# Interdicting Structured Combinatorial Optimization Problems with $\{0,1\}$-Objectives 

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#### Abstract

Interdiction problems ask about the worst-case impact of a limited change to an underlying optimization problem. They are a natural way to measure the robustness of a system, or to identify its weakest spots. Interdiction problems have been studied for a wide variety of classical combinatorial optimization problems, including maximum $s$ - $t$ flows, shortest $s-t$ paths, maximum weight matchings, minimum spanning trees, maximum stable sets, and graph connectivity. Most interdiction problems are NPhard, and furthermore, even designing efficient approximation algorithms that allow for estimating the order of magnitude of a worst-case impact, has turned out to be very difficult. Not very surprisingly, the few known approximation algorithms are heavily tailored for specific problems.

Inspired by an approach of Burch et al. 8, we suggest a general method to obtain pseudoapproximations for many interdiction problems. More precisely, for any $\alpha>0$, our algorithm will return either a $(1+\alpha)$-approximation, or a solution that may overrun the interdiction budget by a factor of at most $1+\alpha^{-1}$ but is also at least as good as the optimal solution that respects the budget. Furthermore, our approach can handle submodular interdiction costs when the underlying problem is to find a maximum weight independent set in a matroid, as for example the maximum weight forest problem. Additionally, our approach can sometimes be refined by exploiting additional structural properties of the underlying optimization problem to obtain stronger results. We demonstrate this by presenting a PTAS for interdicting $b$-stable sets in bipartite graphs.


## 1 Introduction

One way to understand the robustness of a system is to evaluate attack strategies. This naturally leads to interdiction problems; broadly, one is given a set of feasible solutions, along with some rules and a budget for modifying the set, with the goal of inhibiting the solution to an underlying nominal optimization problem. A prominent example that nicely highlights the nature of interdiction problems is maximum flow interdiction. Here, the nominal problem is a maximum $s$ - $t$ flow problem. Given is a directed graph $G=(V, A)$ with arc capacities $u: A \rightarrow \mathbb{Z}_{>0}$, a source $s \in V$ and $\operatorname{sink} t \in V \backslash\{s\}$. Furthermore, each

[^0]arc has an interdiction cost $c: A \rightarrow \mathbb{Z}_{>0}$, and there is a global interdiction budget $B \in \mathbb{Z}_{>0}$. The goal is to find a subset of $\operatorname{arcs} R \subseteq A$ whose cost does not exceed the interdiction budget, i.e., $c(R):=\sum_{a \in R} c(a) \leq B$, such that the value of a maximum $s$ - $t$ flow in the graph $(V, A \backslash R)$ obtained from $G$ by removing $R$ is as small as possible. In particular, if the value of a maximum $s$ - $t$ flow in $G=(V, E)$ is denoted by $\nu((V, E))$, then we can formalize the problem as follows
$$
\min _{R \subseteq A: c(R) \leq B} \nu((V, E \backslash R)) .
$$

A set $R \subseteq A$ with $c(R) \leq B$ is often called an interdiction or removal set. Similarly, one can define interdiction problems for almost any underlying nominal optimization problem.

Interdiction is of practical interest for evaluating robustness and developing attack strategies. Indeed, even the discovery of the famous Max-Flow/Min-Cut Theorem was motivated by a Cold War plan to interdict the Soviet rail network in Eastern Europe [29]. Interdiction has also been studied to find cost-effective strategies to prevent the spread of infection in a hospital [2, to determine how to inhibit the distribution of illegal drugs [32, to prevent nuclear arms smuggling [24], and for infrastructure protection [28, 9], just to name a few applications.

A significant effort has been dedicated to understanding interdiction problems. The list of optimization problems for which interdiction variants have been studied includes maximum flow 31, 32, 26, 34, minimum spanning tree [12, 36], shortest path [3, 20], connectivity of a graph [35], matching [33, 25], matroid rank [17, 18], stable set 4], several variants of facility location [9, 5], and more.

Although one can generate new interdiction problems mechanically from existing optimization problems, there are few general techniques for their solution. The lack of strong exact algorithms for interdiction problems in not surprising in light of the fact that almost all known interdiction problems are NP-hard. However, it is intriguing how little is known about the approximability of interdiction problems. In the context of interdiction problems, the design of approximation algorithms is of particular interest since it often allows accurate estimation of at least the order of magnitude of a potential worst-case impact, which turns out to be a nontrivial task in this context. Polynomial-time approximation schemes (PTASs) are primarily known only when assuming particular graph structures or other special cases. In particular, for planar graphs PTASs have been found for network flow interdiction [26, 34, and matching interdiction [25]. Furthermore, PTASs based on pseudopolynomial algorithms have been obtained for some interdiction problems on graphs with bounded treewidth [33, 4]. Connectivity interdiction is a rare exception where a PTAS is known without any further restrictions on the graph structure [35]. Furthermore, $O(1)$-approximations are known for minimum spanning tree interdiction [36], and for interdicting a class of packing interdiction problems which implies an $O(1)$-approximation for matching interdiction 11. However, for most classical polynomial-time solvable combinatorial optimization problems, like shortest paths, maximum flows and maximum matchings, there is a considerable gap between the approximation quality of the best known interdiction algorithm and the currently strongest hardness result. In particular, among the above-mentioned problems, only the interdiction of shortest $s$ - $t$ paths is known to be APX-hard, and matching interdiction is the only one among these problems for which an $O(1)$-approximation is known. For network flow interdiction, no approximation results are known, even though only strong NP-hardness is known from a complexity point of view.

Burch et al. [8 decided to go for a different approach to attack the network flow interdiction problem, leading to the currently best known solution guarantee obtainable in
polynomial time. Their algorithm solves a linear programming (LP) relaxation to find a fractional interdiction set that lies on an edge of an integral polytope. It is guaranteed that, for any $\alpha>0$, one of the vertices on that edge is either a budget feasible $(1+\alpha)$-approximate solution or a super-optimal solution that overruns the budget by at most a factor of $1+1 / \alpha$. However, one cannot predetermine which objective is approximated and the choice of $\alpha$ biases the outcome. For simplicity we call such an algorithm a 2-pseudoapproximation since, in particular, by choosing $\alpha=1$ one either gets a 2 -approximation or a super-optimal solution using at most twice the budget. In this context, it is also common to use the notion of a $(\sigma, \tau)$-approximate solution, for $\sigma, \tau \geq 1$. This is a solution that violates the budged constraint by a factor of at most $\tau$, and has a value that is at most a factor of $\sigma$ larger than the value of an optimal solution, which is not allowed to violate the budget. Hence, a 2-pseudoapproximation is an algorithm that, for any $\alpha>0$, either returns a $(1+\alpha, 1)$ approximate solution or a $(1,1+1 / \alpha)$-approximate solution.

The main result of this paper is a general technique to get 2-pseudoapproximations for a wide set of interdiction problems. To apply our technique we need three conditions on the nomial problem we want to interdict. First, we need to have an LP description of the nomial problem that has a well-structured dual. In particular, box-total dual integrality (box-TDI) is sufficient. The precise conditions are described in Section 2 Second, the LP description of the nomial problem is a maximization problem whose objective vector only has $\{0,1\}$-coefficients. Third, the LP description of the nomial problem fulfills a downclosedness property, which we call $w$-down-closedness. This third condition is fulfilled by all independence systems, i.e., problems where a subset of a feasible solution is also feasible, like forests, and further problems like maximum $s$ - $t$ flows. Again, a precise description is given in Section 2. In particular, our framework leads to 2-pseudoapproximations for the interdiction of any problem that asks to find a maximum cardinality set in an independence system for which a box-TDI description exists. This includes maximum cardinality independent set in a matroid, maximum cardinality common independent set in two matroids, $b$-stable sets in bipartite graphs, and more. Furthermore, our conditions also include the maximum $s$ - $t$ flow problem, thus implying the result of Burch et al. [8], even though s-t flows do not form an independence system. Apart from its generality, our approach has further advantages. When interdicting independent sets of a matroid, we can even handle general nonnegative objective functions, instead of only $\{0,1\}$-objectives. This is obtained by a reformulation of the weighted problem to a $\{0,1\}$-objective problem over a polymatroid. Also, we can get a 2-pseudoapproximation for interdicting maximum weight independent sets in a matroid with submodular interdiction costs. Submodular interdiction costs allow for modeling economies of scale when interdicting. More precisely, the cost of interdicting an additional element is the smaller the more elements will be interdicted. Additionally, our approach can sometimes be refined by exploiting additional structural properties of the underlying optimization problem to obtain stronger results. We demonstrate this by presenting a PTAS for interdicting $b$-stable sets in bipartite graphs, which is an NP-hard problem. We complete the discussion of $b$-stable set interdiction in bipartite graphs by showing that interdicting classical stable sets in bipartite graphs, which are 1 -stable sets, can be done efficiently by a reduction to matroid intersection. This generalizes a result by Bazgan, Toubaline and Tuza [4] who showed that interdiction of stable sets in a bipartite graph is polynomial-time solvable if all interdiction costs are one.

## Organization of the paper

In Section 2, we formally describe the class of interdiction problems we consider, together with the technical assumptions required by our approach, to obtain a 2 -pseudoapproximation. Furthermore, Section 2 also contains a formal description of our results. Our general approach to obtain 2-pseudoapproximations for a large set of interdiction problems is described in Section 3. In Section 4 we show how, in the context of interdicting independent sets in a matroid, our approach allows for getting a 2-approximation for general nonnegative weights and submodular interdiction costs. Section 5 shows how our approach can be refined for the interdiction of $b$-stable set interdiction in bipartite graphs to obtain a PTAS. Furthermore, we also present an efficient algorithm for stable set interdiction in bipartite graphs in Section 5

## 2 Problem setting and results

We assume that feasible solutions to the nominal problem, like matchings or $s$ - $t$ flows, can be described as follows. There is a finite set $N$, and the feasible solutions can be described by a bounded and nonempty set $\mathcal{X} \subseteq \mathbb{R}_{\geq 0}^{N}$ such that $\operatorname{conv}(\mathcal{X})$ is an integral polytop@ 1 . For example, for matchings we can choose $N$ to be the edges of the given graph $G=(V, E)$, and $\mathcal{X} \subseteq\{0,1\}^{E}$ are all characteristic vectors of matchings $M \subseteq E$ in $G$. Similarly, consider the maximum s-t flow problem on a directed graph $G=(V, A)$, with edge capacities $u: A \rightarrow \mathbb{Z}_{>0}$. Here, we can choose $N=A$ and $\mathcal{X} \subseteq \mathbb{R}_{\geq 0}^{N}$ contains all vectors $f \in \mathbb{R}_{\geq 0}^{N}$ that correspond to $s$ - $t$ flows.

Furthermore, the nominal problem should be solvable by maximizing a linear function $w$ over $\mathcal{X}$. For the case of maximum cardinality matchings one can maximize the linear function with all coefficients being equal to 1 . Finally, we assume that we interdict elements of the ground set $N$, and the interdiction problem can be described by the following min-max mathematical optimization problem:

$$
\begin{array}{lrl}
\min _{R \subseteq N:}^{R \subseteq} \max & w^{T} x &  \tag{1}\\
c(R) \leq B & x & \in \mathcal{X} \\
& x(e) & =0 \quad \forall e \in R,
\end{array}
$$

where $c: N \rightarrow \mathbb{Z}_{>0}$ are interdiction costs on $N$, and $B \in \mathbb{Z}_{>0}$ is the interdiction budget. It is instructive to consider matching interdiction where one can choose $N$ to be all edges and $\mathcal{X} \subseteq\{0,1\}^{N}$ the characteristic vectors of matchings. Imposing $x(e)=0$ then enforces that one has to choose a matching that does not contain the edge $e$ which, as desired, corresponds to interdicting $e$.

Notice that the above way of describing interdiction problems is very general. In particular, it contains a large set of classical combinatorial interdiction problems, like interdicting maximum $s$ - $t$ flows, maximum matchings, maximum cardinality stable sets of a graph, maximum weight forest, and more generally, maximum weight independent set in a matroid or the intersection of two matroids.

Our framework for designing 2-pseudoapproximations for interdiction problems of type (1) requires the following three properties, on which we will expand in the following:

[^1](i) The objective vector $w$ is a $\{0,1\}$-vector, i.e., $w \in\{0,1\}^{N}$,
(ii) the feasible set $\mathcal{X}$ is $w$-down-closed, which is a weaker form of down-closedness that we introduce below, and
(iii) there is a linear description of the convex hull $\operatorname{conv}(\mathcal{X})$ of $\mathcal{X}$ which is box-w-DI solvable. This is a weaker form of being box-TDI equipped with an oracle that returns an integral dual solution to box-constrained linear programs over the description of $\operatorname{conv}(\mathcal{X})$.

In the following we formally define the second and third condition, by giving precise definitions of $w$-down-closedness and box-w-DI solvability. In particular, condition (iii), i.e., box-w-DI solvability, describes how we can access the nominal problem.

## 2.1 w-down-closedness

The notion of $w$-down-closedness is a weaker form of down-closedness. We recall that a set $\mathcal{X} \subseteq \mathbb{R}_{\geq 0}^{N}$ is down-closed if for any $x \in \mathcal{X}$ and $y \in \mathbb{R}_{\geq 0}^{N}$ with $y \leq x$ (componentwise), we have $y \bar{\in} \mathcal{X}$. Contrary to the usual notion of down-closedness, $w$-down-closedness depends on the $\{0,1\}$-objective vector $w$.

Definition 1 (w-down-closedness). Let $w \in\{0,1\}^{N} . \mathcal{X} \subseteq \mathbb{R}_{>0}^{N}$ is $w$-down-closed if for every $x \in \mathcal{X}$ and $e \in N$ with $x(e)>0$, there exists $x^{\prime} \leq x$ such that the following conditions hold:
(i) $x^{\prime} \in \mathcal{X}$;
(ii) $x^{\prime}(e)=0$;
(iii) $w^{T} x^{\prime} \geq w^{T} x-x(e)$.

