofs we use the class $\mathcal{R}^{d}(a)$. The properties of elements of this class are stated in Lemma 3.6. It seems that the class $\mathcal{R}^{d}(a)$ deserves an independent consideration.

Remark 3.10. (Growth of constants.) Let us analyze the growth of the constants in our statements.
$V(d, s, k)$ must be (at least) double exponential in d. It can be chosen to be $k(8 d s)^{d}(8 s+7)^{d 2^{2 d+1}}$ (see [Pik01]).

From the proof of Theorem 3.3 and the above bound on $V(d, s, k)$, it follows that for a polytope $P \in \mathcal{P}_{\mathrm{ifm}}^{d}(s)$ we have $\operatorname{vol}(P) \leq V(d, s, N(d, s)) \leq N(d, s) \cdot(8 d s)^{d}(8 s+7)^{d 2^{2 d+1}}$. By the proof of Lemma 3.8. we can choose $N(d, s)=\left\lceil\frac{1}{\lambda^{*}(d, s)}+1\right\rceil^{d}$.

The best known lower bound on the constant $\lambda^{*}(d, s)$ is $(7(s+1))^{-2^{d+1}}$ (see [LZ91, Lemma 2.1]). Substituting this into the above formula yields

$$
\begin{equation*}
\operatorname{vol}(P) \leq\left\lceil\left(1+(7(s+1))^{2^{d+1}}\right\rceil^{d}(8 d s)^{d}(8 s+7)^{d 2^{2 d+1}}\right. \tag{1}
\end{equation*}
$$

In the asymptotic notation the bound can be expressed as $\operatorname{vol}(P)=(s+1)^{O\left(d 4^{d}\right)}$.
Below we give an example which shows that the maximum volume over all polytopes of $\mathcal{P}_{\mathrm{ifm}}^{d}(s)$ is at least of order $(s+1)^{\Omega\left(2^{d}\right)}$. We use the following sequence considered in [LZ91, Lemma 2.1]. (For the sake of simplicity the dependency on $s$ is not indicated explicitly.) Let us define $y_{1}:=s+1$ and $y_{j}:=1+s \prod_{i=1}^{j-1} y_{i}$ for $j \geq 2$ (equivalently one can use the recurrency $y_{j}=y_{j-1}^{2}-y_{j}+1$ ). For every $d \geq 2$, we introduce the simplex $S_{d}:=\operatorname{conv}\left(\left\{y_{1} e_{1}, \ldots, y_{d-1} e_{d-1},\left(y_{d}-1\right) e_{d}\right\}\right)$. The verification of the fact that $S_{d}$ belongs to $\mathcal{P}_{\mathrm{ifm}}^{d}(s)$ (and even to $\mathcal{P}_{\mathrm{fmi}}^{d}(s)$ ) is left to the reader. The volume of $S_{d}$ can be expressed by

$$
\operatorname{vol}\left(S_{d}\right)=\frac{1}{d!}\left(\prod_{i=1}^{d-1} y_{i}\right)\left(y_{d}-1\right)=\frac{1}{d!} \frac{1}{s}\left(y_{d}-1\right)^{2}
$$

As was noticed in [LZ91, $p$. 1026] one has $y_{d} \geq(s+1)^{2^{d-2}}$ for all $d \geq 2$. This shows $\operatorname{vol}\left(S_{d}\right)=(s+1)^{\Omega\left(2^{d}\right)}$.
Bound (11) does not help to determine all bounded elements of $\mathcal{P}_{\text {ifm }}^{d}(s)$ for fixed values of $d$ and $s$ since the right hand side is tremendously large (for example, more than $10^{500}$ for $d=3$ and $s=1$ ). This is the reason, why our proof of Theorem 2.2 (presented in the following sections) does not rely on (11).

## 4 Tools and proof outline for the explicit description in dimension three

In Sections 5 6, and 7 we use the following additional notation. The area of a set $K \subseteq \mathbb{R}^{2}$ is denoted $\mathrm{A}(K)$ (which is shorter than $\operatorname{vol}(K)$ which we used in the previous sections). Since the only lattice in Sections 5, 6, and 7 is the standard lattice, we write $w(K)$ instead of $w_{\Lambda}(K)$ to denote the lattice width. In this paper, a polygon is a two-dimensional polytope. If $P$ is a polygon with integer vertices we use $i(P)$ and $b(P)$ to denote the number of integer points in the interior and on the boundary of $P$, respectively. The well-known formula of Pick states that $\mathrm{A}(P)=i(P)+\frac{b(P)}{2}-1$.

Let us explain the structure of the proof of Theorem 2.2 and introduce the tools used in the proof.
The proof is essentially based on the following two ideas. We use the parity argument (a rather common tool in the geometry of numbers). Two integer points $x, y \in \mathbb{Z}^{d}$ are said to have the same parity if each component of $x-y$ is even, i.e., $x \equiv y(\bmod 2)$. It is easily seen that the point $\frac{1}{2}(x+y)$ is integer if and only if $x$ and $y$ have the same parity. We will apply this argument to integer points on the boundary of $P \in \mathcal{M}^{3}$ by exploiting the fact that each facet of $P$ contains an integer point in its relative interior
(which is a property of maximal lattice-free convex sets, see Lovász Lov89]. Clearly, there are at most $2^{3}=8$ points of different parity in dimension three. Proofs based on this idea are presented in Section 5

The second idea is the following. We take an arbitrary facet $F$ of $P \in \mathcal{M}^{3}$ and assume without loss of generality that $F \subseteq \mathbb{R}^{2} \times\{0\}$ and $P \subseteq \mathbb{R}^{2} \times \mathbb{R}_{\geq 0}$. Then, we consider the section $F^{\prime}=P \cap\left(\mathbb{R}^{2} \times\{1\}\right)$. Taking into account that $F$ is an integral polygon and contains at least one integer point in its relative interior and that $F^{\prime}$ is lattice-free in $\mathbb{R}^{2} \times\{1\}$ with respect to the lattice $\mathbb{Z}^{2} \times\{1\}$. It follows that either $P$ is "not too high" with respect to $F$ or that $F$ contains a bounded number of integer points. Proofs based on this idea are presented in Sections 6 and 7 .

Let $P, Q$ be polytopes and let $\mathcal{F}(P)$ and $\mathcal{F}(Q)$ be the sets of all faces of $P$ and $Q$, respectively. Then $P$ and $Q$ are said to be combinatorially equivalent (or of the same combinatorial type) if there exists a bijection $T: \mathcal{F}(P) \rightarrow \mathcal{F}(Q)$ satisfying $T\left(F_{1}\right) \subseteq T\left(F_{2}\right)$ for all $F_{1}, F_{2} \in \mathcal{F}(P)$ with $F_{1} \subseteq F_{2}$. Our first lemma dealing with $\mathcal{M}^{3}$ shows that every element of $\mathcal{M}^{3}$ has at most six facets. This yields a quite short list of possible combinatorial types for elements in $\mathcal{M}^{3}$. Our arguments proceed by distinction of different possible combinatorial types. The description of $P \in \mathcal{M}^{3}$ with six facets resp. five facets is given in Sections 5 resp. 6. The description of $P \in \mathcal{M}^{3}$ with four facets (i.e., of simplices in $\mathcal{M}^{3}$ ) can be found in AWW09a. Since the arguments in AWW09a are very lengthy, we present an alternative shorter analysis in Section 7. The following classes of polytopes will be relevant.

- A polytope $P \subseteq \mathbb{R}^{3}$ is said to be a pyramid if $P=\operatorname{conv}(F \cup\{a\})$, where $F$ is a polygon and $a \in \mathbb{R}^{3} \backslash \operatorname{aff}(F)$. In this case $F$ is called the base and $a$ the apex of $P$.
- A polytope $P \subseteq \mathbb{R}^{3}$ is said to be a prism if $P=F+I$, where $F$ is a polygon and $I$ is a segment which is not parallel to $F$. In this case $F+v$ with $v \in \operatorname{vert}(I)$ are called the bases of $P$.
- A polytope $P \subseteq \mathbb{R}^{3}$ is said to be a parallelepiped if $P=I_{1}+I_{2}+I_{3}$ where $I_{1}, I_{2}, I_{3}$ are segments whose directions form a basis of $\mathbb{R}^{3}$.

In the rest of this section we present results which we use as tools. From AW11, Hur90 a relation between area and lattice width is known. In the following theorem, (22) is shown in Hur90 and (3) and (4) in AW11.

Theorem 4.1. Let $K \subseteq \mathbb{R}^{2}$ be a lattice-free closed convex set with $w:=w(K)>1$. Then

$$
\begin{array}{rlrl}
w & \leq 1+\frac{2}{\sqrt{3}}, & \\
\mathrm{~A}(K) & \leq \frac{w^{2}}{2(w-1)} & & \text { if } 1<w \leq 2, \\
\mathrm{~A}(K) & \leq 2, & & \text { if } 2<w \leq 1+\frac{2}{\sqrt{3}} .
\end{array}
$$

The bound (4) is not sharp but sufficient for our purposes (for the sharp upper bound see AW11, Theorem 2.2]). For $h \in \mathbb{Z}$ the set $\mathbb{Z}^{2} \times\{h\}$ in the affine space $\mathbb{R}^{2} \times\{h\}$ can be naturally identified with the lattice $\mathbb{Z}^{2}$ in $\mathbb{R}^{2}$. Such identification will be used several times.

For characterizing faces of maximal lattice-free polytopes we need results on the description of polygons with a given small number of interior integer points. In particular, we need the following result of Rabinowitz Rab89].

Theorem 4.2. (Rab89.) Let $P \subseteq \mathbb{R}^{2}$ be an integral polygon with exactly one interior integer point. Then $P$ is equivalent to one of the polygons shown in Figure 囼,

The only result from the previous sections that is used for the description of $\mathcal{M}^{3}$ is Lemma 3.6 dealing with $\mathcal{R}^{d}(a)$. We use the description of $\mathcal{R}^{2}(a)$ presented in the following remark.
Remark 4.3. With the help of Theorem 4.2, the set $\mathcal{R}^{2}(a)$ can be computed for a given $a \in \mathbb{Z}^{2}$. Let us assume $a=o$, since, by a unimodular transformation, the choice of $a$ is not important. Then, up to $a$ unimodular transformation, every element of $\mathcal{R}^{2}(o)$ coincides with one of the following sets:

$$
\begin{aligned}
R_{1} & :=\operatorname{conv}\left(\left\{o, 2 e_{1}\right\}\right) \\
R_{2} & :=\operatorname{conv}\left(\left\{o, 3 e_{1}, 2 e_{2}\right\}\right) \\
R_{3} & :=\operatorname{conv}\left(\left\{o, 2 e_{1}, e_{1}+2 e_{2}\right\}\right) \\
R_{4} & :=\operatorname{conv}\left(\left\{o, 2 e_{1}+e_{2}, 2 e_{2}+e_{1}\right\}\right)
\end{aligned}
$$



Figure 2: All integral polygons with one interior integer point

This can be seen as follows. By Lemma 3.6[] and III, all elements of $\mathcal{R}^{2}(o)$ are simplices with precisely one relative interior integer point. Thus, up to a unimodular transformation, all two-dimensional elements of $\mathcal{R}^{2}(o)$ appear in Figures 2(a) 2(e), Using Lemma 3.6 II and IV, we end up with $R_{2}, R_{3}$, and $R_{4}$. The fact that $R_{1}$ is the only one-dimensional element of $\mathcal{R}^{2}(o)$ is straightforward to verify.

