Intersection Cuts with Infinite Split Rank

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Abstract

We consider mixed integer linear programs where free integer variables are expressed in terms of nonnegative continuous variables. When this model only has two integer variables, Dey and Louveaux characterized the intersection cuts that have infinite split rank. We show that, for any number of integer variables, the split rank of an intersection cut generated from a bounded convex set P is finite if and only if the integer points on the boundary of P satisfy a certain "2-hyperplane property". The Dey-Louveaux characterization is a consequence of this more general result.

1 Introduction.

In this paper, we consider mixed integer linear programs with equality constraints expressing $m \geq 1$ free integer variables in terms of $k \geq 1$ nonnegative continuous variables.

$$\begin{array}{rcl} x & = & f + \sum_{j=1}^{k} r^j s_j \\ x & \in & \mathbb{Z}^m \\ s & \in & \mathbb{R}^k_+. \end{array} \tag{1}$$

The convex hull R of the solutions to (1) is a corner polyhedron (Gomory [11], Gomory and Johnson [12]). In the remainder we assume $f \in \mathbb{Q}^m \setminus \mathbb{Z}^m$, and $r^j \in \mathbb{Q}^m \setminus \{0\}$. Hence $(x, s) = (f, 0)$ is not a solution of (1). To avoid discussing trivial cases, we assume that $R \neq \emptyset$, and this implies that $\dim(R) = k$. The facets of R are the nonnegativity constraints $s \geq 0$ and intersection cuts (Balas [2]), namely inequalities

$$\sum_{j=1}^{k} \psi(r^j) s_j \geq 1 \tag{2}$$

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obtained from lattice-free convex sets $L \subset \mathbb{R}^m$ containing f in their interior, where ψ denotes the gauge of $L-f$ (Borozan and Cornuéjols [4]). By *lattice-free* convex set, we mean a convex set with no point of $\mathbb{Z}^m$ in its interior. By *gauge* of a convex set L containing the origin in its interior, we mean the function $\gamma_L(r) = \inf\{t > 0 \mid \frac{r}{t} \in L\}$. Intersection cut (2) for a given convex set L is called *L-cut* for short.

When $m = 2$, Andersen, Louveaux, Weismantel and Wolsey [1] showed that the only intersection cuts needed arise from splits (Cook et al. [6]), triangles and quadrilaterals in the plane and a complete characterization of the facet-defining inequalities was obtained in Cornuéjols and Margot [7]. More generally, Borozan and Cornuéjols [4] showed that the only intersection cuts needed in (1) arise from full-dimensional maximal lattice-free convex sets L. Lovász [13] showed that these sets are polyhedra with at most 2^m facets and that they are *cylinders*, i.e., their recession cone is a linear space. These cylinders $L \subset \mathbb{R}^m$ can be written in the form $L = Q + V$ where Q is a polytope of dimension at least one and V is a linear space such that $\dim(Q) + \dim(V) = m$. We say that L is a *cylinder over* Q. A *split* in $\mathbb{R}^m$ is a maximal lattice-free cylinder over a line segment.

Let $L \subset \mathbb{R}^m$ be a polytope containing f in its interior. For $j = 1, \ldots, k$, let the *boundary point* for the ray r^j be the intersection of the half-line $\{f + \lambda r^j \mid \lambda \geq 0\}$ with the boundary of L. We say that L has *rays going into its corners* if each vertex of L is the boundary point for at least one of the rays r^j, $j = 1, \ldots, k$.

The notions of *split closure* and *split rank* were introduced by Cook et al. [6] (precise definitions are in Section 2). They gave an example of (1) with $m = 2$ and $k = 3$ that has an infinite split rank. Specifically, there is a facet-defining inequality for R that cannot be deduced from a finite recursive application of the split closure operation. This inequality is an intersection cut generated from a maximal lattice-free triangle L with integer vertices and rays going into its corners. That triangle has vertices $(0, 0)$, $(2, 0)$ and $(0, 2)$ and has six integer points on its boundary. It is a triangle of *Type 1* according to Dey and Wolsey [9]. Dey and Louveaux [8] showed that, when $m = 2$, an intersection cut has an infinite split rank if and only if it is generated from a Type 1 triangle with rays going into its corners. In this paper we prove a more general theorem, whose statement relies on the following definitions.

A set S of points in $\mathbb{R}^m$ is *2-partitionable* if either $|S| \leq 1$ or there exists a partition of S into *nonempty* sets S_1 and S_2 and a split such that the points in S_1 are on one of its boundary hyperplanes and the points in S_2 are on the other. We say that a polytope is *2-partitionable* if its integer points are 2-partitionable.

Let L be a rational lattice-free polytope in $\mathbb{R}^m$ and let L_I be the convex hull of the integer points in L. We say that L has the *2-hyperplane property* if every face of L_I that is not contained in a facet of L is 2-partitionable. Note that one of the faces of L_I is L_I itself and thus if L has the 2-hyperplane property and L_I is not contained in a facet of L, then there exists a split containing all the integer points of L on its boundary hyperplanes, with at least one integer point of L on each of the hyperplanes.

We illustrate the 2-hyperplane property by giving an example in $\mathbb{R}^3$. Consider the tetrahedron $L \subseteq \mathbb{R}^3$ given in Figure 1. We will show that L has the 2-hyperplane property. Let T_0 be the shaded triangle with corners $(0, 0, 0), (2, 0, 0)$, and $(0, 2, 0)$ and let T_1 be the shaded triangle with corners $(0, 0, 1), (1, 0, 1)$, and $(0, 1, 1)$. For any point $q \in \mathbb{R}^3$ with $1 < q_3 \leq 2$, let $L(q)$ be the tetrahedron obtained as the intersection of the half-space $x_3 \geq 0$ with the cone with apex q and three extreme rays joining it to the corners of T_1. Observe that for

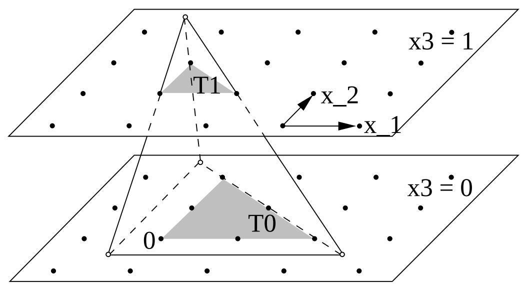

Figure 1: Illustration of the 2-hyperplane property.

$t = (0, 0, 2)$, the vertices of $L(t)$ are t and the corners of T_0. The tetrahedron L depicted in Figure 1 is $L(p)$ for $p = (0.25, 0.25, 1.5)$. As p is in $L(t)$, and the intersection of $L(p)$ with the plane $x_3 = 1$ is the triangle T_1, it follows that the intersection of $L(p)$ with the plane $x_3 = 0$ contains T_0. To check whether L has the 2-hyperplane property or not, we need to check if some of the faces of the convex hull L_I of the integer points in L are 2-partitionable or not. Note that L_I is the convex hull of T_1 and T_0 and that a face of L_I that is not contained in a facet of L is either the triangle T_1 (which is 2-partitionable, using for example the split with boundary hyperplanes $x_1 = 0$ and $x_1 = 1$) or it contains an integer point in the plane $x_3 = 1$ and an integer point in the plane $x_3 = 0$ (and thus is 2-partitionable using these two planes as boundary for the split). As a result, L has the 2-hyperplane property. We now give an example of a polytope L' that does not have the 2-hyperplane property. Define $L'(q)$ as $L(q)$ above, except that we change the half-space $x_3 \geq 0$ to $x_3 \geq -\frac{1}{2}$. Let L' be obtained by truncating $L'(t)$ by $x_3 \leq \frac{3}{2}$, where $t = (0, 0, 2)$ as earlier. Then L'_I is again the convex hull of T_1 and T_0. However T_0 is a face of L'_I not contained in a facet of L'. Furthermore, T_0 is not 2-partitionable because it is not possible to find a split and a partition of the six integer points of T_0 into two nonempty sets S_1, S_2 with the property that S_1 lies on one boundary of the split and S_2 on the other. Therefore L' does not have the 2-hyperplane property.

The main result of this paper is the following theorem.

Theorem 1.1. *Let L be a rational lattice-free polytope in $\mathbb{R}^m$ containing f in its interior and having rays going into its corners. The L-cut has finite split rank if and only if L has the 2-hyperplane property.*

Given a polytope $L \subset \mathbb{R}^m$ containing f in its interior, let L_B be the convex hull of the boundary points for the rays $r^1, \ldots, r^k$. We assume here that nonnegative combinations of the rays r^j in (1) span $\mathbb{R}^m$. This implies that f is in the interior of L_B and that L_B has rays going into its corners. Moreover, the L-cut and L_B-cut are identical. Therefore the previous theorem implies the following.

Corollary 1.2. *Assume that nonnegative combinations of the rays r^j in (1) span $\mathbb{R}^m$. Let L be a rational lattice-free polytope in $\mathbb{R}^m$ containing f in its interior. The L-cut has finite split rank if and only if L_B has the 2-hyperplane property.*

This corollary is a direct generalization of the characterization of Dey and Louveaux for $m = 2$, as triangles of Type 1 do not have the 2-hyperplane property whereas all other lattice-free polytopes in the plane do, as one can check using Lovász' [13] characterization of maximal lattice-free convex sets in the plane.

The paper is organized as follows. In Section 2, we give a precise definition of split inequalities and split rank, as well as useful related results. In Section 3 we give an equivalent formulation of (1) that proves convenient to compute the split rank of L-cuts. We prove one direction of Theorem 1.1 in Section 4 and we prove the other direction in Section 5.

2 Split inequalities and split closure.

Consider a mixed integer set $X = \{(x, y) \mid Ax + By \geq b,\ x \in \mathbb{Z}^p, y \in \mathbb{R}^q\}$, where A and B are respectively $m \times p$ and $m \times q$ rational matrices, and $b \in \mathbb{Q}^m$. Let $Q = \{(x, y) \in \mathbb{R}^{p+q} \mid Ax + By \geq b\}$ be its linear relaxation.

Let $\pi \in \mathbb{Z}^p$ and $\pi_0 \in \mathbb{Z}$. Note that all points in X satisfy the *split disjunction induced* by (π, π_0), i.e.,

$$\pi x \leq \pi_0 \qquad \text{or} \qquad \pi x \geq \pi_0 + 1 \ .$$

The hyperplanes in $\mathbb{R}^{p+q}$ defined by $\pi x = \pi_0$ and $\pi x = \pi_0 + 1$ are the *boundary hyperplanes* of the split. Conversely, two parallel hyperplanes H^1 and H^2 with rational equations, both containing points (x, y) with x integer and such that no points (x, y) with x integer are between them, define a valid split (π, π_0). Indeed, only x variables can have nonzero coefficients in the equation of the planes and we can assume that they are relatively prime integers. An application of Bézout's Theorem [15] shows that if, for $i = 1, 2$, the equation of the hyperplane H^i is given as $\pi x = h_i$, then $|h_1 - h_2| = 1$. Define

$$\begin{aligned} &Q^{\leq} = Q \cap \{(x, y) \in \mathbb{R}^{p+q} \mid \pi x \leq \pi_0\}\ ,\ \ Q^{\geq} = Q \cap \{(x, y) \in \mathbb{R}^{p+q} \mid \pi x \geq \pi_0 + 1\} \\ &Q(\pi, \pi_0) = \operatorname{conv}(Q^{\leq}\ \cup\ Q^{\geq})\ . \end{aligned}$$

As $X \subseteq Q^{\leq}\ \cup\ Q^{\geq}$, any inequality that is valid for $Q(\pi, \pi_0)$ is valid for $\operatorname{conv}(X)$. The facets of $Q(\pi, \pi_0)$ that are not valid for Q are *split inequalities* obtained from the split (π, π_0). As shown by Cook et al. [6], the intersection of all $Q(\pi, \pi_0)$ for all possible splits (π, π_0) yields a polyhedron called the *split closure* of Q. Let the *rank*-0 *split closure* of Q be Q itself. For $t = 1, 2, 3, \ldots$, the *rank-t split closure* of Q is obtained by taking the split closure of the rank-$(t-1)$ split closure of Q.

Note that since $Q(\pi, \pi_0)$ is the convex hull of $Q^{\leq}$ with $Q^{\geq}$, any point $(\bar{x}, \bar{y})$ in $Q(\pi, \pi_0)$ is a (possibly trivial) convex combination of a point $p^1 \in Q^{\leq}$ and a point $p^2 \in Q^{\geq}$. Moreover, if $(\bar{x}, \bar{y})$ is neither in $Q^{\leq}$ nor in $Q^{\geq}$, then the segment p^1p^2 intersects H^i in q^i for $i = 1, 2$. By convexity of Q, we have that $q^1 \in Q \cap H^1 = Q^{\leq} \cap H^1$ and $q^2 \in Q \cap H^2 = Q^{\geq} \cap H^2$. In summary, we have the following.