Notice that if $\mathcal{X} \subseteq \mathbb{R}_{\geq 0}^{N}$ is down-closed, then it is $w$-down-closed for any $w \in\{0,1\}^{N}$, since one can define $x^{\prime} \in \mathcal{X}$ in the above definition by $x^{\prime}(f)=x(f)$ for $f \in N \backslash\{e\}$ and $x^{\prime}(e)=0$. Similarly, $w$-down-closedness also includes all independence systems. We recall that an independence system over a ground set $N$ is a family $\mathcal{F} \subseteq 2^{N}$ of subsets of $N$ such that for any $I \in \mathcal{F}$ and $J \subseteq I$, we have $J \in \mathcal{F}$. In other words, it is closed under taking subsets. Typical examples of independence systems include matchings, forests and stable sets. Naturally, an independence system $\mathcal{F} \subseteq 2^{N}$ can be represented in $\mathbb{R}_{\geq 0}^{N}$ by its characteristic vectors, i.e., $\mathcal{X}=\left\{\chi^{I} \mid I \in \mathcal{F}\right\}$, where $\chi^{I} \in\{0,1\}^{N}$ denotes the characteristic vector of $I$. Clearly, for the same reasons as for down-closed sets, the set $\mathcal{X}$ of characteristic vectors of any independence system is $w$-down-closed for any $w \in\{0,1\}^{N}$.

Hence, many natural combinatorial optimization problems are $w$-down-closed for any $w \in\{0,1\}^{N}$, including matchings, stable sets, independent sets in a matroid or the intersection of two matroids. Furthermore, $w$-down-closedness also captures the maximum $s$ - $t$ flow problem, and a generalization of it, known as polymatroidal network flows, that was introduced independently by Hassin [15] and Lawler and Martel [22]. Loosely speaking, polymatroidal network flows correspond to classic flows with, for every vertex, the addition of submodular packing constraints on the incoming arcs as well as the outgoing ones. See [22] for a formal definition.

Example 2 ( $w$-down-closedness of $s$ - $t$ flow polytope). Let $G=(V, A)$ be a directed graph with two distinct vertices $s, t \in V$ and arc capacities $u: A \rightarrow \mathbb{Z}_{>0}$. Furthermore, we assume
that there are no arcs entering the source $s$, since such arcs can be deleted when seeking maximum s-t flows. The s-t flow polytope $\mathcal{X} \subseteq \mathbb{R}_{\geq 0}^{A}$ can then be described as follows (see, e.g., [21]):

$$
\mathcal{X}=\left\{x \in \mathbb{R}_{\geq 0}^{A} \mid x\left(\delta^{+}(v)\right)-x\left(\delta^{-}(v)\right)=0 \forall v \in V \backslash\{s, t\}\right\}
$$

where $\delta^{+}(v), \delta^{-}(v)$ denote the set of arcs going out of $v$ and entering $v$, respectively; furthermore, $x(U):=\sum_{a \in U} x(a)$ for $U \subseteq A$. A maximum $s$ - $t$ flow can be found by maximizing the linear function $w^{T} x$ over $\mathcal{X}$, where $w=\chi^{\delta^{+}(s)}$, i.e., $w \in\{0,1\}^{A}$ has a 1-entry for each arc $a \in \delta^{+}(s)$, and 0 -entries for all other arcs. This maximizes the total outflow of $s$. Notice that the value of a flow $x \in \mathcal{X}$ is equal to $x\left(\delta^{+}(s)\right)-x\left(\delta^{-}(s)\right)=x\left(\delta^{+}(s)\right)$, since there are no arcs entering $s$; this is indeed the total outflow of $s$.

To see that $\mathcal{X}$ is $w$-down-closed, let $x \in \mathcal{X}$ and $e \in A$, and we construct $x^{\prime} \in \mathcal{X}$ satisfying the conditions of Definition 1 as follows. We compute a path-decomposition of $x$ with few terms. This is a family of s-t paths $P_{1}, \ldots, P_{k} \subseteq A$ with $k \leq|A|$ together with positive coefficients $\lambda_{1}, \ldots, \lambda_{k}>0$ such that $x=\sum_{i=1}^{k} \lambda_{i} \chi^{P_{i}}$ (see [1] for more details). Let $I=\left\{i \in[k] \mid e \in P_{i}\right\}$, where $[k]:=\{1, \ldots, k\}$, and we set $x^{\prime}=\sum_{i \in[k] \backslash I} \lambda_{i} \chi^{P_{i}}$. The flow $x^{\prime} \in \mathcal{X}$ indeed satisfies the conditions of Definition 1. This follows from the fact that $x(e)=\sum_{i \in I} \lambda_{i}$, and each path $P_{i}$ contains precisely one arc of $\delta^{+}(s)$, hence, $x^{\prime}\left(\delta^{+}(s)\right)=x\left(\delta^{+}(s)\right)-\sum_{i \in I} \lambda_{i}$.

The polytope that corresponds to polymatroidal network flows (see [22]), is an $s$ - $t$ flow polytope with additional packing constraints. Its $w$-down-closedness follows therefore from the $w$-down-closedness of the $s$ - $t$ flow polytope.

Furthermore, notice that a non-empty $w$-down-closed system $\mathcal{X}$ always contains the zero vector, independent of $w \in\{0,1\}^{N}$. By $w$-down-closedness we can go through all elements $e \in N$ one-by-one, and replace $x$ by a vector $x^{\prime} \in \mathcal{X}$ with $x^{\prime}(e)=0$, thus proving that the zero vector is in $\mathcal{X}$.

## 2.2 box- $w$-DI solvability

To obtain 2-pseudoapproximations for interdiction problems of type (1), we additionally need to have a good description of the convex hull $\operatorname{conv}(\mathcal{X})$ of $\mathcal{X}$. The type of description we need is a weaker form of box-TDI-ness together with an efficient optimization oracle for the dual that returns integral solutions, which we call box-w-DI solvability, where "DI" stands for 'dual integral".
Definition 3 (box-w-DI solvability). A description $\left\{x \in \mathbb{R}^{N} \mid A x \leq b, x \geq 0\right\}$ of a nonempty polytope $P$ is box-w-DI solvable for some vector $w \in\{0,1\}^{N}$ if the following conditions hold:
(i) For any vector $u \in \mathbb{R}_{\geq 0}^{N}$, the following linear program has an integral dual solution if it is feasible:

$$
\begin{align*}
\max \quad w^{T} x & \\
A x & \leq b \\
x & \leq u  \tag{2}\\
x & \geq 0
\end{align*}
$$

Notice that the dual of the above LP is the following LP:

$$
\min \begin{align*}
b^{T} y+u^{T} r & \\
A^{T} y+r & \geq w  \tag{3}\\
y & \geq 0 \\
& \geq 0
\end{align*}
$$

(ii) For any $u \in \mathbb{R}_{\geq 0}^{N}$, one can decide in polynomial time whether (2) is feasible. Furthermore, if (2) is $\bar{f}$ easible, one can efficiently compute its objective value and an integral vector $r \in \mathbb{Z}_{\geq 0}^{N}$ that corresponds to an optimal integral solution to (3), i.e., there exists an integral vector $y$ such that $y, r$ is an integral optimal solution to (3).

We emphasize that box-w-DI solvability does not assume that the full system $A x \leq$ $b, x \geq 0$ is given as input. In particular, this is useful when dealing with combinatorial problems whose feasible set $\mathcal{X} \subseteq \mathbb{R}_{>0}^{N}$ is such that the polytope $\operatorname{conv}(\mathcal{X})$ has an exponential number of facets, and a description of $\operatorname{conv}(\mathcal{X})$ therefore needs an exponential number of constraints $\mathbb{2}^{2}$. Since the only access to $\mathcal{X}$ that we need is an oracle returning an optimal integral dual solution to (3), we can typically deal with such cases if we have an implicit description of the system $A x \leq b, x \geq 0$ over which we can separate with a separation oracle.

Furthermore, notice that condition (ii) of box-w-DI solvability is a weaker form of boxTDIness due to two reasons. First, our objective vector $w \in\{0,1\}^{N}$ is fixed, whereas in box-TDIness, dual integrality has to hold for any integral objective vector. Second, when dealing with box-TDIness, one can additionally add lower bounds $x \geq \ell$ on $x$ in (2), still getting a linear program with an optimal integral dual solution.

We even have box-TDI descriptions for all problems we discuss here. The only additional property needed for a box-TDI system to be box-w-DI solvable, is that one can efficiently find an optimal integral dual solution. However, such procedures are known for essentially all classical box-TDI systems. In particular, this applies to the classical polyhedral descriptions of the independent sets of a matroid or the intersection of two matroids, stable sets in bipartite graphs, $s$ - $t$ flows, and any problem whose constraint matrix can be chosen to be totally unimodular (TU) and of polynomial size.

Since our only access to the feasible set is via the oracle guaranteed by box-w-DI solvability, we have to be clear about what we consider to be the input size when talking about polynomial time algorithms. In addition to the binary encodings of $B, c$, we also assume that the binary encodings of the optimal value of (3) and the integral optimal vector $r \in \mathbb{Z}_{\geq 0}^{N}$ returned by the box- $w$-DI oracle are part of the input size. This implies that in particular, the binary encoding of $\nu^{*}=\max \left\{w^{T} x \mid A x \leq b, x \geq 0\right\}$ is part of the input size.

### 2.3 Our results

The following theorem summarizes our main result for obtaining 2-pseudoapproximations.
Theorem 4. There is an efficient 2-pseudoapproximation for any interdiction problem of type (1) if the following conditions are satisfied:
(i) The objective function $w$ is a $\{0,1\}$-vector, i.e., $w \in\{0,1\}^{N}$,
(ii) the description of the feasible set $\mathcal{X} \subseteq \mathbb{R}^{N}$ is $w$-down-closed, and
(iii) there is a box-w-DI solvable description of $\operatorname{conv}(\mathcal{X})$.

Using well-known box-TDI description of classical combinatorial optimization problems (see [30]), Theorem 4 leads to 2-pseudoapproximations for the interdiction of many combinatorial optimization problems.

[^2]Corollary 5. There is a 2-pseudoapproximation for interdicting maximum cardinality independent sets of a matroid or the intersection of two matroids, maximum s-t flows, and maximum polymatroidal network flows. Furthermore, there is a 2-pseudoapproximation for all problems where a maximum cardinality set has to be found with respect to down-closed constraints captured by a TU matrix. For example, this includes maximum b-stable sets in bipartite graphs.

We recall that for the maximum $s$ - $t$ flow problem, a 2-pseudoapproximation was already known due to Burch et al. [8].

Furthermore, for interdicting independent sets of a matroid we obtain stronger results by leveraging the strong combinatorial structure of matroids to adapt our approach. Consider a matroid $M=(N, \mathcal{I})$ on ground set $N$ with independent sets $\mathcal{I} \subseteq 2^{N}$. We recall the definition of a matroid, which requires $\mathcal{I}$ to be a nonempty set such that: (i) $\mathcal{I}$ is an independence system, i.e., $I \in \mathcal{I}$ and $J \subseteq I$ implies $J \in \mathcal{I}$, and (ii) for any $I, J \in \mathcal{I}$ with $|I|<|J|$, there exists $e \in J \backslash I$ such that $I \cup\{e\} \in \mathcal{I}$. We typically assume that a matroid is given by an independence oracle, which is an oracle that, for any $I \subseteq N$, returns whether $I \in \mathcal{I}$ or not. See [30, Volume B] for more information on matroids.

For matroids, we can get a 2-pseudoapproximation even for arbitrary nonnegative weight functions $w$, i.e., for interdicting the maximum weight independent set of a matroid. Furthermore, we can also handle monotone nonnegative submodular interdiction costs $c$. A submodular function $c$ defined on a ground set $N$, is a function $c: 2^{N} \rightarrow \mathbb{R}_{\geq 0}$ that assigns a nonnegative value $c(S)$ to each set $S \subseteq N$ and fulfills the following property of economies of scale:

$$
c(A \cup\{e\})-c(A) \geq c(B \cup\{e\})-c(B) \quad A \subseteq B \subseteq N, e \in N \backslash B
$$

In words, the marginal cost of interdicting an element is lower when more elements will be interdicted. Economies of scale can often be a natural property in interdiction problems. It allows for modeling dependencies that are sometimes called cascading failures or chainreactions, depending on the context. More precisely, it may be that the interdiction of a set of elements $S \subseteq N$ will render another element $e \in N$ unusable. This can be described by a submodular interdiction cost $c$ which assigns a marginal cost of 0 to the element $e$, once all elements of $S$ have been removed. Still, removing only $e$ may have a strictly positive interdiction cost. Such effects cannot be captured with linear interdiction costs. A submodular function $c: 2^{N} \rightarrow \mathbb{R}_{\geq 0}$ is called monotone if $c(A) \leq c(B)$ for $A \subseteq B \subseteq N$. We typically assume that a submodular function $f$ is given through a value oracle, which is an oracle that, for any set $S \subseteq N$, returns $f(S)$.

Theorem 6. There is an efficient 2-pseudoapproximation to interdict the problem of finding a maximum weight independent set in a matroid, with monotone nonnegative submodular interdiction costs. The following is a formal description of this interdiction problem:

$$
\min _{R}\left\{\max _{I}\{w(I) \mid I \in \mathcal{I}, I \cap R=\emptyset\} \mid R \subseteq N, c(R) \leq B\right\}
$$

where $c: 2^{N} \rightarrow \mathbb{R}_{\geq 0}$ is a monotone nonegative submodular function, and $w \in \mathbb{Z}_{\geq 0}^{N} 3$. The matroid is given through an independence oracle and the submodular cost function $c$ through a value oracle.

[^3]Finally, we show that our approach can sometimes be refined to obtain stronger approximation guarantees. We illustrate this on the interdiction version of the $b$-stable set problem in bipartite graphs. Here, a bipartite graph $G=(V, E)$ with bipartition $V=I \cup J$, and a vector $b \in \mathbb{Z}_{>0}^{E}$ is given. A $b$-stable set in $G$ is a vector $x \in \mathbb{Z}_{\geq_{0}}^{V}$ such that $x(i)+x(j) \leq b(\{i, j\})$ for $\{i, j\} \in E$. Hence, by choosing $b$ to be the all-ones vector, we obtain the classical stable set problem. Because it can be formulated as a linear program with TU constraints, finding a maximum cardinality $b$-stable set in a bipartite graph is efficiently solvable. However, its interdiction version is easily seen to be NP-hard by a reduction from the knapsack problem. Exploiting the adjacency properties of a polytope that is crucial in our analysis we can even get a true approximation algorithm, which does not violate the budget. More precisely, we obtain a polynomial-time approximation scheme (PTAS), which is an algorithm that, for any $\epsilon>0$, computes efficiently an interdiction set leading to a value of at most $1-\epsilon$ times the optimal value.

