Constant Approximation Algorithm for Non-Uniform Capacitated Multi-Item Lot-Sizing via Strong Covering Inequalities

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Abstract

We study the non-uniform capacitated multi-item lot-sizing (CMILS) problem. In this problem, there is a set of demands over a planning horizon of T time periods and all demands must be satisfied on time. We can place an order at the beginning of each period s, incurring an ordering cost K_s . The total quantity of all products ordered at time s can not exceed a given capacity C_s . On the other hand, carrying inventory from time to time incurs inventory holding cost. The goal of the problem is to find a feasible solution that minimizes the sum of ordering and holding costs.

Levi et al. (Levi, Lodi and Sviridenko, Mathmatics of Operations Research 33(2), 2008) gave a 2-approximation for the problem when the capacities C_s are the same. In this paper, we extend their result to the case of non-uniform capacities. That is, we give a constant approximation algorithm for the capacitated multi-item lot-sizing problem with general capacities.

The constant approximation is achieved by adding an exponentially large set of new covering inequalities to the natural facility-location type linear programming relaxation for the problem. Along the way of our algorithm, we reduce the CMILS problem to two generalizations of the classic knapsack covering problem. We give LP-based constant approximation algorithms for both generalizations, via the iterative rounding technique.

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1 Introduction

Since the seminal papers of Wagner and Whitin [15] and Manne [11] in late 1950's, lot-sizing problems has become one of the most important classes of problems in inventory management and production planning ([14]). Given a sequence of time-varying demands for different products over a time horizon, a lot-sizing problem asks for the time periods for which productions and orders take place and the quantities of products to be produced and ordered, so as to minimize the total production, ordering and inventory holding cost.

In many practical settings, due to the shortage of resources such as manpower, equipments and budget, there are (possibly time-dependent) capacity constraints on the total units of products that can be produced or ordered at a time. Thus, when modeling the lot-sizing problems, it is important to take these capacity constraints into account. Often, the capacity constraints make the problems computationally harder. There has been an immense amount of work on capacitated lot-sizing problems, from the perspective of integer programming, heuristics, as well as tractability of special cases.

In this paper, we study the single-level capacitated multi-item lot-sizing (CMILS) problem, where good decisions have to be made to balance two costs: ordering cost and inventory holding cost. Placing an order at some time s incurs a time-dependent ordering cost K_s , making it too costly to place an order at every time period. On the other hand, placing a few orders will result in holding products in inventory to satisfy future demands, incurring high holding cost. When an order is placed at time s, there is a capacity C_s on the total amount of products that can be ordered.

We study the CMILS problem from the perspective of approximation algorithms. When the capacities are uniform, i.e, the capacities C_s for different time periods s are the same, Levi et al. [8] gave a 2-approximation algorithm for the problem. They used the flow covering inequalities that were introduced in [12], to reduce the unbounded integrality gap of the natural facility-location type LP relaxation for the CMILS problem to 2. However, the flow-covering inequalities heavily used the uniform-capacity property; it seems hard to extend them to the problem with non-uniform capacities.

In this paper, we give a 10-approximation algorithm for non-uniform capacitated multi-item lot-sizing problem. Inspired by the "effective capacity" idea of the knapsack covering inequalities introduced by Carr et al. [4], we introduce a set of covering inequalities that strengthen the natural facility-location type LP relaxation. We believe our covering inequalities can be applied to many other problems with non-uniform capacities. In the cutting-plane method for solving the integer programmings for capacitated problems, our covering inequalities may be used to generate an initial solution that is provably good. Along the way of our algorithm, we reduce the CMILS problem to two generalizations of the classic knapsack covering problem, namely, the interval and laminar knapsack covering problems. We give an iterative rounding algorithm for the laminar knapsack covering problem. The two generalizations and the use of iterative rounding may be of independent interest.

Problem Definition In the capacitated multi-item lot-sizing (CMILS) problem, there is a finite time horizon of T discrete time periods indexed by [T], and a set of N items indexed by [N]. For each item $i \in [N]$, we have a demand that $d_i > 0$ units of item i must be ordered by time $r_i \in [T]$. For each $s \in [T]$, we can place an order at the beginning of period s, incurring an ordering cost $K_s > 0$. If an order is placed at time $s \in [T]$, we can order any subset of items. However, the quantity of total units ordered for all items can not exceed a given capacity $C_s > 0$. Carrying inventory over periods incurs holding costs. For every $i \in [N]$, $s \in [r_i]$, we use $h_i(s)$ to denote the per-unit cost of holding one unit of item i from period s to period s. We assume that s is non-increasing. Also, a unit of item s ordered at the beginning of period s can be used to satisfy the demand for item s immediately; thus we assume s immediately; thus we assume s immediately; thus we assume holding costs.

The mixed integer programming for the problem is given in MIP_{CMILS}, which is a facility-location type programming. For every $i \in [N]$ and $s \in [r_i]$, $x_{s,i} \in [0,1]$ specifies the fraction of demand i that is ordered at time s, and for every $s \in [T]$, $y_s \in \{0,1\}$ indicates whether we place an order at time s or not. The goal of the MIP is to minimize the sum of the ordering cost $\sum_{s \in [T]} y_s K_s$ and the holding cost $\sum_{i \in [N]} d_i \sum_{s \in [r_i]} x_{s,i} h_i(s)$. Constraint (1) requires that the demand for every item i is fully satisfied, Constraint (2) restricts that we can order an item at time s only if we placed an order at time s, and Constraint (3) requires that the total amount of demand ordered at time s does not exceed s. Notice

that in a feasible solution (x^*, y^*) to the CMILS problem, y^* has to be integral, but x^* can be fractional.

$$\min \sum_{s \in [T]} y_s K_s + \sum_{i \in [N]} d_i \sum_{s \in [r_i]} x_{s,i} h_i(s) \quad \text{s.t.} \qquad (\text{MIP}_{\text{CMILS}})$$

$$\sum_{s \in [r_i]} x_{s,i} = 1, \qquad \forall i \in [N]; \qquad (1) \sum_{i \in [N]: r_i \geq s} x_{s,i} d_i \leq y_s C_s, \qquad \forall s \in [T]; \qquad (3)$$

$$x_{s,i} \leq y_s, \qquad \forall i \in [N], s \in [r_i]; \quad (2) \qquad \qquad x_{s,i} \in [0,1], \qquad \forall i \in [N], s \in [r_i]; \quad (4)$$

$$y_s \in \{0,1\}, \qquad \forall s \in [T]. \qquad (5)$$

In the traditional setting for multi-item lot-sizing, there is a demand of value $d_{s,i}$ for each item i at each time period s, and the holding cost is linear for each item i: there will be a per-unit host cost $h_i'(s)$ for carrying one unit of item i from period s to s+1. The way we defined our holding cost functions allows us to assume that there is only one demand for each item: if there are multiple demands for an item at different periods, we can think of that the demands are for different items, by setting the holding cost functions for these items correctly. This assumption simplifies our notation: instead of using an (s,i)-pair to denote a demand, we can use a single index i. Thus, we do not distinguish between demands and items: an element $i \in [N]$ is both a demand and an item. In our setting, the traditional single-item lot-sizing problem corresponds to the case where there is a function $h': [T-1] \to \mathbb{R}_{\geq 0}$ such that $h_i(s) = \sum_{t=s}^{r_i-1} h'(t)$ for every $i \in [N]$ and $s \in [r_i]$.

Know Results For the single-item lot-sizing problem, the seminal paper of Wagner and Whitin [15] gave an efficient dynamic programming algorithm for the uncapacitated version. Later, efficient DP algorithms were also found for the single-item lot-sizing problem with uniform hard capacities ([6]) and uniform soft capacities ([13]). When the capacities are non-uniform, the problem becomes weakly NP-hard ([6]), but admits an FPTAS ([7]). Thus, we have a complete understanding of the status of the single-item lot-sizing problem.