Observation 2.1. *(i) If $(\bar{x}, \bar{y}) \in Q^{\leq}$ or $(\bar{x}, \bar{y}) \in Q^{\geq}$, then $(\bar{x}, \bar{y}) \in Q$ and $(\bar{x}, \bar{y}) \in Q(\pi, \pi_0)$;*

(ii) If $(\bar{x}, \bar{y}) \in Q(\pi, \pi_0) \setminus (Q^{\leq} \cup Q^{\geq})$, then $(\bar{x}, \bar{y})$ is a convex combination of $(x^1, y^1) \in Q \cap H^1 = Q^{\leq} \cap H^1$ and $(x^2, y^2) \in Q \cap H^2 = Q^{\geq} \cap H^2$.

Let $ax + by \geq a_0$ be a valid inequality for $\text{conv}(X)$ and let t be the smallest nonnegative integer such that the inequality is valid for the rank-t split closure of a polyhedron $K \supseteq X$ or $+\infty$ if no such integer exists. The value t is the *split rank* of the inequality with respect to K. It is known that valid inequalities for $\text{conv}(X)$ may have infinite split rank with respect to K (Cook et al. [6]).

The following lemma gives three useful properties of split ranks of inequalities.

Lemma 2.2. *Let Q be the linear relaxation of $X = \{(x,y) \mid Ax + By \geq b,\ x \in \mathbb{Z}^p, y \in \mathbb{R}^q\}$.*

(i) Let $X \subseteq Q_1 \subseteq Q$. The split rank with respect to Q_1 of a valid inequality for conv(X) is at most its split rank with respect to Q;

(ii) Let y' be a subset of the y variables and let $Q(x,y')$ be the orthogonal projection of Q onto the variables (x,y'). Consider a valid inequality $\mathcal{I}$ for conv(X) whose coefficients for the y variables not in y' are all 0. The split rank of inequality $\mathcal{I}$ with respect to $Q(x,y')$ is greater than or equal to its split rank with respect to Q.

(iii) Assume that all points in Q satisfy an equality. Adding any multiple of this equality to a valid inequality for conv(X) does not change the split rank of the inequality with respect to Q.

Proof. (i) Let (π,π_0) be a split on the x variables. We have

$$Q_1^{\leq} \subseteq Q^{\leq}, \qquad \text{and} \qquad Q_1^{\geq} \subseteq Q^{\geq} \ .$$

It follows that the split closure of Q_1 is contained in the split closure of Q and that, for each $t = 0,1,2,\ldots$, the rank-t split closure of Q_1 is contained in the rank-t split closure of Q.

(ii) Let proj be the operation of projecting orthogonally onto the variables (x,y'). It follows from the definitions of projection and convex hull that the operations of taking the projection and taking the convex hull commute. Therefore we have, for any split (π,π_0) on the x variables,

$$\text{proj}\left(\text{conv}(Q^{\leq} \cup Q^{\geq})\right) = \text{conv}\left(\text{proj}(Q^{\leq}) \cup \text{proj}(Q^{\geq})\right) = \text{conv}\left(Q(x,y')^{\leq} \cup Q(x,y')^{\geq}\right) \ ,$$

with the validity of the last equality coming from the fact that none of the variables involved in the disjunction are projected out. Hence, for all $t = 0,1,2,\ldots$, the projection proj of the rank-t split closure of Q is contained in the rank-t split closure of $Q(x,y')$ as the projection of an intersection of polyhedra is contained in the intersection of their projections. The result then follows from the fact that inequality $\mathcal{I}$ is valid for a polyhedron Q' in the (x,y)-space if and only if it is valid for $\text{proj}(Q')$.

(iii) For any $t = 0,1,2,\ldots$, all points in the rank-t split closure of Q satisfy the equality. An inequality is valid for the rank-t split closure of Q if and only if the inequality obtained by adding to it any multiple of the equality is. $\square$

Let $Q_x \subseteq \mathbb{R}^p$ be a polyhedron where $\mathbb{R}^p$ is the space of integer variables of the mixed integer set X. Assume that Q_x is rational and full-dimensional. For each facet F of Q_x there exists a split $(\pi(F), \pi_0(F))$ with boundary hyperplanes H^1 and H^2 parallel to F, with F between H^1 and H^2 and with some points in Q_x strictly between F and H^2. (Note that, if the hyperplane supporting F contains an integer point, then H^1 supports F.) Let the *width of the split* $(\pi(F), \pi_0(F))$ be the Euclidean distance between F and H^2.

As earlier, let $Q = \{(x, y) \in \mathbb{R}^{p+q} \mid Ax + By \geq b\}$. Performing a *round of splits around* Q_x on Q means generating the intersection Q' of $Q(\pi(F), \pi_0(F))$ for all facets F of Q_x. Note that if Q contains the rank-t split closure of an arbitrary polytope Q^*, then Q' contains the rank-$(t+1)$ split closure of Q^*. Define the *width of a round of splits* around Q_x as the minimum of the width of the splits $(\pi(F), \pi_0(F))$ for all facets F of Q_x.

3 Changing space to compute the split rank.

In the remainder of this paper, we will use P to denote the linear relaxation of (1). Given a lattice-free polytope $L \subseteq \mathbb{R}^m$ containing f in its interior, our goal is to compute the split rank of the L-cut with respect to P. We show that this rank can be computed in another space that we find convenient.

Let $P^L(x, s, z)$ be the polyhedron obtained from P by adding one equation corresponding to the L-cut (2) with a free continuous variable z representing the difference between its left and right-hand sides:

$$\begin{array}{rcl} x & = & f + \sum_{j=1}^{k} r^j s_j \\ z & = & 1 - \sum_{j=1}^{k} \psi(r^j) s_j \\ x & \in & \mathbb{R}^m \\ s & \in & \mathbb{R}^k_+ \\ z & \in & \mathbb{R}\ . \end{array} \tag{3}$$

Clearly, P is the orthogonal projection of $P^L(x, s, z)$ onto the (x, s)-space. As the relation between P and $P^L(x, s, z)$ is a bijection projecting or adding a single continuous variable z, the split rank of (2) with respect to P or $P^L(x, s, z)$ are identical. Let $P^L(x, z)$ be the orthogonal projection of $P^L(x, s, z)$ onto the (x, z)-space. By Lemma 2.2 (iii), inequalities (2) and $z \leq 0$ have the same split rank with respect to $P^L(x, s, z)$. By Lemma 2.2 (ii), the split rank of the inequality $z \leq 0$ for $P^L(x, s, z)$ is smaller than or equal to its rank for $P^L(x, z)$. We thus have the following:

Observation 3.1. *Let $L \subseteq \mathbb{R}^m$ be a lattice-free polytope containing f in its interior and let P be the linear relaxation of* (1)*. The split rank of the L-cut (2) with respect to P is smaller than or equal to the split rank of the inequality $z \leq 0$ with respect to $P^L(x, z)$.*

Let $0^k \in \mathbb{R}^k$ be the zero vector and let $e^j \in \mathbb{R}^k$ be the unit vector in direction j. Observe that $P^L(x, s, z)$ is a cone with apex $(f, 0^k, 1)$ and extreme rays $\{(r^j, e^j, -\psi(r^j)) \mid j = 1, 2, \ldots, k\}$. As $(f, 1)$ is a vertex of $P^L(x, z)$, the latter is also a pointed cone. Its apex is $(f, 1)$ and its extreme rays are among $\{(r^j, -\psi(r^j)) \mid j = 1, 2, \ldots, k\}$. Note also that if we embed L in the hyperplane $z = 0$, then, for all $j = 1, 2, \ldots, k$, the point $p^j = f + \frac{1}{\psi(r^j)} r^j$ is on the boundary of L. The intersection of $P^L(x, z)$ with the hyperplane $z = 0$ is the convex hull of the points p^j for $j = 1, \ldots, k$.

Consider the pointed cone P^L with apex $(f, 1)$ and extreme rays joining f to the vertices of L embedded in the hyperplane $z = 0$ (Figure 2). Note that $P^L(x, z)$ and P^L have the same apex and that all the extreme rays of $P^L(x, z)$ are convex combinations of those of P^L. Thus, $P^L(x, z) \subseteq P^L$ and $P^L(x, z) = P^L$ if and only if L has rays going into its corners.

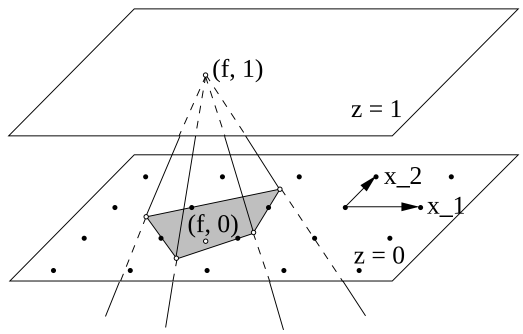

Figure 2: Illustration for the relation between P^L, $P^L(x, z)$ and L. Polytope L (shaded) is embedded in the plane $z = 0$ and contains $(f, 0)$; cone P^L has apex $(f, 1)$ and extreme rays joining $(f, 1)$ to the corners of L; cone $P^L(x, z)$ (not drawn) has apex $(f, 1)$ and extreme rays joining $(f, 1)$ to points on the boundary of L.

Let H be a polyhedron in the (x, z)-space. For any $\bar{x} \in \mathbb{R}^m$, define the *height* of $\bar{x}$ with respect to H as $\max\{\bar{z} \mid (\bar{x}, \bar{z}) \in H\}$, with the convention that this number may be $+\infty$ if $\{(\bar{x}, z) \in H\}$ is unbounded in the direction of the z-unit vector or $-\infty$ if the maximum is taken over an empty set. Define the *height* of H as the maximum height of $\bar{x} \in \mathbb{R}^m$ in H. Observe that if the height of H is $< +\infty$ then the height with respect to H is a concave function over $\mathbb{R}^m$. Note that the split rank of the inequality $z \leq 0$ with respect to H is at most t if and only if the rank-t split closure of H has height at most zero.

By Lemma 2.2 (i), the split rank of $z \leq 0$ with respect to $P^L(x, z)$ is at most its split rank with respect to P^L. With the help of Observation 3.1, we get:

Observation 3.2. *Let $L \subseteq \mathbb{R}^m$ be a lattice-free polytope containing f in its interior. If the rank-t split closure of P^L has height at most zero, then the split rank of the L-cut (2) with respect to P is at most t.*

4 Proof of necessity.

In this section, we prove the "only if" part of Theorem 1.1. For a polyhedron Q, we denote by $\text{int}(Q)$ (resp. $\text{relint}(Q)$) the interior (resp. relative interior) of Q.

Theorem 4.1. *Let L be a rational lattice-free polytope in $\mathbb{R}^m$ containing f in its interior and having rays going into its corners. If the L-cut has finite split rank with respect to P, then every face of L_I that is not contained in a facet of L is 2-partitionable.*

Proof. Assume that the L-cut has split rank k for some finite k. Thus, the height of the rank-k split closure of $P^L(x, s, z)$ is at most zero. By a theorem of Cook et al. [6], the split closure of a polyhedron is a polyhedron. Therefore there exists a finite number t of splits

$(\pi^1, \pi_0^1), \ldots, (\pi^t, \pi_0^t)$ such that applying these splits to P^L in that order reduces its height to zero or less. Let $Q^0 := P^L$ and $Q^j := Q^{j-1}(\pi^j, \pi_0^j)$ for $j = 1, 2, \ldots, t$.

Suppose for a contradiction that there exists a face F of L_I that is not 2-partitionable and F not contained in a facet of L. As F is not 2-partitionable, it must contain at least two integer points and thus $\text{relint}(F) \neq \emptyset$. We claim that any point $\bar{x} \in \text{relint}(F)$ has positive height with respect to Q^j for $j = 0, 1, \ldots, t$, a contradiction.

We prove the claim by induction on j. For $j = 0$, the result follows from the fact that $\bar{x} \in \text{int}(L)$ and every point in $\text{int}(L)$ has a positive height with respect to P^L. Indeed, any point $\bar{x} \in \text{int}(L)$ can be written as $\bar{x} = \lambda f + (1-\lambda)x^*$ where x^* is a point on the boundary of L with $0 < \lambda \leq 1$. By convexity of P^L, the height of $\bar{x}$ with respect to P^L is at least $\lambda \cdot 1 + (1-\lambda) \cdot 0 > 0$, as the height of f (resp. x^*) with respect to P^L is 1 (resp. 0).

Suppose now that $j > 0$ and that the claim is true for $j - 1$. If F is contained in one of the boundary hyperplanes of (π^j, π_0^j) then F is contained in $Q^{j-1\leq}$ or $Q^{j-1\geq}$ and Observation 2.1 (i) shows that the height of any $\bar{x} \in \text{relint}(F)$ with respect to Q^j and Q^{j-1} is identical, proving the claim for j. Otherwise, as F is not 2-partitionable, there exists an integer point $\bar{p}$ of F that is not on the boundary hyperplanes of (π^j, π_0^j). Since $\bar{p}$ is integer, it is strictly on one of the two sides of the split disjunction implying that there exists a point $x^* \in \text{relint}(F)$ that is also strictly on one of the two sides of the split disjunction. The height of x^* with respect to Q^j and Q^{j-1} is identical and positive by induction hypothesis. All points on the relative boundary of F have non-negative height with respect to Q^j as they are convex combinations of vertices of F that have height zero, and the height is a concave function. As any point $\bar{x} \in \text{relint}(F)$ is a convex combination of x^* and a point on the boundary of F with a positive coefficient for x^*, the height of $\bar{x}$ with respect to Q^j is positive. This proves the claim. □

5 Proof of sufficiency.

Recall that P denotes the linear relaxation of (1). In this section, we prove the following theorem.