Theorem 7. There is a PTAS for the interdiction of b-stable sets in bipartite graphs.
We complete this discussion of interdicting $b$-stable sets in bipartite graphs by showing that the special case of interdicting stable sets in bipartite graph, i.e., $b=1$, is efficiently solvable. This is done through a reduction to a polynomial number of efficiently solvable matroid intersection problems.

Theorem 8. The problem of interdicting the maximum cardinality stable set in a bipartite graph can be solved efficiently.

The above theorem generalizes a result by Bazgan, Toubaline and Tuza 4 who showed that interdiction of stable sets in bipartite graphs can be done efficiently when all interdiction costs are one. Our result applies to arbitrary interdiction costs.

## 3 General approach to obtain 2-pseudoapproximations

Consider an interdiction problem that fulfills the conditions of Theorem 4 As usual, let $N$ be the ground set of our problem, $w \in\{0,1\}^{N}$ be the objective vector, and we denote by $\mathcal{X} \subseteq \mathbb{R}_{\geq 0}^{N}$ the set of feasible solutions. Furthermore, let $\left\{x \in \mathbb{R}^{N} \mid A x \leq b, x \geq 0\right\}=\operatorname{conv}(\mathcal{X})$ be a box-w-DI solvable description of $\operatorname{conv}(\mathcal{X})$. We denote by $m$ the number of rows of $A$.

One key ingredient in our approach is to model interdiction as a modification of the objective instead of a restriction of sets that can be chosen. This is possible due to $w$-downclosedness. More precisely, we replace the description of the interdiction problem given by (11) with the following min-max problem.

$$
\begin{align*}
\min _{r} \max _{x} \quad(w-r)^{T} x & \\
A x & \leq b \\
x & \geq 0  \tag{4}\\
c^{T} r & \leq B \\
r & \in\{0,1\}^{N}
\end{align*}
$$

We start by showing that (11) and (4) are equivalent in the following sense. For any interdiction set $R \subseteq N$, let

$$
\begin{aligned}
\phi(R): & =\max \left\{w^{T} x \mid x \in \operatorname{conv}(\mathcal{X}), x(e)=0 \forall e \in R\right\} \\
& =\max \left\{w^{T} x \mid A x \leq b, x \geq 0, x(e)=0 \forall e \in R\right\}
\end{aligned}
$$

Hence, $\phi(R)$ is the value of the problem (1) for a fixed set $R$. Similarly, we define for any characteristic vector $r \in\{0,1\}^{N}$ of an interdiction set

$$
\begin{aligned}
\psi(r): & =\max \left\{(w-r)^{T} x \mid x \in \operatorname{conv}(\mathcal{X})\right\} \\
& =\max \left\{(w-r)^{T} x \mid A x \leq b, x \geq 0\right\}
\end{aligned}
$$

Thus, $\psi(r)$ is the value of (4) for a fixed vector $r \in\{0,1\}^{N}$.
Lemma 9. For every interdiction set $R \subseteq N$, we have $\phi(R)=\psi\left(\chi^{R}\right)$. In particular, this implies that (11) and (4) have the same optimal value, and optimal interdiction sets $R$ to (1) correspond to optimal characteristic vectors $\chi^{R}$ to (4) and vice versa.

We show Lemma 9 based on another lemma stated below that highlights an important consequence of $w$-down-closedness, which we will use later again.
Lemma 10. Let $r \in \mathbb{R}_{\geq 0}^{N}$ and $U=\{e \in N \mid r(e) \geq 1\}$. Then there exists $x \in \mathbb{R}_{\geq 0}^{N}$ with $x(e)=0 \forall e \in U$, such that $x$ is an optimal solution to the following linear program.

$$
\begin{align*}
\max _{x} \quad(w-r)^{T} x & \\
A x & \leq b  \tag{5}\\
x & \geq 0
\end{align*}
$$

Proof. Among all optimal solutions to the above linear program, let $x^{*}$ be one that minimizes $x^{*}(U)$. Notice that $x^{*}$ can be chosen to be a vertex of $\operatorname{conv}(\mathcal{X})=\left\{x \in \mathbb{R}^{N} \mid A x \leq b, x \geq 0\right\}$, since $x^{*}$ can be obtained by minimizing the objective $\chi^{U}$ over the face of all optimal solutions to the above LP. We have to show $x^{*}(U)=0$. Assume for the sake of contradiction that there is an element $e \in U$ such that $x^{*}(e)>0$. Since $x^{*}$ is a vertex of $\operatorname{conv}(\mathcal{X})$, we have $x^{*} \in \mathcal{X}$. By $w$-down-closedness of $\mathcal{X}$, there is a vector $x^{\prime} \in \mathcal{X}$ with $x^{\prime} \leq x^{*}, x^{\prime}(e)=0$, and $w^{T} x^{\prime} \geq w^{T} x^{*}-x^{*}(e)$. We thus obtain

$$
\begin{aligned}
(w-r)^{T} x^{\prime} & \geq w^{T} x^{*}-x^{*}(e)-r^{T} x^{\prime} \\
& \geq w^{T} x^{*}-x^{*}(e)-\left(r^{T} x^{*}-x^{*}(e)\right) \\
& =(w-r)^{T} x^{*}
\end{aligned}
$$

where in the second inequality we used $r^{T} x^{\prime} \leq r^{T} x^{*}-x^{*}(e)$, which follows from $x^{\prime} \leq x^{*}$ together with $x^{\prime}(e)=0$ and $r(e) \geq 1$. Hence, $x^{\prime}$ is an optimal solution to the LP with $x^{\prime}(U)<x^{*}(U)$, which violates the definition of $x^{*}$ and thus finishes the proof.

Proof of Lemma 9, Let $R \subseteq N$ be an interdiction set, and $r=\chi^{R}$ its characteristic vector. Let $x \in \mathcal{X}$ be an optimal solution to the maximization problem defining $\phi(R)$, i.e., $w^{T} x=$ $\phi(R)$ and $x(e)=0 \forall e \in R$. We have

$$
\psi(r) \geq(w-r)^{T} x=w^{T} x=\phi(R)
$$

where the first equality follows from $x(e)=0$ for $e \in R$. Hence, $\psi(r) \geq \phi(R)$.
Conversely, let $x \in \operatorname{conv}(\mathcal{X})$ be an optimal solution to the maximization problem defining $\psi(r)$, i.e., $\psi(r)=(w-r)^{T} x$. By Lemma 10, $x$ can be chosen such that $r^{T} x=x(R)=0$. Hence,

$$
\psi(r)=(w-r)^{T} x=w^{T} x \leq \phi(R)
$$

and thus $\phi(R)=\psi(r)$.

Hence, (4) is an alternative description of the interdiction problem (11) in which we are interested. In a next step we relax the integrality of $r$ to obtain the following mathematical program.

$$
\begin{align*}
\min _{r \in \mathbb{R}^{n}} \max _{x \in \mathbb{R}^{n}} \quad(w-r)^{T} x & \\
A x & \leq b \\
x & \geq 0  \tag{6}\\
c^{T} r & \leq B \\
1 \geq r & \geq 0
\end{align*}
$$

As we will show next, the constraint $1 \geq r$ can be dropped due to $w$-down-closedness without changing the objective. This leads to the following problem.

$$
\begin{align*}
\min _{r \in \mathbb{R}^{n}} \max _{x \in \mathbb{R}^{n}} \quad(w-r)^{T} x & \\
A x & \leq b \\
x & \geq 0  \tag{7}\\
c^{T} r & \leq B \\
r & \geq 0
\end{align*}
$$

The following lemma not only highlights that the objective values of (6) and (7) match, but also that any optimal interdiction vector $r$ of (7) can easily be transformed to an optimal interdiction vector of (6). Thus, we can restrict ourselves to (7). We recall that $\psi(r)$ corresponds to the inner maximization problem of both (6) and (77) for a fixed vector $r$.
Lemma 11. We have

$$
\psi(r)=\psi(r \wedge 1) \quad \forall r \in \mathbb{R}_{\geq 0}^{N}
$$

where $r \wedge 1$ is the component-wise minimum between $r$ and the all-ones vector $1 \in \mathbb{R}^{N}$. This implies that (6) and (7) have the same optimal value, and if $r$ is optimal for (7) then $r \wedge 1$ is optimal for (6).
Proof. Let $r \in \mathbb{R}_{\geq 0}^{N}$ and consider the maximization problem that defines $\psi(r)$, which is the same as the linear program desribed by (5). Furthermore, let $r^{\prime}=r \wedge 1$, and let $U=\{e \in N \mid r(e) \geq 1\}$. In particular, $r$ and $r^{\prime}$ are identical on $N \backslash U$. We clearly have $\psi\left(r^{\prime}\right) \geq \psi(r)$ by monotonicity of $\psi$. Therefore only $\psi\left(r^{\prime}\right) \leq \psi(r)$ has to be shown.

By Lemma 10, there exists an optimal vector $x$ to the maximization problem defining $\psi\left(r^{\prime}\right)$ that satisfies $x(e)=0 \forall e \in U$. Furthermore, by using that $r$ and $r^{\prime}$ are identical on $N \backslash U$ we obtain

$$
\begin{aligned}
\psi\left(r^{\prime}\right) & =\left(w-r^{\prime}\right)^{T} x \\
& =w^{T} x-\sum_{e \in N \backslash U} r^{\prime}(e) x(e)-\sum_{e \in U} r^{\prime}(e) x(e) \\
& =w^{T} x-\sum_{e \in N \backslash U} r(e) x(e)-\sum_{e \in U} r^{\prime}(e) x(e) \quad\left(r \text { and } r^{\prime} \text { are identical on } N \backslash U\right) \\
& =w^{T} x-\sum_{e \in N \backslash U} r(e) x(e)-\sum_{e \in U} r(e) x(e) \quad(x(e)=0 \text { for } e \in U) \\
& =(w-r)^{T} x \\
& \leq \psi(r)
\end{aligned}
$$

as desired.

Interestingly, problem (7) has already been studied in a different context. It can be interpreted as the problem to inhibit a linear optimization problem by a continuous and limited change of the objective vector $w$. In particular, Frederickson and Solis-Oba [12, 13 present efficient algorithms to solve this problem when the underlying combinatorial problem is the maximum weight independent set problem in a matroid. Jüttner [18] presents efficient procedures for polymatroid intersection and minimum cost circulation problem. Also, Jüttner provides an excellent discussion how such problems can be solved efficiently using parametric search techniques.

However, our final goal is quite different from their setting since, eventually, we need to find a $\{0,1\}$-vector $r$. This difference is underlined by the fact that without integrality, problem (7) can often be solved efficiently, whereas the interdiction problems we consider are NP-hard.

Still, we continue to further simplify (7) in a similar way as it was done by Jüttner [18]. For a fixed $r$, the inner maximization problem in (7) is a linear program with a finite optimum, since $\mathcal{X}$ is bounded and nonempty by assumption, and therefore also $\operatorname{conv}(\mathcal{X})=$ $\left\{x \in \mathbb{R}^{N} \mid A x \leq b, x \geq 0\right\}$ is bounded and nonempty. Hence, we can leverage strong duality to dualize the inner maximization into a minimization problem. We thus end up with a problem where we first minimize over $r$ and then over the dual variables, which we can rewrite as a single minimization, thus obtaining the following LP.

$$
\begin{align*}
& \min \quad b^{T} y \\
& A^{T} y+r \geq w \\
& y \geq 0  \tag{8}\\
& c^{T} r \leq B \\
& r \geq 0
\end{align*}
$$

Hence, by strong duality, the optimal value of (8) is the same as the optimal value of (7). This reduction also shows why problem (7), which has no integrality constraints on $r$, can often be solved efficiently. This can often be achieved by obtaining an optimal vector $r \in \mathbb{R}_{\geq 0}^{N}$ by solving the LP (8) with standard linear programming techniques.

What we will do in the following is to show that there is an optimal solution $(r, y)$ for (7) which can be written as a convex combination of two integral solutions $\left(r^{1}, y^{1}\right)$ and $\left(r^{2}, y^{2}\right)$ that may violate the budget constraint. Similar to a reasoning used in Burch et al. 8] this then implies than one of $r^{1}$ and $r^{2}$ is a 2-pseudoapproximation.

To compute $r^{1}$ and $r^{2}$, we move the constraint $c^{T} r \leq B$ in (8) into the objective via Lagrangian duality, by introducing a multiplier $\lambda \geq 0$ (see [7] for more details). We do this in two steps to highlight that the resulting Lagrangian dual problem can be solved via the oracle guaranteed by box-w-DI solvability. First, we dualize (8) to the obtain the following linear program, which is nicely structured in the sense that for any fixed $\lambda \geq 0$, it corresponds to optimizing a linear function over $\operatorname{conv}(\mathcal{X})$ with upper box constraints.

$$
\begin{align*}
\max \begin{aligned}
w^{T} z-\lambda B & \\
A z & \leq b \\
z-\lambda c & \leq 0 \\
z & \geq 0 \\
& \geq 0
\end{aligned} r=0 .
\end{align*}
$$

Consider the above LP as a problem parameterized by $\lambda \geq 0$. Since $\left\{x \in \mathbb{R}^{N} \mid A x \leq\right.$ $b, x \geq 0\}$ is box- $w$-DI solvable, the LP obtained from (9) by fixing $\lambda \geq 0$ has an optimal
integral dual solution. Furthermore, such an optimal integral dual solution can be found efficiently by box-w-DI solvability. The dual problem of (9) for a fixed $\lambda \geq 0$ is the problem $\mathrm{LP}(\lambda)$ below with optimal objective value $L(\lambda)$.

$$
L(\lambda)=\min \quad b^{T} y-\lambda\left(B-c^{T} r\right) ~ 子 \begin{gather*}
A^{T} y+r \geq w \\
y \quad  \tag{LP}\\
\\
y \geq 0
\end{gather*}
$$

Notice that LP( $\lambda$ ) is indeed the problem obtained from (8) by moving the constraint $c^{T} r \leq B$ into the objective using $\lambda$ as Lagrangian multiplier.