For the multi-item lot-sizing problem, the dynamic programming in [15] carries over if the problem is uncapacitated. Levi et al. [8] showed that the uniform capacitated multi-item lot-sizing problem is already strongly NP-hard, and gave a 2-approximation algorithm for the problem. Special cases of the CMILS problem have been studied in the literature. Anily and Tzur [2] gave a dynamic programming algorithm for the special case where the capacities and ordering costs are both uniform, and the number of items is a constant. (In our model, this means that there is a family of O(1) functions $h': [T-1] \to R_{\geq 0}$, such that for every $i \in [N]$, there is a function h' from the family such that $h_i(s) = \sum_{t=s}^{r_i-1} h'(t)$ for every $s \in [r_i]$.) Anily et al. [3] considered the CMILS problem with uniform capacities and the monotonicity assumption for the holding cost functions. This assumption says that there is an ordering of items according to their importance. For every time period s, the cost of holding an unit of item i from period s to s+1, is higher than (or equal to) the cost of holding one unit of a less important item from period s to s+1. [3] gave a small size linear programming formulation to solve this special case exactly. Later, Even et al. [5] gave a dynamic programming for the problem when demands are polynomially bounded and the holding cost functions satisfy the monotonicity assumption. The assumption of polynomially bounded demands is necessary even with the monotonicity assumption, since otherwise the problem is weakly NP-hard ([6]).

1.1 Our Reults and Techniques

Our main result is a constant approximation algorithm for the capacitated multi-item lot-sizing problem.

Theorem 1.1. There is a 10-approximation algorithm for the multi-item lot-sizing problem with non-uniform capacities.

We give an overview of our techniques used for proving Theorem 1.1. We start with the natural facility-location type linear programming relaxation for the CMILS problem, which is obtained from MIP_{CMILS} by relaxing Constraint (5) to $y_s \in [0,1], \forall s \in [T]$. Since Constraint (1) requires $\sum_{s \in [r_i]} x_{s,i} = 1$

for every $i \in [N]$, $\{x_{s,i}\}_{s \in [r_i]}$ form a distribution over $[r_i]$. We define the tail of the distribution to be the set of some latest periods in $[r_i]$ that contributes a constant mass to the distribution.

For simplicity, let us first assume that the capacities for the orders are soft capacities. That is, we are allowed to place multiple orders at each period s in our final solution, by paying a cost K_s for each order. In this case, the integrality gap of the natural LP relaxation is O(1). In our rounding algorithm, we scale up the y variables by a large constant. With the scaling, we require that each item $i \in [N]$ is only ordered in the tail of the distribution for i. In this way, the holding cost can be bounded by a constant times the holding cost of the LP solution.

With these requirements, the remaining problem becomes an "interval knapsack covering" (Interval-KC) problem, where we view each period $s \in [T]$ as a knapsack of capacity C_s and cost K_s . If we place an order at period s, then we select the knapsack s. For every $a,b \in \mathbb{Z}$ such that $0 \le a < b < T$, we require that the total capacity of selected knapsacks in (a,b] is at least $R_{a,b}$. The goal of the Interval-KC problem is to select a set of knapsacks of the minimum cost that satisfies all the requirements. We give a constant approximation algorithm for Interval-KC based on iterative rounding. Specifically, we first reduce the Interval-KC instance to an instance of a more restricted problem, called "laminar knapsack covering" (Laminar-KC) problem, in which the intervals (a,b] with positive $R_{a,b}$ form a laminar family. The laminar structure allows us to apply the iterative rounding technique to solve the problem: we maintain an LP and in each iteration the LP is solved to obtain a vertex-point solution; as the algorithm proceeds, more and more variables will become integral and finally the algorithm will terminate. Overall, the LP solution to the CMILS instance leads to a good LP solution to the Interval-KC instance, which in turn leads to a good LP solution to the Laminar-KC instance. Therefore, the final ordering cost can be bounded.

However, when the capacities are hard capacities, the integrality gap of the natural LP becomes unbounded. The issue with the above algorithm is that some y_s might have value more than a constant, and scaling it up will make it more than 1. So our final solution needs to include many orders at period s. Indeed, the same issue occurs for many other problems such as knapsack covering and capacitated facility location, for which the linear programming has O(1) integrality gap for the soft capacitated version but unbounded gap for the hard capacitated version. For both problems, stronger LP relaxations are known to overcome the gap instances ([1, 4]). Our idea behind the strengthened LP is similar to those in [4] and [1]. If for some $s \in [T]$, y_s is larger than a constant, then we can afford to place an order at time s. So, we break [T] into two sets: S^+ contains the periods in which we already placed an order, and $[T] \setminus S^+$ contains the periods s with small y_s value. We consider the residual instance obtained from the input CMILS instance by pre-selecting the orders in S^+ . In this residual instance, all periods s other than the ones with pre-selected orders have small y_s values; thus we hope to run the algorithm for the soft-capacitated multi-item lot-sizing problem. The challenge is that we can not remove S^+ from the residual instance since we do not know what items are assigned to the orders in S^+ . To overcome this issue, we introduce a set of strong covering inequalities. These inequalities use the "effective capacity" idea from the knapsack covering inequalities introduced in [4]: if we want to use some orders to satisfy d units of demand, then the "effective capacity" of an order at time s is min $\{C_s, d\}$, instead of C_s .

2 Preliminaries

Notations Throughout this paper, C and K will always be vectors in $\mathbb{R}^T_{\geq 0}$. For any set $S \subseteq [T]$, we define $C(S) := \sum_{s \in S} C_s$ and $K(S) := \sum_{s \in S} K_s$. For every set $I \subseteq [N]$ of items, we use $d(I) := \sum_{i \in I} d_i$ to denote the total demand for items in I. Since we shall use intervals of integers frequently, we use (a,b),(a,b],[a,b),[a,b] to denote sets of integers in these intervals; the only exception is that [0,1] will still denote the set of real numbers between 0 and 1 (including 0 and 1). In particular, an interval over [T] is some (a,b], for some integers a,b such that $0 \leq a < b \leq T$.

Knapsack Covering Inequalities In the classic knapsack covering (KC) problem, we are given a set [T] of knapsacks, where each knapsack $s \in [T]$ has a capacity $C_s > 0$ and a cost $K_s \ge 0$. The goal of the problem is to select a subset $S^* \subseteq [T]$ with $C(S^*) \ge R$, so as to minimize $K(S^*)$. That is, we want to select a set of knapsacks with total capacity at least R so as to minimize the total cost of selected knapsacks. In the CMILS problem, if we only have one item 1 with demand $d_1 = R$ and $r_1 = T$ and the holding cost is identically 0, then the problem is reduced to the KC problem.

It is well-known that the KC problem is weakly NP-hard and admits an FPTAS based on dynamic programming. However, for many problems that involve capacity covering constraints, it is hard to incorporate the dynamic programming technique. This motivates the study of LP relaxations for KC. The naive LP relaxation, which is $\min \sum_{s \in [T]} y_s K_s$ subject to $\sum_{s \in [T]} y_s C_s \ge R$ and $y \in [0,1]^T$, has unbounded integrality gap, even if we assume that $C_s \le R$ for every $s \in [T]$. Consider the instance with $T=2, C_1=R-1, C_2=R, K_1=0$ and $K_2=1$, where R is a very large number. The LP solution $(y_1,y_2)=(1,1/R)$ has cost 1/R; however, the optimum solution to the problem has cost 1.

To overcome the above gap instance, Carr et al. [4] introduced a set of valid inequalities, that are satisfied by all integral solutions to the given KC instance. The inequalities are called knapsack covering (KC) inequalities, which are defined as follows:

$$\sum_{s \in [T] \setminus S} \min \left\{ C_s, R - C(S) \right\} y_s \geq R - C(S), \quad \forall S \subseteq [T] \text{ s.t. } C(S) < R. \text{ (Knapsack Covering Inequality)}$$

For each $S \subseteq [T]$ with C(S) < R, the above inequality requires the selected knapsacks in $[T] \setminus S$ to have a total capacity at least R - C(S). In this residual problem, if the capacity of a knapsack $s \in [T] \setminus S$ is more than R - C(S), its "effective capacity" is only R - C(S). So the KC inequalities are valid. For the above gap instance, the KC inequality for $S = \{1\}$ requires min $\{C_2, R - (R - 1)\}$ $y_2 \ge R - (R - 1)$, i.e, $y_2 \ge 1$. Thus the KC inequalities can handle the gap instance. Indeed, it is shown in [4] that the LP with all the KC inequalities has integrality gap 2.