## 5 Elements in $\mathcal{M}^{3}$ with six facets

In this section we show that there exists, up to unimodular transformation, only one $P \in \mathcal{M}^{3}$ with six facets.

Lemma 5.1. Let $P \in \mathcal{M}^{3}$. Then, $P$ has at most six facets. Furthermore, if $P$ has six facets, then each facet of $P$ is either the parallelogram shown in Figure 2(g) or the triangle shown in Figure 2(c).
Proof. We first show that $P$ has at most six facets. Let $\mathcal{F}$ be the set of all facets of $P$. We choose two integer points $p_{1}, p_{2}$ on an edge of $P$ with $\left[p_{1}, p_{2}\right] \cap \mathbb{Z}^{3}=\left\{p_{1}, p_{2}\right\}$. For each $F \in \mathcal{F}$ we fix an integer point $p_{F}$ in the relative interior of $F$ in the following way. If $F \in \mathcal{F}$ and $p_{1}, p_{2} \in F$ let $p_{F}$ be a point in $\operatorname{relint}(F) \cap \mathbb{Z}^{3}$ such that the triangle with vertices $p_{1}, p_{2}, p_{F}$ has minimal area. This ensures $\left[p_{F}, p_{i}\right] \cap \mathbb{Z}^{3}=\left\{p_{F}, p_{i}\right\}$ for $i=1,2$. If $F \in \mathcal{F}$ and $F \cap\left\{p_{1}, p_{2}\right\}=\left\{p_{i}\right\}$ for some $i=1,2$, let $p_{F}$ be a point in relint $(F) \cap \mathbb{Z}^{3}$ with $\left[p_{F}, p_{i}\right] \cap \mathbb{Z}^{3}=\left\{p_{F}, p_{i}\right\}$. If $F \in \mathcal{F}$ and $F \cap\left\{p_{1}, p_{2}\right\}=\emptyset$, we choose $p_{F}$ to be any point in $\operatorname{relint}(F) \cap \mathbb{Z}^{3}$. Let $X:=\left\{p_{1}, p_{2}\right\} \cup\left\{p_{F}: F \in \mathcal{F}\right\}$. By construction, all points in $X$ have different parity. Hence, $|\mathcal{F}|=|X|-2 \leq\left|\mathbb{Z}^{d} / 2 \mathbb{Z}^{d}\right|-2=2^{3}-2=6$.

Let us now show the second part of the assertion. For that, we first show that each facet of $P$ contains exactly one integer point in its relative interior. Assume there exists a facet $F_{1}$ containing at least two integer points in its relative interior. Choose a vertex $v_{1}$ of $F_{1}$ and two integer points $p_{1}, p_{2} \in \operatorname{relint}\left(F_{1}\right) \cap \mathbb{Z}^{3}$ such that the triangle with vertices $v_{1}, p_{1}, p_{2}$ has minimal area. Let $e=\left[v_{1}, v_{2}\right]$ be an edge of $P$ which is not contained in $F_{1}$ and let $\bar{v}_{2}$ be the integer point on the edge $e$ which is closest to $v_{1}$. Let $F_{2}$ and $F_{3}$ be the two facets containing both $v_{1}$ and $\bar{v}_{2}$. Let $p_{3}$ (resp. $p_{4}$ ) be an integer point in the relative interior of $F_{2}$ (resp. $F_{3}$ ) such that $\left[v_{1}, p_{i}\right] \cap \mathbb{Z}^{3}=\left\{v_{1}, p_{i}\right\}$ and $\left[\bar{v}_{2}, p_{i}\right] \cap \mathbb{Z}^{3}=\left\{\bar{v}_{2}, p_{i}\right\}$ for $i=3,4$ (this can again be achieved by choosing triangles with minimal area). In the remaining three facets choose arbitrary relative interior integer points $p_{5}, p_{6}, p_{7}$ such that $\left[\bar{v}_{2}, p_{i}\right] \cap \mathbb{Z}^{3}=\left\{\bar{v}_{2}, p_{i}\right\}$ for $i=5,6,7$. By construction, the points $v_{1}, \bar{v}_{2}, p_{1}, \ldots, p_{7}$ must have different parity which is a contradiction.

Let now $F$ be an arbitrary facet of $P$. It follows that $F$ is one of the polygons shown in Figure 2, If $F$ is different from the quadrilateral $2(\mathrm{~g})$ and the triangle 2(c), then it contains four integer points with different parity. These four integer points together with the five interior integer points of the other five facets of $P$ are nine points of different parity which is a contradiction.

The next lemma shows that all facets of a polytope $P \in \mathcal{M}^{3}$ with six facets are quadrilaterals as pictured in Figure $2(\mathrm{~g})$ and thus, the shape of $P$ is uniquely determined.

Lemma 5.2. Each $P \in \mathcal{M}^{3}$ with six facets is a parallelepiped where each of the six facets is a parallelogram as depicted in Figure 2(g).
Proof. By Lemma 5.1 $P$ has only two types of facets. Since quadrangular facets do not contain edges with relative interior integer points, it follows that $P$ has an even number of triangular facets and that these facets are pairwise attached. In [Grü03, Sections 6.2 and 6.3] all possible combinatorial types of three-dimensional polytopes with six facets are enumerated (there are exactly seven such types). Since all facets of $P$ are quadrilaterals shown in Figure 2(g) or triangles shown in Figure 2(c), and since triangular facets occur pairwise, we deduce that $P$ is one of the three combinatorial types in Figure 3 .

(a) Type A

(b) Type B

(c) Type C

Figure 3: Possible combinatorial types of $P$
First assume that $P$ is of combinatorial type B, having only triangular facets. Since all facets contain exactly one edge with exactly one relative interior integer point, only two different polytopes $P$ are possible as depicted in Figure 4(a), where the gray nodes represent integer points on edges. In both cases, the three integer points represented in gray together with the six relative interior integer points of the six facets of $P$ are nine points of different parity which is a contradiction. Thus, $P$ cannot be of combinatorial type B.


Figure 4: Polytopes $P$ of combinatorial types B and C
Now assume that $P$ is of combinatorial type $C$, having two quadrangular and four triangular facets. Then, the location of the two relative interior integer points on its edges is already determined by the structure of the facets of $P$ as illustrated in Figure 4(b). These two points together with a particular vertex of $P$ (the gray nodes in Figure 4(b) and the six relative interior integer points of the six facets of $P$ are nine points of different parity. Thus, $P$ cannot be of combinatorial type C.

It follows that $P$ must be of combinatorial type A. This implies that all facets of $P$ are quadrangular and therefore $P$ has the shape depicted in Figure 1(1).

## 6 Elements in $\mathcal{M}^{3}$ with five facets

By Grü03, Section 6.1], there are exactly two combinatorial types of three-dimensional polytopes with five facets. These are quadrangular pyramids (i.e., pyramids having a quadrangular base) and triangular
prisms (i.e., prisms having triangular bases). We will analyze both combinatorial types separately.

### 6.1 Quadrangular pyramids

Let $P \in \mathcal{M}^{3}$ be a quadrangular pyramid. Using a unimodular transformation, base $F$ and apex $a=$ $\left(a_{1}, a_{2}, a_{3}\right)$ of $P$ can be assumed to satisfy $F \subseteq \mathbb{R}^{2} \times\{0\}$ and $a_{3}>0$. We can further assume that $a_{3} \geq 2$ since for $a_{3}=1, P$ is contained in $\mathbb{R}^{2} \times[0,1]$ which is a contradiction to its maximality.

We first show that there is only one quadrangular pyramid $P \in \mathcal{M}^{3}$ with $a_{3}=2$ and $a_{3}=3$, respectively, up to a unimodular transformation.

Lemma 6.1. Let $P \in \mathcal{M}^{3}$ be a quadrangular pyramid with base $F \subseteq \mathbb{R}^{2} \times\{0\}$ and apex $a=\left(a_{1}, a_{2}, a_{3}\right)$, where $a_{3}=2$. Then $P$ is equivalent to the pyramid $M_{8}$.

Proof. Let $F^{\prime}:=P \cap\left(\mathbb{R}^{2} \times\{1\}\right)$. Since each triangular facet of $P$ contains an integer point in its relative interior, it follows that $F^{\prime}$ is a maximal lattice-free quadrilateral and contains precisely four integer points, one in the relative interior of each of the edges of $F^{\prime}$. Without loss of generality assume $F^{\prime} \cap \mathbb{Z}^{3}=\{0,1\}^{2} \times\{1\}$. By convexity, $\operatorname{vert}\left(F^{\prime}\right)$ lies in the union of $(0,1) \times \mathbb{R} \times\{1\}$ and $\mathbb{R} \times(0,1) \times\{1\}$. On the other hand $\operatorname{vert}\left(F^{\prime}\right)=\frac{1}{2} a+\frac{1}{2} \operatorname{vert}(F) \subseteq \frac{1}{2} \mathbb{Z}^{3}$. Hence vert $\left(F^{\prime}\right)$ lies in the union of $\left\{\frac{1}{2}\right\} \times \frac{1}{2} \mathbb{Z} \times\{1\}$ and $\frac{1}{2} \mathbb{Z} \times\left\{\frac{1}{2}\right\} \times\{1\}$. Clearly, $\operatorname{vert}\left(F^{\prime}\right)$ is disjoint with $[0,1]^{2} \times\{1\}$. It follows that $F^{\prime}$ contains the set $B:=\frac{1}{2} e_{1}+\frac{1}{2} e_{2}+e_{3}+\operatorname{conv}\left(\left\{ \pm e_{1}, \pm e_{2}\right\}\right)$. If $B$ were a proper subset of $F^{\prime}$, then one of the points from the set $\{0,1\}^{2} \times\{1\}$ would be in the relative interior of $F^{\prime}$, a contradiction. Hence $F^{\prime}=B$. We have determined that, up to a unimodular transformation, $F$ is a translate of $\operatorname{conv}\left(\left\{ \pm 2 e_{1}, \pm 2 e_{2}\right\}\right)$ and $F^{\prime}$ is a translate of $B$ by an integer vector. This implies the assertion.