The following lemma summarizes the relationships between the different problems we introduced.

Lemma 12. The optimal values of (6), (7), (8), and (9) are all the same and equal to $\max _{\lambda \geq 0} L(\lambda)$.

Furthermore, the common optimal value of the above-mentioned problems are a lower bound to OPT, the optimal value of the considered interdiction problem (1).

Proof. Problem (6) and (7) have identical optimal values due to Lemma 11. The LP (8) was obtained from (7) by dualizing the inner maximization problem. Both problems have the same optimal value due to strong duality, which holds since $\operatorname{conv}(\mathcal{X})=\left\{x \in \mathbb{R}^{N} \mid A x \leq\right.$ $b, x \geq 0\}$ is a nonempty polytope and thus, the inner maximization problem of (7) has a finite optimum value for any $r \in \mathbb{R}^{n}$. This also shows that the optimum value of (7), and hence also of (8) and (6), is finite. Problems (9) and (8) are a primal-dual pair of linear programs. For this pair of LPs, strong duality holds because (8), and therefore also (9), has a finite optimum value. Finally $\max _{\geq 0} L(\lambda)$ is the same as the optimum value of (8) by Lagrangian duality.

It remains to observe that the optimal value of the above problems is a lower bound to OPT. We recall that by Lemma 9 problem (4) is a rephrasing of the original interdiction problem (11), and thus also has optimal value OPT. Finally, (6) is obtained from (4) by relaxation the integrality condition on $r$. Thus, the optimum value of (6) -which is also the optimum value of (7), (8), (9), and $\max _{\lambda \geq 0} L(\lambda)$-is less or equal to OPT, as claimed.

The following theorem shows that we can efficiently compute an optimal dual multiplier $\lambda^{*}$ together with two integral vectors $r^{1}, r^{2}$ that are optimal solutions to $L P\left(\lambda^{*}\right)$, one of which will turn out to be a 2 -pseudoapproximation to the considered interdiction problem (1).

Theorem 13. There is an efficient algorithm to compute a maximizer $\lambda^{*}$ of $\max _{\lambda \geq 0} L(\lambda)$, and two vectors $r^{1}, r^{2} \in \mathbb{Z}_{\geq 0}^{N}$ such that:
(i) $\exists$ integral $y^{1}, y^{2} \in \mathbb{Z}^{m}$ such that both $\left(r^{1}, y^{1}\right)$ and $\left(r^{2}, y^{2}\right)$ are optimal solutions to $L P\left(\lambda^{*}\right)$.
(ii) $c^{T} r^{1} \geq B \geq c^{T} r^{2}$.

Before proving Theorem 13, we show that it implies our main result, Theorem 4.
Theorem 14. Let $\lambda^{*}$ be a maximizer of $\max _{\lambda \geq 0} L(\lambda)$, let $\left(r^{1}, y^{1}\right),\left(r^{2}, y^{2}\right)$ be two optimal solutions to $L P\left(\lambda^{*}\right)$ with $c^{T} r^{1} \geq B \geq c^{T} r^{2}$, and let $\alpha>0$. Then at least one of the following two conditions holds:
(i) $c^{T} r^{1} \leq\left(1+\frac{1}{\alpha}\right) B$, or
(ii) $b^{T} y^{2} \leq(1+\alpha) L\left(\lambda^{*}\right)$.

Furthermore, if (ii) holds, then $r^{1} \wedge 1$ is the characteristic vector of a $\left(1,1+\frac{1}{\alpha}\right)$-approximation to (11). If (iii) holds, then $r^{2} \wedge 1$ is the characteristic vector of a $(1+\alpha, 1)$-approximation to (1).
Proof. Before showing that either (ii) or (iii) holds, we show the second part of the theorem. Assume first that (ii) holds. We recall that problem (1) and (4) are equivalent due to Lemma 9 Thus, the objective value of the interdiction problem (1) that corresponds to $r^{1} \wedge 1$ is given by $\psi\left(r^{1} \wedge 1\right)$ which, by Lemma 11 is equal to $\psi\left(r^{1}\right)$. Hence, to show that $r^{1} \wedge 1$ is a $\left(1,1+\frac{1}{\alpha}\right)$-approximation, it suffices to prove $\psi\left(r^{1}\right) \leq L\left(\lambda^{*}\right)$, because $L\left(\lambda^{*}\right) \leq$ OPT by Lemma 12 ,

Indeed, $\psi\left(r^{1}\right) \leq L\left(\lambda^{*}\right)$ holds due to:

$$
\begin{aligned}
L\left(\lambda^{*}\right) & =b^{T} y^{1}-\lambda^{*}\left(B-c^{T} r^{1}\right) & & \left(\left(r^{1}, y^{1}\right) \text { is a maximizer of } L P\left(\lambda^{*}\right)\right) \\
& \geq b^{T} y^{1} & & \left(B \leq c^{T} r^{1} \text { and } \lambda^{*} \geq 0\right) \\
& \geq \psi\left(r^{1}\right) & & \left(y^{1} \text { is a feasible solution to the dual of the LP defining } \psi\left(r^{1}\right)\right) .
\end{aligned}
$$

Similarly, if (iii) holds then the objective value corresponding to $r^{2} \wedge 1$ is

Since $r^{2}$ satisfies $c^{T} r^{2} \leq B$, the characteristic vector $r^{2} \wedge 1$ is therefore indeed a $(1+\alpha, 1)$ approximation to (11).

Hence, it remains to show that at least one of (ii) and (iii) holds. Assume for the sake of contradiction that both do not hold. Because both $\left(r^{1}, y^{1}\right)$ and $\left(r^{2}, y^{2}\right)$ are maximizers of $L P\left(\lambda^{*}\right)$, also any convex combination of these solutions is a maximizer. In particular let $\mu=\frac{\alpha}{1+\alpha}$ and consider the maximizer $\left(r_{\mu}, y_{\mu}\right)$ of $L P\left(\lambda^{*}\right)$, where $r_{\mu}=\mu r^{1}+(1-\mu) r^{2}$ and $y_{\mu}=\mu y^{1}+(1-\mu) y^{2}$. We obtain $L\left(\lambda^{*}\right)=b^{T} y_{\mu}-\lambda^{*}\left(B-c^{T} r_{\mu}\right)$

$$
\left.\geq(1-\mu) b^{T} y^{2}-\lambda^{*}\left(B-\mu c^{T} r^{1}\right) \quad \text { (ignoring } \mu b^{T} y^{1} \text { and }(1-\mu) \lambda^{*} c^{T} r^{2}, \text { which are both } \geq 0\right)
$$

$$
=\frac{1}{1+\alpha} b^{T} y^{2}-\lambda^{*}\left(B-\frac{\alpha}{1+\alpha} c^{T} r^{1}\right) \quad\left(\operatorname{using} \mu=\frac{\alpha}{1+\alpha}\right)
$$

$$
>L\left(\lambda^{*}\right)
$$

(using that both (ii) and (iii) do not hold),
thus leading to a contradiction and proving the theorem.

Theorem 13 together with Theorem 14 imply our main result, Theorem 4, due to the following. Theorem [13 guarantees that we can compute efficiently $\lambda^{*}, r^{1}, r^{2}$ as needed in Theorem 14 Then, depending whether condition (ii) or (iii) holds, we either return $r^{1} \wedge 1$ or $r^{2} \wedge 1$ as our 2-pseudoapproximation. Notice that to check whether 囵 holds, we have to compute $L\left(\lambda^{*}\right)$. This can be done efficiently due to the fact that our description of $\operatorname{conv}(\mathcal{X})$ is box-w-DI solvable. More precisely, as already discussed, $L P\left(\lambda^{*}\right)$ is the dual of (9) for $\lambda=\lambda^{*}$ whose optimal value can be computed by box- $w$-DI solvability. Hence, it remains to prove Theorem 13,

$$
\begin{aligned}
& \psi\left(r^{2}\right) \leq b^{T} y^{2} \quad\left(y^{2} \text { is a feasible solution to the dual of the LP defining } \psi\left(r^{2}\right)\right) \\
& \leq(1+\alpha) L\left(\lambda^{*}\right) \quad \text { (by (iiil) } \\
& \leq(1+\alpha) \text { OPT (by Lemma 12). }
\end{aligned}
$$

### 3.1 Proof of Theorem 13

First we discuss some basic properties of $L(\lambda)$. We start by observing that $L(\lambda)$ is finite for any $\lambda>0$. This follows by the fact that $L(\lambda)$ is the optimal value of (9) when $\lambda$ is considered fixed. More precisely, for any fixed $\lambda \geq 0$, the problem (9) is feasible and bounded. It is feasible because $z=0$ is feasible since $b \geq 0$. Furthermore, it is bounded since by assumption $\operatorname{conv}(\mathcal{X})=\left\{z \in \mathbb{R}^{N} \mid A z \leq b, z \geq 0\right\}$ is a polytope. Additionally, $L(\lambda)$ has the following properties, which are true for any Lagrangian dual of a finite LP (see [7] for more details):

- $L(\lambda)$ is piecewise linear.
- Let $\left[\lambda_{1}, \lambda_{2}\right]$ be one of the linear segments of $L(\lambda)$, let $t \in\left(\lambda_{1}, \lambda_{2}\right)$, and $\left(r_{t}, y_{t}\right)$ be an optimal solution to $L(t)$. Then, $\left(r_{t}, y_{t}\right)$ is an optimal solution for the whole segment, i.e., for any $L P(\lambda)$ with $\lambda \in\left[\lambda_{1}, \lambda_{2}\right]$. As a consequence, the slope of the segment is $c^{T} r_{t}-B$.
Also, we recall that $L(\lambda)$ can be evaluated efficiently for any $\lambda \geq 0$; since (9) is box-w-DI solvable, it can be solved for any fixed $\lambda \geq 0$.

We will find an optimal multiplier $\lambda^{*} \geq 0$ using bisection. For this, we start by showing two key properties of $L(\lambda)$. First, we show that any optimal multiplier $\lambda^{*}$ to $L(\lambda)$ is not larger than some upper bound with polynomial input length. Second, we show that each linear segment of $L(\lambda)$ has some minimal width, which makes it possible to reach it with a polynomial number of iterations using bisection.

We recall that

$$
\nu^{*}=\max \left\{w^{T} x \mid A x \leq b, x \geq 0\right\}=\min \left\{b^{T} y \mid A^{T} y \geq w, y \geq 0\right\}
$$

is the optimal value of the nominal problem without interdiction, and that $\log \left(\nu^{*}\right)$ is part of the input size.

Lemma 15. If $\lambda^{*}$ is a maximizer of $L(\lambda)$, then $\lambda^{*} \leq \nu^{*}$. Furthermore, for every $\lambda \geq \nu^{*}$, $r=0$ is an optimal solution to $L P(\lambda)$.
Proof. Let $r=0 \in \mathbb{Z}^{N}$ and $y^{*}$ be a minimizer of $\min \left\{b^{T} y \mid A^{T} y \geq w, y \geq 0\right\}$. Hence, in particular, $b^{T} y^{*}=\nu^{*}$. We first show that for any $\lambda \geq \nu^{*}$, the pair $\left(r, y^{*}\right)$ is a minimizer of $L P(\lambda)$. Assume for the sake of contradiction that there is some $\lambda \geq \nu^{*}$ such that $\left(r, y^{*}\right)$ is not a minimizer of $L P(\lambda)$. Let $\left(r^{\prime}, y^{\prime}\right)$ be a minimizer of $L P(\lambda)$ which, because the dual of $L P(\lambda)$ is box- $w$-DI, can be assumed to be integral. Clearly, we must have $r^{\prime} \neq 0=r$, since for $r=0$, the vector $y^{*}$ attains by definition the smallest value in $L P(\lambda)$. Hence, we obtain

$$
\begin{aligned}
b^{T} y^{*}-\lambda B & >b^{T} y^{\prime}-\lambda B+\lambda c^{T} r^{\prime} & & \left(\left(r^{\prime}, y^{\prime}\right) \text { attains a smaller value than }\left(r, y^{*}\right) \text { in } L P(\lambda)\right) \\
& \geq-\lambda B+\lambda c^{T} r^{\prime} & & \left(b^{T} y^{\prime} \geq 0 \text { since } b \geq 0 \text { and } y^{\prime} \geq 0\right),
\end{aligned}
$$

which implies

$$
\nu^{*}>\lambda c^{T} r^{\prime} .
$$

However, this is a contradiction since $\lambda \geq \nu^{*}$, and $c^{T} r^{\prime} \geq 1$ because $c \in \mathbb{Z}_{>0}^{N}$ and $r^{\prime} \in \mathbb{Z}_{>0}^{N}$ is nonzero. Thus, $\left(r, y^{*}\right)$ is indeed a minimizer of $L P(\lambda)$ for any $\lambda \geq \nu^{*}$. However, since $B>0$, this implies

$$
L\left(\nu^{*}\right)=b^{T} y-\nu^{*} B>b^{T} y-\lambda B=L(\lambda) \quad \forall \lambda>\nu^{*},
$$

thus implying the lemma.

Hence, Lemma 15 implies that to find a maximizer $\lambda^{*}$ of $L(\lambda)$, we only have to search within the interval $\left[0, \nu^{*}\right]$.
Lemma 16. Each segment of the piecewise linear function $L(\lambda)$ has width at least $\frac{1}{(c(N))^{2}}$.
Proof. We start by deriving a property of the kinks of $L(\lambda)$, namely that they correspond to a rational value $\lambda$ whose denominator is at most $\frac{1}{c(N)}$. Later we will derive from this property that the distance between any two kinks is at least $\frac{1}{(c(N))^{2}}$.