Rounding Procedure That Detects Violated Inequalities Although the KC inequalities can strengthen the LP for KC, the LP with all these inequalities can not be solved efficiently. This issue can be circumvented by using a rounding algorithm that detects violated inequalities. Given a vector $y \in [0,1]^T$, there is a rounding algorithm that either returns a feasible solution to the given KC instance, whose cost is at most $2\sum_{s\in[T]}y_sK_s$, or returns a knapsack covering inequality that is violated by y. This rounding algorithm can be used as a separation oracle for the Ellipsoid method. We keep on running the Ellipsoid method, as long as the rounding algorithm is returning a violated constraint. When the algorithm fails to give a violated constraint, it returns a feasible solution whose cost is guaranteed to be at most twice the cost of the optimum LP solution. This technique has been used in many previous results, e.g. [1, 4, 8, 9, 10]. In particular, [8] used this technique to obtain their 2-approximation for the uniform-capacitated multi-item lot-sizing problem. In this paper, we also apply the technique to obtain our 10-approximation for the problem with non-uniform capacities.

Laminar and Interval Knapsack Covering Along the way of our algorithm, we introduce two generalizations of the knapsack covering problem. In the laminar knapsack covering (Laminar-KC) problem, we are given a set [T] of knapsacks, each $s \in [T]$ with a capacity $C_s > 0$ and a cost $K_s \ge 0$. We are also given a laminar family \mathcal{S} of intervals of [T]. That is, for every two distinct intervals $(a,b], (a',b'] \in \mathcal{S}$, we have either $(a,b] \cap (a',b'] = \emptyset$, or $(a,b] \subsetneq (a',b']$, or $(a',b'] \subsetneq (a,b]$. For each $(a,b] \in \mathcal{S}$, we are given a requirement $R_{a,b} > 0$. The goal of the problem is to find a set $S^* \subseteq [T]$ such that $C(S^* \cap (a,b]) \ge R_{a,b}$ for every $(a,b] \in \mathcal{S}$, so as to minimize $K(S^*)$. W.l.o.g we assume that $[T] \in \mathcal{S}$. We use (T,C,K,\mathcal{S},R) to denote a Laminar-KC instance.

The interval knapsack covering (Interval-KC) problem is a more general problem, in which we are still given a set [T] of knapsacks, each $s \in [T]$ with a capacity $C_s > 0$ and a cost $K_s \ge 0$. However now we have a requirement $R_{a,b} \ge 0$ for every interval (a,b] over T. The goal of the problem is to select a set S^* of knapsacks with the minimum cost, such that $C(S^* \cap (a,b]) \ge R_{a,b}$, for every interval (a,b] over [T]. We use (T,C,K,R) to denote an Interval-KC instance.

We remark that the dynamic programming for KC can be easily extended to give an FPTAS for the Laminar-KC problem. However, we do not know how it can be applied to the more general Interval-KC problem, as well as the CMILS problem. Our algorithm for CMILS heavily uses the LP technique. We reduce the CMILS problem to the Interval-KC problem, which will be further reduced to the Laminar-KC problem. Both reductions require the use of linear programming. After the reductions, we obtain a fractional solution to some LP relaxation for the Laminar-KC and round it to an integral solution. We now state the two main theorems for the two problems, that will be used in our algorithm for CMILS.

Theorem 2.1 (Main Theorem for Laminar-KC). Let (T, C, K, S, R) be a Laminar-KC instance and $y \in [0,1]^T$. Let $S^+ = \{s \in [T] : y_s = 1\}$ and $R_{a,b} := \max\{R_{a,b} - C(S^+ \cap (a,b]), 0\}$ for every $(a,b] \in \mathcal{S}$. Suppose for every $(a,b] \in \mathcal{S}$ with $\widetilde{R}_{a,b} > 0$, we have

either
$$\sum_{s \in (a,b] \setminus S^{+}} \min \left\{ C_{s}, \widetilde{R}_{a,b} \right\} y_{s} \geq 2\widetilde{R}_{a,b},$$
or
$$\sum_{s \in (a,b] \setminus S^{+}: C_{s} \geq \widetilde{R}_{a,b}} y_{s} \geq 1.$$

$$(7)$$

$$\sum_{s \in (a,b] \setminus S^+: C_s \ge \widetilde{R}_{a,b}} y_s \ge 1. \tag{7}$$

Then, we can efficiently find a feasible solution $S^* \supseteq S^+$ to the instance such that $K(S^*) \leq \sum_{s \in [T]} y_s K_s$.

Theorem 2.2 (Main Theorem for Interval-KC). Let (T, C, K, R) be an Interval-KC instance and $y \in [0,1]^T$. Let $S^+ = \{s \in [T] : y_s = 1\}$ and $\tilde{R}_{a,b} := \max\{R_{a,b} - C(S^+ \cap (a,b]), 0\}$ for every interval (a,b] over [T]. Suppose for every interval (a,b] with $R_{a,b} > 0$, we have

either
$$\sum_{s \in (a,b] \setminus S^{+}} \min \left\{ C_{s}, \widetilde{R}_{a,b} \right\} y_{s} \ge 10 \widetilde{R}_{a,b},$$
or
$$\sum_{s \in (a,b] \setminus S^{+}: C_{s} \ge \widetilde{R}_{a,b}} y_{s} \ge 6.$$
(9)

$$\sum_{s \in (a,b] \setminus S^+: C_s \ge \widetilde{R}_{a,b}} y_s \ge 6. \tag{9}$$

Then, we can efficiently find a feasible solution $S^* \supseteq S^+$ to the instance such that $K(S^*) \le \sum_{s \in [T]} y_s K_s$.

In both theorems, we are given a fractional vector $y \in [0,1]^T$ to the given instance. S^+ is the set of knapsacks with y values being 1 and thus can be included in our final solution S^* . $\widetilde{R}_{a,b}$ can be viewed as the residual requirement for the interval (a, b] after we selected knapsacks in S^+ . Notice that Inequality (6) (resp. Inequality (8)) is the knapsack covering inequality with ground set (a, b], $S_1 = (a, b] \cap S^+$, and the right side multiplied by 2 (resp. 10).

With Theorem 2.2, we can give an LP-based 10-approximation for the Interval-KC problem. The rounding algorithm takes a vector $y \in [0,1]^T$ as input. We let $\hat{y}_s = \min\{10y_s,1\}$ for every $s \in [T]$, $S^{+} = \{s \in [T] : \widehat{y}_{s} = 1\}$ and $\widetilde{R}_{a,b} = \max\{R_{a,b} - C((a,b] \cap S^{+}), 0\}$ for every interval (a,b]. Focus on every interval (a, b] with $\widetilde{R}_{a,b} > 0$. If the KC inequality is satisfied for the ground set (a, b] and $S_1 =$ $S^+ \cap (a,b]$, then Inequality (8) with y replaced by \widehat{y} holds for this (a,b]; otherwise, we return this KC inequality. Then, we can apply Theorem 2.2 to obtain a feasible solution S^* whose cost is at most $\sum_{s \in [T]} \widehat{y}_s K_s \le 10 \sum_{s \in [T]} y_s K_s$; this gives us a 10-approximation algorithm for the Interval-KC problem.

Corollary 2.3. There is a 10-approximation algorithm for the Interval-KC problem.

Organization The remaining part of the paper is organized as follows. In Section 3, we give our algorithm for the CMILS problem, using Theorem 2.2 as a black box. In Section 4, we describe our iterative rounding algorithm for Laminar-KC, which proves Theorem 2.1. In Appendix A, we use Theorem 2.1 to prove Theorem 2.2, by reducing the given Interval-KC instance to a Laminar-KC instance.