Lemma 6.2. Let $P \in \mathcal{M}^{3}$ be a quadrangular pyramid with base $F \subseteq \mathbb{R}^{2} \times\{0\}$ and apex $a=\left(a_{1}, a_{2}, a_{3}\right)$, where $a_{3}=3$. Then $P$ is equivalent to the pyramid $M_{9}$.

Proof. If $p \in P \cap\left(\mathbb{R}^{2} \times\{2\}\right)$ is an integer point in the relative interior of a facet of $P$, then $2 p-a \in$ $P \cap\left(\mathbb{R}^{2} \times\{1\}\right)$ is also an integer point in the relative interior of the same facet of $P$. Consequently, $F^{\prime}:=P \cap\left(\mathbb{R}^{2} \times\{1\}\right)$ contains precisely four integer points, one in the relative interior of each of its edges. Without loss of generality assume $F^{\prime} \cap \mathbb{Z}^{3}=\{0,1\}^{2} \times\{1\}$. By convexity, vert $\left(F^{\prime}\right)$ lies in the union of $(0,1) \times \mathbb{R} \times\{1\}$ and $\mathbb{R} \times(0,1) \times\{1\}$. On the other hand $\operatorname{vert}\left(F^{\prime}\right)=\frac{1}{3} a+\frac{2}{3} \operatorname{vert}(F) \subseteq \frac{1}{3} \mathbb{Z}^{3}$. Hence vert $\left(F^{\prime}\right)$ lies in the union of $\left\{\frac{1}{3}, \frac{2}{3}\right\} \times \frac{1}{3} \mathbb{Z} \times\{1\}$ and $\frac{1}{3} \mathbb{Z} \times\left\{\frac{1}{3}, \frac{2}{3}\right\} \times\{1\}$. Clearly, vert $\left(F^{\prime}\right)$ is disjoint with $[0,1]^{2} \times\{1\}$. A simple analysis of all possible cases reveals that, by a unimodular transformation, only one $F^{\prime}$ is possible and we can assume that $F^{\prime}:=\frac{1}{3} e_{1}+\frac{1}{3} e_{2}+e_{3}+\operatorname{conv}\left(\left\{\frac{4}{3} e_{1},-\frac{2}{3} e_{1}, \frac{4}{3} e_{2},-\frac{2}{3} e_{2}\right\}\right)$. Thus, up to a unimodular transformation, $F$ is a translate of $\operatorname{conv}\left(\left\{2 e_{1},-e_{1}, 2 e_{2},-e_{2}\right\}\right)$. This implies the assertion.

In the following we assume that $a_{3} \geq 4$ and show that no further maximal lattice-free quadrangular pyramid $P$ exists. The proof consists of the following steps. First, we construct all bases which are possible for such a pyramid $P$. Second, we argue that only two of them can appear as bases for $a_{3} \geq 11$ and analyze these two separately. Third, the other bases are ruled out by a computer enumeration.

We start with a lemma which shall be used later for simplices in Section 7 as well.
Lemma 6.3. Let $P \in \mathcal{M}^{3}$ be a simplex or a quadrangular pyramid with base $F \subseteq \mathbb{R}^{2} \times\{0\}$ and apex $a=\left(a_{1}, a_{2}, a_{3}\right)$, where $h:=a_{3} \geq 4$. Then $w(F)=2$ and the following inequalities hold:

$$
\begin{equation*}
2 i(F)+b(F) \leq\left\lfloor\frac{6 h-4}{h-2}\right\rfloor \leq 10 . \tag{5}
\end{equation*}
$$

In particular, if $P$ is a simplex (resp. a quadrangular pyramid), then $(i(F), b(F)) \in Z_{S}$ (resp. $Z_{Q}$ ), where

$$
\begin{aligned}
& Z_{S}:=\{(1, j): j=3, \ldots, 8\} \cup\{(2, j): j=3, \ldots, 6\}, \\
& Z_{Q}:=\{(1, j): j=4, \ldots, 8\} \cup\{(2, j): j=4, \ldots, 6\} .
\end{aligned}
$$

Proof. Let $F^{\prime}:=P \cap\left(\mathbb{R}^{2} \times\{1\}\right)$. Since $F$ contains an integer point in its relative interior we have $w(F) \geq 2$. Assume that $w(F) \geq 3$. Then $h \geq 4$ implies $w:=w\left(F^{\prime}\right)=w(F) \frac{h-1}{h} \geq \frac{9}{4}>1+\frac{2}{\sqrt{3}}$. Hence, by Theorem 4.1 $F^{\prime}$ is not lattice-free which is a contradiction. Thus, we have $w(F)=2$ and it follows $2>w=w(F) \frac{h-1}{h} \geq \frac{3}{2}$. Applying Theorem 4.1 to $F^{\prime}$, we obtain

$$
\begin{equation*}
\mathrm{A}(F)=\left(\frac{h}{h-1}\right)^{2} \mathrm{~A}\left(F^{\prime}\right) \leq\left(\frac{h}{h-1}\right)^{2} \frac{w^{2}}{2(w-1)}=\frac{2 h}{h-2} \tag{6}
\end{equation*}
$$

where the last equality follows from $w=2 \frac{h-1}{h}$. Consequently, combining (6) and Pick's formula and using the fact that $\left\lfloor\frac{6 h-4}{h-2}\right\rfloor$ is monotonically nonincreasing for $h \geq 4$, we arrive at the stated inequalities.

We now show that $i(F) \leq 2$. Assume the contrary, i.e., $i(F) \geq 3$. Performing an appropriate unimodular transformation to $P$ we can assume that $\pi(F)=\left[o, 2 e_{1}\right]$. For $x \in \pi(P)$ let $f(x)$ be the length of the line segment $\pi^{-1}(x) \cap P$.

The conditions $w(F)=2$ and $i(F) \geq 3$ imply $f\left(e_{1}\right) \geq 3$. By Lemma 3.7 $\pi(P)$ contains an integer point in its interior. The relative interior of $\left[e_{1}, \pi(a)\right]$ does not contain integer points, since otherwise the value of $f$ at the integer point in $\left[e_{1}, \pi(a)\right] \backslash\left\{e_{1}\right\}$ closest to $e_{1}$ would be $>1$ yielding a contradiction to the lattice-freeness of $P$. Thus, the interior of $\operatorname{conv}\left(\left\{o, e_{1}, \pi(a)\right\}\right)$ or $\operatorname{conv}\left(\left\{e_{1}, 2 e_{1}, \pi(a)\right\}\right)$ contains an integer point. By symmetry reasons, we may assume that for $T:=\operatorname{conv}\left(\left\{o, e_{1}, \pi(a)\right\}\right)$ one has $\operatorname{int}(T) \cap \mathbb{Z}^{2} \neq \emptyset$.

Let $R$ be an element of $\mathcal{R}^{2}\left(e_{1}\right)$ contained in $T$ and such that the relative interior of $R$ contains an interior integer point of $T$. Note that $R$ is equivalent to one of the polygons $R_{1}, \ldots, R_{4}$ in Remark 4.3.

Case 1: $R \equiv R_{1}\left(\bmod \operatorname{Aff}\left(\mathbb{Z}^{2}\right)\right)$. Then $R=\left[e_{1}, p\right]$ for some $p \in T \cap \mathbb{Z}^{2}$ and such that the point $\frac{1}{2}\left(e_{1}+p\right)$ is integer and in the interior of $T$. By the concavity of $f$, one has

$$
f\left(\frac{1}{2}\left(e_{1}+p\right)\right) \geq \frac{1}{2} f\left(e_{1}\right)+\frac{1}{2} f(p) \geq \frac{1}{2} f\left(e_{1}\right) \geq \frac{3}{2}>1
$$

Thus, a contradiction to the lattice-freeness of $P$.
Case 2: $R \equiv R_{4}\left(\bmod \operatorname{Aff}\left(\mathbb{Z}^{2}\right)\right)$. Then $R=\operatorname{conv}\left\{e_{1}, p, q\right\}$ for some $p, q \in T \cap \mathbb{Z}^{2}$ and $\frac{1}{3}\left(e_{1}+p+q\right)$ is integer and in the interior of $T$. By the concavity of $f$, we have

$$
f\left(\frac{1}{3}\left(e_{1}+p+q\right)\right) \geq \frac{1}{3}\left(f\left(e_{1}\right)+f(p)+f(q)\right) \geq \frac{1}{3} f\left(e_{1}\right) \geq 1
$$

It follows that $f(p)=f(q)=0$, since otherwise one has $f\left(\frac{1}{3}\left(e_{1}+p+q\right)\right)>1$ yielding a contradiction to the lattice-freeness of $P$. Then, in view of the choice of $T$, we have $p, q \in[o, \pi(a)]$. The equality $\{p, q\}=\{o, \pi(a)\}$ would imply that $a_{3}=3$ contradicting the assumption. Thus, one of the points $p, q$ (say $p$ ) lies in the relative interior of $[o, \pi(a)]$. We use the point $2 p-q$, which is the integer point on $[o, \pi(a)] \backslash[p, q]$ closest to $p$.

We shall use the following property of $R_{4}$. Let $r_{1}, r_{2}, r_{3}$ be the vertices of $R_{4}$. Then the segment joining $r_{1}$ and $2 r_{2}-r_{3}$ (the reflection of $r_{3}$ with respect to $r_{2}$ ) contains precisely two integer points in its relative interior. Consider the subcase that the point $2 p-q$ lies in the relative interior of $[o, \pi(a)]$. Then the relative interior of $\left[e_{1}, 2 p-q\right]$ is contained in the interior of $T$. Taking into account the indicated property of $R_{4}$ we see that the relative interior of $\left[e_{1}, 2 p-q\right]$ contains two integer points. Thus, applying the arguments as in Case 1, we arrive at a contradiction. For the subcase that the point $2 p-q$ coincides with $o$ or $\pi(a)$, the fact that the relative interior of $\left[e_{1}, 2 p-q\right]$ contains two integer points contradicts the fact that the segments $\left[o, e_{1}\right]$ and $\left[e_{1}, \pi(a)\right]$ do not contain integer points in their relative interiors.

Case 3: $R \equiv R_{i}\left(\bmod \operatorname{Aff}\left(\mathbb{Z}^{2}\right)\right)$ for $i \in\{2,3\}$. Then there exists an edge $e$ of $R$ incident to $e_{1}$ which contains at least three integer points. Since the edge $\left[o, 2 e_{1}\right]$ of $\pi(P)$ contains three integer points and the integer point $e_{1}$ is between the two remaining integer points, it follows that the edge $e$ is not contained in the boundary of $\pi(P)$. Thus, on $e$ we can find an integer point $p$ such that $\frac{1}{2}\left(e_{1}+p\right)$ is integer and in the interior of $\pi(P)$. But then, applying the same arguments as in Case 1 we arrive at a contradiction.