Theorem 5.1. *Let L be a rational lattice-free polytope in $\mathbb{R}^m$ containing f in its interior. If L has the 2-hyperplane property, then the L-cut has finite split rank with respect to P.*

Before giving the details of the proof, we present the main ideas. The first one is presented in Section 5.1, where we prove that Theorem 5.1 holds when there is a sequence of "intersecting splits" followed by an "englobing split". These notions are defined as follows.

Let Q be a polytope in $\mathbb{R}^m$ and let (π, π_0) be a split. The part of Q contained between or on the boundary hyperplanes $\pi x = \pi_0$ and $\pi x = \pi_0 + 1$ is denoted by $\overline{Q(\pi, \pi_0)}$. The split (π, π_0) is a *Q-intersecting* split if both of its boundary hyperplanes have a nonempty intersection with Q. The split is *Q-englobing* if $Q = \overline{Q(\pi, \pi_0)}$. Note that a split can be simultaneously englobing and intersecting, as illustrated in Figure 3.

Let $(\pi^1, \pi_0^1), (\pi^2, \pi_0^2), \ldots, (\pi^t, \pi_0^t)$ be a finite sequence of splits. Recall the definition of $Q(\pi, \pi_0)$ from Section 2. The polytopes $Q^0 := Q$, and $Q^j := Q^{j-1}(\pi^j, \pi_0^j)$ for $j = 1, \ldots, t$ are the *polytopes obtained from the sequence* and Q^t is the *polytope at the end of the sequence.* When S denotes the sequence $(\pi^1, \pi_0^1), (\pi^2, \pi_0^2), \ldots, (\pi^t, \pi_0^t)$ of splits, we will use the notation

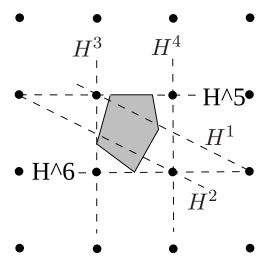

Figure 3: Illustration for intersecting and englobing splits. Polytope Q is shaded. Split with boundary hyperplanes H^1 and H^2 is Q-intersecting; split with boundary hyperplanes H^3 and H^4 is Q-englobing; split with boundary hyperplanes H^5 and H^6 is simultaneously Q-intersecting and Q-englobing.

$Q(S)$ to denote the polytope Q^t. We say that this sequence is a *sequence of Q-intersecting splits* if, for all $j = 1, \dots, t$, we have that (π^j, π_0^j) is Q^{j-1}-intersecting.

Our approach to proving Theorem 5.1 is to work with P^L, as introduced at the end of Section 2, instead of with P directly. By Observation 3.2, to show that an L-cut has finite split rank with respect to P, it is sufficient to show that the height of P^L can be reduced to at most 0 in a finite number of split operations. Lemma 5.7 guarantees a reduction of the height of P^L when applying repeatedly a sequence of intersecting splits followed by an englobing split. A key result to proving this lemma is Lemma 5.5, showing that, for an intersecting split, we can essentially guarantee a constant reduction in height for the points cut off by the split. A consequence of these results is Corollary 5.8 that provides a sufficient condition for proving that the L-cut obtained from a bounded rational lattice-free set L has finite split rank. It is indeed enough to exhibit a finite sequence of L-intersecting cuts followed by a final englobing cut.

Sections 5.2-5.4 deal with the unfortunate fact that it is not obvious that a sequence of intersecting splits followed by an englobing split always exists. However, since it is possible to reduce L to L_I using Chvátal cuts, the result is proved by replacing each of the Chvátal cuts by a finite collection of intersecting splits for enlarged polytopes, and using the 2-hyperplane property for proving that a final englobing split exists. Section 5.2 shows how to enlarge polytopes so that they have desirable properties. Section 5.3 proves a technical lemma about Chvátal cuts. In Section 5.4, induction on the dimension is used to prove that the region removed by a Chvátal cut can also be removed by a finite number of intersecting splits for enlarged polytopes.

We now present details of the proof.

5.1 Intersecting splits.

The Euclidean distance between two points x^1, x^2 is denoted by $d(x^1, x^2)$. For a point x^1 and a set S, we define $d(x^1, S) = \inf\{d(x^1, y) \mid y \in S\}$. The *diameter* of a polytope Q, denoted by $\mathrm{diam}(Q)$, is the maximum Euclidean distance between two points in Q.

Let Q_x be a rational polytope in $\mathbb{R}^m$ of dimension at least 1, and let $M_0 < M$ be two finite numbers. Define $R(Q_x, M, M_0)$ (see Figure 4) as the nonconvex region in the (x, z)-space $\mathbb{R}^m \times \mathbb{R}$ containing all points $(\bar{x}, \bar{z})$ such that $\bar{x}$ is in the affine subspace $\mathrm{aff}(Q_x)$ spanned by Q_x and

$$\bar{z} \leq \begin{cases} M & \text{if } \bar{x} \in \text{relint}(Q_x) \\ M_0 - \frac{d(\bar{x}, Q_x)}{\text{diam}(Q_x)} (M - M_0) & \text{otherwise.} \end{cases}$$

Note that this definition implies that $\bar{z} \leq M_0$ if $\bar{x}$ is on the boundary of Q_x.

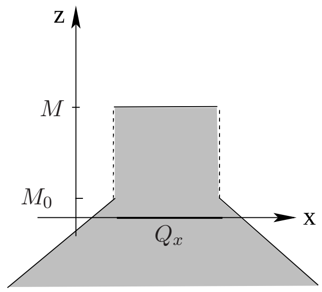

Figure 4: Illustration for the definition of $R(Q_x, M, M_0)$ (shaded area).

The purpose of $R(Q_x, M, M_0)$ is to provide an upper bound on the height of points outside Q_x with respect to any polyhedron Q of height M having the property that the height with respect to Q of any $\bar{x} \notin \text{relint}(Q_x)$ is at most M_0, as shown in the next lemma.

Lemma 5.2. *Let $M_0 < M$ be two finite numbers, let $Q^0 \in \mathbb{R}^{m+1}$ be a rational polyhedron of height M in the (x, z)-space, and let $Q_x \in \mathbb{R}^m$ be a rational polytope containing $\{x \mid \exists z > M_0 \text{ with } (x, z) \in Q^0\}$ in its relative interior. Then $Q^0 \subseteq R(Q_x, M, M_0)$.*

Proof. Let $(\bar{x}, \bar{z}) \in Q^0$. As the height of Q^0 is M, we have $\bar{z} \leq M$. If $\bar{x} \in \text{relint}(Q_x)$ then it follows that $(\bar{x}, \bar{z}) \in R(Q_x, M, M_0)$. Otherwise $\bar{x} \notin \text{relint}(Q_x)$ and therefore $z \leq M_0$ by definition of Q_x. Let x^M be a point in Q_x with height M with respect to Q^0. Let x^0 be the intersection of the half-line starting at x^M and going through $\bar{x}$ with the boundary of Q_x. Notice that this intersection is on the segment $x^M\bar{x}$. As the height of a point with respect to Q^0 is a concave function, the height of x^M is M, and the height of x^0 is at most M_0 by choice of Q_x, we have that

$$\bar{z} \leq M_0 - \frac{(M - M_0)}{d(x^M, x^0)} d(\bar{x}, x^0) .$$

The result follows from $d(x^M, x^0) \leq \text{diam}(Q_x)$ and $d(\bar{x}, x^0) \geq d(\bar{x}, Q_x)$. $\square$

We define a $(Q_x, H^1, \bar{H})$ *triplet* as follows: Q_x is a full-dimensional rational polytope in $\mathbb{R}^m$ with $m \geq 2$; $H^1 \subseteq \mathbb{R}^m$ is a hyperplane with $Q_x^1 := H^1 \cap Q_x \neq \emptyset$; $H \neq H^1$ is a hyperplane in $\mathbb{R}^m$ supporting a nonempty face of Q_x^1 and $\bar{H}$ is a closed half-space bounded by H not containing Q_x^1 (Figure 5). Such a triplet naturally occurs when using a Q_x-intersecting split (π, π_0) when both Q_x and $Q_x(\pi, \pi_0)$ are full dimensional: H^1 is one of the two boundary hyperplanes of the split, H is a hyperplane supporting a facet of $Q_x(\pi, \pi_0)$ that is not a facet of Q_x, and $\bar{H}$ is the half-space bounded by H that does not contain $Q_x(\pi, \pi_0)$.

Given a $(Q_x, H^1, \bar{H})$ triplet, we claim that there exists a hyperplane H^* separating $\text{int}(Q_x)$ from $\bar{H} \cap H^1$ with $H^* \cap H^1 = H \cap H^1$ and maximizing the angle between H^1 and H^*. This maximum value is denoted by $\theta(Q_x, H^1, \bar{H})$.

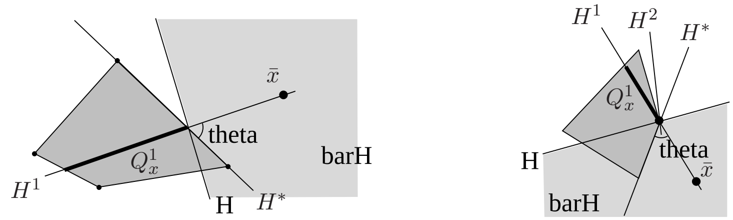

Figure 5: Illustration of the definition of $\theta(Q_x, H^1, \bar{H})$ for $m = 2$. Polytope Q_x is shaded, polytope Q_x^1 is the bold segment, part of $\bar{H}$ is lightly shaded. On the left, rotating H^1 around $H \cap H^1$ to get a hyperplane supporting a face of Q_x yields a facet of Q_x and $H^2 = H^*$. On the right, such a rotation could give the depicted hyperplane H^2, and an additional rotation gives H^*.

To see that the claim holds, let $\bar{x}$ be a point in the relative interior of $\bar{H} \cap H^1$. As $H \cap H^1$ is a hyperplane of H^1 separating $\bar{x}$ from Q_x^1, we can rotate H^1 around $H \cap H^1$ to get a hyperplane H^2 supporting a face of Q_x. Then, we can possibly rotate H^2 around $H \cap H^1$ to increase the angle between H^2 and H^1 while keeping the resulting hyperplane H^* supporting a face of Q_x. This rotation is stopped either if an angle of $\frac{\pi}{2}$ is obtained, or if H^* gains a point of Q_x outside of $H \cap H^1$.

Lemma 5.3.

$$d(\bar{x}, Q_x) \geq d(\bar{x}, H \cap H^1) \cdot \sin\theta(Q_x, H^1, \bar{H}) \ . \tag{4}$$

Proof. The observation follows from

$$d(\bar{x}, Q_x) \geq d(\bar{x}, H^*) = d(\bar{x}, H^* \cap H^1) \cdot \sin\theta(Q_x, H^1, \bar{H}) = d(\bar{x}, H \cap H^1) \cdot \sin\theta(Q_x, H^1, \bar{H}) \ . \tag{5}$$

Indeed, the first inequality comes from the fact that H^* separates $\bar{x}$ from Q_x, the first equality is pictured in Figure 6, and the last equality follows from $H^* \cap H^1 = H \cap H^1$. $\square$

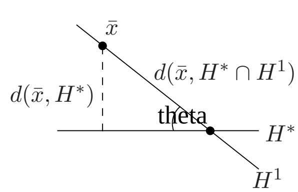

Figure 6: Illustration for the proof of Lemma 5.3.

Lemma 5.4. *Consider a $(Q_x, H^1, \bar{H})$ triplet. Let $M_0^* < M^*$ be two finite numbers and let $Q \subseteq R(Q_x, M^*, M_0^*)$ be a rational polyhedron of height $M \leq M^*$. Let $\bar{x} \in H^1 \cap \bar{H}$. The height $\bar{z}$ of $\bar{x}$ with respect to Q satisfies*

$$\bar{z} \leq M_0^* - \sin\theta(Q_x, H^1, \bar{H}) \cdot \frac{(M^* - M_0^*)}{diam(Q_x)} \cdot d(\bar{x}, H \cap H^1) \ .$$

Proof. As $Q \subseteq R(Q_x, M^*, M_0^*)$ and $\bar{x} \notin \text{relint}(Q_x)$, we have that

$$\bar{z} \leq M_0^* - \frac{d(\bar{x}, Q_x)}{\text{diam}(Q_x)} (M^* - M_0^*) \ . \tag{6}$$

If $\bar{x}$ is on the boundary of Q_x, then $\bar{z} \leq M_0^*$, and the result holds since $d(\bar{x}, H \cap H^1) = 0$. Otherwise, the result follows from using Lemma 5.3 to replace the term $d(\bar{x}, Q_x)$ in (6) by $d(\bar{x}, H \cap H^1) \cdot \sin\theta(Q_x, H^1, \bar{H})$. □

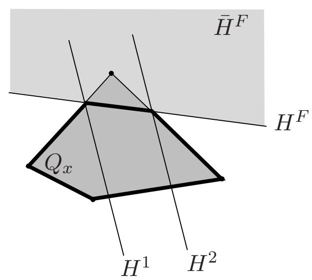

Figure 7: Illustration of the definition of $\delta(Q_x, (\pi, \pi_0), \bar{H}^F)$ for $m = 2$. Polytope Q_x is shaded, part of $\bar{H}^F$ is lightly shaded, and $Q_x(\pi, \pi_0)$ is depicted in bold line.