3 Approximation Algorithm for Capacitated Multi-Item Lot-Sizing

In this section, we describe our 10-approximation algorithm for the CMILS problem, using Theorem 2.2 as a black box. We first present our LP relaxation with strong covering inequalities. Then we define an Interval-KC instance, where knapsacks correspond to orders in the CMILS problem. Any feasible solution to Interval-KC instance gives a set of orders, for which there is a way to satisfy the demands with small holding cost. Finally, we obtain a small-cost solution to the Interval-KC instance, using Theorem 2.2; this leads a solution to the CMILS instance, with small holding and ordering costs.

The LP obtained from MIP_{CMILS} by replacing Constraint (5) with $y_s \in [0,1], \forall s \in [T]$ has unbounded integrality gap, which can be derived from the gap instance for KC. To overcome this integrality gap,

$$\min \quad \sum_{s \in [T]} y_s K_s + \sum_{i \in [N]} d_i \sum_{s \in [r_i]} x_{s,i} h_i(s) \qquad \text{s.t.} \tag{\mathsf{LP}}_{\mathsf{new}})$$

$$\sum_{s \in [r_i]} x_{s,i} = 1, \forall i \in [N]; \quad (10) \qquad x_{s,i} \in [0,1], \forall i \in [N], s \in [r_i]; \quad (11) \qquad y_s \in [0,1], \forall s \in [T]; \quad (12)$$

$$C(S_1) + \sum_{s \in S_2} \min \{C_s, d(I) - C(S_1)\} y_s + \sum_{i \in I} x_{[T] \setminus (S_1 \cup S_2), i} d_i \ge d(I),$$

$$\forall S_1, S_2 \subseteq [T], I \subseteq [N] \text{ s.t. } S_1 \cap S_2 = \emptyset, C(S_1) < d(I).$$
(13)

we introduce a set of new inequalities. The inequalities are described in Constraint (13), where for convenience, we define $x_{s,i} = 0$ if $s > r_i$ and let $x_{S,i} = \sum_{s \in S} x_{s,i}$ for every $S \subseteq [T]$ and $i \in [N]$.

convenience, we define $x_{s,i}=0$ if $s>r_i$ and let $x_{S,i}=\sum_{s\in S}x_{s,i}$ for every $S\subseteq [T]$ and $i\in [N]$. We now show the validity of Constraint (13); focus on some sets $S_1,S_2\subseteq [T]$ and $I\subseteq [N]$ such that $S_1\cap S_2=\emptyset$ and $C(S_1)< d(I)$. We break [T] into three sets: S_1,S_2 and $[T]\setminus (S_1\cup S_2)$, and consider the quantities for the units of demands in I that are satisfied by each of the 3 sets. Orders in S_1 satisfies at most $C(S_1)$ units of demand, orders in S_2 satisfies at most $\sum_{s\in S_2}C_sy_s$ units, and orders in $[T]\setminus (S_1\cup S_2)$ satisfy exactly $\sum_{i\in I}x_{[T]\setminus (S_1\cup S_2),i}d_i$ units. Then, Constraint (13) is valid if we replace $\min\{C_s,d(I)-C(S_1)\}$ by C_s . In an integral solution we have $y_s\in\{0,1\}$ for every $s\in S_2$. If some $s\in S_2$ has $y_s=1$ and $C_s>d(I)-C(S_1)$, then Constraint (13) already holds. Thus, even if we replace C_s with $\min\{C_s,d(I)-C(S_1)\}$, the constraint still holds. In other words, we use orders in S_2 to cover at most $d(I)-C(S_1)$ units of demands; an order at time $s\in S_2$ has "effective capacity" $\min\{C_s,d(I)-C(S_1)\}$. Indeed, it is the threshold that gives the power of Constraint (13); without it, Constraint (13) is implied by the constraints in the natural LP relaxation, and thus does not strengthen the LP.

If we let $S_1 = \emptyset$, $S_2 = \{s\}$ and $I = \{i\}$ for some $s \in [T], i \in [N]$, then Constraint (13) becomes $\min \{C_s, d_i\} y_s + x_{[T] \setminus \{s\}, i} d_i \geq d_i$. Since $x_{[T], i} = 1$ by Constraint (10), the inequality implies $\min \{C_s, d_i\} y_s \geq x_{s,i} d_i$. Thus, $x_{s,i} \leq \frac{\min\{C_s, d_i\}}{d_i} y_s \leq y_s$, implying Constraint (2). If we let $S = \emptyset$, $S_2 = \{s\}$ and I = [N] for some $s \in [T]$, then Constraint (13) becomes $\min \{C_s, d([N])\} y_s + \sum_{i \in [N]} x_{[T] \setminus \{s\}, i} d_i \geq d([N])$. Since $x_{[T], i} = 1$ for every i by Constraint (10), the inequality implies $\min \{C_s, d([N])\} y_s \geq \sum_{i \in [N]} x_{s,i} d_i$, which implies Constraint (3). Thus, in our strengthened LP, we do not need Constraints (2) and (3). Our final strengthened LP is $\mathsf{LP}_{\mathsf{new}}$.

Rounding a Fractional Solution to $\mathsf{LP}_{\mathsf{new}}$ Let (x,y) be a fixed fractional solution satisfying Constraints (10) to (12). We give a rounding algorithm that returns either an inequality of form (13) that is violated by y, or a feasible solution (x^*,y^*) to the CMILS instance whose cost is at most $10\left(\sum_{s\in[T]}y_sK_s+\sum_{i\in[N]}d_i\sum_{s\in[r_i]}x_{s,i}h_i(s)\right)$.

We define $\mathsf{hcost}(x') := \sum_{i \in [N]} d_i \sum_{s \in [r_i]} x_{s,i}' h_i(s)$, for every $x' \in [0,1]^{[T] \times [N]}$ to be the holding cost of any fractional solution (x',y'). We now define the requirement vector R for our Interval-KC instance. If all the requirements are satisfied, then the demands can be satisfied at a small holding cost. Indeed, the approximation ratio for the holding cost is only $\frac{5}{2}$, better than the ratio 10 for the ordering cost. For every interval (a,b] over [T], the requirement $R_{a,b}$ is defined as:

$$R_{a,b} := \sum_{i \in [N]: r_i \in (a,b]} \max \left\{ 1 - \frac{5}{2} x_{[a],i}, 0 \right\} d_i.$$
 (14)

Lemma 3.1. Let $y^* \in \{0,1\}^T$ and $S^* = \{s \in [T] : y_s^* = 1\}$. If for every interval (a,b] over [T], we have $C(S^* \cap (a,b]) \ge R_{a,b}$, then there is an x^* such that (x^*,y^*) is a feasible solution to the CMILS instance and $\mathsf{hcost}(x^*) \le \frac{5}{2}\mathsf{hcost}(x)$.

The lemma says that if all the requirements are satisfied, then we have an x^* with small $\mathsf{hcost}(x^*)$. In the proof, we reduce the problem of finding a good x^* to the problem of finding a perfect matching in a fractional b-matching instance. We show that the perfect matching exists if all the requirements are satisfied. We defer the proof of the lemma to Appendix B.

Now, our goal becomes to find a set $S^* \subseteq [T]$ satisfying all the requirements. This is exactly an Interval-KC instance, and we shall apply Theorem 2.2 to solve the instance. To guarantee the conditions

of Theorem 2.2, it suffices to guarantee that a small number of inequalities in Constraint (13) are satisfied for (x, y).