So far, we have shown that $i(F) \in\{1,2\}$ and $2 i(F)+b(F) \leq 10$. If $P$ is a simplex, then $b(F) \geq 3$. Thus, $(i(F), b(F)) \in Z_{S}$ in this case. If $P$ is a quadrangular pyramid, then $b(F) \geq 4$. Thus, we have $(i(F), b(F)) \in Z_{Q}$.

In order to analyze quadrangular pyramids $P \in \mathcal{M}^{3}$ further we need a list of all integral quadrilaterals $Q$ in the plane with $w(Q)=2$ and $(i(Q), b(Q)) \in Z_{Q}$ since these quadrilaterals are candidates for the base of $P$. By (5), it follows that $2 i(F)+b(F) \leq 6$ for $a_{3} \geq 11$ which implies that the base $F$ of such a pyramid has exactly one integer point in its relative interior and exactly the four vertices as the only integer points on its boundary. From Figure 2 it follows that only two quadrilaterals qualify as a base for $P$ in this case (Figure $2(\mathrm{f})$ and $2(\mathrm{~g})$. We will analyze these two possible bases separately from the others. However, we will first prove the following lemma.

Lemma 6.4. Let $Q \subseteq \mathbb{R}^{2}$ be an integral quadrilateral with $w(Q)=2, i(Q)=2$, and $b(Q) \in\{4,5,6\}$. Then, up to a unimodular transformation, $Q$ is one of the quadrilaterals depicted in Figure 5 .


Figure 5: All integral quadrilaterals $Q$ with $w(Q)=2, i(Q)=2$, and $b(Q) \in\{4,5,6\}$

Proof. Let $Q$ be an integral quadrilateral in the plane satisfying $w(Q)=2$ and $i(Q)=2$. We divide the proof according to the number of integer points on the boundary of $Q$.

Case 1: $i(Q)=2$ and $b(Q)=4$. Pick's formula gives $\mathrm{A}(Q)=3$ in this case. Without loss of generality we assume that the two interior integer points are placed at $(1,0)$ and $(2,0)$. This implies that for any $u \in \mathbb{Z}^{2} \backslash\left\{o, \pm e_{2}\right\}$ we have $w(Q, u) \geq 3$ and therefore it must hold $v_{2} \in\{0, \pm 1\}$ for each vertex $v=\left(v_{1}, v_{2}\right)$ of $Q$. We distinguish three subcases based on the number of vertices of $Q$ that lie on the line $y=0$.

Subcase 1a: Two vertices of $Q=\operatorname{conv}(\{a, b, c, d\})$ lie on the line $y=0$. Then, one vertex is $a=(0,0)$ and the other $c=(3,0)$. Let the remaining two vertices $b$ and $d$ satisfy $d_{2}=1=-b_{2}$. We can assume that $d=(0,1)$ for if $d=\left(d_{1}, 1\right)$ we apply the unimodular transformation $(x, y) \mapsto\left(x-d_{1} y, y\right)$. For convexity reasons it follows that $b \in\{(1,-1),(2,-1),(3,-1),(4,-1),(5,-1)\}$. Choices $b=(1,-1)$ and $b=(5,-1)$ are equivalent and lead to the quadrilateral shown in Figure 5(a) $b=(2,-1)$ and $b=(4,-1)$ lead to Figure 5(b) and $b=(3,-1)$ leads to Figure 5(c),

Subcase 1b: One vertex of $Q=\operatorname{conv}(\{a, b, c, d\})$ lies on the line $y=0$. Without loss of generality assume that $a=(0,0)$ and $b, c$, and $d$ satisfy $b_{2}=1=-c_{2}=-d_{2}$. It follows that $c_{1}=d_{1}+1$ since $b(Q)=4$, by assumption. Without loss of generality we can place $b$ at $(0,1)$. By convexity of $Q$ and since $(1,0)$ and $(2,0)$ are the only interior integer points of $Q$ we obtain $c=(5,-1)$ and $d=(4,-1)$ giving the quadrilateral shown in Figure 5(d).

Subcase 1c: No vertex of $Q=\operatorname{conv}(\{a, b, c, d\})$ lies on the line $y=0$. Without loss of generality let $a_{2}=b_{2}=1=-c_{2}=-d_{2}$. It follows that $b_{1}=a_{1}+1$ and $c_{1}=d_{1}+1$. Thus, $\mathrm{A}(Q)=2$ which contradicts Pick's formula.

Case 2: $i(Q)=2$ and $b(Q)=5$. Pick's formula gives $\mathrm{A}(Q)=3.5$. Placing the two interior integer points of $Q$ at $(1,0)$ and $(2,0)$ as above implies again that $v_{2} \in\{0, \pm 1\}$ for each vertex $v=\left(v_{1}, v_{2}\right)$ of $Q$. If two vertices of $Q$ lie on the line $y=0$, then $Q$ has no edge with a relative interior integer point, a contradiction to $b(Q)=5$. If no vertex of $Q$ lies on the line $y=0$, then $\mathrm{A}(Q)=3$, a contradiction to Pick's formula. Thus, precisely one vertex of $Q=\operatorname{conv}(\{a, b, c, d\})$ lies on the line $y=0$. Place it at $a=(0,0)$. Without loss of generality let $b_{2}=c_{2}=-1=-d_{2}$. Using an appropriate unimodular transformation we can assume that $d=(0,1)$. Thus, either the edge connecting $b$ and $c$ or the edge connecting $c$ and $d$ has a relative interior integer point which is $\frac{1}{2}(b+c)$ or $\frac{1}{2}(c+d)$, respectively. In the first case we end up with $b=(3,-1)$ and $c=(5,-1)$ (Figure 5(e) , whereas the latter leads to $b=(5,-1)$ and $c=(6,-1)$ (Figure $5(\mathrm{f})$.

Case 3: $i(Q)=2$ and $b(Q)=6$. Pick's formula gives $\mathrm{A}(Q)=4$. Placing the two interior integer points of $Q$ at $(1,0)$ and $(2,0)$ as above implies again that $v_{2} \in\{0, \pm 1\}$ for each vertex $v=\left(v_{1}, v_{2}\right)$ of $Q$. If two vertices of $Q$ lie on the line $y=0$, then $Q$ has no edge with a relative interior integer point, a contradiction to $b(Q)=6$. We consider two subcases.

Subcase 3a: No vertex of $Q=\operatorname{conv}(\{a, b, c, d\})$ lies on the line $y=0$. Without loss of generality assume that $a_{2}=b_{2}=1=-c_{2}=-d_{2}$. We either have $b_{1}=a_{1}+2$ and $c_{1}=d_{1}+2$ or $b_{1}=a_{1}+1$ and $c_{1}=d_{1}+3$. Using an appropriate unimodular transformation we can assume that $a=(0,1)$. Then, the first case leads to $b=(2,1), c=(3,-1)$, and $d=(1,-1)$ (Figure $5(\mathrm{j})$ ), whereas the latter leads to $b=(1,1), c=(4,-1)$, and $d=(1,-1)$ (Figure $5(\mathrm{~g})$ ).

Subcase 3b: One vertex of $Q=\operatorname{conv}(\{a, b, c, d\})$ lies on the line $y=0$. Without loss of generality assume that $a=(0,0)$ and $b, c$, and $d$ satisfy $b_{2}=1=-c_{2}=-d_{2}$. Using an appropriate unimodular transformation we can assume that $b=(0,1)$. Then, the edge connecting $c$ and $d$ has either two or one relative interior integer points. In the first case we obtain $c=(5,-1)$ and $d=(2,-1)$ (Figure 5(h)). In
the second case both edges, the one connecting $c$ and $d$ and the one connecting $b$ and $c$ have each one relative interior integer point and it follows $c=(6,-1)$ and $d=(4,-1)$ (Figure 5(i)).

Lemma 6.4 completes the list of the possible bases of a quadrangular pyramid $P \in \mathcal{M}^{3}$ : precisely the quadrilaterals shown in Figure $2(\mathrm{f}) 2(\mathrm{l})$ and 5 qualify for a base of $P$. We will now show that there is no quadrangular pyramid $P \in \mathcal{M}^{3}$ with $a_{3} \geq 11$.

Lemma 6.5. Let $P \subseteq \mathbb{R}^{3}$ be a pyramid with base $\operatorname{conv}\left(\left\{ \pm e_{1}, \pm e_{2}\right\}\right)$ and apex $a=\left(a_{1}, a_{2}, a_{3}\right) \in \mathbb{Z}^{3}$, where $a_{3} \geq 4$. Then $P$ is not maximal lattice-free.

Proof. By an appropriate unimodular transformation we can assume that $0 \leq a_{i}<a_{3}$ for $i=1,2$. We represent the base by $F:=\operatorname{conv}\left(\left\{ \pm e_{1}, \pm e_{2}\right\}\right)=\left\{y \in \mathbb{R}^{3}:\left|y_{1}\right|+\left|y_{2}\right| \leq 1, y_{3}=0\right\}$. Then, $P=\left\{x \in \mathbb{R}^{3}:\right.$ $x=(1-\lambda) y+\lambda a$ for $0 \leq \lambda \leq 1$ and $y \in F\}$ and therefore

$$
\begin{aligned}
\operatorname{int}(P) & =\left\{x \in \mathbb{R}^{3}: x=(1-\lambda) y+\lambda a \text { for } 0<\lambda<1 \text { and } y \in \operatorname{relint}(F)\right\} \\
& =\left\{x \in \mathbb{R}^{3}: \frac{1}{1-\lambda} x-\frac{\lambda}{1-\lambda} a \in \operatorname{relint}(F) \text { for some } 0<\lambda<1\right\} \\
& =\left\{x \in \mathbb{R}^{3}:\left|x_{1}-\lambda a_{1}\right|+\left|x_{2}-\lambda a_{2}\right|<1-\lambda \text { and } x_{3}=\lambda a_{3} \text { for some } 0<\lambda<1\right\} .
\end{aligned}
$$

It follows

$$
\begin{equation*}
\mathbb{Z}^{3} \cap \operatorname{int}(P)=\left\{x \in \mathbb{Z}^{3}:\left|a_{3} x_{1}-a_{1} x_{3}\right|+\left|a_{3} x_{2}-a_{2} x_{3}\right|<a_{3}-x_{3}, x_{3} \in\left\{1, \ldots, a_{3}-1\right\}\right\} \tag{7}
\end{equation*}
$$

From (7) we derive the following equivalences:

- $(0,0,1) \in \operatorname{int}(P)$ if and only if $a_{1}+a_{2}<a_{3}-1$;
- $(1,1,1) \in \operatorname{int}(P)$ if and only if $a_{1}+a_{2}>a_{3}+1$;
- $(1,0,1) \in \operatorname{int}(P)$ if and only if $a_{1}-a_{2}>1$;
- $(0,1,1) \in \operatorname{int}(P)$ if and only if $a_{2}-a_{1}>1$.