Let Q_x be a full-dimensional rational polytope in $\mathbb{R}^m$ with $m \geq 2$ and let (π, π_0) be a Q_x-intersecting split with boundary hyperplanes H^1 and H^2 (see Figure 7). Assume first that $Q_x(\pi, \pi_0)$ is full-dimensional and strictly contained in Q_x, and that the width w of a round of splits around Q_x satisfies $w < \text{diam}(Q_x)$. Recall that the width of a round of splits is defined at the end of Section 2. Let H^F be a hyperplane supporting a facet F of $Q_x(\pi, \pi_0)$ that is not a facet of Q_x and let $\bar{H}^F$ be the closed half-space not containing $Q_x(\pi, \pi_0)$ bounded by H^F. As mentioned earlier, we have that, for $i = 1, 2$, $(Q_x, H^i, \bar{H}^F)$ is a triplet. Let

$$\delta(Q_x, (\pi, \pi_0), \bar{H}^F) = \frac{w}{\text{diam}(Q_x)} \cdot \min\{\sin\theta(Q_x, H^1, \bar{H}^F),\ \sin\theta(Q_x, H^2, \bar{H}^F)\} \ .$$

Define the *reduction coefficient* for $(Q_x, (\pi, \pi_0))$, denoted by $\delta(Q_x, (\pi, \pi_0))$, as the minimum of $\delta(Q_x, (\pi, \pi_0), \bar{H}^F)$ taken over all hyperplanes H^F supporting a facet F of $Q_x(\pi, \pi_0)$ that is not a facet of Q_x. As $Q_x(\pi, \pi_0)$ has a finite number of facets, this minimum is well-defined and its value is positive and at most one. Assume now that $Q_x(\pi, \pi_0)$ is not full-dimensional or that $Q_x = Q_x(\pi, \pi_0)$ or that $w \geq \text{diam}(Q_x)$. The reduction coefficient for $(Q_x, (\pi, \pi_0))$ is then defined as the value 1. Note that the reduction coefficient depends only on Q_x and (π, π_0) and always has a positive value smaller than or equal to 1.

Lemma 5.4 can be used to prove a bound on the height of some points $\bar{x}$ after applying an intersecting split. Given a set $S \subseteq \mathbb{R}^n$ we denote its closure by $\text{cl}(S)$. (We mean the topological closure here, not to be confused with the split closure.)

Lemma 5.5. *Let Q_x be a full-dimensional rational polytope in $\mathbb{R}^m$ with $m \geq 2$. Let (π, π_0) be a Q_x-intersecting split and let S be the sequence of a round of splits around Q_x followed by (π, π_0).*

Then, for any two finite numbers $M_0^* < M^*$ and for any rational polyhedron $Q \subseteq R(Q_x, M^*, M_0^*)$, the height with respect to $Q(S)$ of any point in $cl(Q_x \setminus Q_x(\pi, \pi_0))$ is at most $\max\{M_0^*,\ M -\ \delta(Q_x, (\pi, \pi_0)) \cdot (M^* - M_0^*)\}$, *where M is the height of Q.*

Proof. Let w be the width of the round of splits around Q_x. Observe that if $w \geq \mathrm{diam}(Q_x)$ then all the splits used during the round of splits around Q_x are Q_x-englobing. It follows that the height of $Q(S)$ is at most M_0^* and the result holds (recall that if $Q(S) = \emptyset$ then its height is $-\infty$). Similarly, if $Q_x(\pi, \pi_0)$ is not full-dimensional, then (π, π_0) is Q_x-englobing and the result holds. Finally, if $Q_x(\pi, \pi_0) = Q_x$ then the result trivially holds.

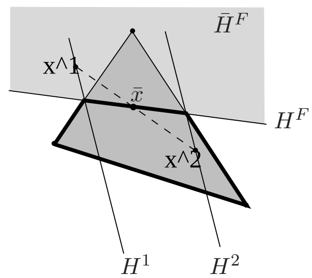

Figure 8: Illustration of the proof of Lemma 5.5. Polytope $Q_x \subseteq \mathbb{R}^2$ is shaded, part of $\bar{H}^F$ is lightly shaded, and $Q_x(\pi, \pi_0)$ is depicted in bold line. Point $\bar{x}$ is a convex combination of x^1 and x^2.

We can thus assume that $w < \mathrm{diam}(Q_x)$ and that $Q_x(\pi, \pi_0)$ is full-dimensional and strictly contained in Q_x. Let H^1 and H^2 be the boundary hyperplanes of (π, π_0). Let $\bar{x} \in \mathrm{cl}(Q_x \setminus Q_x(\pi, \pi_0))$ with maximum height $\bar{z}$ with respect to $Q(S)$. If $\bar{x}$ is in a facet F of $Q_x(\pi, \pi_0)$, let H^F be a hyperplane supporting F. Otherwise, let H^F be a hyperplane supporting a facet of $Q_x(\pi, \pi_0)$ separating $\bar{x}$ from $Q_x(\pi, \pi_0)$. Let $\bar{H}^F$ be the half-space bounded by H^F not containing $Q_x(\pi, \pi_0)$. We may assume $\bar{z} > M_0^*$ as otherwise the result trivially holds. Thus $\bar{x}$ is in $\mathrm{int}(Q_x)$ and strictly between H^1 and H^2 (Figure 8). Therefore, as shown in Observation 2.1 (ii), $(\bar{x}, \bar{z})$ is a convex combination of a point $(x^1, z^1) \in Q(S)$ with $x^1 \in H^1$ and a point $(x^2, z^2) \in Q(S)$ with $x^2 \in H^2$, namely

$$(\bar{x}, \bar{z}) = \frac{d(x^2, \bar{x})}{d(x^1, x^2)} \cdot (x^1, z^1) + \frac{d(x^1, \bar{x})}{d(x^1, x^2)} \cdot (x^2, z^2) \ . \tag{7}$$

As all points in $\bar{H}^F \cap H^i$ for $i = 1, 2$ are not in $\mathrm{int}(Q_x)$ and thus have height at most M_0^*, one of the points x^i is in $Q_x \setminus \bar{H}^F$. Moreover, as $\bar{x} \in \bar{H}^F$, the other one is in $\bar{H}^F$. Without loss of generality, we assume that $x^2 \in Q_x \setminus \bar{H}^F$. For $i = 1, 2$, let p^i be the closest point to x^i in $H^F \cap H^i$ and let $d_i = d(x^i, p^i)$. We thus have $d_2 > 0$. As $z^1 \leq M_0^* < \bar{z}$, we have $z^2 > z^1$ and points on the segment joining (x^1, z^1) to (x^2, z^2) have increasing height as they get closer to (x^2, z^2). We thus have that $\bar{x}$ is on H^F, implying that $d_1 > 0$. Using (7) and the similarity of triangles $x^1 p^1 \bar{x}$ and $x^2 p^2 \bar{x}$, implying $\frac{d(x^i, \bar{x})}{d(x^1, x^2)} = \frac{d^i}{d_1 + d_2}$, we get

$$(\bar{x}, \bar{z}) = \frac{d_2}{d_1 + d_2} \cdot (x^1, z^1) + \frac{d_1}{d_1 + d_2} \cdot (x^2, z^2) \ . \tag{8}$$

To simplify notation in the remainder of the proof, we use δ instead of $\delta(Q_x, (\pi, \pi_0))$. By Lemma 5.4 and the definition of δ, the height z with respect to Q of any point $x \in \bar{H}^F \cap H^i$ is at most $M_0^* - \frac{\delta}{w} \cdot (M^* - M_0^*) \cdot d(x, H^F \cap H^i)$.

It follows that we have

$$z^1 \leq M_0^* - \frac{\delta}{w} \cdot (M^* - M_0^*) \cdot d_1 \ . \tag{9}$$

Using (9) in (8), we get

$$\bar{z} \ \leq \ \frac{d_2}{d_1 + d_2} \cdot \left(M_0^* - \frac{\delta}{w} \cdot (M^* - M_0^*) \cdot d_1 \right) + \frac{d_1}{d_1 + d_2} \cdot z^2 \ . \tag{10}$$

Assume first that $d_2 \geq w$. As $z^2 \leq M = M_0^* + (M - M_0^*)$, (10) becomes

$$\bar{z} \ \leq \ M_0^* + \frac{d_1}{d_1 + d_2} \left((M - M_0^*) - \delta \cdot \frac{d_2}{w} \cdot (M^* - M_0^*) \right) \ .$$

As we have that $\bar{z} > M_0^*$, the expression in brackets above is positive. Since the fraction in front of it is at most 1, we get

$$\bar{z} \ \leq \ M_0^* + (M - M_0^*) - \delta \cdot \frac{d_2}{w}(M^* - M_0^*) \leq M - \delta \cdot (M^* - M_0^*) \ ,$$

which proves the result.

Assume now that $d_2 < w$. We claim that

$$z^2 \leq M_0^* + \frac{d_2}{w} \cdot (M - M_0^*) \ . \tag{11}$$

To show this, we first claim that there exists a facet F^1 of Q_x with $d(x^2, F^1) \leq d^2$. Recall that p^2 is the point in $H^F \cap H^2$ closest to x^2. If p^2 is on the boundary of Q_x, then any facet F^1 of Q_x containing p^2 proves the claim. Otherwise, as $H^F \cap H^2$ supports $Q_x \cap H^2$, the segment x^2p^2 intersects the boundary of $Q_x \cap H^2$ in a point that is on a facet F^1 of Q_x, proving the claim. Let $H(F^1)$ and $H'(F^1)$ be the boundary hyperplanes of the split $(\pi(F^1), \pi_0(F^1))$ such that $H(F^1) \cap \text{int}(Q_x) = \emptyset$ (Figure 9).

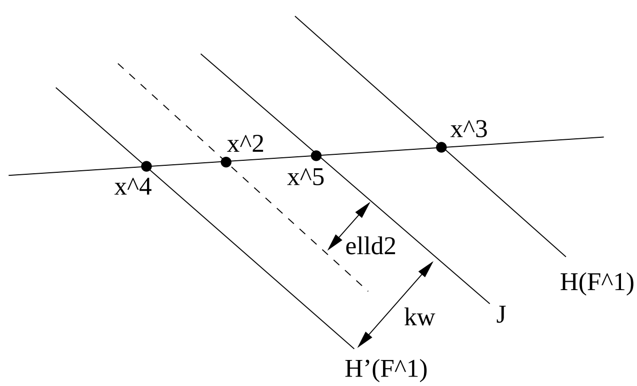

Figure 9: Illustration of the proof of Lemma 5.5.

By definition of the width of a round of splits around Q_x, the distance k between F^1 and $H'(F^1)$ is at least w. By Observation 2.1 (ii), the point (x^2, z^2) is a convex combination of a point $(x^3, z^3) \in Q \cap H(F^1)$ and a point $(x^4, z^4) \in Q \cap H'(F^1)$. The hyperplane J in $\mathbb{R}^m$ supporting F^1 can be lifted in the (x, z)-space to the hyperplane $J' = \{(x, z) \in \mathbb{R}^{m+1} \mid x \in J\}$ orthogonal to the x-space. Similarly, the hyperplane $H'(F^1) \in \mathbb{R}^m$ can be lifted to a hyperplane $H''(F^1)$ in $\mathbb{R}^{m+1}$ orthogonal to the x-space. The segment joining (x^3, z^3) to (x^4, z^4) intersects J' in a point $(x^5, z^5) \in Q$ and thus

$$(x^2, z^2) = \frac{d(x^2, x^5)}{d(x^4, x^5)} \cdot (x^4, z^4) + \frac{d(x^2, x^4)}{d(x^4, x^5)} \cdot (x^5, z^5) \ . \tag{12}$$

Let q' (resp. q'') be the closest point to (x^2, z^2) in J' (resp. $H''(F^1)$) and let $\ell = d((x^2, z^2), q')$. Note that ℓ is at most the distance between x^2 and F^1 and recall that we have shown above that this is at most d^2. Using similar triangles $(x^2, z^2)q'(z^4, z^4)$ and $(x^2, z^2)q''(z^5, z^5)$, we have

$$\frac{d(x^2, x^5)}{d(x^4, x^5)} = \frac{\ell}{k} \qquad \text{and} \qquad \frac{d(x^2, x^4)}{d(x^4, x^5)} = \frac{k - \ell}{k} \ . \tag{13}$$

Note that, trivially, $z^4 \leq M$ and that $z^5 \leq M_0^*$ as all points x on J are either on the boundary or outside of Q_x. Using these inequalities and (13) in (12), we get

$$z^2 \leq \frac{\ell}{k} \cdot M + \frac{k - \ell}{k} \cdot M_0^* = \frac{\ell}{k} \cdot (M - M_0^*) + M_0^* \leq \frac{d^2}{w} \cdot (M - M_0^*) + M_0^* \ , \tag{14}$$

proving (11). Using (11) in (10), we obtain

$$\bar{z} \quad \leq \quad M_0^* + \frac{d_1}{d_1 + d_2} \cdot \frac{d_2}{w} \cdot ((M - M_0^*) - \delta \cdot (M^* - M_0^*)) \ .$$

Since $\bar{z} > M_0^*$, the expression in brackets above is positive and each of the two fractions in front of it are positive and at most 1. We thus get

$$\bar{z} \quad \leq \quad M - \delta \cdot (M^* - M_0^*) \ ,$$

proving the result. □

The next lemma plays an important role in the proof of Theorem 5.1. Before stating the lemma, we make the following observation, which will be used in its proof.