Let $\bar{\lambda}>0$ be the value of a kink of $L(\lambda)$, i.e., there is one segment of the piecewise linear function $L(\lambda)$ that ends at $\bar{\lambda}$ and one that starts at $\bar{\lambda}$. We call the segment ending at $\bar{\lambda}$ the left segment and the one starting at $\bar{\lambda}$ the right segment. Let $\left(r^{1}, y^{1}\right)$ be an optimal solution for all $L P(\lambda)$ where $\lambda$ is within the left segment. Similarly, let $\left(r^{2}, y^{2}\right)$ be an optimal solution for the right segment. By box- $w$-DI solvability, we can choose $\left(r^{i}, y^{i}\right)$ for $i \in\{1,2\}$ to be integral. We start by showing that $r^{1}$ and $r^{2}$ are $\{0,1\}$-vectors, i.e, $r^{1}, r^{2} \in\{0,1\}^{N}$. We can rewrite $L(\lambda)$ as follows:

$$
\begin{align*}
L(\lambda) & =\min \left\{b^{T} y-\lambda\left(B-c^{T} r\right) \mid A^{T} y+r \geq w, y \geq 0, r \geq 0\right\} \\
& =\min _{r \geq 0}\left(-\lambda B+c^{T} r+\max \left\{(w-r)^{T} x \mid A x \leq b, x \geq 0\right\}\right)  \tag{10}\\
& =\min _{r \geq 0}\left(-\lambda B+c^{T} r+\psi(r)\right)
\end{align*}
$$

where the second equality follows by dualizing the LP of the first line for a fixed $r \geq 0$, and the third equality follows by the definition of $\psi$. By Lemma 11 we have $\psi(r)=\psi(r \wedge 1)$, and since $c \in \mathbb{Z}_{>0}$, this implies that a minimizing $r$ is such that $r=r \wedge 1$. Thus an integral minimizing $r$ satisfies $r \in\{0,1\}^{N}$, as desired.

The slope of the left segment is $\beta_{1}=-B+c^{T} r^{1}$ and the slope of the right segment is $\beta_{2}=-B+c^{T} r^{2}$. Let $\alpha_{1}=b^{T} y^{1}$ and $\alpha_{2}=b^{T} y^{2}$. Again using (10) we have $\alpha_{i}=\psi\left(r^{i}\right)$ for $i \in\{1,2\}$. This implies that $\alpha_{i}$ for $i \in\{1,2\}$ is integral because $\psi\left(r^{i}\right)$ is defined as the optimum of an LP with integral objective vector over an integral polytope $\left\{x \in \mathbb{R}^{N} \mid A x \leq\right.$ $b, x \geq 0\}$.

Because $L(\lambda)$ is concave, the slope decreases strictly at each kink, i.e., $\beta_{1}>\beta_{2}$. Furthermore, since the left and right segment touch at $\bar{\lambda}$, we have

$$
\alpha_{1}+\bar{\lambda} \beta_{1}=\alpha_{2}+\bar{\lambda} \beta_{2}
$$

Because $\beta_{1}>\beta_{2}$ and $\bar{\lambda}>0$, this implies $\alpha_{1}<\alpha_{2}$, and $\bar{\lambda}$ can be written as

$$
\bar{\lambda}=\frac{\alpha_{2}-\alpha_{1}}{\beta_{1}-\beta_{2}}
$$

Notice that

$$
\beta_{1}-\beta_{2}=c^{T}\left(r^{1}-r^{2}\right) \leq\|c\|_{1}=c(N)
$$

where we use the fact that $r^{1}, r^{2} \in\{0,1\}^{N}$ for the inequality. In summary, any kink $\bar{\lambda}$ is a rational number $\frac{p}{q}$ with $p, q \in \mathbb{Z}_{>0}$ and $q \leq c(N)$. In particular, this implies that the first segment, which goes from $\lambda=0$ to the first kink, has width at least $\frac{1}{c(N)} \geq \frac{1}{(c(N))^{2}}$. The last segment clearly has infinite width. Any other segment is bordered by two kinks $\lambda_{1}=\frac{p_{1}}{q_{1}}$
and $\lambda_{2}=\frac{p_{2}}{q_{2}}$ with $\lambda_{1}<\lambda_{2}$ and has therefore a width of

$$
\begin{array}{rlrl}
\lambda_{1}-\lambda_{2} & =\frac{p_{1} q_{2}-p_{2} q_{1}}{q_{1} q_{2}} & \\
& \geq \frac{1}{q_{1} q_{2}} & & \left(\text { since } \lambda_{1}-\lambda_{2}>0\right) \\
& \geq \frac{1}{(c(N))^{2}} & & \left(\text { since } q_{1}, q_{2} \leq c(N)\right)
\end{array}
$$

We use the bisection procedure Algorithm 1 to compute $\lambda^{*}, r^{1}$ ，and $r^{2}$ as claimed by Theorem 13．Notice that $L\left(\lambda^{1}\right)$ and $L\left(\lambda^{2}\right)$ ，as needed by Algorithm 1 to determine $\lambda^{*}$ ，can be computed due to box－w－DI solvability．Algorithm 1 is clearly efficient；it remains to show its correctness．

```
    for \(i=1, \ldots, 1+\left\lfloor\log _{2}\left(\nu^{*}(c(N))^{2}\right)\right\rfloor\) do
        \(\lambda=\frac{1}{2}\left(\lambda^{1}+\lambda^{2}\right)\).
        if \(-B+c^{T} r \geq 0\) then
                \(\lambda^{1}=\lambda\).
                \(r^{1}=r\).
        else
            \(\lambda^{2}=\lambda\).
            \(r^{2}=r\).
        end
    end
```

Algorithm 1: Computing $\lambda^{*}, r^{1}$ and $r^{2}$ as claimed by Theorem[13
Initialization: $\lambda^{1}=0, \lambda^{2}=\nu^{*}, r^{1}=\chi^{N}, r^{2}=0$
Use box-w-DI solvability oracle to compute integral $r \in \mathbb{Z} \geq 0$ satisfying that there
is a $y$ such that $(r, y)$ is an optimal solution to $\operatorname{LP}(\lambda)$.

Compute $\lambda^{*}$ as the intersection of the two segments at $\lambda^{1}$ and $\lambda^{2}$ ：

$$
\lambda^{*}=\frac{L\left(\lambda^{2}\right)-L\left(\lambda^{1}\right)-\lambda^{2}\left(-B+c^{T} r^{2}\right)+\lambda^{1}\left(-B+c^{T} r^{1}\right)}{c^{T}\left(r^{1}-r^{2}\right)} .
$$

return $\lambda^{*}, r^{1}, r^{2}$ ．

Lemma 17．$\lambda^{*}, r^{1}$ and $r^{2}$ as returned by Algorithm $⿴ 囗 十$ fulfill the properties required by Theorem［13．

Proof．Notice that throughout the algorithm the following invariant is maintained：$r^{i}$ is an optimal solution to $L P\left(\lambda^{i}\right)$ for $i \in\{1,2\}$ ．Furthermore，$-B+c^{T} r^{1} \geq 0$ and $-B+c^{T} r^{2}<0$ ． We highlight that after initialization，these two invariants are maintained because $-B+$ $c^{T} \chi^{N}=-B+c(N)>0$ because we assumed $B<c(N)$ to avoid the trivial special case when everything is interdicted．Additionally，$-B+c^{T} 0=-B<0$ ．Also note that $r^{2}=0$ is an optimal solution to $L P\left(\nu^{*}\right)$ by Lemma 15

Due to this invariant and the fact that $L(\lambda)$ is concave, we know that there is a maximizer $\lambda^{*}$ of $L(\lambda)$ within $\left[\lambda^{1}, \lambda^{2}\right)$.

Observe that the distance $\lambda^{2}-\lambda^{1}$ halves at every iteration of the for loop. Consider now $\lambda^{1}$ and $\lambda^{2}$ after the for loop. Their distance is bounded by

$$
\lambda^{2}-\lambda^{1}=\nu^{*}\left(\frac{1}{2}\right)^{1+\left\lfloor\log _{2}\left(\nu^{*}(c(N))^{2}\right)\right\rfloor}<\nu^{*}\left(\frac{1}{2}\right)^{\log _{2}\left(\nu^{*}(c(N))^{2}\right)}=\frac{1}{(c(N))^{2}}
$$

Hence, the distance between $\lambda^{2}$ and $\lambda^{1}$ is less then the width of any segment of the piecewise linear function $L(\lambda)$, due to Lemma 16. This leaves the following options. Either one of $\lambda^{1}$ or $\lambda^{2}$ is a maximizer of $L(\lambda)$, and the other one is in the interior of the segment to the left or right, respectively. Or, neither $\lambda^{1}$ nor $\lambda^{2}$ is a maximizer of $L(\lambda)$. In this case $\lambda^{1}$ and $\lambda^{2}$ are in the interior of the segment to the left and right, respectively, of the unique maximizer $\lambda^{*}$. In all of these cases, the solutions $r^{1}$ and $r^{2}$ are both optimal with respect to some maximizer $\lambda^{*}$ of $L(\lambda)$, since they are on two segments that meet on an optimal multiplier $\lambda^{*}$.

It remains to prove that the returned $\lambda^{*}$ is correct. Since both $r^{1}$ and $r^{2}$ are optimal solutions to $L\left(\lambda^{*}\right)$ for some maximizer $\lambda^{*}$, we have

$$
\begin{equation*}
L\left(\lambda^{*}\right)=b^{T} y^{1}-\lambda^{*}\left(B-c^{T} r^{1}\right)=b^{T} y^{2}-\lambda^{*}\left(B-c^{T} r^{2}\right) \tag{11}
\end{equation*}
$$

Furthermore,

$$
\begin{aligned}
& L\left(\lambda^{1}\right)=b^{T} y^{1}-\lambda^{1}\left(B-c^{T} r^{1}\right), \text { and } \\
& L\left(\lambda^{2}\right)=b^{T} y^{2}-\lambda^{2}\left(B-c^{T} r^{2}\right),
\end{aligned}
$$

By replacing $b^{T} y^{i}=L\left(\lambda^{i}\right)+\lambda^{i}\left(B-c^{T} r^{i}\right)$, for $i \in\{1,2\}$, in (11) and solving for $\lambda^{*}$, we obtain

$$
\lambda^{*}=\frac{L\left(\lambda^{2}\right)-L\left(\lambda^{1}\right)-\lambda^{2}\left(-B+c^{T} r^{2}\right)+\lambda^{1}\left(-B+c^{T} r^{1}\right)}{c^{T}\left(r^{1}-r^{2}\right)}
$$

thus showing that the returned $\lambda^{*}$ is indeed optimal.
Hence, even the somewhat limited access through box-w-DI solvability that we assume to our optimization problem is enough to obtain an efficient 2-pseudoapproximation for the interdiction problem due to the efficiency of the bisection method described in Algorithm 1 However, in many concrete settings, more efficient methods can be employed to get an optimal multiplier $\lambda^{*}$ and optimal integral dual solutions $r^{1}, r^{2}$. In particular, often one can even obtain strongly polynomial procedure by employing Megiddo's parametric search technique [23]. We refer the interested reader to [18] for a technical details of how this can be done in a very similar context.

## 4 Matroids: weighted case and submodular costs

In this section we consider the problem of interdicting a feasible set $\mathcal{X} \subseteq\{0,1\}^{N}$ that corresponds to the independent sets of a matroid. It turns out that we can exploit structural properties of matroids to solve natural generalization of the interdiction problem considered in Theorem 4 In particular, even for arbitrary nonnegative weight functions $w \in \mathbb{Z}_{>0}^{N}$, we can obtain a 2-pseudoapproximation for the corresponding interdiction problem. Whhat's
more is that we can achieve this when the interdiction costs are submodular, rather than just linear.

For clarity, we first discuss in Section 4.1 a technique to reduce arbitrary nonnegative weights to the case of $\{0,1\}$-objectives that was mentioned previously. In Section 4.2, we then build up and extend this technique to also deal with submodular interdiction costs.

### 4.1 Weighted case

Let $M=(N, \mathcal{I})$ be a matroid, and let $w: N \rightarrow \mathbb{Z}_{\geq 0}$. The canonical problem we want to interdict is the problem of finding a maximum weight independent set, i.e., $\max \{w(I) \mid I \in$ $\mathcal{I}\}$. Let $r_{w}: 2^{N} \rightarrow \mathbb{Z}_{\geq 0}$ be the weighted rank function, i.e.,

$$
r_{w}(S)=\max \{w(I) \mid I \subseteq S, I \in \mathcal{I}\}
$$

In words, $r_{w}(S)$ is the weight of a heaviest independent set that is contained in $S$. We recall a basic fact on weighted rank functions [30, Section 44.1a].

One key observation we exploit is that the maximum weight independent set can be rephrased as maximizing an all-ones objective function over the following polymatroid:

$$
\begin{equation*}
P_{w}=\left\{x \in \mathbb{R}_{\geq 0}^{N} \mid x(S) \leq r_{w}(S) \forall S \subseteq N\right\} \tag{12}
\end{equation*}
$$

Even more importantly, we do not only have $\max \left\{x(N) \mid x \in P_{w}\right\}=\max \{w(I) \mid I \in \mathcal{I}\}$, but we also have that the problem of interdicting the maximum weight independent set problem of a matroid maps to the problem of interdicting the corresponding all-ones maximization problem on the polymatroid. This is formalized through the lemma below.

Lemma 18. For any $R \subseteq N$, we have

$$
\max \left\{x(N) \mid x \in P_{w}, x(R)=0\right\}=\max \{w(I) \mid I \in \mathcal{I}, I \subseteq N \backslash R\}
$$

Proof. Observe that the right-hand side of the above equality is, by definition, equal to $r_{w}(N \backslash R)$.
lhs $\leq$ rhs: Let $x^{*}$ be a maximizer of $\max \left\{x(N) \mid x \in P_{w}, x(R)=0\right\}$. We have

$$
\begin{array}{rrr}
x^{*}(N) & =x^{*}(N \backslash R) & \left(x^{*}(R)=0\right) \\
& \leq r_{w}(N \backslash R) & \left(\text { since } x^{*} \in P_{w}\right),
\end{array}
$$

thus showing the desired inequality.
lhs $\geq$ rhs: Conversely, let $I^{*}$ be a maximizer of $\max \{w(I) \mid I \in \mathcal{I}, I \subseteq N \backslash R\}$. Hence, $w\left(I^{*}\right)=r_{w}(N \backslash R)$. Define $y \in \mathbb{R}_{\geq 0}^{N}$ by

$$
y(e)= \begin{cases}w(e) & \text { if } e \in I^{*} \\ 0 & \text { if } e \in N \backslash I^{*}\end{cases}
$$

Clearly, $y(N)=w\left(I^{*}\right)=r_{w}(N \backslash R)$. Thus, to show that the left-hand side of the equality of Lemma 18 is at least as large as the right-hand side, it suffices to show that $y$ is feasible to the maximization problem on the left-hand side, i.e., $y(R)=0$ and $y \in P_{w}$. We have $y(R)=0$ since $I^{*} \subseteq N \backslash R$. Furthermore,

$$
y(S)=w\left(S \cap I^{*}\right) \leq r_{w}(S) \quad \forall S \subseteq N
$$

where the inequality follows from $S \cap I^{*} \in \mathcal{I}$. Hence, this implies $y \in P_{w}$ and completes the proof.