If one of the above mentioned conditions is fulfilled, $P$ is not lattice-free. We can therefore assume that the following two inequalities are satisfied:

$$
\begin{align*}
\left|a_{1}+a_{2}-a_{3}\right| & \leq 1,  \tag{8}\\
\left|a_{1}-a_{2}\right| & \leq 1 \tag{9}
\end{align*}
$$

It can be verified directly that for $a_{1}, a_{2}, a_{3}$ satisfying (8) and (9) one has $\left|a_{3}-2 a_{1}\right|+\left|a_{3}-2 a_{2}\right| \leq$ 2. In view of (7), $(1,1,2) \in \operatorname{int}(P)$ if and only if $\left|a_{3}-2 a_{1}\right|+\left|a_{3}-2 a_{2}\right|<a_{3}-2$. Hence, when (8) and (9) are fulfilled, $(1,1,2)$ is an interior point of $P$ if $a_{3}>4$. It remains to exclude the case $a_{3}=4$. Integer vectors $a=\left(a_{1}, a_{2}, a_{3}\right)$ satisfying (8), (9) and $a_{3}=4$ are precisely vectors from the set $\{(2,2,4),(2,1,4),(3,2,4),(1,2,4),(2,3,4)\}$. All these vectors do not correspond to maximal lattice-free pyramids.

Lemma 6.6. Let $P \subseteq \mathbb{R}^{3}$ be a pyramid with base $\operatorname{conv}\left(\left\{e_{1}, e_{2}, \pm\left(e_{1}+e_{2}\right)\right\}\right)$ and apex $a=\left(a_{1}, a_{2}, a_{3}\right) \in \mathbb{Z}^{3}$, where $a_{3} \geq 4$. Then $P$ is not maximal lattice-free.

Proof. By an appropriate unimodular transformation we can assume that $0 \leq a_{i}<a_{3}$ for $i=1,2$. The set $\operatorname{conv}\left(\left\{e_{1}, e_{2}, \pm\left(e_{1}+e_{2}\right)\right\}\right)$ is the set of all $y=\left(y_{1}, y_{2}, y_{3}\right) \in \mathbb{R}^{3}$ satisfying

$$
\begin{array}{ll}
y_{1} \leq 1, & y_{1}-2 y_{2} \leq 1 \\
y_{2} \leq 1, & y_{3}=0 \\
y_{2}-2 y_{1} \leq 1
\end{array}
$$

By this, $\operatorname{int}(P)$ is the set of all $x=\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}$ satisfying

$$
\begin{array}{lll}
x_{1}-\lambda a_{1}<1-\lambda, & x_{1}-\lambda a_{1}-2\left(x_{2}-\lambda a_{2}\right)<1-\lambda, & x_{3}=\lambda a_{3}, \\
x_{2}-\lambda a_{2}<1-\lambda, & x_{2}-\lambda a_{2}-2\left(x_{1}-\lambda a_{1}\right)<1-\lambda &
\end{array}
$$

for some $0<\lambda<1$. Consequently, $\mathbb{Z}^{3} \cap \operatorname{int}(P)$ is the set of all $x=\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{Z}^{3}$ satisfying

$$
\begin{array}{ll}
a_{3} x_{1}+\left(1-a_{1}\right) x_{3}<a_{3}, & a_{3} x_{1}-2 a_{3} x_{2}+\left(1-a_{1}+2 a_{2}\right) x_{3}<a_{3}, \\
a_{3} x_{2}+\left(1-a_{2}\right) x_{3}<a_{3}, & a_{3} x_{2}-2 a_{3} x_{1}+\left(1-a_{2}+2 a_{1}\right) x_{3}<a_{3} .
\end{array}
$$

From these inequalities we obtain that $(1,1,1) \in \operatorname{int}(P)$ if and only if $a_{1}>1$ and $a_{2}>1$. Hence, latticefreeness requires that $a_{1} \in\{0,1\}$ or $a_{2} \in\{0,1\}$. By symmetry, it suffices to consider the cases $a_{1}=0$ and $a_{1}=1$.

Case 1: $a_{1}=0$. If $a_{2}>1$, then $(0,1,1) \in \operatorname{int}(P)$. Otherwise $(0,0,1) \in \operatorname{int}(P)$.
Case 2: $a_{1}=1$. If $a_{2}>3$, then $(0,1,1) \in \operatorname{int}(P)$. Thus, we have $a_{2} \leq 3$. If $2 a_{2}<a_{3}$, then $(0,0,1) \in$ $\operatorname{int}(P)$. So we have $2 a_{2} \geq a_{3}$ and it follows $a_{3} \in\{4,5,6\}$. Hence, $a \in\{(1,2,4),(1,3,4),(1,3,5),(1,3,6)\}$. All these vectors do not correspond to maximal lattice-free pyramids.

Lemmas 6.5 and 6.6 restrict potential quadrangular pyramids $P \in \mathcal{M}^{3}$ to satisfy $4 \leq a_{3} \leq 10$. Since, in addition, the set of possible bases is known from Figures $2(\mathrm{~h}) \mid 2(\mathrm{l})$ and 5 we are left with a finite list of quadrangular candidate pyramids. Computer enumeration shows that none of them is maximal lattice-free.

### 6.2 Triangular prisms

Let $P \in \mathcal{M}^{3}$ be a triangular prism. We first show that the two triangular bases of $P$ are translates.
Lemma 6.7. Let $P \in \mathcal{M}^{3}$ be combinatorially equivalent to a triangular prism. Then $P$ is a prism, i.e., the two bases of $P$ are parallel translates.

Proof. Let $H_{1}, H_{2}$, and $H_{3}$ be the hyperplanes containing the quadrilateral facets of $P$. We show that $H_{1}, H_{2}$, and $H_{3}$ do not share a point. Assume the contrary and choose $p \in H_{1} \cap H_{2} \cap H_{3}$. Let $T_{2}$ be the triangular facet of $P$ such that the pyramid $S$ with base $T_{2}$ and apex $p$ contains $P$. Let $T_{1}$ be the triangular facet of $P$ distinct from $T_{2}$. Let $q$ be a vertex of $T_{2}$ closest to aff $\left(T_{1}\right)$ and let $H$ be the hyperplane parallel to $\operatorname{aff}\left(T_{1}\right)$ and passing through $q$. If $T_{1}$ and $T_{2}$ are not parallel, then the relative interior of $P \cap H$ is contained in the interior of $P$. On the other hand $T_{1}+q-r$, where $r$ is the integer point $r=T_{1} \cap[p, q]$, is contained in $P \cap H$. Hence the relative interior of $P \cap H$ contains an integer point, a contradiction. Thus, $T_{1}$ and $T_{2}$ are parallel. Then, since $T_{2}$ is a base of $P$ and $T_{1}$ is a section of $S$ parallel to $T_{2}$, we infer that $T_{1}$ and $T_{2}$ are homothetic. By construction, $T_{1}$ is strictly smaller than $T_{2}$. Since $T_{1}$ is an integral triangle which contains at least one integer point in its relative interior we have $w\left(T_{1}\right) \geq 2$. Therefore, since $T_{2}$ is integer and strictly larger, $w\left(T_{2}\right) \geq 3$. Without loss of generality we assume that $T_{2} \subseteq \mathbb{R}^{2} \times\{0\}$ and $T_{1} \subseteq \mathbb{R}^{2} \times\{h\}$ with $h \geq 2(h=1$ do not need to be considered since the quadrangular facets of $P$ contain integer points in their relative interior). Let now $T^{\prime}:=P \cap\left(\mathbb{R}^{2} \times\{1\}\right)$. It follows that

$$
w\left(T^{\prime}\right)=\frac{h-1}{h} w\left(T_{2}\right)+\frac{1}{h} w\left(T_{1}\right) \geq \frac{3(h-1)+2}{h}=3-\frac{1}{h} \geq \frac{5}{2}>1+\frac{2}{\sqrt{3}},
$$

a contradiction to (2) in Theorem 4.1 since $T^{\prime}$ is a lattice-free polygon in $\mathbb{R}^{2} \times\{1\}$ with respect to the lattice $\mathbb{Z}^{2} \times\{1\}$. Hence $H_{1}, H_{2}$, and $H_{3}$ do not share a point and $P$ is a prism.

According to Lemma 6.7 it suffices to investigate triangular prisms $P \in \mathcal{M}^{3}$ whose triangular facets are parallel translates. Without loss of generality we assume that the triangular facets $T_{1}, T_{2}$ of $P$ satisfy $T_{2} \subseteq \mathbb{R}^{2} \times\{0\}$ and $T_{1} \subseteq \mathbb{R}^{2} \times\{h\}$ with $h \geq 2$. From Theorem 4.1 and the fact that $P$ is latticefree, it follows that the hyperplane $H:=\mathbb{R}^{2} \times\{1\}$ satisfies $w(P \cap H) \leq 1+\frac{2}{\sqrt{3}}$. Hence, $1+\frac{2}{\sqrt{3}} \geq$ $w(P \cap H)=w\left(T_{2}\right) \geq 2$ and since $w\left(T_{2}\right) \in \mathbb{Z}$ we obtain $2=w\left(T_{2}\right)=w(P \cap H)$. Theorem 4.1 yields $2 \geq \mathrm{A}(P \cap H)=\mathrm{A}\left(T_{2}\right)$ and Pick's formula gives $2 i\left(T_{2}\right)+b\left(T_{2}\right) \leq 6$ implying $i\left(T_{2}\right)=1$ and $b\left(T_{2}\right) \in\{3,4\}$. Thus, by Figure 2, $P$ has two triangular facets which are either the triangle shown in Figure 2(e) or the triangle shown in Figure 2(c). We prove that for each of these two cases there exists exactly one maximal lattice-free triangular prism, up to a unimodular transformation.

Lemma 6.8. Let $P \in \mathcal{M}^{3}$ be a triangular prism whose triangular facets are the triangle shown in Figure 2(e). Then, $P$ is equivalent to $M_{10}$.