Observation 5.6. *Let C_1 be a full-dimensional convex set and let C_2 be a closed set in $\mathbb{R}^m$. Then $C_1 \setminus C_2$ is either full-dimensional or empty.*

Proof. Suppose that $C_1 \setminus C_2$ is not empty and let $x \in C_1 \setminus C_2$. Since C_2 is closed, there exists $\epsilon > 0$ such that the closed ball $B(x, \epsilon) = \{y \in \mathbb{R}^m \,|\, d(y, x) \leq \epsilon\}$ does not intersect C_2. Therefore, $C_1 \cap B(x, \epsilon) \subseteq C_1 \setminus C_2$. But $C_1 \cap B(x, \epsilon)$ is full-dimensional, as $x \in C_1$, $x \in \text{int}(B(x, \epsilon))$, and C_1 is convex and full-dimensional. □

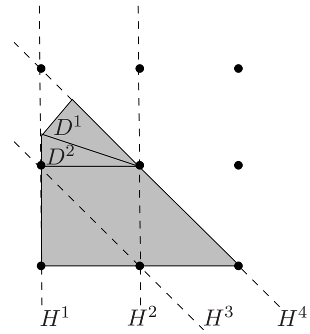

Figure 10: Illustration of the statement of Lemma 5.7 in a simple case. Polytope $L = L^1$ is shaded, $L^2 = L \setminus D^1$, $L^3 = L \setminus (D^1 \cup D^2)$, the boundary hyperplanes of (π^1, π_0^1) are H^1 and H^2, and the boundary hyperplanes of (π^2, π_0^2) are H^3 and H^4. The statement of the lemma is more general, as L^i must merely contain $L \setminus (D^1 \cup \ldots \cup D^{i-1})$ instead of being equal to it, as in this illustration.

The next lemma applies Lemma 5.5 iteratively n times, with a polytope $L^i \subseteq \mathbb{R}^m$ playing the role of Q_x for $i = 1, \ldots, n$. For application i, this creates a region $D^i = \mathrm{cl}(L^i \setminus L^i(\pi^i, \pi_0^i))$ for which a guaranteed height reduction is obtained. This guarantee is applied to a polyhedron $Q \subseteq R(L, M^*, M_0^*)$ where L is a polytope contained in L^1. The statement of the lemma is illustrated in Figure 10.

Lemma 5.7. *Let L be a full-dimensional rational polytope in $\mathbb{R}^m$ with $m \geq 2$. For $i = 1, \ldots n$, let L^i be a rational polytope in $\mathbb{R}^m$, let (π^i, π_0^i) be an L^i-intersecting split and let $D^i = cl(L^i \setminus L^i(\pi^i, \pi_0^i))$ for $i = 1, \ldots n$. Assume that $L \subseteq L^1$ and that $L \setminus (D^1 \cup \ldots \cup D^i) \subseteq L^{i+1}$, for $i = 1, \ldots, n-1$. Then there exists a finite sequence S of splits and a value $\Delta > 0$ such that, for any two finite numbers $M_0^* < M^*$ and any rational polyhedron $Q \subseteq R(L, M^*, M_0^*)$, one of the following cases holds*

(i) the height of all points in D^1 is at most M_0^* with respect to $Q(S)$;

(ii) the height of all points in $D^1 \cup \ldots \cup D^n$ is at most $M - \Delta \cdot (M^* - M_0^*)$ with respect to $Q(S)$, where M is the height of Q.

Proof. Let q be the smallest index $\{1, \ldots, n-1\}$ such that $L \setminus (D^1 \cup \ldots \cup D^q)$ is not full-dimensional and let $q = n$ if no such index exists. This ensures that L^j is full-dimensional for $j = 1, \ldots, q$. For $j = 1, \ldots, q$, let R^j denote the round of splits around L^j. Let the sequence S be $S = (R^1, (\pi^1, \pi_0^1), R^2, (\pi^2, \pi_0^2), \ldots, R^q, (\pi^q, \pi_0^q))$.

Let $Q^1 = Q$ and let $Q^{j+1} = Q^j(R^j, (\pi^j, \pi_0^j))$, and let $0 < \delta^j \leq 1$ be the reduction coefficient for $(L^j, (\pi^j, \pi_0^j))$ for $j = 1, \ldots, q$. Let $\Delta^1 = \delta^1$, let $\Delta^j = \delta^j \cdot \Delta^{j-1}$ for $j = 2, \ldots, q$, and observe that $\Delta^1 \geq \Delta^2 \geq \ldots \geq \Delta^q > 0$. Let $\Delta := \Delta^q$.

Since $Q \subseteq R(L, M^*, M_0^*)$ and $L \subseteq L^1$, we trivially have $Q \subseteq R(L^1, M^*, M_0^*)$. If $M_0^* \geq M - \Delta^1 \cdot (M^* - M_0^*)$, then Lemma 5.5 applied to L^1, (π^1, π_0^1), and Q proves that (i) holds for Q^2, proving the result for $Q(S)$ as $Q(S) \subseteq Q^2$. Therefore we may assume that $M_0^* < M - \Delta^1 \cdot (M^* - M_0^*)$.

We claim that the height of any point in $D^1 \cup \ldots \cup D^j$ is at most $M - \Delta^j \cdot (M^* - M_0^*)$ with respect to Q^{j+1} for $j = 1, \ldots, q$. We prove this claim by induction on j.

For $j = 1$, this is implied by Lemma 5.5 as it shows that the height of all points in D^1 is at most $M - \Delta^1 \cdot (M^* - M_0^*)$ with respect to Q^2.

Assume now that the claim is true for some $1 \leq j < q$ and we prove it for $j+1$. Let M^{j+1} be the height of Q^{j+1}. If $M^{j+1} \leq M - \Delta^j \cdot (M^* - M_0^*)$, then the claim holds for $j+1$ as $\Delta^j \geq \Delta^{j+1}$. Otherwise, as $L \setminus (D^1 \cup \ldots \cup D^j) \subseteq L^{j+1}$, the induction hypothesis implies that all points on the boundary or outside of L^{j+1} have height at most $M - \Delta^j \cdot (M^* - M_0^*)$ with respect to Q^{j+1}. Therefore Lemma 5.2 shows that $Q^{j+1} \subseteq R(L^{j+1}, M^{j+1}, M - \Delta^j \cdot (M^* - M_0^*))$. We can thus use Lemma 5.5 to get that the height of any point in D^{j+1} with respect to Q^{j+2} is at most

$$\max\{M - \Delta^j \cdot (M^* - M_0^*),\ M^{j+1} - \delta^{j+1} \cdot (M^{j+1} - (M - \Delta^j \cdot (M^* - M_0^*)))\}\ . \qquad (15)$$

The second term can be rewritten as

$$\begin{aligned} & (1 - \delta^{j+1}) \cdot M^{j+1} + \delta^{j+1} \cdot (M - \Delta^j \cdot (M^* - M_0^*)) \\ \leq\ & (1 - \delta^{j+1}) \cdot M + \delta^{j+1} \cdot (M - \Delta^j \cdot (M^* - M_0^*)) \\ =\ & M - \delta^{j+1} \cdot \Delta^j \cdot (M^* - M_0^*). \end{aligned}$$

As $0 < \delta^{j+1} \leq 1$, we obtain that the maximum in (15) is at most $M - \delta^{j+1} \cdot \Delta^j \cdot (M^* - M_0^*)$. Thus, the height of any point in $D^1 \cup \ldots \cup D^{j+1}$ is at most $M - \Delta^{j+1} \cdot (M^* - M_0^*)$ with respect to Q^{j+2}, proving the claim for $j+1$. This completes the proof of the claim.

If $q = n$, then point (ii) in the statement of the lemma is satisfied, as we have shown that the height of any point in $D^1 \cup \ldots \cup D^q$ is at most $M - \Delta^q \cdot (M^* - M_0^*) = M - \Delta \cdot (M^* - M_0^*)$ with respect to $Q^{q+1} = Q(S)$. If $q < n$ then, by definition of q, we have that $L \setminus (D^1 \cup \ldots \cup D^q)$ is not full-dimensional. Observe that, as the D^i's are closed sets, $(D^1 \cup \ldots \cup D^q)$ is a closed set. Then Observation 5.6 implies that $L \setminus (D^1 \cup \ldots \cup D^q)$ is empty. Since $M_0^* \leq M - \Delta^q \cdot (M^* - M_0^*)$ and since the height of all points outside L is at most M_0^* with respect to $Q(S)$, this shows that the height of any point in the x-space is at most $M - \Delta^q \cdot (M^* - M_0^*)$ with respect to $Q(S)$, proving that (ii) is satisfied. □

This lemma is sufficient to prove the following corollary.

Corollary 5.8. *Consider a mixed integer set of the form* (1) *with* $m = 2$. *Let* $L \subseteq \mathbb{R}^2$ *be a rational lattice-free polytope containing* f *in its interior. Let* $(\pi^1, \pi_0^1), \ldots, (\pi^n, \pi_0^n)$ *be a sequence of* L*-intersecting splits and let* L^n *be the polytope obtained by applying the sequence on* L. *Let* (π, π_0) *be an* L^n*-englobing split. Then there exists a finite number* q *such that the height of the rank-*q *split closure of* P^L *is at most zero.*

Proof. We prove the following more general result. Let Q_x be a full-dimensional rational polytope in $\mathbb{R}^m$ with $m \geq 2$ and let $M_0^* < M^*$ be two finite numbers. Let $Q \subseteq R(Q_x, M^*, M_0^*)$ be a rational polyhedron of height M such that the height of any integer point in Q_x is at most M_0^*. Let $(\pi^1, \pi_0^1), \ldots, (\pi^n, \pi_0^n)$ be a sequence of Q_x-intersecting splits and let Q_x^n be the polytope obtained by applying the sequence on Q_x. Let (π, π_0) be a Q_x^n-englobing split. Then there exists a finite number q such that the height of the rank-q split closure of Q is at most M_0^*.

Let Q_x^i for $i = 1, \ldots, n$ be the polytopes obtained from applying the sequence of splits on Q_x. Let S and Δ be obtained by applying Lemma 5.7 for $L := Q_x$, $L^1 := Q_x$, $L^{i+1} := Q_x^i$

for $i = 1, \ldots, n-1$, M^*, M_0^* and Q. Assume that $M - M_0^* \geq \Delta \cdot (M^* - M_0^*)$. Then all points on the boundary of Q_x^n have height at most $M_n := \max\{M_0^*,\ M - \Delta \cdot (M^* - M_0^*)\}$ with respect to $Q(S)$. Thus the height with respect to $Q(S)$ of any point on one of the boundary hyperplanes of the disjunction (π, π_0) is at most M_n, and the height of $Q(S)(\pi, \pi_0)$ is at most M_n. Applying the sequence $(S, (\pi, \pi_0))$ on Q at most $\lceil \frac{M - M_0^*}{\Delta \cdot (M^* - M_0^*)} \rceil$ times, we get a polyhedron Q^1 of height M^1 and for which all points in $\mathrm{cl}(Q_x \setminus Q_x^1)$ have height at most M_0^*. We can then iterate the above argument, applying Lemma 5.7 in iteration $j = 1, \ldots, n-1$ to $L := Q_x^j$, $L^i := Q_x^{i+j-1}$ for $i = 1, \ldots, n-j$, $M^* := M^j$, M_0^* and $Q := Q^j$. After these $n-1$ iterations, we get a polyhedron Q^n whose height is at most M_0^*.

The proof of the statement of the corollary follows by setting in the above proof $Q_x := L$, $Q := Q^L$, $M^* := 1$, and $M_0^* := 0$. $\square$

We note that this result can be used to prove the Dey-Louveaux theorem stating that the split rank of any L-cut is finite whenever $L \subseteq \mathbb{R}^2$ is a maximal lattice-free rational polytope distinct from a triangle of Type 1 with rays going into its corners. Indeed, for each case (quadrilateral, triangles of Type 2 or 3, and triangles of Type 1 with at least one corner ray missing), one can exhibit explicitly the intersecting splits (at most two of them) and the englobing split required by Corollary 5.8. However, to handle the general case where $L \subset \mathbb{R}^m$ with $m \geq 3$, a generalization of this corollary is presented in Section 5.4.