We therefore can focus on the problem $\max \left\{x(N) \mid x \in P_{w}, x(R)=0\right\}$ to which we can now apply Theorem 4. For this it remains to observe that $P_{w}$ is 1-down-closed because it is down-closed. Furthermore, the description of $P_{w}$ given by (12) is box-1-DI solvable since it is well-known to be even box-TDI, a property that holds for all polymatroids [30, Section 44.3], and one can efficiently find an optimal integral dual solution to the problem of finding a maximum size point over (12) with upper box constraints. In fact, this problem can be interpreted as a maximum cardinality polymatroid intersection problem, one polymatroid being $P_{w}$ and the other one being defined by the upper box constraints. An optimal integral dual solution to the maximum cardinality polymatroid intersection problem can be found in strongly polynomial time by standard techniques (for clarity we provide some more details about this in Section 4.2). In summary, our technique presented in Section 3 to obtain 2-pseudoapproximations therefore indeed applies to this setting.

### 4.2 Submodular costs

In this section, we show how to obtain a 2-pseudoapproximation for the interdiction of the maximum weight independent set of a matroid with submodular interdiction costs. When dealing with submodular interdiction costs, we assume that the interdiction costs $\kappa$ are a nonnegative and monotone submodular function $\kappa: 2^{N} \rightarrow \mathbb{R}_{\geq 0}$. As before, a removal set $R \subseteq N$ has to satisfy the budget constraint, i.e., $\kappa(R) \leq B$. We assume that the submodular function $\kappa$ is given by a value oracle.

To design a 2-pseudoapproximation, we will describe a way to formulate the problem such that it can be attacked with essentially the same techniques as described in Section 3 . For simplicity of presentation, and to avoid replicating reasonings introduced in Section 3 we focus on the key differences in this section, and refer to Section 3 for proofs that are essentially identical.

We extend the model for the weighted case. A variable $q(S)$ is introduced for each set $S \subseteq N$. In the non-relaxed mathematical program, we have $q \in\{0,1\}^{2^{N}}$, and only one variable $q(S)$ is equal to one, which indicates the set $S$ of elements we interdict. Below is a mathematical description of a relaxation, where we allow the variables $q(S)$ to take real values. If instead of allowing $q(S) \in \mathbb{R}_{\geq 0}$, we set $q(S) \in\{0,1\}$, then the mathematical program below would be an exact description of the interdiction problem with submodular interdiction costs.

$$
\begin{array}{rlrl}
\min _{q \in \mathbb{R}^{N}} \max _{x \in \mathbb{R}^{n}}\left(1-\sum_{S \subseteq N} \chi^{S} \cdot q(S)\right)^{T} x & & \\
x(S) & \leq r_{w}(S) & \forall S \subseteq N \\
x & \geq 0 &  \tag{13}\\
\sum_{S \subseteq N} \kappa(S) \cdot q(S) & \leq B & & \\
\sum_{S \subseteq N} q(S) & \leq 1 & \\
q(S) & \geq 0 & \forall S \subseteq N
\end{array}
$$

We start by dropping the constraint $\sum_{S \subseteq N} q(S) \leq 1$. As we will see later, this does not change the objective value. This step is similar to dropping the constraint $r \leq 1$ when going from (6) to (7) in the standard setting of our framework without submodular interdiction costs. We thus obtain the following mathematical program.

$$
\begin{array}{rlrl}
\min _{q \in \mathbb{R}^{N}} \max _{x \in \mathbb{R}^{n}}\left(1-\sum_{S \subseteq N} \chi^{S} \cdot q(S)\right)^{T} x & & \\
x(S) & \leq r_{w}(S) \quad \forall S \subseteq N \\
x & \geq 0 & &  \tag{14}\\
\sum_{S \subseteq N} \kappa(S) \cdot q(S) & \leq B & & \\
q(S) & \geq 0 & \forall S \subseteq N
\end{array}
$$

Now, by dualizing the inner problem we get the following LP.

$$
\min \begin{array}{rlr}
\sum_{S \subseteq N} r_{w}(S) y(S) & & \\
\left(\sum_{S \subseteq N: e \in S} y(S)\right)+\left(\sum_{S \subseteq N: e \in S} q(S)\right) & \geq 1 & \forall e \in N \\
y(S) & \geq 0 \quad \forall S \subseteq N  \tag{15}\\
q(S) & \geq 0 \quad \forall S \subseteq N \\
& \sum_{S \subseteq N} \kappa(S) \cdot q(S) & \leq B
\end{array}
$$

As in the case with linear interdiction costs, we dualize the budget constraint with a Lagrangian multiplier $\lambda$ to obtain the following family of LPs, parameterized by $\lambda$ :

$$
\begin{aligned}
& L(\lambda)=\min \quad \sum_{S \subseteq N} r_{w}(S) y(S)+\lambda\left(\sum_{S \subseteq N} \kappa(S) \cdot q(S)\right)-\lambda B \\
&\left(\sum_{S \subseteq N: e \in S} y(S)\right)+\left(\sum_{S \subseteq N: e \in S} q(S)\right) \geq 1 \quad \forall e \in N \quad(\operatorname{LP}(\lambda)) \\
& y(S) \geq 0 \quad \forall S \subseteq N \\
& q(S) \geq 0 \quad \forall S \subseteq N
\end{aligned}
$$

It remains to observe that for any $\lambda \geq 0, \mathrm{LP}(\lambda)$ is the dual of a maximum cardinality polymatroid intersection problem-when forgetting about the constant term $-\lambda B$-where the two polymatroids are defined by the submodular functions $r_{w}$ and $\lambda \cdot \kappa$, respectively. A key result in this context is that there is a set $A \subseteq N$ such that the optimal primal value, which is equal to the optimal dual value by strong duality, is equal to $\lambda \kappa(A)+r_{w}(N \backslash A)$ (see [30, Section 46.2]). This implies that defining $q(A)=1, y(N \backslash A)=1$, and setting all other entries of $q$ and $y$ to zero is an optimal solution to $(\mathrm{LP}(\lambda))$. Furthermore, such a set $A$ can be found in strongly polynomial time [30, Section 47.1]. Note that this fact also implies that dropping the constraint $\sum_{S \subseteq N} q(S) \leq 1$ when going from (13) to (14) did not change the objective value of the mathematical program. Furthermore, we can evaluate $L(\lambda)$ efficiently for any $\lambda \geq 0$.

From this point on, the approach is identical to the one presented in Section 3 for linear interdiction costs. More precisely, we determine the optimal dual multiplier $\lambda^{*}$ and two optimal dual solutions $\left(q^{1}, y^{1}\right),\left(q^{2}, y^{2}\right)$ to $L P\left(\lambda^{*}\right)$ such that
(i) The dual solutions have the above-mentioned property that all four vectors $y^{1}, q^{1}, y^{2}, q^{2}$ only have 0 -entries with the exception of a single 1-entry. Let $R_{1}, R_{2} \subseteq N$ be the sets such that $q^{1}\left(R^{1}\right)=q^{2}\left(R^{2}\right)=1$.
(ii) One solution has interdiction cost that is upper bounded by the budget and one has an interdiction cost that is lower bounded by the budget, i.e., $\kappa\left(A^{1}\right) \leq B \leq \kappa\left(A^{2}\right)$.
The value $\lambda^{*}$ and vectors $q^{1}, y^{1}, q^{2}, y^{2}$ can either be found by bisection, as described in Section 3, or they can be obtained in strongly polynomial time via Megiddo's parametric search technique (see [18] for details). An identical reasoning as used in Theorem 14 shows that one of $R^{1}$ or $R^{2}$ is a 2 -pseudoapproximation.

## 5 Refinements for bipartite $b$-stable set interdiction

This section specializes our approach to the interdiction of $b$-stable sets in a bipartite graph. We recall that given is a bipartite graph $G=(V, E)$ with bipartition $V=I \cup J$ and edge capacities $b \in \mathbb{Z}_{\geq 0}^{E}$. A $b$-stable set is a vector $x \in \mathbb{Z}_{\geq 0}^{N}$ such that $x(i)+x(j) \leq b(\{i, j\})$ for each $\{i, j\} \in E$. The value of a $b$-stable set $x$ is given by $x(V)$. The maximum $b$-stable set problem asks to find a $b$-stable set of maximum value. Furthermore, we are given an interdiction cost $c: V \rightarrow \mathbb{Z}_{>0}$ for each vertex, and an interdiction budget $B \in \mathbb{Z}_{>0}$. As usual, the task is to remove a subset $R \subseteq V$ with $c(R) \leq B$ such that value of a maximum $b$-stable set in the graph obtained from $\bar{G}$ by removing $R$ is as small as possible.

In Section 5.1 we show how our approach can be adapted to get a PTAS for $b$-stable set interdiction, thus proving Theorem 7. In Section 5.2 we complete the discussion on $b$-stable set interdiction by presenting an exact algorithm to solve the interdiction problem of the classical stable set problem in bipartite graphs, which corresponds to the case when $b$ is the all-ones vector.

Before presenting these results, we remark that $b$-stable set problem has also a wellknown vertex-capacitated variant. In this case an additional vector $u \in \mathbb{Z}_{\geq 0}^{V}$ is given and constraints $x \leq u$ are imposed. The vertex-capacitated problem can easily be reduced to the uncapacitated problem by adding two additional vertices $v_{I}, v_{J}$, where $v_{I}$ is added to $I$ and $v_{J}$ to $J$, and connecting $v_{I}$ to all vertices in $J$ and $v_{J}$ to all vertices in $I$. Finally, by choosing $b\left(\left\{v_{I}, j\right\}\right)=u(j)$ for $j \in J$ and $b\left(\left\{v_{J}, i\right\}\right)=u(i)$ for $i \in I$, one obtains a $b$-stable set problem that is equivalent to the vertex-capacitated version. Furthermore, a vertex interdiction strategy for minimizing the maximum $b$-independent set problem in this auxiliary graph carries over exactly to the vertex-capacitated variant. Thus, the approach we present can also deal with vertex capacities.

### 5.1 PTAS by exploiting adjacency structure

As in our general approach, we start with the relaxation (6). Below, we adapt the description of the relaxation to this specialized setting highlight some structural aspects of the problem.

$$
\begin{align*}
\min _{r \in \mathbb{R}^{V}} \max _{x \in \mathbb{R}^{V}}(1-r)^{T} x & \\
A x & \leq b \\
x & \geq 0  \tag{16}\\
c^{T} r & \leq B \\
0 \leq r & \geq 1
\end{align*}
$$

Notice that the matrix $A \in\{0,1\}^{E \times V}$ is the incidence matrix of the bipartite graph $G$, i.e., $A(e, v)=1$ if and only if $v \in V$ is one of the endpoints of $e \in E$. This matrix is well known to be totally unimodular (TU) [21. Similar to our general approach, we could now drop the constraint $r \leq 1$. However, since this does not lead to a further simplification in this setting, we will keep this constraint. Following our general approach, we dualize the inner maximization problem to obtain the following linear program.

$$
\min \begin{align*}
& b^{T} y \\
& A^{T} y+r \geq 1 \\
& y \geq 0  \tag{17}\\
& c^{T} r \leq B \\
& 0 \leq r \leq 1
\end{align*}
$$

Observe that $\{0,1\}$-solutions to (17) have a nice combinatorial interpretation. More precisely, they correspond to a subset $R \subseteq V$ of the vertices (where $\chi^{R}=r$ ) with $c(R) \leq B$ and an edge set $F \subseteq E$ (where $\chi^{F}=y$ ) such that $F$ is an edge cover in the graph obtained from $G$ by removing the vertices $R$.

Not surprisingly, apart from the budget constraint $c^{T} r \leq B$, the feasible region of the above LP closely resembles the bipartite edge cover polytope. We will make this link more explicit in the following with the goal to exploit well-known adjacency properties of the bipartite edge cover polytope. First, notice that for any feasible solution ( $y, r$ ) to (8), the vector $(y \wedge 1, r)$ is also feasible with equal or lower objective value. This follows from the fact that $A$ is a $\{0,1\}$-matrix. Hence, we can add the constraint $y \leq 1$ without changing the problem to obtain the following LP.

$$
\begin{align*}
& \min \quad b^{T} y \\
& A^{T} y+r \geq 1 \\
& c^{T} r \leq B  \tag{18}\\
& 0 \leq y \leq 1 \\
& 0 \leq r \leq 1
\end{align*}
$$

The feasible region of the above LP is given by intersection the polyope

$$
P=\left\{\left.\binom{y}{r} \in \mathbb{R}^{|E|+|V|} \right\rvert\, A^{T} y+r \geq 1,0 \leq y \leq 1,0 \leq r \leq 1\right\}
$$

with the half-space $\left\{(y, r) \in \mathbb{R}^{|E|+|V|} \mid c^{T} r \leq B\right\}$. Notice that $P$ is integral because the matrix $A$ is TU. The key property we exploit is that $P$ has very well-structured adjacency properties, because it can be interpreted as a face of a bipartite edge cover polytope, a polytope whose adjacency structure is well known. More precisely, it turns out that any two adjacent vertices of $P$ represent solutions that do not differ much in terms of cost and objective function. Hence, similar to our general approach, we compute two vertex solutions of $P$, one over budget but with a good objective value and the other one under budget, with the additional property that they are adjacent on $P$. We then return the one solution that is budget-feasible. This procedure as stated does not yet lead to a PTAS, but it can be transformed into one by a classical preprocessing technique that we will briefly mention at the end.