Proof. Without loss of generality we assume that the two triangular facets of $P$, denoted $F$ and $F^{\prime}$, are given by $F:=\operatorname{conv}\left(\left\{e_{1}, e_{2},-\left(e_{1}+e_{2}\right)\right\}\right)$ and $F^{\prime}:=a+F$, where $a=\left(a_{1}, a_{2}, a_{3}\right)$ is the integer point in the relative interior of $F^{\prime}$. By applying an appropriate unimodular transformation we can further assume that $0 \leq a_{i}<a_{3}$ for $i=1,2$. Since the quadrangular facets of $P$ need to contain integer points in their relative interior it holds $a_{3} \geq 2$. By symmetry, we assume $a_{1} \leq a_{2}$. In particular, we have $a_{2} \geq 1$, otherwise $(0,0,1) \in \operatorname{int}(P)$. We now set up the facet description of $P$ which is only dependent on the parameters $a_{1}, a_{2}$, and $a_{3}$. It follows that $\mathbb{Z}^{3} \cap \operatorname{int}(P)$ is the set of all $x=\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{Z}^{3}$ satisfying

$$
\begin{array}{cc}
a_{3} x_{1}-2 a_{3} x_{2}+\left(2 a_{2}-a_{1}\right) x_{3}<a_{3}, & a_{3} x_{1}+a_{3} x_{2}-\left(a_{1}+a_{2}\right) x_{3}<a_{3}, \\
a_{3} x_{2}-2 a_{3} x_{1}+\left(2 a_{1}-a_{2}\right) x_{3}<a_{3}, & x_{3} \in\left\{1, \ldots, a_{3}-1\right\} .
\end{array}
$$

From these inequalities we obtain the following equivalences:

- $(0,0,1) \in \operatorname{int}(P)$ if and only if $-a_{1}+2 a_{2}<a_{3} ;$
- $(0,1,1) \in \operatorname{int}(P)$ if and only if $2 a_{1}<a_{2}$;
- $(1,1,1) \in \operatorname{int}(P)$ if and only if $a_{3}<a_{1}+a_{2}$.

This implies that the following inequalities hold:

$$
\begin{align*}
a_{1}+a_{3} & \leq 2 a_{2},  \tag{10}\\
a_{2} & \leq 2 a_{1},  \tag{11}\\
a_{1}+a_{2} & \leq a_{3} . \tag{12}
\end{align*}
$$

Adding (10) and (12) yields $2 a_{1} \leq a_{2}$ and together with (11) we obtain $a_{2}=2 a_{1}$. Substituting this into (10) and (12) leads to $a_{3} \leq 3 a_{1}$ and $3 a_{1} \leq a_{3}$ which means that $a_{3}=3 a_{1}$. It follows that $a=\left(a_{1}, 2 a_{1}, 3 a_{1}\right)$ for some $a_{1} \geq 1$. We infer that $(1,2,3) \in \operatorname{int}(P)$ if $a_{1} \geq 2$. Thus, we have $a=(1,2,3)$ and end up with the triangular prism $M_{10}$.

Lemma 6.9. Let $P \in \mathcal{M}^{3}$ be a triangular prism whose triangular facets are the triangle shown in Figure 2(c), Then, $P$ is equivalent to $M_{11}$.

Proof. Without loss of generality we assume that the two triangular facets of $P$, denoted $F$ and $F^{\prime}$, are given by $F:=\operatorname{conv}\left(\left\{ \pm e_{1}, 2 e_{2}\right\}\right)$ and $F^{\prime}:=a+F$, where $a=\left(a_{1}, a_{2}, a_{3}\right)$ is the integer point in the relative interior of $F^{\prime}$. By applying an appropriate unimodular transformation we can further assume that $0 \leq a_{i}<a_{3}$ for $i=1,2$. Since the quadrangular facets of $P$ need to contain integer points in their relative interior it holds $a_{3} \geq 2$. We now set up the facet description of $P$ which is only dependent on the parameters $a_{1}, a_{2}$, and $a_{3}$. It follows that $\mathbb{Z}^{3} \cap \operatorname{int}(P)$ is the set of all $x=\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{Z}^{3}$ satisfying

$$
\begin{array}{rr}
2 a_{3} x_{1}+a_{3} x_{2}-\left(2 a_{1}+a_{2}-1\right) x_{3}<2 a_{3}, & -a_{3} x_{2}+\left(a_{2}-1\right) x_{3}<0 \\
-2 a_{3} x_{1}+a_{3} x_{2}+\left(2 a_{1}-a_{2}+1\right) x_{3}<2 a_{3}, & x_{3} \in\left\{1, \ldots, a_{3}-1\right\} .
\end{array}
$$

From these inequalities we obtain the following equivalences:

- $(0,1,1) \in \operatorname{int}(P)$ if and only if $2 a_{1}+1<a_{2}+a_{3}$;
- $(1,1,1) \in \operatorname{int}(P)$ if and only if $a_{3}+1<2 a_{1}+a_{2}$.

This implies that the following inequalities hold:

$$
\begin{align*}
a_{2}+a_{3} & \leq 2 a_{1}+1,  \tag{13}\\
2 a_{1}+a_{2} & \leq a_{3}+1 \tag{14}
\end{align*}
$$

Adding (13) and (14) yields $a_{2} \leq 1$ and therefore $a_{2} \in\{0,1\}$. We distinguish into two cases.
Case 1: $a_{2}=0$. If $2 a_{1}>1$, then $(1,0,1) \in \operatorname{int}(P)$. Thus, we have $2 a_{1} \leq 1$ implying $a_{1}=0$. Substituting this into (13) leads to $a_{3} \leq 1$ which is a contradiction.

Case 2: $a_{2}=1$. From (13) and (14), we obtain $a_{3}=2 a_{1}$, i.e., $a=\left(a_{1}, 1,2 a_{1}\right)$ for some $a_{1} \geq 1$. If $a_{1} \geq 2$ we have $(1,1,2) \in \operatorname{int}(P)$. Thus, it holds $a=(1,1,2)$ which leads to the triangular prism $M_{11}$.

## 7 Elements in $\mathcal{M}^{3}$ with four facets

Let $P \in \mathcal{M}^{3}$ be a simplex and let $F$ be an arbitrary facet of $P$. Using a unimodular transformation we can assume that $F \subseteq \mathbb{R}^{2} \times\{0\}$. Throughout this section we refer to $F$ as the base of $P$ and denote the vertex $a=\left(a_{1}, a_{2}, a_{3}\right)$ of $P$ which is not contained in $\operatorname{aff}(F)$ as the apex of $P$, where we assume $a_{3}>0$. We can further assume that $a_{3} \geq 2$ since for $a_{3}=1, P$ is contained in the split $\left\{x \in \mathbb{R}^{3}: 0 \leq x_{3} \leq 1\right\}$ which is a contradiction to its maximality.

We first consider simplices $P \in \mathcal{M}^{3}$ with $a_{3}=2$ and $a_{3}=3$, respectively. Let $F^{\prime}:=P \cap\left(\mathbb{R}^{2} \times\{1\}\right)$. Since each facet of $P$ contains an integer point in its relative interior, it follows that $F^{\prime}$ is a maximal lattice-free triangle. Indeed, if $a_{3}=2$, then any integer point $w=\left(w_{1}, w_{2}, w_{3}\right)$ in the relative interior of one of the three facets different from $F$ satisfies $w_{3}=1$. On the other hand, if $a_{3}=3$, then any integer point $w=\left(w_{1}, w_{2}, w_{3}\right)$ in the relative interior of one of the three facets different from $F$ with $w_{3}=2$ guarantees that the point $2 w-a \in F^{\prime}$ is also an integer point in the relative interior of the same facet as $w$. According to Dey and Wolsey DW08 the maximal lattice-free triangles can be partitioned into three types:

- a type 1 triangle, i.e., a triangle with integer vertices and exactly one integer point in the relative interior of each edge,
- a type 2 triangle, i.e., a triangle with at least one fractional vertex $v$, exactly one integer point in the relative interior of the two edges incident to $v$ and at least two integer points on the third edge,
- a type 3 triangle, i.e., a triangle with exactly three integer points on the boundary, one in the relative interior of each edge.


Figure 6: All types of maximal lattice-free triangles in dimension two

Lemma 7.1. Let $P \in \mathcal{M}^{3}$ be a simplex with base $F \subseteq \mathbb{R}^{2} \times\{0\}$ and apex $a=\left(a_{1}, a_{2}, a_{3}\right)$, where $a_{3} \in\{2,3\}$. Then $P$ is equivalent to one of the simplices $M_{1}, M_{2}, M_{3}, M_{6}$, or $M_{7}$.

Proof. We distinguish into three cases according to the type of triangle of $F^{\prime}:=P \cap\left(\mathbb{R}^{2} \times\{1\}\right)$.
Case 1: $F^{\prime}$ is a triangle of type 1. Without loss of generality assume $F^{\prime}=\operatorname{conv}\left(\left\{e_{3}, 2 e_{1}+e_{3}, 2 e_{2}+e_{3}\right\}\right)$. Thus, if $a_{3}=2, F$ is a translate of $\operatorname{conv}\left(\left\{o, 4 e_{1}, 4 e_{2}\right\}\right)$ which leads to $M_{2}$. If $a_{3}=3, F$ is a translate of $\operatorname{conv}\left(\left\{o, 3 e_{1}, 3 e_{2}\right\}\right)$ which leads to $M_{3}$.

Case 2: $F^{\prime}$ is a triangle of type 2. Without loss of generality assume that the edge of $F^{\prime}$ having at least two relative interior integer points contains the points $(0,0,1)$ and $(0,1,1)$ in its relative interior, and let the vertex $w=\left(w_{1}, w_{2}, 1\right)$ of $F^{\prime}$ opposite to this edge satisfy $w_{1}>1$. By an appropriate unimodular transformation we can assume that the remaining two edges pass through the points $(1,0,1)$ and $(1,1,1)$. First assume $a_{3}=2$. Then $\operatorname{vert}\left(F^{\prime}\right)=\frac{1}{2} a+\frac{1}{2} \operatorname{vert}(F) \subseteq \frac{1}{2} \mathbb{Z}^{3}$. Hence, the three vertices of $F^{\prime}$ lie in $\frac{1}{2} \mathbb{Z} \times\left\{\frac{1}{2}\right\} \times\{1\}$ and $\{0\} \times \frac{1}{2} \mathbb{Z} \times\{1\}$. It follows that $F^{\prime}=$ $\operatorname{conv}\left(\left\{\left(0, \frac{3}{2}, 1\right),\left(0,-\frac{1}{2}, 1\right),\left(2, \frac{1}{2}, 1\right)\right\}\right)$ or $F^{\prime}=\operatorname{conv}\left(\left\{(0,2,1),(0,-1,1),\left(\frac{3}{2}, \frac{1}{2}, 1\right)\right\}\right)$. Thus, in the former case, $F$ is a translate of $\operatorname{conv}(\{(0,3,0),(0,-1,0),(4,1,0)\})$ leading to $M_{7}$, whereas in the latter case $F$ is a translate of $\operatorname{conv}(\{(0,4,0),(0,-2,0),(3,1,0)\})$ leading to $M_{1}$. Now assume $a_{3}=3$. Then $\operatorname{vert}\left(F^{\prime}\right)=$ $\frac{1}{3} a+\frac{2}{3} \operatorname{vert}(F) \subseteq \frac{1}{3} \mathbb{Z}^{3}$. Hence, two vertices of $F^{\prime}$ lie in $\{0\} \times \frac{1}{3} \mathbb{Z} \times\{1\}$ and the third vertex lies either in $\frac{1}{3} \mathbb{Z} \times\left\{\frac{1}{3}\right\} \times\{1\}$ or $\frac{1}{3} \mathbb{Z} \times\left\{\frac{2}{3}\right\} \times\{1\}$. By symmetry, we can assume that the third vertex lies in $\frac{1}{3} \mathbb{Z} \times\left\{\frac{2}{3}\right\} \times\{1\}$. It follows that $F^{\prime}=\operatorname{conv}\left(\left\{(0,2,1),(0,-2,1),\left(\frac{4}{3}, \frac{2}{3}, 1\right)\right\}\right)$ or $F^{\prime}=\operatorname{conv}\left(\left\{\left(0, \frac{4}{3}, 1\right),\left(0,-\frac{2}{3}, 1\right),\left(2, \frac{2}{3}, 1\right)\right\}\right)$. Thus, in the former case, $F$ is a translate of $\operatorname{conv}(\{(0,3,0),(0,-3,0),(2,1,0)\})$ leading to $M_{1}$, whereas in the latter case $F$ is a translate of $\operatorname{conv}(\{(0,2,0),(0,-1,0),(3,1,0)\})$ leading to $M_{6}$.