Lemma 3.2. Let (a,b] be an interval with $R_{a,b} > 0$, and $I = \{i \in [N] : r_i \in (a,b], x_{[a],i} < \frac{2}{5}\}$. Let (S_1, S_2) be an arbitrary partition of (a,b] into two parts such that $C(S_1) < R_{a,b}$. If Constraint (13) is satisfied for S_1, S_2 and I, then we have

either $\sum_{s \in S_2} \min \{C_s, R_{a,b} - C(S_1)\} y_s \ge R_{a,b} - C(S_1), \tag{15}$

or $\sum_{s \in S_2: C_s \ge R_{a,b} - C(S_1)} y_s \ge \frac{3}{5}. \tag{16}$

Proof. Notice that $R_{a,b} = \sum_{i \in [N]: r_i \in (a,b]} \max \left\{ 1 - \frac{5}{2} x_{[a],i}, 0 \right\} d_i = \sum_{i \in I} \left(1 - \frac{5}{2} x_{[a],i} \right) d_i$. Since $x_{[r_i],i} = 1$ and $x_{s,i} = 0$ for every $i \in I$, $s > r_i$, we have $x_{(a,b],i} = 1 - x_{[a],i}$ for every $i \in I$. Constraint (13) for S_1, S_2 and I implies

$$\sum_{s \in S_2} \min \{C_s, d(I) - C(S_1)\} y_s \ge \sum_{i \in I} x_{S_1 \cup S_2, i} d_i - C(S_1) = \sum_{i \in I} (1 - x_{[a], i}) d_i - C(S_1).$$
 (17)

If Inequality (16) holds then we are done. Thus, we assume Inequality (16) does not hold. We consider the decrease of the sum on the left-side of Inequality (17) after we change d(I) to $R_{a,b} = \sum_{i \in I} \left(1 - \frac{5}{2}x_{[a],i}\right) d_i$. For each $s \in S_2$, if $C_s \geq R_{a,b} - C(S_1)$, then the decrease of the coefficient of y_s is at most $d(I) - R_{a,b} = \frac{5}{2} \sum_{i \in I} x_{[a],i} d_i$. Otherwise $C_s < R_{a,b} - C(S_1)$ and there is no decrease for the coefficient of y_s . Since Inequality (16) does not hold, the decrease of the left side is at most $\sum_{s \in S_2: C_s \geq R_{a,b} - C(S_1)} y_s \times \frac{5}{2} \sum_{i \in I} x_{[a],i} d_i \leq \frac{3}{5} \times \frac{5}{2} \sum_{i \in I} x_{[a],i} d_i = \frac{3}{2} \sum_{i \in I} x_{[a],i} d_i$. So, we have

$$\sum_{s \in S_2} \min \left\{ C_s, R_{a,b} - C(S_1) \right\} y_s \ge \sum_{i \in I} \left(1 - x_{[a],i} \right) d_i - C(S_1) - \frac{3}{2} \sum_{i \in I} x_{[a],i} d_i$$

$$= \sum_{i \in I} \left(1 - \frac{5}{2} x_{[a],i} \right) d_i - C(S_1) = R_{a,b} - C(S_1),$$

which is exactly Inequality (15).

Let $\widehat{y}_s = \min\{10y_s, 1\}$ for every $s \in [T]$ and $S^+ = \{s \in [T], \widehat{y}_s = 1\} = \{s \in [T] : y_s \ge 1/10\}$ be the set of elements with y values at least 1/10. For every interval (a, b], define $\widetilde{R}_{a,b} = \max\{R_{a,b} - C((a, b] \cap S^+), 0\}$ to be the residual requirement for the interval (a, b], as in Theorem 2.2. For every interval (a, b] with $\widetilde{R}_{a,b} > 0$, we define $I = \{i \in [N] : r_i \in (a, b], x_{[a],i} < \frac{2}{5}\}$ as in Lemma 3.2. Let $S_1 = (a, b] \cap S^+$ and $S_2 = (a, b] \setminus S^+$. Since $C(S_1) < R_{a,b}$, we can check if Constraint (13) is satisfied for I, S_1 and S_2 . If not, we return this violated constraint. So, we assume the condition is satisfied. Then by Lemma 3.2, we have either Inequality (15) or (16). Notice that $y_s < 1/10$ and $\widehat{y}_s = 10y_s$ for every $s \in S_2$. Multiplying the two inequalities by 10, and replacing $R_{a,b} - C(S_1)$ with $\widetilde{R}_{a,b}$ and S_2 with $(a,b] \setminus S^+$, we have

either
$$\sum_{s\in(a,b]\backslash S^+} \min\left\{C_s,\widetilde{R}_{a,b}\right\} \widehat{y}_s \geq 10\widetilde{R}_{a,b},$$
 or
$$\sum_{s\in(a,b]\backslash S^+:C_s\geq\widetilde{R}_{a,b}} \widehat{y}_s \geq 6.$$

The above property holds for every time interval (a, b] such that $R_{a,b} > 0$. Since there are only $O(T^2)$ intervals and for each interval (a, b] we only need to check one constraint of form (13), our algorithm runs in polynomial time.

Thus, the Interval-KC instance (T,C,K,R) and the vector $\widehat{y} \in [0,1]^T$ satisfy the condition of Theorem 2.2. We can apply the theorem to find a set $S^* \supseteq S^+$ such that $K(S^*) \le \sum_{s \in [T]} \widehat{y}_s K_s \le 10 \sum_{s \in [T]} y_s K_s$ and $C([a,b) \cap S^*) \ge R_{a,b}$ for every interval (a,b]. Let $y^* \in \{0,1\}^T$ be the indicator vector for S^* ; by Lemma 3.1 there is an x^* such that (x^*,y^*) is a feasible solution to the CMILS instance and $\mathsf{hcost}(x^*) \le \frac{5}{2}\mathsf{hcost}(x)$. Thus, we obtain a feasible solution (x^*,y^*) to the CMILS instance whose cost is at most $10\sum_{s \in [T]} y_s K_s + \frac{5}{2}\mathsf{hcost}(x)$; this finishes the proof of Theorem 1.1.

4 Approximation Algorithm for Laminar Knapsack Covering via Iterative Rounding

This section is dedicated to the proof of Theorem 2.1, by describing our iterative rounding algorithm. Recall that we are given a Laminar-KC instance $(T, C, K, \mathcal{S}, R)$, a vector $y \in [0, 1]^T$ and set $S^+ = \{s \in [T] : y_s = 1\}$. Let $\widetilde{R}_{a,b} := \max\{R_{a,b} - C((a,b] \cap S^+), 0\}$ for every $(a,b] \in \mathcal{S}$. For every $(a,b] \in \mathcal{S}$ with $\widetilde{R}_{a,b} > 0$, either Inequality (6) or Inequality (7) holds.

We now give an overview of the rounding algorithm. We maintain a set $S^* \subseteq [T]$ of knapsacks that we already selected, and a set $S^\circ \subseteq [T]$ of knapsacks we discarded. There is a sub-family $\mathcal{S}^1 \cup \mathcal{S}^2 \subseteq \mathcal{S}$ of "active" intervals from the laminar family \mathcal{S} , where $\mathcal{S}^1 \cap \mathcal{S}^2 = \emptyset$. Each set $(a,b] \in \mathcal{S}^1 \cup \mathcal{S}^2$ has positive residual requirement $\widehat{R}_{a,b}$, after we selected knapsacks in S^* ; that is, $\widehat{R}_{a,b} := R_{a,b} - C(S^* \cap (a,b]) > 0$. We maintain an LP relaxation (LP_{IterRound}) in the rounding algorithm, in which we have a constraint for every interval $(a,b] \in \mathcal{S}^1 \cup \mathcal{S}^2$. We maintain a feasible solution y to the LP. In each iteration of the rounding algorithm, we update y to be an optimum vertex-point solution of LP_{IterRound}. Then, we show that there must be some knapsack $s \notin S^\circ \cup S^*$ such that $y_s \in \{0,1\}$. Depending on whether $y_s = 0$ or $y_s = 1$, we discard or select s, by adding s to s0 or s1. Then we can update s1, s2 and the residual requirement vector s2 accordingly. To analyze the algorithm, we prove two key lemmas. First, we show that the feasibility of s2 is maintained, after we update s3, s5 and s6. Second, we show that the algorithm makes progress in each iteration: some new knapsack s3 is added to s5 or s5.