We start by introducing a bipartite edge cover polytope $P^{\prime}$ such that $P$ is a face of $P^{\prime}$. To simplify the exposition, we do a slight change to the above sketch of the algorithm. More precisely, we will restate (18) in terms of a problem on $P^{\prime}$ and then work on the polytope $P^{\prime}$ instead of $P$. We will define $P^{\prime}$ with a system of linear constraints. It has two new rows and one new variable $r_{I J}$ in addition to the constraints $A^{T} y+r \geq 1$ of $P$. The rows correspond to two new vertices in the graph, one in $I$ and one in $J$, and the new variable is for an edge between the two new vertices. The updated constraints are

$$
\underbrace{\left(\begin{array}{ccc}
A^{T} & I & 0  \tag{19}\\
0 & \left(\chi^{J}\right)^{T} & 1 \\
0 & \left(\chi^{I}\right)^{T} & 1
\end{array}\right)}_{D:=}\left(\begin{array}{c}
y \\
r \\
r_{\mathrm{IJ}}
\end{array}\right) \geq 1
$$

where $\chi^{I}, \chi^{J} \in\{0,1\}^{V}$ are the characteristic vectors of $I \subseteq V$ and $J \subseteq V$, respectively. Let $D$ be the $\{0,1\}$-matrix on the left-hand side of the constraint (19). Notice that $D$ is the
vertex-edge incidence matrix of a bipartite graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$, where $G^{\prime}$ is obtained from $G$ as follows: add one new vertex $w_{I}$ to $I$ and one new vertex $w_{J}$ to $J$; then connect $w_{I}$ to all vertices in $J \cup\left\{w_{J}\right\}$ and $w_{J}$ to all vertices in $I$. Hence, $I^{\prime}=I \cup\left\{w_{I}\right\}$ and $J^{\prime}=J \cup\left\{w_{J}\right\}$ is a bipartition of $V^{\prime}$. Since $D$ is an incidence matrix of a bipartite graph, it is TU. For easier reference to the different types of edges in $G^{\prime}$ we partition $E^{\prime}$ into the edge $E$, the edge set

$$
E_{R}=\left\{\left\{w_{I}, j\right\} \mid j \in J\right\} \cup\left\{\left\{i, w_{J}\right\} \mid i \in I\right\}
$$

and the single edge $f=\left\{w_{I}, w_{J}\right\}$, i.e., $E^{\prime}=E \cup E_{R} \cup\{f\}$.
Now consider the edge cover polytope that corresponds to $D$ :

$$
\left.\left.P^{\prime}=\left\{\left(\begin{array}{c}
y \\
r \\
r_{I J}
\end{array}\right) \in \mathbb{R}^{|E|+|V|+1}\right) \right\rvert\, D \cdot\left(\begin{array}{c}
y \\
r \\
r_{I J}
\end{array}\right) \geq 1,0 \leq y, r, r_{I J} \leq 1\right\}
$$

Notice that $P$ is obtained from $P^{\prime}$ by considering the face of $P^{\prime}$ defined by $r_{I J}=1$, and projecting out the variable $r_{I J}$. Every vertex $y, r, r_{I J}$ of $P^{\prime}$ is a characteristic vector of an edge cover in $G^{\prime}$, where $y$ represents the characteristic vector of the edges in $E$, the vector $r$ is the characteristic vector of the edges in $E_{R}$, and $r_{I J}=1$ indicates that $f$ is part of the edge cover.

We can now restate (18) as follows in terms of $P^{\prime}$ :

$$
\min \left\{b^{T} y \left\lvert\,\left(\begin{array}{c}
y  \tag{20}\\
r \\
r_{I J}
\end{array}\right) \in P^{\prime}\right., c^{T} r \leq B\right\} .
$$

Indeed, one can always choose for free $r_{I J}=1$ in the above LP, since $r_{I J}$ does not appear in the objective. Furthermore, when setting $r_{I J}=1$, the LP (20) has the same feasible vectors $(y, r)$ as (18). We can thus focus on (20) instead of (18).

One can interpret an edge cover $F$ in $G^{\prime}$ as an interdiction strategy of the original problem as follows. Every vertex $v \in V$ that is incident with either $w_{I}$ or $w_{J}$ through an edge of $F$ will be interdicted. To obtain a better combinatorial interpretation of 20, we extend the vectors $b$ and $c$ to all edges $E^{\prime}$. More precisely, $b$ is only defined for edges in $E$. We set $b(e)=0$ for $e \in E^{\prime} \backslash E$. Furthermore, the vector $c$ can be interpreted as a vector on the edges $E_{R}$, where $c\left(\left\{w_{I}, j\right\}\right):=c(j)$ and $c\left(\left\{i, w_{J}\right\}\right):=c(i)$ for $i \in I$ and $j \in J$. For $e \in E^{\prime} \cup\{f\}$ we set $c(e)=0$. Using this notation, the best $\{0,1\}$-solution to (20) can be interpreted as an edge cover $F$ of $G^{\prime}$ that minimizes $b(F)$ under the constraint $c(F) \leq B$. One can observe that the best $\{0,1\}$-solution to (20) corresponds to an optimal interdiction set for the original non-relaxed interdiction problem.

Also, we want to highlight that problems of this type, where a combinatorial optimization problem has to be solved under an additional linear packing constraint with nonnegative coefficients are also known as budgeted optimization problems or restricted optimization problems and have been studied for various problem settings, like spanning trees, matchings, and shortest paths (see [14] and references therein for more details). The way we adapt our procedure to exploit adjacency properties of the edge cover polytope is inspired by procedures to find budgeted matchings and spanning trees [27, 6, 14].

We compute an optimal vertex solution $p^{*}=\left(y^{*}, r^{*}, r_{I J}^{*}=1\right)$ to (20) via standard linear programming techniques. If $r^{*}$ is integral, i.e., $r^{*} \in\{0,1\}^{V}$, then $r^{*}$ corresponds to an optimal interdiction set since it is optimal for the relaxation and integral. Hence, assume $r^{*}$ not to be integral from now on. This implies that $p^{*}$ is in the interior of an edge of $P^{\prime}$, since
it is a vertex of the polytope obtained by intersecting $P^{\prime}$ with a single additional constraint. This edge of the polytope $P^{\prime}$ is described by looking at the constraints of $P^{\prime}$ that are tight with respect to the optimal vertex solution. From this description of the edge, we can efficiently compute its two endpoints $y^{1}, r^{1}, r_{I J}^{1}=1$ and $y^{2}, r^{2}, r_{I J}^{2}=1$, which are vertices of $P^{\prime}$ and therefore integral. These two solutions correspond to edge covers $F^{1}, F^{2} \subseteq E^{\prime}$ in $G^{\prime}$ with $f \in F^{1} \cap F^{2}$. For simplicity, we continue to work with these edge covers $F^{1}$ and $F^{2}$. One of these edge covers will violate the budget constraint and be superoptimal, say the first one, i.e., $c\left(F^{1}\right)>B$ and $b\left(F^{1}\right)<b^{T} y^{*}$, and the other one strictly satisfies the budget constraint and is suboptimal, i.e., $c\left(F^{2}\right)<B$ and $b\left(F^{2}\right)>b^{T} y^{*}$. Hence, this is just a particular way to obtain two solutions as required by our general approach, with the additional property that they are adjacent on the polytope $P^{\prime}$.

The key observation is that $F^{2}$ is not just budget-feasible, but almost optimal. We prove this by exploiting the following adjacency property of edge cover polytopes shown by Hurkens.

Lemma 19 (Hurkens [16). Two edge covers $U_{1}$ and $U_{2}$ of a bipartite graph are adjacent if and only if $U_{1} \Delta U_{2}$ is an alternating cycle or an alternating path with endpoints in $V\left(U_{1} \cap U_{2}\right)$, where $V\left(U_{1} \cap U_{2}\right)$ denotes all endpoints of the edges in $U_{1} \cap U_{2}$.
Lemma 20. $b\left(F^{2}\right) \leq b^{T} y^{*}+2 b_{\max }$, where $b_{\max }=\max _{e \in E} b(e)$.
Proof. We will prove the statement by constructing a new edge cover $Z \subseteq E^{\prime}$ of $G^{\prime}$ with the following two properties:
(i) $c(Z) \leq c\left(F^{2}\right)$, and
(ii) $b(Z) \leq b\left(F^{1}\right)+2 b_{\text {max }}$.

We claim that this implies the result due to the following. First observe that there can be no edge cover $W$ of $G^{\prime}$ such that $c(W) \leq c\left(F_{2}\right)$ and $b(W)<b\left(F_{2}\right)$. If such an edge cover existed, then $p^{*}$ would not be an optimal solution to (20), because $p^{*}$ is a convex combination of $\chi^{F^{1}}$ and $\chi^{F^{2}}$, and by replacing $F^{2}$ by $W$ one would obtain a new budget-feasible solution with lower objective value. Hence, if (ii) then $b(Z) \geq b\left(F^{2}\right)$, which in turn implies

$$
b\left(F^{2}\right) \leq b(Z) \stackrel{(\text { iii) }}{\leq} b\left(F^{1}\right)+2 b_{\max } \leq b^{T} y^{*}+2 b_{\max }
$$

Hence, it remains to prove the existence of an edge cover $Z \subseteq E^{\prime}$ satisfying (ii) and (iii).
By Lemma $19 U=F^{1} \Delta F^{2}$ is either an alternating path or cycle. In both cases, $U$ contains at most 4 edges of $E_{R}$, at most 2 in $E_{R} \cap F^{1}$ and at most 2 in $E_{R} \cap F^{2}$. Let $E_{R}^{1}=E_{R} \cap U \cap F^{1}$ be the up to two edges of $U$ in $E_{R} \cap F^{1}$. Consider $X=F^{1} \backslash E_{R}^{1}$. $X$ is not necessarily an edge cover because we removed up to two edges of $E_{R}$. Hence, there may be up to 4 vertices not covered by $X$. However, the up to two edges of $E_{R}$ that we removed to obtain $X$ from $F^{1}$ are both incident with one of the two vertices $w_{I}$ and $w_{J}$. Since $f \in X$ because $f \in F^{1}$, the two vertices $w_{I}, w_{J}$ remain covered by $X$. Hence, there are at most two vertices $i, j \in V$ that are not covered by $X$. These two vertices are covered by the edge cover $F^{2}$. Thus, there are up to two edges $g, h \in F^{2} \backslash F^{1}$ that touch $i$ and $j$. Now consider the edge cover $Z=X \cup\{g, h\}$. Observe that $Z \cap E_{R}=\left(F_{1} \cap F_{2} \cap E_{R}\right)$. Hence, $c(Z) \leq c\left(F_{2}\right)$ and condition (i) holds. Furthermore, $X \subseteq F^{1}$, and thus $b(Z) \leq b\left(F^{1}\right)+b(g)+b(h) \leq$ $b\left(F^{1}\right)+2 b_{\max }$, implying (iii) and finishing the proof.

Hence, $F^{2}$ corresponds to an interdiction strategy that is optimal up to $2 b_{\text {max }}$. From here, it is not hard to obtain a PTAS. Let $\epsilon>0$. If $2 b_{\max } \leq \epsilon b^{T} y^{*}$, then $F^{2}$ corresponds to an interdiction strategy that is an $(1-\epsilon)$-approximation. Otherwise, we use the following well-known guessing technique (see [27, 14). Consider an optimal integral solution $\bar{y}, \bar{r}, \bar{r}_{I J}$ of (20). The vector $\bar{r}$ of such a solution is the characteristic vector of an optimal interdiction set, and OPT $=b^{T} \bar{y}$ is the optimal value of our interdiction problem. We guess the $\left\lceil\frac{2}{\epsilon}\right\rceil$ heaviest edges $W$ of $\{e \in E \mid \bar{y}(e)=1\}$, i.e., the ones with highest $b$-values. This can be done by going through all subsets of $E$ of size $\left\lceil\frac{2}{\epsilon}\right\rceil$, which is a polynomial number of subsets for a fixed $\epsilon>0$. For each such guess we consider the resulting residual version of problem (20), where we set $y(e)=1$ for each guessed edge and remove all edges of strictly higher $b$-values than the lowest $b$-value of the guessed edges. Hence, we end up with a residual problem where $b_{\text {max }}$ is less than or equal to the $b$-value of any guessed edge. For the right guess $W$, we have $b(W) \leq$ OPT and thus get

$$
b_{\max } \leq \frac{\epsilon}{2} b(W) \leq \frac{\epsilon}{2} \mathrm{OPT},
$$

implying that the set $F^{2}$ for the right guess is indeed a $(1-\epsilon)$-approximation.
Notice that if $b_{\max }$ is sufficiently small with respect to $b^{T} y^{*}$, i.e., $2 b_{\max } \leq \epsilon b^{T} y^{*}$, then the expensive guessing step can be skipped.

### 5.2 Efficient algorithm for stable set interdiction in bipartite graphs

We complete the discussion on bipartite $b$-stable set interdiction by showing that the problem of interdicting stable sets, which are the same as 1 -stable sets, in a bipartite graph can be solved in polynomial time.

We reuse the notation of the previous section. Hence, $G=(V, E)$ is a bipartite graph with bipartition $V=I \cup J, c: E \rightarrow \mathbb{Z}_{>0}$ are the interdiction costs, and $B \in \mathbb{Z}_{>0}$ is the interdiction budget. Furthermore, we denote by $\alpha(G)$ the size of a maximum cardinality stable set in $G$ and by $\nu(G)$ the size of a maximum cardinality matching. It is well-known from König's Theorem that for any bipartite graph $G=(V, E)$,

$$
\alpha(G)=|V|-\nu(G) .
$$

Hence, the objective value of some interdiction set $R \subseteq V$ with $c(R) \leq B$ is equal to

$$
\alpha(G[V \backslash R])=|V|-|R|-\nu(G[V \backslash R]),
$$

where $G[W]$ for any $W \subseteq V$ is the induced subgraph of $G$ over the vertices $W$, i.e., the graph obtained from $G$ by removing $V \backslash W$.

We start by discussing some structural properties that can be assumed to hold for at least one optimal solution. Let $R^{*}$ be an optimal solution to the interdiction problem, and let $M^{*} \subseteq E$ be a maximum cardinality matching in $G[V \backslash R]$. By the above discussion, the value of the interdiction set $R^{*}$ is

$$
\begin{equation*}
\alpha\left(G\left[V \backslash R^{*}\right]\right)=|V|-\left|R^{*}\right|-\left|M^{*}\right| . \tag{21}
\end{equation*}
$$

In the following, we will focus on finding an optimal matching $M^{*}$, and then derive $R^{*}$ from this matching. We start with a lemma that shows how $R^{*}$ can be obtained from $M^{*}$. For this we need some additional notation. We number the vertices $V=\left\{v_{1}, \ldots, v_{n}\right\}$ such that $c\left(v_{1}\right) \leq c\left(v_{2}\right) \leq \cdots \leq c\left(v_{n}\right)$. For $\ell \in\{0, \ldots, n\}$ let $V_{\ell}=\left\{v_{1}, \ldots, v_{\ell}\right\}$ with $V_{0}=\emptyset$. Furthermore, for any subset of edges $U$, we denote by $V(U)$ the set $\cup_{e \in U} e$ of all endpoints of edges in $U$.