Case 3: $F^{\prime}$ is a triangle of type 3. Without loss of generality assume that $F^{\prime}=\operatorname{conv}(\{u, v, w\})$ with $u_{1}<0,1<u_{2}, 1<v_{1}, 0<v_{2}<1,0<w_{1}<1, w_{2}<0$, and $u_{3}=v_{3}=w_{3}=1$, see Figure 7 First


Figure 7: Triangle of type 3
assume $a_{3}=2$. Then $\operatorname{vert}\left(F^{\prime}\right)=\frac{1}{2} a+\frac{1}{2} \operatorname{vert}(F) \subseteq \frac{1}{2} \mathbb{Z}^{3}$. Thus, it follows $v_{2}=w_{1}=\frac{1}{2}$ and hence we obtain $v=\left(\frac{3}{2}, \frac{1}{2}, 1\right)$ and $w=\left(\frac{1}{2},-\frac{1}{2}, 1\right)$. This implies $u=\left(-\frac{3}{2}, \frac{3}{2}, 1\right)$. However, the edge connecting $u$ and $w$ contains the two integer points $(0,0,1)$ and $(-1,1,1)$ in its relative interior which is a contradiction to the fact that $F^{\prime}$ is of type 3. Now assume $a_{3}=3$. Then $\operatorname{vert}\left(F^{\prime}\right)=\frac{1}{3} a+\frac{2}{3} \operatorname{vert}(F) \subseteq \frac{1}{3} \mathbb{Z}^{3}$. Thus, it follows $v_{2} \in\left\{\frac{1}{3}, \frac{2}{3}\right\}$ and $w_{1} \in\left\{\frac{1}{3}, \frac{2}{3}\right\}$. Since the edge connecting $v$ and $w$ goes through the point $(1,0,1)$, the following cases are possible:

$$
\begin{aligned}
& v=\left(\frac{5}{3}, \frac{1}{3}, 1\right), w=\left(\frac{1}{3},-\frac{1}{3}, 1\right) \quad \Longrightarrow u=\left(-\frac{5}{3}, \frac{5}{3}, 1\right) \quad \Longrightarrow \quad F^{\prime} \text { is of type 2, } \\
& v=\left(\frac{7}{3}, \frac{2}{3}, 1\right), w=\left(\frac{1}{3},-\frac{1}{3}, 1\right) \quad \Longrightarrow u=\left(-\frac{7}{6}, \frac{7}{6}, 1\right) \quad \Longrightarrow \quad u \notin \frac{1}{3} \mathbb{Z}^{3}, \\
& v=\left(\frac{4}{3}, \frac{1}{3}, 1\right), w=\left(\frac{1}{3},-\frac{2}{3}, 1\right) \quad \Longrightarrow u=\left(-\frac{2}{3}, \frac{4}{3}, 1\right), \\
& v=\left(\frac{5}{3}, \frac{2}{3}, 1\right), w=\left(\frac{1}{3},-\frac{2}{3}, 1\right) \quad \Longrightarrow u=\left(-\frac{5}{9}, \frac{10}{9}, 1\right) \quad \Longrightarrow \quad u \notin \frac{1}{3} \mathbb{Z}^{3}, \\
& v=\left(\frac{4}{3}, \frac{2}{3}, 1\right), w=\left(\frac{1}{3},-\frac{4}{3}, 1\right) \quad \Longrightarrow u=\left(-\frac{4}{15}, \frac{16}{15}, 1\right) \quad \Longrightarrow \quad u \notin \frac{1}{3} \mathbb{Z}^{3}, \\
& v=\left(\frac{4}{3}, \frac{1}{3}, 1\right), w=\left(\frac{2}{3},-\frac{1}{3}, 1\right) \quad \Longrightarrow \quad F^{\prime} \text { is no triangle, } \\
& v=\left(\frac{5}{3}, \frac{2}{3}, 1\right), w=\left(\frac{2}{3},-\frac{1}{3}, 1\right) \quad \Longrightarrow u=\left(-\frac{10}{3}, \frac{5}{3}, 1\right) \quad \Longrightarrow \quad(-1,1,1) \in \operatorname{relint}\left(F^{\prime}\right), \\
& v=\left(\frac{4}{3}, \frac{2}{3}, 1\right), w=\left(\frac{2}{3},-\frac{2}{3}, 1\right) \quad \Longrightarrow \quad u=\left(-\frac{4}{3}, \frac{4}{3}, 1\right) \quad \Longrightarrow \quad F^{\prime} \text { is of type } 2 .
\end{aligned}
$$

In seven of these eight cases, it follows that $F^{\prime}$ is not a valid triangle. In the open case where $v=\left(\frac{4}{3}, \frac{1}{3}, 1\right)$, $w=\left(\frac{1}{3},-\frac{2}{3}, 1\right)$, and $u=\left(-\frac{2}{3}, \frac{4}{3}, 1\right)$ we infer that $F$ is a translate of $\operatorname{conv}\left(\left\{\left(2, \frac{1}{2}, 0\right),\left(\frac{1}{2},-1,0\right),(-1,2,0)\right\}\right)$. However, such a translate does never have all three vertices integer.

In the following we assume that $a_{3} \geq 4$. Our proof consists of the following steps. Firstly, we construct all bases which are possible for such a simplex $P \in \mathcal{M}^{3}$. Secondly, we argue that all simplices $P \in \mathcal{M}^{3}$ satisfy $a_{3} \leq 12$. This gives a finite set of simplices that need to be checked for maximal lattice-freeness. Finally, the ultimate list of maximal lattice-free simplices is obtained by computer enumeration.

By Lemma 6.3 all integral triangles $T$ in the plane with $w(T)=2$ and $(i(T), b(T)) \in Z_{S}$ are potential bases for a maximal lattice-free simplex $P \in \mathcal{M}^{3}$ with $a_{3} \geq 4$. From (5), it follows that $2 i(F)+b(F) \leq 6$ for $a_{3} \geq 11$ and therefore $(i(F), b(F))=(1,3)$ or $(i(F), b(F))=(1,4)$. If $(i(F), b(F))=(1,3)$, then $F$ is, up to a unimodular transformation, the triangle shown in Figure 2(e). In Lemma 7.3 we shall show that $a_{3} \leq 12$ in this case since otherwise $P$ is not lattice-free. If $(i(F), b(F))=(1,4)$, then $F$ is, up to a unimodular transformation, the triangle shown in Figure 2(c) In Lemma 7.4 we shall show that $a_{3} \leq 8$ in this case since otherwise $P$ is not lattice-free. Thus, we can use computer enumeration to find all simplices $P \in \mathcal{M}^{3}$.

Lemma 7.2. Let $T \subseteq \mathbb{R}^{2}$ be an integral triangle with $w(T)=2, i(T)=2$, and $b(T) \in\{3,4,5,6\}$. Then, up to a unimodular transformation, $T$ is one of the triangles depicted in Figure 8 .

Proof. Let $T$ be an integral triangle in the plane satisfying $w(T)=2$ and $i(T)=2$. We divide the proof according to the number of integer points on the boundary of $T$.

Case 1: $i(T)=2$ and $b(T)=3$. Without loss of generality we assume that the two interior integer points are placed at $(1,0)$ and $(2,0)$. This implies that for any $u \in \mathbb{Z}^{2} \backslash\left\{o, \pm e_{2}\right\}$ we have $w(T, u) \geq 3$ and


Figure 8: All integral triangles $T$ with $w(T)=2, i(T)=2$, and $b(T) \in\{3,4,5,6\}$
therefore it must hold $v_{2} \in\{0, \pm 1\}$ for each vertex $v=\left(v_{1}, v_{2}\right)$ of $T$. Observe that exactly one vertex of $T=\operatorname{conv}(\{a, b, c\})$ lies on the line $y=0$, say $a=(0,0)$. Let the remaining two vertices $b$ and $c$ satisfy $b_{2}=1=-c_{2}$. Using an appropriate unimodular transformation we can assume that $b=(0,1)$. For convexity reasons it follows that $c=(5,-1)$ which leads to the triangle shown in Figure 8(a).

Case 2: $i(T)=2$ and $b(T)=4$. Pick's formula gives $\mathrm{A}(T)=3$ in this case. Placing the two interior integer points of $T$ at $(1,0)$ and $(2,0)$ as above implies again that $v_{2} \in\{0, \pm 1\}$ for each vertex $v=\left(v_{1}, v_{2}\right)$ of $T$. Let $T=\operatorname{conv}(\{a, b, c\})$. Clearly, we cannot have two vertices on the line $y=0$. If none of the vertices is on the line $y=0$, then assume without loss of generality that $a_{2}=b_{2}=1=-c_{2}$. It follows that either $b_{1}=a_{1}+2$ with $\mathrm{A}(T)=2$, or $b_{1}=a_{1}+1$ with $\mathrm{A}(T)=1$. In both cases this is a contradiction to Pick's formula. Thus, exactly one vertex lies on the line $y=0$, say $a=(0,0)$. Let the remaining two vertices $b$ and $c$ satisfy $b_{2}=1=-c_{2}$. As above, we can assume that $b=(0,1)$ which implies $c=(6,-1)$. This gives the triangle shown in Figure 8(b).