Now we describe the LP relaxation LP_{IterRound} used in the iterative rounding. In the LP, y_s indicates whether s is included in the final solution. Thus, we require $y_s = 0$ for every $s \in S^{\circ}$ (Constraint (20)) and $y_s = 1$ for every $s \in S^{*}$ (Constraint (21)). For a set $(a, b] \in \mathcal{S}^1$, we require Constraint (18) to hold; for a set $(a, b] \in \mathcal{S}^2$, we require Constraint (19) to hold. Notice that the two constraints are respectively Inequality (6) and Inequality (7), with S^+ replaced by S^* and \widetilde{R} replaced by \widehat{R} .

$$\min \sum_{s \in [T]} y_s K_s \quad \text{s.t.} \qquad (\mathsf{LP}_{\mathsf{IterRound}})$$

$$\sum_{s \in (a,b] \setminus S^*} \min \left\{ C_s, \widehat{R}_{a,b} \right\} y_s \ge 2 \widehat{R}_{a,b}, \quad \forall (a,b] \in \mathcal{S}^1; \quad (18) \quad y_s = 0, \quad \forall s \in S^\circ; \quad (20)$$

$$y_s = 1, \quad \forall s \in S^*; \quad (21)$$

$$\sum_{s \in (a,b] \setminus S^*: C_s \ge \widehat{R}_{a,b}} y_s \ge 1, \quad \forall (a,b] \in \mathcal{S}^2; \quad (19) \quad y_s \in [0,1], \quad \forall s \notin S^\circ \cup S^*. \quad (22)$$

The pseudo-code for the rounding algorithm is given in Algorithm 1. We initialize $S^*, S^\circ, \widehat{R}, \mathcal{S}^1$ and \mathcal{S}^2 in Statements 1 to 3. In each iteration of the outer loop (the loop beginning with Statement 4), we first repeatedly remove intervals (a,b] from $\mathcal{S}^1 \cup \mathcal{S}^2$ if the requirement for (a,b] is implied by the requirement for some other interval $(a',b'] \in \mathcal{S}^1 \cup \mathcal{S}^2$ (Statements 5 and 6). Then, we solve $\mathsf{LP}_{\mathsf{IterRound}}$ to obtain an optimum vertex point solution y (Statement 7). Some knapsacks $s \in [T] \setminus S^\circ$ may have $y_s = 0$, in which case we permanently discard s by adding s to S° (Statement 8); some knapsacks $s \in [T] \setminus S^*$ may have $y_s = 1$, in which case we add s to our final solution S^* (Statement 10). With S^* updated, we shall update \widehat{R} , \mathcal{S}^1 and \mathcal{S}^2 accordingly (Statements 11 to 14). In particular, we remove intervals (a,b] from $\mathcal{S}^1 \cup \mathcal{S}^2$ if the requirement for (a,b] is already satisfied by S^* (Statement 13). We move sets (a,b] from \mathcal{S}^1 to \mathcal{S}^2 if (a,b] satisfies Constraint (19) (Statement 14). The algorithm terminates when $\mathcal{S}^1 \cup \mathcal{S}^2 = \emptyset$.

Since we removed intervals from $S^1 \cup S^2$ in Statements 5, 6 and Statement 13, the requirements for intervals (a,b] in $S \setminus (S^1 \cup S^2)$ are irrelevant. For such an interval (a,b], either $C(S^* \cap (a,b]) \geq R_{a,b}$, or there exists some set $(a',b'] \in S^1 \cup S^2$ such that $(a',b'] \setminus (S^\circ \cup S^*) = (a,b] \setminus (S^\circ \cup S^*)$ and $\widehat{R}_{a',b'} = R_{a',b'} - C(S^* \cap (a',b']) \geq R_{a,b} - C(S^* \cap (a,b])$. In the former case, the requirement for (a,b] is already satisfied; in the later case, the requirement for (a,b] is implied by the requirement for some interval $(a',b'] \in S^1 \cup S^2$, conditioned on that we must choose knapsacks in S^* and not choose knapsacks in S° . So, when $S^1 \cup S^2 = \emptyset$ becomes empty, the requirements for all sets in S are satisfied.

Algorithm 1 Iterative Rounding Algorithm for the Proof of Theorem 2.1.

```
Input: (T, C, K, \mathcal{S}, R), y, S^+ and (\widetilde{R}_{a,b})_{(a,b] \in \mathcal{S}} as in Theorem 2.1.

Output: a feasible solution S^* \subseteq [T] to the instance with K(S^*) \leq \sum_{s \in [T]} y_s K_s.
  1: let S^{\circ} \leftarrow \emptyset, S^* \leftarrow S^+, \widehat{R}_{a,b} \leftarrow \widetilde{R}_{a,b} for every (a,b] \in \mathcal{S}
  2: let S^2 \leftarrow \left\{ (a, b] \in S : \widehat{R}_{a,b} > 0, \text{Constraint (19) is satisfied for } (a, b] \right\}
  3: let S^1 \leftarrow \{(a,b] \in S : \widehat{R}_{a,b} > 0\} \setminus S^2
  4: while S^1 \cup S^2 \neq \emptyset do
                                                                                                                                                                   (outer loop)
             while there exist two distinct intervals (a, b], (a', b') \in S^1 \cup S^2
  5:
                           s.t. (a,b] \setminus (S^{\circ} \cup S^{*}) = (a',b'] \setminus (S^{\circ} \cup S^{*}) and \widehat{R}_{a',b'} \geq \widehat{R}_{a,b} do
                 S^1 \leftarrow S^1 \setminus \{(a,b]\}, S^2 \leftarrow S^2 \setminus \{(a,b]\}
  6:
            let y to be an optimal vertex point solution of \mathsf{LP}_{\mathsf{IterRound}}
  7:
            while there exists s \in [T] \setminus S^{\circ} such that y_s = 0 do: add s to S^{\circ}
  8:
            while there exists s \in [T] \setminus S^* such that y_s = 1 do
  9:
                                                                                                                                                                (middle loop)
 10:
                for every (a, b] \in S^1 \cup S^2 such that s \in (a, b] do
                                                                                                                                                                   (inner loop)
11:
                     R_{a,b} \leftarrow R_{a,b} - C_s
12:
                     if \widehat{R}_{a,b} \leq 0 then S^1 \leftarrow S^1 \setminus \{(a,b]\}, S^2 \leftarrow S^2 \setminus \{(a,b]\} if (a,b] \in S^1 and Constraint (19) holds for (a,b] then S^1 \leftarrow S^1 \setminus \{(a,b]\}, S^2 \leftarrow S^2 \cup \{(a,b]\}
 13:
14:
```

Before formally proving the two key lemmas we need to prove Theorem 2.1, we make some simple observations about Algorithm 1, assuming Statement 7 always finds a feasible solution y.

Observation 4.1. (4.1a) After the initialization of S° and S^{*} in Statement 1, we always have $y_{s} = 0$ for every $s \in S^{\circ}$ and $y_{s} = 1$ for every $s \in S^{*}$.

- (4.1b) After the initialization of S^1 and S^2 in Statements 2 and 3, we always have $S^1 \cap S^2 = \emptyset$.
- (4.1c) At the beginning of each iteration of the outer loop, we have $\widehat{R}_{a,b} = R_{a,b} C(S^* \cap (a,b]) > 0$ for every $(a,b] \in S^1 \cup S^2$.

Observation (4.1a) holds since we add s to S° only if $y_s = 0$, to S^{*} only if $y_s = 1$. In LP_{IterRound}, we have Constraints (20) and (21). Thus the two constraints hold when we update y in Statement 7. After Statement 3, $S^{1} \cap S^{2} = \emptyset$. The only place we add an interval to either S^{1} or S^{2} is Statement 14, in which we move the interval from S^{1} to S^{2} . Thus Observation (4.1b) holds. After Statement 3, we have that for every interval $(a,b] \in S^{1} \cup S^{2}$, $\widehat{R}_{a,b} = \widetilde{R}_{a,b} = \max\{R_{a,b} - C(S^{+} \cap (a,b]), 0\} = \max\{R_{a,b} - C(S^{*} \cap (a,b]), 0\} > 0$. Thus, $\widehat{R}_{a,b} = R_{a,b} - C(S^{*} \cap (a,b]) > 0$. Every time we add a knapsack s to S^{*} in Statement 10, for every $(a,b] \in S^{1} \cup S^{2}$ such that $s \in (a,b]$, we decrease $\widehat{R}_{a,b}$ by C_{s} in Statement 12. Then if $\widehat{R}_{a,b} \leq 0$ after the decrease, we remove (a,b] from $S^{1} \cup S^{2}$ in Statement 13. Thus, Observation (4.1c) holds.