5.2 Enlarging the polyhedron.

The purpose of this section is to prove Theorem 5.12 showing that it is possible to enlarge a lattice-free rational polytope $L \subset \mathbb{R}^m$ to a rational lattice-free polytope L' such that, for all facet F of L', the split $(\pi(F), \pi_0(F))$ is L'-intersecting and such that L has the 2-hyperplane property if and only if L' does. This is a useful result, as this allows us to show that the effect of a Chvátal split on a polytope can be obtained by a sequence of intersecting splits for the enlarged polytope. This is developed in Section 5.3.

A *lattice subspace* of $\mathbb{R}^m$ is an affine space $x + V$ where $x \in \mathbb{Z}^m$ and V is a linear space generated by rational vectors. Equivalently, an affine space $\mathcal{A} \subseteq \mathbb{R}^m$ is a lattice subspace if it is spanned by the integer points in $\mathcal{A}$ (see Barvinok [3] for instance). We need the following technical result.

Lemma 5.9. *Let L be a full-dimensional rational polytope in $\mathbb{R}^m$ with $m \geq 2$ given by $\{x \mid Ax \leq b\}$, where A is an integral matrix and b is an integral vector. The rows of A are denoted by $a_1, \ldots, a_n$. Suppose $a_1 x \leq b_1$ defines a facet of L with no integer point contained in its affine hull $\{x \mid a_1 x = b_1\}$. Let $\tilde{A}$ be the matrix obtained from A by removing row a_1 and let $\tilde{b}$ be the vector obtained from b by removing its first component b_1. Then there exists a rational inequality $a'x \leq b'$ such that $\tilde{L} := \{x \mid \tilde{A}x \leq \tilde{b}, a'x \leq b'\}$ contains the same set of integer points as L, $L \subseteq \tilde{L}$, and the hyperplane $\{x \mid a'x = b'\}$ contains integer points.*

Proof. Let G be the greatest common divisor of the coefficients in a_1. As $\{x \mid a_1 x = b_1\}$ does not contain any integer points, $\frac{b_1}{G}$ is fractional by Bézout's Theorem [15]. Consider the set $L' = \{x \mid \tilde{A}x \leq \tilde{b},\ a_1 x = G \cdot \lceil \frac{b_1}{G} \rceil\}$. As $\mathrm{rec}(L') \subseteq \mathrm{rec}(L)$ (where $\mathrm{rec}(L)$ denotes the recession cone of L), L' is a polytope contained in the hyperplane $\{x \mid a_1 x = G \cdot \lceil \frac{b_1}{G} \rceil\}$. Since $\{x \mid a_1 x = G \cdot \lceil \frac{b_1}{G} \rceil\}$ is an $m-1$ dimensional lattice subspace, its integer points can be partitioned into infinitely many parallel lattice subspaces of dimension $m-2$. As L' is bounded, one of them is an $m-2$ dimensional lattice subspace $\mathcal{A}$ that does not intersect L'. (See Figure 11.) This implies that we can choose a rational hyperplane $H \subseteq \mathbb{R}^m$ containing $\mathcal{A}$ and such that H separates L from L' and does not contain a point in L' (note that if L' is empty, then we choose H containing $\mathcal{A}$ and such that L is contained in one of the two half-space bounded by H). Let $H = \{x \mid a'x = b'\}$ and such that the half-space $a'x \leq b'$ contains L.

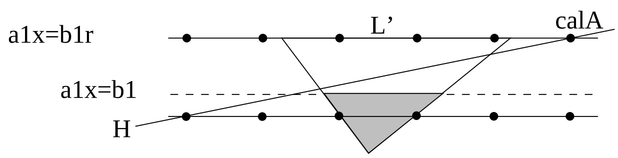

Figure 11: Illustration of the proof of Lemma 5.9. Polytope L is shaded, polytope L' is a segment.

By construction, the hyperplane $\{x \mid a'x = b'\}$ contains integer points from $\mathcal{A}$.

Suppose for a contradiction that there exists an integer point $p \in \tilde{L} \setminus L$. Then p satisfies $a_1 x \geq G \cdot \lceil \frac{b_1}{G} \rceil$ and $a'p \leq b'$. This implies that the segment joining p to any point $\bar{p}$ in L intersects the hyperplane $a_1 x = G\lceil \frac{b_1}{G} \rceil$ at some point p' (with, possibly, $p' = p$). Then $p' \in L'$ and, as all points in L' violate the inequality $a'x \leq b'$, we have $p' \notin \tilde{L}$. This is a contradiction with the fact that the segment $p\bar{p}$ is contained in $\tilde{L}$. □

We next make a couple of observations that are used in the remainder of the paper. The proof of the first observation can be found in Eisenbrand and Schulz [10].

Observation 5.10. *Let $\mathcal{M}$ be a unimodular transformation of $\mathbb{R}^m$. Then integer points are mapped to integer points and (π, π_0) is a split if and only if it is mapped to a split.*

Observation 5.11. *Let L and L' be two polytopes having the same dimension, such that $L \subseteq L'$ and $L_I = L'_I$. If L' has the 2-hyperplane property, then so does L.*

Proof. Consider any face F of L_I that is not contained in a facet of L. Face F is not contained in a facet of L', as the intersection of any facet of L' with L is contained in a facet of L. Since L' has the 2-hyperplane property, F is 2-partitionable. □

So far, we essentially dealt with full-dimensional lattice-free polytopes. For the remainder of the paper, we need to extend a couple of definitions to non-full dimensional polytopes. Let L be a convex set in $\mathbb{R}^m$. We say that L is *lattice-free in its affine hull* if L does not contain an integer point in its relative interior. Note that when L is full-dimensional, this definition is equivalent to L being lattice-free.

Recall the definition of the split $(\pi(F), \pi_0(F))$ given at the beginning of Section 5 for a facet F of a full-dimensional rational lattice-free polytope $L \subset \mathbb{R}^m$. When L is not full-dimensional, we consider the same definition restricted to the affine hull A of L. This split is uniquely defined in A and it will be denoted by $(\pi^A(F), \pi_0^A(F))$. Note that the split $(\pi^A(F), \pi_0^A(F))$ can be extended (in a non-unique way) to a split $(\pi(F), \pi_0(F))$ of $\mathbb{R}^m$.

Theorem 5.12. *Let L be a rational polytope in $\mathbb{R}^m$ that is lattice-free in its affine hull A. Assume that $\dim(A) \geq 1$ and that A contains integer points. Then it is possible to enlarge L to a rational lattice-free polytope $L' \subseteq A$ such that L has the 2-hyperplane property if and only if L' does, and such that for each facet F of L'*

(i) the affine hull of F contains integer points;

(ii) the split $(\pi^A(F), \pi_0^A(F))$ is L'-intersecting.

Proof. The affine space A is a lattice-subspace of dimension $t = \dim(A) \geq 1$ and therefore by choosing a lattice basis to define new coordinates, we may assume that L is full-dimensional in $\mathbb{R}^t$. Observe that if $t = 1$, then L always has the 2-hyperplane property and the result holds when L' is taken as the smallest segment with integer endpoints containing L. Hence, in the remainder of the proof, we assume $t \geq 2$. We obtain L' in two phases.

Phase 1: We first exhibit a rational lattice-free polytope L^1 containing L, such that $L \cap \mathbb{Z}^t = L^1 \cap \mathbb{Z}^t$ and every L^1-englobing split is L^1-intersecting.

If L is such that every L-englobing split is L-intersecting, then L^1 is trivially taken to be L. Otherwise, consider an L-englobing split that is not L-intersecting and let H^1 and H^2 be its two boundary hyperplanes. We assume without loss of generality that $H^2 \cap L = \emptyset$. Let $C \subseteq \mathbb{R}^t$ be the image of the unit hypercube $\bar{C}$ in $\mathbb{R}^t$ under a unimodular transformation $\mathcal{M}$, such that all its vertices are in $H^1 \cup H^2$, and such that the following condition (*) holds.

(*) If $L \cap H^1$ has dimension at least 1, then C is chosen such that $C \cap L \cap H^1$ has dimension at least 1. If $L \cap H^1$ is a single point, then C is chosen such that $C \cap L \cap H^1 = L \cap H^1$.

Note that if $L \cap H^1$ is empty, we are free to choose $\mathcal{M}$ as any unimodular transformation mapping the vertices of $\bar{C}$ to integer points in $H^1 \cup H^2$. We now apply the inverse of the unimodular transformation $\mathcal{M}$ so that C is transformed back into $\bar{C}$, L is transformed into a polytope Q, and H^i is transformed into hyperplane $\bar{H}^i$ for $i = 1, 2$. Without loss of generality, we can assume that $\bar{H}^1$ and $\bar{H}^2$ are the hyperplanes $\{x \in \mathbb{R}^t \mid x_1 = 0\}$ and $\{x \in \mathbb{R}^t \mid x_1 = 1\}$. Observation 5.10 shows that condition (*) still holds for $\bar{C}$ and Q. Let $\bar{v}$ be the center of the hypercube $\bar{C}$, i.e. $\bar{v} = (\frac{1}{2}, \ldots, \frac{1}{2})$. If $Q \cap \bar{H}^1$ has dimension at least 1, let p^1 be a rational fractional point in $\bar{C} \cap Q \cap \bar{H}^1$. If $Q \cap \bar{H}^1$ is of dimension less than 1, let p^1 be any rational fractional point in $\bar{C} \cap \bar{H}^1$. Let q^1 be the point in $\bar{C} \cap \bar{H}^2$ such that $\frac{p^1+q^1}{2} = \bar{v}$. Moreover we consider the set of the $2(t-1)$ centers of the facets of $\bar{C}$ other than the two supported by $\bar{H}^1$ and $\bar{H}^2$, i.e.,

$$p^i_j = \begin{cases} \frac{1}{2} & j \neq i \\ 0 & j = i \end{cases} \qquad q^i_j = \begin{cases} \frac{1}{2} & j \neq i \\ 1 & j = i \end{cases} \qquad \text{for } \ i = 2, \ldots, t, \ j = 1, \ldots, t.$$

Let S be the set of $2t$ points p^i, q^i for $i = 1, \ldots, t$ and let $Q_1 = \operatorname{conv}(S \cup Q)$. Clearly, $Q \subseteq Q_1$ and Q_1 is rational. We now show that $Q \cap \mathbb{Z}^t = Q_1 \cap \mathbb{Z}^t$. Notice that integer points in Q_1 are either on $\bar{H}^1$ or $\bar{H}^2$. Since $Q_1 \cap \bar{H}^2$ is reduced to a single fractional point q^1, $Q_1 \cap \bar{H}^2$ does not contain integer points. When $Q \cap \bar{H}^1$ has dimension at least 1, our choice of p^1 implies that $Q \cap \bar{H}^1 = Q_1 \cap \bar{H}^1$. When $Q \cap \bar{H}^1$ has dimension less than 1, our choice of p^1

guarantees that the only integer point (if any) in $Q_1 \cap \bar{H}^1$ is also in $Q \cap \bar{H}^1$. Therefore, in both cases, $Q \cap \mathbb{Z}^t = Q_1 \cap \mathbb{Z}^t$.

We claim that the split closure of Q_1 contains the point $\bar{v}$ defined above. Since $\bar{v}$ is an interior point of Q_1, this shows that every Q_1-englobing split is Q_1-intersecting.

We prove the claim by showing that for any split (π, π_0), we have $\bar{v} \in Q_1(\pi, \pi_0)$. This is obvious if $\pi \cdot \bar{v} \leq \pi_0$ or $\pi \cdot \bar{v} \geq \pi_0 + 1$, so we may assume that $\pi_0 < \pi \cdot \bar{v} < \pi_0 + 1$. Since π is integer and $\bar{v}$ has all coordinates equal to $\frac{1}{2}$, we have that $\pi \cdot \bar{v} = \pi_0 + 0.5$. If $\pi_j \neq 0$ for some $j \neq 1$, then

$$\pi \cdot p^j = \pi \cdot \bar{v} - 0.5 \cdot \pi_j = \pi_0 + 0.5 - 0.5 \cdot \pi_j$$

$$\pi \cdot q^j = \pi \cdot \bar{v} + 0.5 \cdot \pi_j = \pi_0 + 0.5 + 0.5 \cdot \pi_j \ .$$

The integrality of π implies that $|\pi_j| \geq 1$, and thus $p^j \in Q_1(\pi, \pi_0)$ and $q^j \in Q_1(\pi, \pi_0)$ and $\bar{v} = \frac{p^j + q^j}{2} \in Q_1(\pi, \pi_0)$. On the other hand, if $\pi_j = 0$ for all $j \neq 1$, then the split disjunction must be $\{x_1 \leq 0 \vee x_1 \geq 1\}$ (since $\pi_0 < \pi \cdot \bar{v} < \pi_0 + 1$). But then, $p^1 \in Q_1(\pi, \pi_0)$ and $q^1 \in Q_1(\pi, \pi_0)$ and $\bar{v} = \frac{p^1 + q^1}{2} \in Q_1(\pi, \pi_0)$. This completes the proof of the claim.

Let $L^1 := \mathcal{M}(Q_1)$. Since $\mathcal{M}$ is a one-to-one map of splits into splits by Observation 5.10, any L^1-englobing split is L^1-intersecting.