Case 3: $i(T)=2$ and $b(T)=5$. Pick's formula gives $\mathrm{A}(T)=3.5$. Placing the two interior integer points of $T$ at $(1,0)$ and $(2,0)$ as above implies again that $v_{2} \in\{0, \pm 1\}$ for each vertex $v=\left(v_{1}, v_{2}\right)$ of $T$. Clearly, we cannot have two vertices on the line $y=0$. If no vertex of $T$ lies on the line $y=0$, then with similar arguments as above we infer that $\mathrm{A}(T) \leq 3$, a contradiction to Pick's formula. Thus, precisely one vertex of $T=\operatorname{conv}(\{a, b, c\})$ lies on the line $y=0$, say $a=(0,0)$. Without loss of generality let $b_{2}=1=-c_{2}$. Note that the two edges connecting $a$ and $b$, resp. connecting $a$ and $c$, do not have integer points in their relative interior. The edge connecting $b$ and $c$ has at most one relative interior integer point. Therefore, we have at most four integer points on the boundary of $T$ which is a contradiction to $b(T)=5$.

Case 4: $i(T)=2$ and $b(T)=6$. Pick's formula gives $\mathrm{A}(T)=4$. Placing the two interior integer points of $T$ at $(1,0)$ and $(2,0)$ as above implies again that $v_{2} \in\{0, \pm 1\}$ for each vertex $v=\left(v_{1}, v_{2}\right)$ of $T$. Clearly, we cannot have two vertices on the line $y=0$. If exactly one vertex of $T$ lies on the line $y=0$, say $a=(0,0)$, then using the same arguments as above we infer that $T$ has at most four integer points on its boundary, a contradiction to $b(T)=6$. Thus, no vertex of $T$ is on the line $y=0$. Without loss of generality let $a_{2}=b_{2}=1=-c_{2}$. It follows that $b_{1}=a_{1}+4$, otherwise Pick's formula is violated. Using an appropriate unimodular transformation, we obtain $a=(0,1), b=(4,1)$ and $c=(1,-1)$, see Figure 8(c).

From Lemma 6.3 and Lemma 7.2 it follows that any facet of a simplex $P \in \mathcal{M}^{3}$ with $a_{3} \geq 4$ has the structure shown in Figures 2(a) $2(\mathrm{e})$ and 8 Furthermore, inequalities (5) imply that only $2(\mathrm{c})$ and $2(\mathrm{e})$ are possible if $a_{3} \geq 11$. In the following two lemmas we will show that simplices having those two bases are not lattice-free for $a_{3} \geq 13$. Thus, by computer enumeration over all potential bases and values for $a_{3}$ ranging from 4 to 12 , we obtain a finite list of simplices. Screening those which are not maximal lattice-free we end up with the simplices $M_{4}$ and $M_{5}$.

Lemma 7.3. Let $P \subseteq \mathbb{R}^{3}$ be a simplex with one facet being $\operatorname{conv}\left(\left\{e_{1}, e_{2},-\left(e_{1}+e_{2}\right)\right\}\right)$ and apex $a=$ $\left(a_{1}, a_{2}, a_{3}\right) \in \mathbb{Z}^{3}$, where $a_{3} \geq 13$. Then, $P$ is not lattice-free.

Proof. By applying an appropriate unimodular transformation we can assume that $0 \leq a_{i}<a_{3}$ for $i=1,2$. By symmetry, we assume $a_{1} \leq a_{2}$. We now set up the facet description of $P$ which is only dependent on the parameters $a_{1}, a_{2}$, and $a_{3}$. It follows that $\mathbb{Z}^{3} \cap \operatorname{int}(P)$ is the set of all $x=\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{Z}^{3}$ satisfying

$$
\begin{array}{cc}
a_{3} x_{1}-2 a_{3} x_{2}+\left(1+2 a_{2}-a_{1}\right) x_{3}<a_{3}, & a_{3} x_{1}+a_{3} x_{2}+\left(1-a_{1}-a_{2}\right) x_{3}<a_{3}, \\
a_{3} x_{2}-2 a_{3} x_{1}+\left(1+2 a_{1}-a_{2}\right) x_{3}<a_{3}, & x_{3} \in\left\{1, \ldots, a_{3}-1\right\} .
\end{array}
$$

From these inequalities, it follows that

$$
\begin{equation*}
a_{3}+1 \geq a_{1}+a_{2} \tag{15}
\end{equation*}
$$

since otherwise $(1,1,1) \in \operatorname{int}(P)$. Assume $a_{1}=0$. If $a_{2} \leq 1$, we have $(0,0,1) \in \operatorname{int}(P)$, otherwise $(0,1,1) \in \operatorname{int}(P)$. Therefore, we must have $a_{1} \geq 1$. It follows that

$$
\begin{equation*}
2 a_{1}+1 \geq a_{2} \tag{16}
\end{equation*}
$$

since otherwise $(0,1,1) \in \operatorname{int}(P)$. Observe that $(0,0,1) \in \operatorname{int}(P)$ if and only if $a_{1}+a_{3}-2 a_{2}>1$ and $a_{2}+a_{3}-2 a_{1}>1$. Assume $a_{1} \leq 3$. Then $(0,0,1) \in \operatorname{int}(P): a_{1}+a_{3}-2 a_{2} \stackrel{16}{\geq} a_{3}-3 a_{1}-2 \geq 2>1$; $a_{2}+a_{3}-2 a_{1}=\left(a_{2}-a_{1}\right)+a_{3}-a_{1} \geq a_{3}-a_{1} \geq 10>1$. Thus, we have $a_{1} \geq 4$. Using (15) this implies $a_{3} \geq a_{2}+3$ and therefore $a_{2}+a_{3}-2 a_{1} \geq 2\left(a_{2}-a_{1}\right)+3>1$. Hence, we have

$$
\begin{equation*}
2 a_{2}+1 \geq a_{1}+a_{3} \tag{17}
\end{equation*}
$$

since otherwise $(0,0,1) \in \operatorname{int}(P)$. However, using inequalities (15)-17) it can now be shown that $(1,2,3) \in \operatorname{int}(P):$

$$
\begin{aligned}
(1,2,3) \in \operatorname{int}(P) \Longleftrightarrow \quad & 3 a_{1}+3 a_{2}-2 a_{3}>3 \\
& 3 a_{1}-6 a_{2}+4 a_{3}>3 \\
- & 6 a_{1}+3 a_{2}+a_{3}>3
\end{aligned}
$$

$3 a_{1}+3 a_{2}-2 a_{3} \stackrel{\text { (17) }}{\geq} 5 a_{1}-a_{2}-2 \stackrel{\text { (16) }}{\geq} 3 a_{1}-3=3\left(a_{1}-1\right)>3 ; 3 a_{1}-6 a_{2}+4 a_{3} \stackrel{\text { (15) }}{\geq} 7 a_{1}-2 a_{2}-4 \stackrel{\text { (16) }}{\geq} 3\left(a_{1}-2\right)>3$; $-6 a_{1}+3 a_{2}+a_{3} \stackrel{(15)}{\geq}-5 a_{1}+4 a_{2}-1 \stackrel{(17)}{\geq} 2 a_{3}-3 a_{1}-3 \stackrel{(15)}{\geq} 2 a_{2}-a_{1}-5 \stackrel{\text { (177 }}{\geq} a_{3}-6>3$.
Lemma 7.4. Let $P \subseteq \mathbb{R}^{3}$ be a simplex with one facet being $\operatorname{conv}\left(\left\{ \pm e_{1}, 2 e_{2}\right\}\right)$ and apex $a=\left(a_{1}, a_{2}, a_{3}\right) \in$ $\mathbb{Z}^{3}$, where $a_{3} \geq 9$. Then, $P$ is not lattice-free.
Proof. By applying an appropriate unimodular transformation we can assume that $0 \leq a_{i}<a_{3}$ for $i=1,2$. We now set up the facet description of $P$ which is only dependent on the parameters $a_{1}, a_{2}$, and $a_{3}$. It follows that $\mathbb{Z}^{3} \cap \operatorname{int}(P)$ is the set of all $x=\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{Z}^{3}$ satisfying

$$
\begin{array}{rr}
2 a_{3} x_{1}+a_{3} x_{2}+\left(2-2 a_{1}-a_{2}\right) x_{3}<2 a_{3}, & -a_{3} x_{2}+a_{2} x_{3}<0, \\
-2 a_{3} x_{1}+a_{3} x_{2}+\left(2+2 a_{1}-a_{2}\right) x_{3}<2 a_{3}, & x_{3} \in\left\{1, \ldots, a_{3}-1\right\} .
\end{array}
$$

From these inequalities we obtain the following equivalences:

- $(0,1,1) \in \operatorname{int}(P)$ if and only if $2 a_{1}+2<a_{2}+a_{3}$;
- $(1,1,1) \in \operatorname{int}(P)$ if and only if $a_{3}+2<2 a_{1}+a_{2}$.

This implies that the following inequalities hold:

$$
\begin{align*}
2 a_{1}+2 & \geq a_{2}+a_{3}  \tag{18}\\
a_{3}+2 & \geq 2 a_{1}+a_{2} . \tag{19}
\end{align*}
$$

Adding (18) and (19) yields $a_{2} \leq 2$. Using (18), (19) and $a_{2} \leq 2$ it can be shown that $(1,1,2) \in \operatorname{int}(P)$ :

$$
\begin{gathered}
(1,1,2) \in \operatorname{int}(P) \Longleftrightarrow \begin{array}{r}
4 a_{1}+2 a_{2}-a_{3}>4, \\
-4 a_{1}+2 a_{2}+3 a_{3}>4, \\
\\
-2 a_{2}+a_{3}>0 .
\end{array} \\
4 a_{1}+2 a_{2}-a_{3} \stackrel{\text { 目 }}{\geq} 4 a_{2}+a_{3}-4>4 ;-4 a_{1}+2 a_{2}+3 a_{3} \stackrel{\text { 19) }}{\geq} 4 a_{2}+a_{3}-4>4 ;-2 a_{2}+a_{3} \stackrel{a_{2} \leq 2, a_{3} \geq 9}{>} 4 .
\end{gathered}
$$

## 8 Remarks on the computer enumeration

In view of the results in Sections 47 for proving Theorem 2.2 it remains to verify the following.

- The integral quadrangular pyramids with bases as in Figures $2(\mathrm{~h}) 2(\mathrm{l})$ and 5 of height $h$ with $4 \leq h \leq 10$ are not in $\mathcal{M}^{3}$.
- The integral simplices with bases as in Figures 2(a) 2(e) and 8 of height $h$ with $4 \leq h \leq 12$ belonging to $\mathcal{M}^{3}$ are equivalent to $M_{4}$ or $M_{5}$.
This can be done by a computer enumeration which involves less than 15000 polytopes.


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