The first key lemma is Lemma 4.2. The heart of the proof of the lemma is in the proof of Lemma 4.3. We prove Lemma 4.3 now, while deferring the proof of Lemma 4.2 to Appendix B.

Lemma 4.2. At the beginning of each iteration of the outer loop, y is a feasible solution to LP_{IterRound}.

Lemma 4.3. If y is a feasible solution to $\mathsf{LP}_{\mathsf{IterRound}}$ at the beginning of an iteration of the middle loop (the loop beginning with Statement 9), then it is also feasible at the end of the iteration.

Proof. For notational purposes, we use s^* to denote the knapsack that will be added to S^* in this iteration. After this iteration, Constraint (20) is unaffected, and Constraints (21) and (22) remain satisfied since $y_{s^*} = 1$. Thus, we only need to focus on Constraints (18) and (19). If some $(a, b] \in \mathcal{S}^1 \cup \mathcal{S}^2$ does not contain s^* , then $(a, b] \setminus S^*$ and $\widehat{R}_{a,b}$ do not change in the iteration. Thus the constraint for (a, b] (either Constraint (18) or Constraint (19)) is unaffected. Thus, we can focus on an interval $(a, b] \in \mathcal{S}^1 \cup \mathcal{S}^2$ that contains s^* . In the iteration of the inner loop (the loop beginning with Statement 11) for this interval (a, b], Statement 12 decreases $\widehat{R}_{a,b}$ by C_{s^*} , Statement 13 removes (a, b] from $\mathcal{S}^1 \cup \mathcal{S}^2$ if $\widehat{R}_{a,b}$ becomes at most 0, and Statement 14 moves (a, b] from \mathcal{S}^1 to \mathcal{S}^2 if (a, b] satisfies Constraint (19).

To the end of this proof, S^* will refer to the set S^* before we add s^* , $\widehat{R}_{a,b}$ will refer to the value of $\widehat{R}_{a,b}$ before we run Statement 12 and $\widehat{R}_{a,b}^{\mathsf{new}} = \widehat{R}_{a,b} - C_{s^*}$ will be the value of $\widehat{R}_{a,b}$ after Statement 12. If $\widehat{R}_{a,b}^{\mathsf{new}} \leq 0$, then (a,b] will be removed from $S^1 \cup S^2$ and there will be no Constraint for (a,b] in $\mathsf{LP}_{\mathsf{IterRound}}$. Thus, we can assume $\widehat{R}_{a,b}^{\mathsf{new}} > 0$.

Consider the first case where we have $(a,b] \in \mathcal{S}^1$ at the beginning of the iteration of the middle loop. If $\sum_{s \in (a,b] \backslash S^* \backslash \{s^*\} : C_s \geq \widehat{R}_{a,b}^{\text{new}}} y_s \geq 1$, then (a,b] will be moved to \mathcal{S}^2 and Constraint (19) for (a,b] will be satisfied. So, we can assume that $\sum_{s \in (a,b] \backslash S^* \backslash \{s^*\} : C_s \geq \widehat{R}_{a,b}^{\text{new}}} y_s < 1$. (a,b] will remain in \mathcal{S}^1 after the iteration. We consider how Constraint (18) for (a,b] is affected by adding s^* to S^* and changing $\widehat{R}_{a,b}$ to $\widehat{R}_{a,b}^{\text{new}}$. The right side of the inequality is decreased by exactly $2(\widehat{R}_{a,b} - \widehat{R}_{a,b}^{\text{new}}) = 2C_{s^*}$. We now consider the decrease of the left side. First, adding s^* to S^* will decrease the left side by $y_{s^*} \min \left\{ C_{s^*}, \widehat{R}_{a,b} \right\} = C_{s^*}$ since $y_{s^*} = 1$ and $C_{s^*} < \widehat{R}_{a,b}$. Second, some knapsack $s \in (a,b] \backslash S^* \backslash \{s^*\}$ will have $\min \left\{ C_s, \widehat{R}_{a,b}^{\text{new}} \right\} < \min \left\{ C_s, \widehat{R}_{a,b} \right\}$. This happens only if $C_s \geq \widehat{R}_{a,b}^{\text{new}}$. Moreover, if this happens, the decrease of the left-side due to this s is at most $y_s(\widehat{R}_{a,b} - \widehat{R}_{a,b}^{\text{new}}) = y_sC_{s^*}$. Since we have $\sum_{s \in (a,b] \backslash S^* \backslash \{s^*\} : C_s \geq \widehat{R}_{a,b}^{\text{new}}} y_s < 1$, the overall decrease of the left-side of Constraint (18) for (a,b] is at most $C_{s^*} + \sum_{s \in (a,b] \backslash S^* \backslash \{s^*\} : C_s \geq \widehat{R}_{a,b}^{\text{new}}} y_sC_{s^*} < 2C_{s^*}$, which is the decrease of its right-side. Thus, the constraint for (a,b] remains satisfied at the end of the iteration for the middle loop.

Then assume that $(a,b] \in S^2$ at the beginning of the iteration of the inner loop. Notice that we have assumed that $\widehat{R}_{a,b}^{\mathsf{new}} > 0$, which implies $C_{s^*} < \widehat{R}_{a,b}$. The left-side of Constraint (19) can only increase: (i) though we will add the knapsack s^* to S^* , we have $C_{s^*} < \widehat{R}_{a,b}$ and thus it does not contribute to the left-side at the beginning of the iteration of the middle loop; (ii) for a knapsack $s \in (a,b] \setminus S^* \setminus \{s^*\}$, $C_s \geq \widehat{R}_{a,b}$ implies that $C_s \geq \widehat{R}_{a,b}^{\mathsf{new}}$. Thus, Constraint (19) for (a,b] remains true at the end of the iteration of the middle loop. This finishes the proof of the lemma.

We defer the proof of the second key lemma to Appendix B. The proof uses the fact that S is a laminar family and the properties of vertex-point solutions to $\mathsf{LP}_{\mathsf{IterRound}}$.

Lemma 4.4. The outer loop will terminate in at most T iterations.

With the two key lemmas, we can complete the proof of Theorem 2.1. By Observation (4.1a), $y_s = 1$ for every $s \in S^*$. So, we always have $K(S^*) \leq \sum_{s \in [T]} y_s K_s$. The only statement that changes y is Statement 7. Since y is a feasible solution to $\mathsf{LP}_{\mathsf{IterRound}}$ before running the statement, and the LP tries to minimize $\sum_{s \in [T]} y_s K_s$, we have that $\sum_{s \in [T]} y_s K_s$ can only decrease over the course of the algorithm. Thus the returned solution S^* has cost at most $\sum_{s \in [T]} y_s K_s$, for the initial y-vector.