If $L = L^1$ then, trivially, L has the 2-hyperplane property if and only if L^1 does. Otherwise, consider the L-englobing split defined by H^1, H^2. Recall that this split is not L-intersecting, therefore L_I is contained in H^1 and as H^1 defines a face of L and a face of L^1, both L and L^1 have the 2-hyperplane property. Thus, L has the 2-hyperplane property if and only if L^1 does.

Phase 2: At the end of Phase 1, we obtain L^1 such that any L^1-englobing split is L^1-intersecting and $L \cap \mathbb{Z}^t = L^1 \cap \mathbb{Z}^t$. Let $L' := L^1$ and apply the following algorithm to L'.

1. If there exists a facet F of L' such that the affine hull A^F of F does not contain integer points, do step 2 below. Otherwise stop.

2. Using Lemma 5.9, enlarge L' to a polytope L''. Observe that the integer points in L'' are exactly those in L' and that, compared to L', L'' has fewer facets whose affine hull does not contain integer points. Rename $L' := L''$ and go to step 1.

Since L^1 has a finite number of facets, the above algorithm terminates. The polytope at the end of the algorithm satisfies the statement of the lemma. Indeed, first notice that $L^1 \subseteq L'$, where L^1 is the polytope obtained at the end of Phase 1. Second, by construction, every facet F of L' satisfies condition (i) in the statement of the lemma. Moreover, if $(\pi(F), \pi_0(F))$ is not L'-intersecting, then it is L'-englobing. As $L^1 \subseteq L'$, this split is also L^1-englobing. By construction of L^1, it is L^1-intersecting and thus also L'-intersecting, a contradiction. Therefore (ii) holds.

Finally, we show that L^1 satisfies the 2-hyperplane property if and only if L' does. As $L^1 \subseteq L'$, $L^1_I = L'_I$ and $\dim(L^1) = \dim(L')$, Observation 5.11 shows one of the two implications. For the converse, suppose that L^1 has the 2-hyperplane property. Observe that any facet of L^1 containing integer points is contained in a facet of L'. This shows that if F is a face of L'_I that is not contained in a facet of L', then F is not contained in a facet of L^1 and is thus 2-partitionable. It follows that L' has the 2-hyperplane property. ☐

5.3 Chvátal splits.

Given a polytope Q in $\mathbb{R}^m$, a *Chvátal split* is a split (π, π_0) such that $Q \cap \{x \in \mathbb{R}^m \mid \pi x \geq \pi_0 + 1\} = \emptyset$. Note that $Q_I \subset Q \cap \{x \in \mathbb{R}^m \mid \pi x \leq \pi_0\}$.

The goal of this section is to prove a technical lemma about Chvátal splits. The statement of the lemma is illustrated in Figure 12.

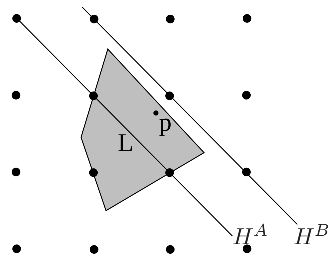

Figure 12: Illustration for the statement of Lemma 5.13 with $m = 2$. Polytope Q is shaded and polytope L is a segment.

Lemma 5.13. *Let Q be a full-dimensional rational lattice-free polytope in $\mathbb{R}^m$ with $m \geq 2$. Let (π, π_0) be a Chvátal split and let $H^A := \{x \mid \pi x = \pi_0\}$ and $H^B := \{x \mid \pi x = \pi_0 + 1\}$ be its boundary hyperplanes, with $H^B \cap Q$ empty. Assume that $L := H^A \cap Q$ has dimension* $\dim(L) = m-1$, *and for each facet F of L the affine hull of F contains integer points and the split $(\pi^{H^A}(F), \pi_0^{H^A}(F))$ is L-intersecting. Let p be any point strictly between H^A and H^B. Then there exists a finite sequence S of L-intersecting splits such that $Q(S) \cap \{x \in \mathbb{R}^m \mid \pi x \geq \pi_0\}$ is contained in the pyramid conv$(p \cup L)$.*

Proof. Let F be a facet of L. By hypothesis, there exists a split (π^F, π_0^F) with boundary hyperplanes H_0^F and H_1^F such that $H_0^F \cap H^A$ contains F and $H_1^F \cap L \neq \emptyset$. All the integer points in $\mathbb{R}^m$ can be partitioned on equally spaced hyperplanes parallel to H^A, and the integer points in any of these hyperplanes can themselves be partitioned on equally spaced affine subspaces parallel to $H_0^F \cap H^A$. Let A^k with $k \in \mathbb{Z}$ denote these affine subspaces in H^A, where $A^0 := H_0^F \cap H^A$, $A^1 := H_1^F \cap H^A$, and A^j is between A^i and A^k if and only if $i < j < k$. Let B^0 be an affine subspace in H^B parallel to A^0. Let $a_0 \in A^0$, $b_0 \in B^0$ and let v be the vector $b_0 - a_0$. Define $B^k = A^k + v$ with $k \in \mathbb{Z}$ to be the translate of A^k. (See an illustration in Figure 13.)

For all $i, j \in \mathbb{Z}$, define $C^{i,j}$ as the hyperplane containing $A^i \cup B^j$. For all $k \in \mathbb{Z}$, we have that $C^{0,k}$ and $C^{1,k+1}$ are the boundary hyperplanes of a split (π^k, π_0^k). If $Q \subseteq \{x \in \mathbb{R}^m \mid \pi x \leq \pi_0\}$, then the lemma holds. So we assume that $\operatorname{int}\left(\overline{Q(\pi, \pi_0)}\right) \neq \emptyset$. Let ℓ be the largest index k such that $C^{0,k} \cap \operatorname{int}\left(\overline{Q(\pi, \pi_0)}\right) = \emptyset$. Let t be the finite index such that $p \in \operatorname{conv}(A^0 \cup B^t) \cup \operatorname{int}\left(\operatorname{conv}(A^0 \cup B^t \cup B^{t+1})\right)$. Consider the sequence of L-intersecting splits (π^ℓ, π_0^ℓ), ..., (π^t, π_0^t). Let Q^k for $k = \ell, \ldots, t$ be the polytopes obtained by applying this sequence of splits to Q. We claim that $p \notin Q^t$ and that, in fact, p and $\overline{Q^t(\pi, \pi_0)}$ are on opposite sides of the hyperplane $C^{0,t+1}$. If the claim is correct, applying the same reasoning to each facet of L in succession yields the lemma.

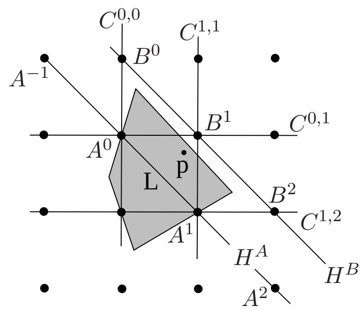

Figure 13: Illustration for the proof of Lemma 5.13 with $m = 2$, $\ell = 0$ and $t = 1$. Polytope Q is shaded.

Let $U^k := \overline{Q^k(\pi, \pi_0)}$ for $k = \ell, \ldots, t$ and let $\bar{H}^B$ be the half-space bounded by H^B containing H^A. We prove the claim by induction on k by showing that Q^k is contained on one side of $C^{0,k+1}$ and that $C^{0,k+1} \cap U^k = F$ for $k = \ell, \ldots, t$.

For $k = \ell$, observe that $C^{0,\ell} \cap \overline{Q(\pi, \pi_0)} = F$ and thus $C^{0,\ell} \cap Q$ is contained on one side of $C^{0,\ell+1}$. Observe also that $C^{1,\ell+1} \cap Q$ is contained in the interior of $\bar{H}^B$ and thus it is on the same side of $C^{0,\ell+1}$ as $C^{0,\ell} \cap Q$. It follows that Q^ℓ is contained on one side of $C^{0,\ell+1}$ and that the only points of U^ℓ on $C^{0,\ell+1}$ are points in F. This proves the claim for $k = \ell$.

Suppose now that $k > \ell$ and that the induction hypothesis is true for $k-1$. Observe that $C^{0,k} \cap U^{k-1} = F$ and that $C^{1,k+1} \cap Q^{k-1}$ is contained in the interior of $\bar{H}^B$ and thus is on the same side of $C^{0,k+1}$ as $C^{0,k} \cap Q^{k-1}$. It follows that Q^k is contained on one side of $C^{0,k+1}$ and that the only points of U^k on $C^{0,k+1}$ are points in F. This proves the claim for $k > \ell$. $\square$

5.4 Proof of the main theorem.

Consider a full-dimensional polytope $L \subset \mathbb{R}^m$. By a theorem of Chvátal [5], there exists a finite sequence of Chvátal splits $S := ((\pi^1, \pi_0^1), \ldots, (\pi^c, \pi_0^c))$ such that, when applying the sequence S to L, the last polytope L^c is the convex hull L_I of the integer points in L. If $L = L_I$, we define $c := 0$. Otherwise, if $L \neq L_I$, define $L^0 := L$ and, for $j = 1, \ldots, c$, let L^j be the sequence of polytopes obtained from S. We may assume that $L^j \neq L^{j-1}$ for $j = 1, \ldots, c$. If L_I is not full-dimensional, we set t to be the smallest index in $\{0, 1, \ldots, c-1\}$ such that (π^{t+1}, π_0^{t+1}) is L^t-englobing. If no such englobing split exists in the sequence, we set $t := c$. It follows that L^j is full-dimensional for $j = 0, \ldots, t$. We say that L has *Chvátal-index* t if t is the smallest possible value over all possible sequences of Chvátal splits as described above.

Lemma 5.14. *If $L \subset \mathbb{R}^m$ is a full-dimensional polytope with the 2-hyperplane property and Chvátal-index t, then there exists a sequence S of t Chvátal splits together with a split (π^{t+1}, π_0^{t+1}) that is L^t-englobing.*

Proof. If L_I is full-dimensional then $L_I = L^t$ and L_I is not contained in a facet of L. Since L has the 2-hyperplane property, L_I is 2-partitionable and therefore an L_I-englobing split exists. This is the split (π^{t+1}, π_0^{t+1}).

If L_I is not full-dimensional, we must have $c > t$ and the definition of t shows that (π^{t+1}, π_0^{t+1}) is L^t-englobing. $\square$

We now prove a result that implies Theorem 5.1.

Theorem 5.15. *Let L be a full-dimensional rational lattice-free polytope in $\mathbb{R}^m$ such that L has the 2-hyperplane property, and let $M_0 < M$ be finite numbers. For any polyhedron $Q \subset R(L, M, M_0)$, there exists a finite number q such that the height of the rank-q split closure of Q is at most M_0.*

Proof. If there exists an L-englobing split (π, π_0), then for any polyhedron $Q \subset R(L, M, M_0)$, the height of $Q(\pi, \pi_0)$ is at most M_0 and the theorem holds.

We prove the theorem by induction on the dimension m of L. If $m = 1$, L cannot contain more than two integer points and thus an L-englobing split exists and the result holds.

Assume now that L has dimension $m \geq 2$ and that the result holds for polytopes with strictly smaller dimension. Let t be the Chvátal-index of L. We make a second induction on t. More precisely, the induction hypothesis is that the theorem holds for any full-dimensional rational lattice-free polytope $K \subset \mathbb{R}^k$ with either $k < m$, or $k = m$ and Chvátal-index $t' < t$, for any finite numbers $M_0^* < M^*$, and for any $Q^* \subset R(K, M^*, M_0^*)$.

Assume first that $t = 0$. Lemma 5.14 shows that there exists an L-englobing split and therefore the theorem holds. Assume now that $t > 0$. We consider the first Chvátal split (π^1, π_0^1) in a sequence (π^i, π_0^i), $i = 1, \ldots, t$ leading from L to L_I. The following claim, if true, shows that the effect of applying (π^1, π_0^1) on L can be obtained by applying a finite sequence of intersecting splits for polytopes satisfying the statement of Lemma 5.7.

Claim 1. Let (π^1, π_0^1) be a Chvátal split for L such that $L(\pi^1, \pi_0^1) \neq L$. Then there exist a finite number $n \geq 1$, polytopes L^i, and splits (μ^i, μ_0^i) such that (μ^i, μ_0^i) is L^i-intersecting for $i = 1, \ldots, n$, and the following properties hold, where $D^i = \mathrm{cl}(L^i \setminus L^i(\mu^i, \mu_0^i))$:

(i) $L \subseteq L^1$;

(ii) $L \setminus (D^1 \cup \ldots \cup D^i) \subseteq L^{i+1}$ for $i = 1, \ldots, n-1$;

(iii) $\mathrm{cl}(L \setminus L(\pi^1, \pi_0^1)) \subseteq D^1 \cup \ldots \cup D^n$.

Before proving the claim, we show that it implies the theorem. We use a third level of induction, this time on the number n from the above claim. We assume that the theorem holds for any full-dimensional rational lattice-free polytope $K \subset \mathbb{R}^k$, for any finite numbers $M_0^* < M^*$, for any $Q^* \subset R(K, M^*, M_0^*)$ with either

1. $k < m$, or
2. $k = m$ and K has Chvátal-index $t' < t$, or
3. $k = m$, K has Chvátal-index t and there is a sequence of length $n' < n$ satisfying Claim 1 for the first of the t Chvátal splits.