Lemma 21. Let $R$ be an optimal interdiction set and let $M^{*}$ be a maximum cardinality matching in the graph $G[V \backslash R]$. Then the set $R^{*} \subseteq V$ defined below is also an optimal solution to the interdiction problem.

$$
R^{*}=V_{\ell} \backslash V\left(M^{*}\right)
$$

where $\ell \in\{0, \ldots, n\}$ is the largest value such that $c\left(V_{\ell} \backslash V\left(M^{*}\right)\right) \leq B$.
Proof. The interdiction set $R^{*}$ is budget feasible by assumption, and $R, R^{*} \subseteq V \backslash V\left(M^{*}\right)$ so $|R| \leq\left|R^{*}\right|$ by the construction of $R^{*}$. Let $M^{\prime}$ be a maximum cardinality matching in the graph $G\left[V \backslash R^{*}\right]$. Since $M^{*}$ is a matching in $G\left[V \backslash R^{*}\right]$ it holds that $\left|M^{\prime}\right| \geq\left|M^{*}\right|$. Thus $R^{*}$ is also an optimal interdiction set because

$$
\alpha\left(G\left[V \backslash R^{*}\right]\right)=|V|-\left|R^{*}\right|-\left|M^{\prime}\right| \leq|V|-|R|-\left|M^{*}\right|=\alpha(G[V \backslash R])
$$

One of our key observations is that we can find an optimal matching $M^{*}$ of Lemma 21 efficiently by matroid intersection techniques if we know the following four quantities that depend on $M^{*}$ :
(i) The maximum value $\ell \in\{0, \ldots, n\}$ such that $R^{*}=V_{\ell} \backslash V\left(M^{*}\right)$ satisfies $c\left(R^{*}\right) \leq B$;
(ii) $\beta_{I}=\left|V_{\ell} \cap V\left(M^{*}\right) \cap I\right|$;
(iii) $\beta_{J}=\left|V_{\ell} \cap V\left(M^{*}\right) \cap J\right|$;
(iv) $\gamma=\left|M^{*}\right|$.

There may be different quadruples $\left(\ell, \beta_{I}, \beta_{J}, \gamma\right)$ that correspond to different optimal matchings $M^{*}$. However, we need any such set of values that corresponds to an optimal $M^{*}$. Before showing how an optimal quadruple ( $\ell, \beta_{I}, \beta_{J}, \gamma$ ) can be used to find $M^{*}$ by matroid intersection, we highlight that there is only a polynomial number of possible quadruples. This follows since $\ell \in\{0, \ldots, n\}$ can only take $n+1$ different values, $\beta_{I}$ and $\beta_{J}$ only take at most $|I|+1$ and $|J|+1$ different values, respectively, and the cardinality of $M^{*}$ is between 0 and $\nu(G) \leq \min \{|I|,|J|\}$. Hence, each possible quadruple $\left(\ell, \beta_{I}, \beta_{J}, \gamma\right)$ is element of the set

$$
\mathcal{Q}=\{0, \ldots, n\} \times\{0, \ldots,|I|+1\} \times\{0, \ldots,|J|+1\} \times\{0, \ldots, \nu(G)\}
$$

We will go through all quadruples in $\mathcal{Q}$ and try to construct a corresponding mathcing $M^{*}$ by the matroid intersection technique that we introduce below. Thus, we will consider at least once an optimal quadruple, for which we will obtain an optimal $M^{*}$, which will then lead to an optimal $R^{*}$ through Lemma 21. Hence, our task reduces to find a matching that "corresponds" to a given quadruple in $\mathcal{Q}$. We define formally what this means in the following.

Definition 22. We say that a matching $M$ in $G$ corresponds to $\left(\ell, \beta_{I}, \beta_{J}, \gamma\right) \in \mathcal{Q}$ if the following conditions are fulfilled:
(i) $c\left(V_{\ell} \backslash V(M)\right) \leq B$,
(ii) $\beta_{I}=\left|V_{\ell} \cap V(M) \cap I\right|$,
(iii) $\beta_{J}=\left|V_{\ell} \cap V(M) \cap J\right|$,
(iv) $\gamma=|M|$.

We call a quadruple in $\mathcal{Q}$ feasible if there exists a matching that corresponds to it. Furthermore, a quadruple is called optimal if there is a matching $M^{*}$ corresponding to it such that $R^{*}=V_{\ell} \backslash V\left(M^{*}\right)$ is an optimal interdiction set.

Notice that our definition of a matching $M$ corresponding to a quadruple $\left(\ell, \beta_{I}, \beta_{J}, \gamma\right) \in$ $\mathcal{Q}$ does not require that $\ell$ is the maximum value such that $c\left(V_{\ell} \backslash M\right) \leq B$ since we obtain the properties we need without requiring this condition in our correspondence, as shown by the next lemma.

Lemma 23. Let $\left(\ell, \beta_{I}, \beta_{J}, \gamma\right) \in \mathcal{Q}$ be a feasible quadruple with $M$ corresponding to it. Then the set $R=V_{\ell} \backslash V(M)$ is an interdiction set of objective value

$$
\begin{equation*}
\alpha(G[V \backslash R]) \leq|V|-|R|-|M|=|V|-\gamma-\ell+\beta_{I}+\beta_{J} \tag{22}
\end{equation*}
$$

Furthermore, if $\left(\ell, \beta_{I}, \beta_{J}, \gamma\right) \in \mathcal{Q}$ is an optimal quadruple, then

$$
\alpha(G[V \backslash R])=|V|-\gamma-\ell+\beta_{I}+\beta_{J} .
$$

Proof. The inequality in (22) follows immediately from (21) since

$$
\begin{aligned}
\alpha(G[V \backslash R]) & =|V|-|R|-\nu(G[V \backslash R]) & & (\text { by (21) }) \\
& \leq|V|-|R|-|M| & & (\text { since } M \text { is a matching in } G[V \backslash R]),
\end{aligned}
$$

with equality if and only if $M$ is a maximum cardinality matching in $G[V \backslash R]$. To obtain the equality in (22), we observe that $|M|=\gamma$. Furthermore,

$$
|R|=\left|V_{\ell} \backslash V(M)\right|=\left|V_{\ell}\right|-\left|V_{\ell} \cap V(M) \cap I\right|-\left|V_{\ell} \cap V(M) \cap J\right|=\ell-\beta_{I}-\beta_{J},
$$

which implies the desired equality.
The main consequence of Lemma 23 is that the value of an optimal solution is determined entirely by its quadruple. Thus one can find an optimal quadruple by testing the feasibility of every quadruple in $\mathcal{Q}$ and choosing one that minimizes the right hand side of (22). We conclude the following.

Corollary 24. Let $\left(\ell, \beta_{I}, \beta_{J}, \gamma\right) \in \mathcal{Q}$ be a feasible quadruple that minimizes $|V|-\gamma-\ell+$ $\beta_{I}+\beta_{J}$, and let $M^{*}$ be a matching that corresponds to it. Then $R^{*}=V_{\ell} \backslash V\left(M^{*}\right)$ is an optimal interdiction set to bipartite stable set interdiction problem.

Hence, all that remains to be done to obtain an efficient algorithm is to design a procedure that, for a quadruple $\left(\ell, \beta_{I}, \beta_{J}, \gamma\right) \in \mathcal{Q}$ decides whether it is feasible, and if so, finds a corresponding matching $M$. Using this procedure we check all quadruples to determine a feasible quadruple $\left(\ell, \beta_{I}, \beta_{J}, \gamma\right)$ that minimizes $|V|-\gamma-\ell-\beta_{I}-\beta_{J}$, and then return $R^{*}=V_{\ell} \backslash V\left(M^{*}\right)$, where $M^{*}$ is a matching corresponding to such a quadruple.

Hence, let $q=\left(\ell, \beta_{I}, \beta_{J}, \gamma\right) \in Q$ and we show how to check feasibility of $q$ and find a corresponding matching $M$ if $q$ is feasible. Let $c^{\prime}: E \rightarrow \mathbb{Z}_{\geq 0}$ be an auxiliary cost function defined by

$$
c^{\prime}\left(v_{k}\right)= \begin{cases}c\left(v_{k}\right) & \text { if } k \leq \ell \\ 0 & \text { if } k>\ell\end{cases}
$$

Based on $c^{\prime}$ we define weights $w: E \rightarrow \mathbb{Z}_{\geq 0}$, where

$$
w(\{i, j\})=c^{\prime}(i)+c^{\prime}(j) \quad \forall\{i, j\} \in E .
$$

Our goal is to determine a maximum weight matching $M$ in $G$ such that $\left|V_{\ell} \cap V(M) \cap I\right|=\beta_{I}$, $\left|V_{\ell} \cap V(M) \cap J\right|=\beta_{J}$, and $\gamma=|M|$. Notice that maximizing $w$ corresponds to maximizing $c\left(V(M) \cap V_{\ell}\right)$. Hence, a maximizer $M$ will be a matching in $G$ that satisfies conditions (iii)-(iv) of Definition 22 and subject to fulfilling these three conditions, it maximizes $c\left(V(M) \cap V_{\ell}\right)$, which is the same as minimizing $c\left(V_{\ell} \backslash V(M)\right)$. Hence, if $c\left(V_{\ell} \backslash V(M)\right) \leq B$, then the quadruple ( $\ell, \beta_{I}, \beta_{J}, \gamma$ ) is feasible and $M$ corresponds to it, otherwise, the quadruple is not feasible. It remains to show how to find efficiently a maximum weight matching $M$ in $G$ such that $\left|V_{\ell} \cap V(M) \cap I\right|=\beta_{I},\left|V_{\ell} \cap V(M) \cap J\right|=\beta_{J}$, and $\gamma=|M|$.

This optimization problem corresponds to maximizing $w$ over a face of a matroid intersection polytope. Indeed, define one laminar matroid $M_{1}=\left(E, \mathcal{F}_{1}\right)$ such that a set $U \subseteq E$ is independent in $M_{1}$, i.e., $U \in \mathcal{F}_{1}$, if $U$ contains at most one edge incident with $i \in I$ for each $v \in I$, at most $\beta_{I}$ edges incident with vertices in $V_{\ell} \cap I$ and at most $\gamma$ edges in total. Similarly, define $M_{2}=\left(E, \mathcal{F}_{2}\right)$ such that $U \in \mathcal{I}_{2}$ if $U$ contains am most one edge incident with any vertex $j \in J$, at most $\beta_{J}$ edges incident with $V_{\ell} \cap J$ and at most $\gamma$ edges in total. The problem we want to solve is to find a set $M \in \mathcal{F}_{1} \cap \mathcal{F}_{2}$ such that the constraints $\left|M \cap V_{\ell} \cap I\right| \leq \beta_{I},\left|M \cap V_{\ell} \cap J\right| \leq \beta_{J}$ and $|M| \leq \gamma$ are fulfilled with equality. Hence, this is indeed the problem of maximizing $w$ over a particular face of the matroid intersection polytope corresponding to $M_{1}$ and $M_{2}$. This problem can be solved in strongly polynomial time by matroid intersection algorithms. Alternatively, one can also find a vertex solution to the following polynolmial-sized LP, which describes this face of the matroid intersection polytope, and is therefore integral when feasible.

$$
\max \begin{aligned}
w^{T} x & \\
x(\delta(v)) & \leq 1 \quad \forall v \in V \\
x\left(\delta\left(V_{\ell} \cap I\right)\right) & =\beta_{I} \\
x\left(\delta\left(V_{\ell} \cap J\right)\right) & =\beta_{J} \\
x(E) & =\gamma
\end{aligned}
$$

For more details on optimization over the matroid intersection polytope, we refer the interested reader to [30, Chapter 41].

## 6 Conclusions

We present a framework to obtain 2-pseudoapproximations for a wide set of combinatorial interdiction problems, including maximum cardinality independent set in a matroid or the intersection of two matroids, maximum $s$ - $t$ flows, and packing problems defined by a constraint matrix that is TU. Our approach is inspired by a technique of Burch et al. [8, who presented a 2 -pseudoapproximation for maximum $s$ - $t$ flows. Furthermore, we show that our framework can also be adapted to more general settings involving matroid optimization. More precisely, we also get a 2-pseudpapproximation for interdicting the maximum weight independent set problem in a matroid with submodular interdiction costs. Submodularity is a natural property for interdiction costs since it models economies of scale. Our framework for 2-pseudoapproximations is polyhedral and sometimes we can exploit polyhedral properties of well-structured interdiction problems to obtain stronger results. We demonstrate this on the problem of interdicting $b$-stable sets in bipartite graphs. For this setting we obtain a

PTAS, by employing ideas from multi-budgeted optimization. Furthermore, we show that the special case of stable set interdiction in bipartite graphs can be solved efficiently by matroid intersection techniques.

Many interesting open questions remain in the field of interdicting combinatorial optimization problems. It particular, it remains open whether stronger pseudoapproximations can be obtained for the considered problems. Also in terms of "true" approximation algorithms, large gaps remain.

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[^1]:    ${ }^{1}$ The discussion that follows also works for feasible sets $\mathcal{X}$ such that $\operatorname{conv}(\mathcal{X})$ is not integral. However, integrality of $\mathcal{X}$ simplifies parts of our discussion and is used to show that our 2-pseudoapproximation is efficient. Furthermore, all problems we consider naturally have the property that conv $(\mathcal{X})$ is integral.

[^2]:    ${ }^{2}$ In some cases one can get around this problem by using an extended formulation. This is a lifting of a polytope in a higher dimension with the goal to obtain a lifted polytope with an inequality description of only polynomial size (see [19, 10]).

[^3]:    ${ }^{3}$ Notice that the integrality requirement for $w$ is not restrictive. Any $w \in \mathbb{Q}_{\geq 0}^{N}$ can be scaled up to an integral weight vector without changing the problem.