As above, we assume that L has Chvátal index $t > 0$ and $Q \subset R(L, M, M_0)$. Let (π^1, π_0^1) be a Chvátal split such that $L(\pi^1, \pi_0^1)$ has Chvátal index $t - 1$. Consider the sets defined in Claim 1. By Lemma 5.7 for L, L^i, $(\pi^i, \pi_0^i) := (\mu^i, \mu_0^i)$ for $i = 1, \ldots, n$, $M^* := M$, $M_0^* := M_0$ and Q, we obtain $\Delta > 0$ and a finite sequence of splits S such that after applying the sequence S to Q, one of the following cases holds:

(i) the height of all points in $D^1 = \mathrm{cl}(L^1 \setminus L^1(\mu^1, \mu_0^1))$ is at most M_0 with respect to $Q(S)$.

(ii) the height of all points in $D^1 \cup \ldots \cup D^n$ is at most $H - \Delta \cdot (M - M_0)$ with respect to $Q(S)$, where H denotes the height of Q;

Case (i): We use the induction hypothesis on $K := \mathrm{cl}(L \setminus D^1)$. Note that, by Lemma 5.2, $Q(S) \subseteq R(K, H', M_0)$, where H' is the height of $Q(S)$, since the height of all points in D^1 is at most M_0 with respect to $Q(S)$, and this is also the case for all points not in the interior of L.

If $n = 1$, then $K \subseteq L(\pi^1, \pi_0^1)$ by (iii) of Claim 1. Since $L(\pi^1, \pi_0^1)$ has Chvátal index $t - 1$, it satisfies the second induction hypothesis and the result holds.

If $n \geq 2$, observe that L^i and (μ^i, μ_0^i) for $i = 2, \ldots, n$ show that the claim is satisfied with a sequence of $n - 1$ splits for K. By definition of the Chvátal index, K is full-dimensional and therefore K satisfies the third induction hypothesis.

Therefore, by induction, there exists a finite q' such that the height of the rank-q' split closure Q^1 of $Q(S)$ is at most M_0. Since the split closure of a polyhedron is a polyhedron (Cook et al. [6]), this implies the existence of a finite sequence S' of splits that gives Q^1 when applied to $Q(S)$. Applying the sequence S followed by the sequence S' on the polyhedron Q, we reduce its height to at most M_0, proving the theorem.

Case (ii): If $M_0 \geq H - \Delta \cdot (M - M_0)$, then in particular the height of all points in D^1 is at most M_0 with respect to $Q(S)$ and Case (i) applies. Therefore we may assume that $M_0 \leq H - \Delta \cdot (M - M_0)$. Case (ii) combined with conclusion (iii) in Claim 1 that $\mathrm{cl}(L \setminus L(\pi^1, \pi_0^1)) \subseteq D^1 \cup \ldots \cup D^n$, implies that the height of all points in $\mathrm{cl}(L \setminus L(\pi^1, \pi_0^1))$ is at most $H - \Delta \cdot (M - M_0)$ with respect to $Q(S)$. Recall that all points not in the interior of L have height at most $M_0 \leq H - \Delta \cdot (M - M_0)$ with respect to Q and thus also with respect to $Q(S)$. Let $H(S)$ be the height of $Q(S)$. Then, by Lemma 5.2, $Q(S) \subseteq R(L(\pi^1, \pi_0^1), H(S), H - \Delta \cdot (M - M_0))$. Since the Chvátal-index of $L(\pi^1, \pi_0^1)$ is $t - 1$, we can apply the second induction hypothesis to $Q(S)$. Therefore, there exists a finite sequence S^1 of splits that gives a polyhedron Q^1 of height at most $H - \Delta \cdot (M - M_0)$ when applied to $Q(S)$. We can get Q^1 from Q by applying the sequence S followed by S^1. Note that $Q^1 \subseteq Q \subseteq R(L, M, M_0)$ so Lemma 5.7 can be applied to Q^1 with the same Δ and sequence of splits S, and Case (i) or (ii) will hold, where the only change in Case (ii) is the height of Q^1, which is at most $H - \Delta \cdot (M - M_0)$. Therefore let us apply the sequence S on Q^1. If we end up in Case (i) then we are done by the arguments in Case (i). Otherwise we have the situation of Case (ii) where the height of all points in $D^1 \cup \ldots \cup D^n$ is at most $H - 2\Delta \cdot (M - M_0)$ with respect to $Q^1(S)$. If $M_0 \geq H - 2\Delta \cdot (M - M_0)$, the theorem follows as in Case (i). If $M_0 \leq H - 2\Delta \cdot (M - M_0)$, we can apply the second induction hypothesis to $Q^1(S)$. Therefore, there exists a finite sequence S^2 of splits that we can apply to $Q^1(S)$ to obtain Q^2 with height at most $H - 2\Delta \cdot (M - M_0)$. We can get Q^2 from Q by applying the sequence S followed by S^1 followed by S followed by S^2. Continuing this process at most $\lceil \frac{H - M_0}{\Delta \cdot (M - M_0)} \rceil$ times, we end up in Case (i) at some point, proving the theorem.

It thus remains to prove the claim.

Proof of Claim 1. Let H^A and H^B be the boundary hyperplanes of the Chvátal split (π^1, π_0^1) such that $L^* := H^A \cap L \neq \emptyset$. By assumption, (π^1, π_0^1) is not L-englobing and $L(\pi^1, \pi_0^1) \neq L$.

Therefore, L^* is not a face of L and thus $\dim(L^*) = m - 1$ and L^* is lattice-free in its affine space. Moreover, both H^A and H^B are lattice subspaces.

Let W be obtained using Theorem 5.12 on L^*, and let $L_s := \text{conv}(L \cup W)$. Let d be a vector joining an integer point in H^A to an integer point in H^B. Let $q \in \text{relint}(W)$ and let p be a point on the boundary of L_s between H^A and H^B such that the line joining q and p is in the direction d (Figure 14, left). Apply Lemma 5.13 to $Q := L_s$, $L := W$, p, and (π^1, π_0^1). This yields a sequence $(\mu^1, \mu_0^1), \dots, (\mu^r, \mu_0^r)$ of W-intersecting splits with the following property. Let $L^1 = L_s$ and, for $i = 1, \dots, r$, define L^{i+1} as the polytope obtained from L^i by applying the split (μ^i, μ_0^i). Then Lemma 5.13 shows that $L^{r+1} \cap \{x \mid \pi^1 x \geq \pi_0^1\}$ is contained in $\text{conv}(p \cup W)$.

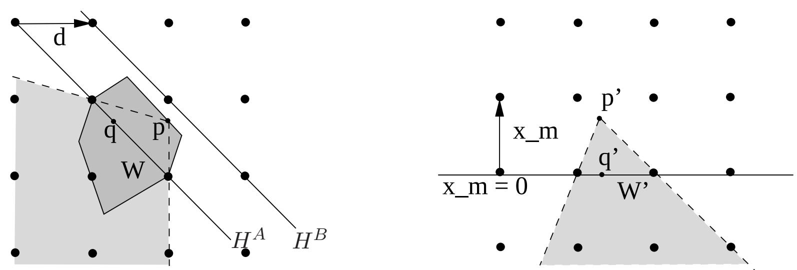

Figure 14: Illustration for the proof of Theorem 5.15. On the left, polytope $L_s \subseteq \mathbb{R}^2$ is shaded, W is a segment, and part of cone U is lightly shaded with its boundary in dashed lines. On the right, the images of W, U, p, and q after applying the unimodular transformation τ are depicted.

Let U be the cone with apex p and rays joining p to the vertices of W. Observe that $U \supseteq L^{r+1}$. Let $\mathcal{T} : \mathbb{R}^m \to \mathbb{R}^m$ be a unimodular transformation mapping the vector d to $(0, \dots, 0, 1) \in \mathbb{R}^m$ and mapping H^A to $\mathbb{R}^m \cap \{x \mid x_m = 0\}$ (Figure 14, right). Note that $\mathcal{T}(\mathbb{Z}^m \cap H^A) = \mathbb{Z}^m \cap \{x \mid x_m = 0\}$ by Corollary 4.3a in Schrijver [14].

Let $p' := \mathcal{T}(p)$, $U' := \mathcal{T}(U)$, and $W' := \mathcal{T}(W)$. As L has the 2-hyperplane property, and as L^* is the intersection of L with one boundary hyperplane of a Chvátal split, the convex hull G of the integer points in L^* is a face of L_I. Moreover, if a face of G is contained in a facet of L then it is contained in a facet of L^*. It follows that L^* has the 2-hyperplane property and by applying Theorem 5.12 to L^* we obtain that so does W. As $\mathcal{T}$ is unimodular, the transformation $\mathcal{T}$ maps splits in H^A to splits in $\mathbb{R}^m \cap \{x \mid x_m = 0\}$. Therefore, W' also has the 2-hyperplane property. We will now apply the first induction hypothesis to $K := W' \subseteq \mathbb{R}^{m-1}$, $Q^* := U' \subseteq \mathbb{R}^m$, M^* equals the height of U' and $M_0^* = 0$. Here, the crucial point to follow this induction step is to understand that variable x_m plays the role of variable z. Thus, all splits obtained from this induction are parallel to the x_m-axis. Note that, by Lemma 5.2, $Q^* \subseteq R(K, M^*, M_0^*)$. By induction, there exists a finite sequence of splits, all parallel to the x_m-axis, whose application on U' reduces its height to 0, or equivalently, removes $\text{conv}(p' \cup W')$. Using the inverse of $\mathcal{T}$, this yields a sequence $(\mu^{r+1}, \mu_0^{r+1}), \dots, (\mu^n, \mu_0^n)$ of splits whose boundary hyperplanes are all parallel to d and such that its application to U removes $\text{conv}(p \cup W)$. Since $-d$ is in the interior of the full-dimensional recession cone of U, the sequence $(\mu^{r+1}, \mu_0^{r+1}), \dots, (\mu^n, \mu_0^n)$ is U-intersecting. One last hurdle remains, as U is unbounded and we need to construct polytopes L^i such that (μ^i, μ_0^i) is L^i-intersecting. To this end, let $U^{r+1}, \ldots, U^n$ be the polyhedra obtained by applying the sequence $(\mu^{r+1}, \mu_0^{r+1}), \ldots, (\mu^n, \mu_0^n)$ to U. For $j = r+1, \ldots, n$, let $w^{1,j}$ and $w^{2,j}$ be two integer points in U^j contained in each of the two boundary hyperplanes of (μ^j, μ_0^j) respectively. It is possible to truncate U into a polytope U^* by intersecting U with a half-space bounded by a hyperplane parallel to H^A, such that L^{r+1} and all the points $w^{1,j}, w^{2,j}$ are contained in U^*. As all these points are integer, they are contained in the polytope obtained by applying the sequence $(\mu^{r+1}, \mu_0^{r+1}), \ldots, (\mu^n, \mu_0^n)$ of splits to U^*, proving that the sequence is a U^*-intersecting sequence.

Redefine L^{r+1} to be equal to U^* and note that this redefinition is an enlargement. For $i = r+1, \ldots, n$, define L^{i+1} as the polytope obtained from L^i by applying the split (μ^i, μ_0^i). The sequence (μ^i, μ_0^i) and polytopes L^i for $i = 1, \ldots, n$ are those used to show that Claim 1 is satisfied. Indeed, by construction, $L \subseteq L_s = L^1$, therefore point (i) holds. Observe that since $L \subseteq L^1$, $L \setminus (D^1 \cup \ldots \cup D^i) \subseteq L^1 \setminus (D^1 \cup \ldots \cup D^i) = L^{i+1}$ for $i = 1, \ldots, r-1$. Moreover, because the redefinition of L^{r+1} is an enlargement, $L^r \setminus D^r \subseteq L^{r+1}$. Consequently, $L^1 \setminus (D^1 \cup \ldots \cup D^i) \subseteq L^{i+1}$ for $i = 1, \ldots, r$. Also note that $L^i \setminus D^i = L^{i+1}$ for $i = r+1, \ldots, n$. Therefore, point (ii) holds. Finally, point (iii) holds since $L^1 \setminus (D^1 \cup \ldots \cup D^r)$ is contained in $\text{conv}(p \cup W)$ and $D^{r+1} \cup \ldots \cup D^n$ contains $\text{conv}(p \cup W)$. Therefore $D^1 \cup \ldots \cup D^n$ contains $\text{cl}(L \setminus L(\pi^1, \pi_0^1))$. □

This completes the proof of Theorem 5.15 □

We can now prove Theorem 5.1, completing the proof of Theorem 1.1.

Proof of Theorem 5.1. Since $f \in \text{int}(L)$ and L satisfies the assumptions of Theorem 5.15, we have that $P^L \subset R(L, 1, 0)$. Theorem 5.15 implies that there exists a finite number q such that the height of the rank-q split closure of P^L is at most zero. By Observation 3.2, this implies that the split rank with respect to P of the L-cut is at most q. □

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