It remains to show that S^* is a feasible solution to the instance (T, C, K, S, R). Notice that we remove intervals (a,b] from $S^1 \cup S^2$ in Statement 6 and Statement 13. Assume towards the contradiction that S^* is not a feasible solution. Let $(a,b] \in S$ be an interval with $C((a,b] \cap S^*) < R_{a,b}$; if there are many such intervals (a,b], we choose the one that is removed from $S^1 \cup S^2$ the latest. If we removed (a,b] from $S^1 \cup S^2$ in Statement 13, then at that time we already have $C((a,b] \cap S^*) \geq R_{a,b}$. Thus (a,b] can only be removed from $S^1 \cup S^2$ in Statement 6. By Observation (4.1c), $\widehat{R}_{a',b'} = R_{a',b'} - C((a',b'] \cap S^*) > 0$ for every $(a',b'] \in S^1 \cup S^2$, at the beginning of an iteration of the outer loop. At the time of the removal there exists some other $(a',b'] \in S^1 \cup S^2$ such that $(a,b] \setminus (S^\circ \cup S^*) = (a',b'] \setminus (S^\circ \cup S^*)$ and $R_{a',b'} - C((a',b'] \cap S^*) \geq R_{a,b} - C((a,b] \cap S^*)$. The inequality remains true as the algorithm proceeds since whenever we add a new knapsack $s \notin S^\circ \cup S^*$ to S^* , $s \in (a,b]$ if and only if $s \in (a',b']$. By our choice of (a,b], at the end of the algorithm we have $C((a',b'] \cap S^*) \geq R_{a',b'}$, implying that $C((a,b] \cap S^*) \geq R_{a,b}$, a contradiction. Thus, S^* is a feasible solution and we proved Theorem 2.1.

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A Reduction of Interval Knapsack Covering to Laminar Knapsack Covering

In this section, we prove Theorem 2.2 via a reduction of the given Interval-KC instance to a Laminar-KC instance. Recall the given Interval-KC instance is (T, C, K, R). We are also given a vector $y \in [0, 1]^{[T]}$ and $S^+ = \{s \in [T] : y_s = 1\}$. For every interval (a, b], we have $\widetilde{R}_{a,b} = \max\{R_{a,b} - C((a, b] \cap S^+), 0\}$; if $\widetilde{R}_{a,b} > 0$, then either Inequality (8) or Inequality (9) holds.

We start by defining the requirement function for the Laminar-KC instance. For every interval (a, b], we define

$$\widetilde{R}'_{a,b} := \sup \left\{ W \ge 0 : \sum_{s \in (a,b] \setminus S^+} \min \left\{ C_s, W \right\} y_s \ge 2W, \text{ or } \sum_{s \in (a,b] \setminus S^+: C_s \ge W} y_s \ge 1 \right\}.$$
(23)

That is, $\widetilde{R}'_{a,b}$ is the largest possible value that makes either Inequality (6) or Inequality (7) hold for the interval (a,b]. To construct a laminar family \mathcal{S} of intervals for the Laminar-KC instance, we run a simple recursive procedure: let $\mathcal{S} \leftarrow \emptyset$ initially and then we call construct-laminar-family(0,T).

Algorithm 2 construct-laminar-family(a, b)

- 1: add (a, b] to S
- 2: **if** a + 1 < b **then**
- 3: find the $c \in (a,b)$ that maximizes min $\left\{ \widetilde{R}'_{a,c}, \widetilde{R}'_{c,b} \right\}$
- 4: construct-laminar-family (a, c)
- 5: construct-laminar-family (c, b)

The $\mathcal S$ we constructed naturally defines a laminar tree, in which every non-leaf interval (a,b] has exactly two children (a,c] and (c,b], for some $c\in(a,b)$; for every leaf (a,b] in the tree, we have b-a=1. The next lemma shows that the \widetilde{R}' -requirements for intervals in $\mathcal S$ will capture the \widetilde{R} -requirements for all intervals.

Lemma A.1. For every interval (a,b] such that $\widetilde{R}_{a,b} > 0$, there exist an interval $(a',b'] \in \mathcal{S}$ such that $(a',b'] \subseteq (a,b]$ and $\widetilde{R}'_{a',b'} \geq \widetilde{R}_{a,b}$.

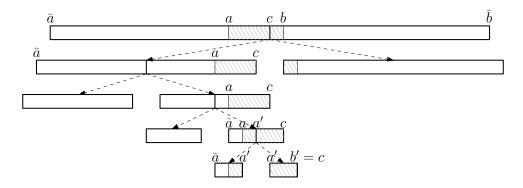


Figure 1: Illustration for the Proof of Lemma A.1.

Proof. We first give the high-level idea behind the proof; see Figure 1 for illustration. We consider the inclusive-minimal interval $(\widehat{a}, \widehat{b}] \in \mathcal{S}$ that contains (a, b]. Let $(\widehat{a}, c]$ and $(c, \widehat{b}]$ be the two child-intervals of $(\widehat{a}, \widehat{b}]$. Then, $c \in (a, b)$ by our choice of $(\widehat{a}, \widehat{b})$. One of the two intervals in (a, c] and (c, b] must contain enough total capacity. Assume this interval is $(a, c] \subseteq (\widehat{a}, c)$. Then, we start from $(\widetilde{a}, c] = (\widehat{a}, c]$; if the right-child interval (a', c] of $(\widetilde{a}, c]$ is a superset of (a, c], then we let $(\widetilde{a}, c] = (a', c]$ and repeat. So, eventually, we can find an interval $(\widetilde{a}, c]$ in the tree, and its right child (a', c], such that $(\widetilde{a}, c] \supseteq (a, c) \supseteq (a', c]$. Then, we can show that the interval (a', b'] = (a', c] satisfies the requirement of the lemma.

Now we prove the lemma formally. We consider the first case in which we have Inequality (8) for (a,b]. Then $10\widetilde{R}_{a,b} \leq \sum_{s \in (a,b] \backslash S^+} \min \left\{ C_s, \widetilde{R}_{a,b} \right\} y_s \leq \sum_{s \in (a,b] \backslash S^+} \widetilde{R}_{a,b} \cdot y_s$. Since $\widetilde{R}_{a,b} > 0$ and $y_s \leq 1$ for every $s \in (a,b] \backslash S^+$, we have $b-a \geq 10$.

Consider the laminar tree defined by \mathcal{S} . Let $(\widehat{a}, \widehat{b}]$ be the inclusive-minimal set in \mathcal{S} such that $(a, b] \subseteq (\widehat{a}, \widehat{b}]$. Such a set exists since $[T] \in \mathcal{S}$. Since $\widehat{b} - \widehat{a} \ge b - a \ge 10$, there will be two children $(\widehat{a}, c]$ and $(c, \widehat{b}]$ of $(\widehat{a}, \widehat{b}]$ in the laminar tree. By our choice of $(\widehat{a}, \widehat{b})$, neither $(\widehat{a}, c]$ nor (c, \widehat{b}) is a superset of

(a,b]. Thus, a < c < b. By Inequality (8), we have $\sum_{s \in (a,b] \setminus S^+} \min \left\{ C_s, \widetilde{R}_{a,b} \right\} y_s \ge 10 \widetilde{R}_{a,b}$. Thus, we have either $\sum_{s \in (a,c] \setminus S^+} \min \left\{ C_s, \widetilde{R}_{a,b} \right\} y_s \ge 5 \widetilde{R}_{a,b}$ or $\sum_{s = (c,b] \setminus S^+} \min \left\{ C_s, \widetilde{R}_{a,b} \right\} y_s \ge 5 \widetilde{R}_{a,b}$. Without loss of generality, we assume the first inequality holds; this implies that $c - a \ge 5$.

We run the following procedure to find two sets $(\tilde{a},c], (a',c] \in \mathcal{S}$. Let $\tilde{a}=\bar{a}$ initially; thus $(\tilde{a},c] \in \mathcal{S}$ and $(\tilde{a},c] \supseteq (a,c]$. While the right child (a',c] of $(\tilde{a},c]$ is a superset of (a,c], we let $\tilde{a}=a'$ and repeat. At the end of the process, we find a set $(\tilde{a},c] \in \mathcal{S}$ and its right child $(a',c] \in \mathcal{S}$ such that $(\tilde{a},c] \supseteq (a',c] \supseteq (a',c]$. (Notice that the right child of $(\tilde{a},c]$ always exists during the procedure since $c-a \ge 5$). Notice that $\sum_{s \in (\tilde{a},c] \setminus S^+} \min \left\{ C_s, \tilde{R}_{a,b} \right\} y_s \ge 5\tilde{R}_{a,b}$ and $\min \left\{ C_s, \tilde{R}_{a,b} \right\} y_s \le \tilde{R}_{a,b}$ for every $s \in (\tilde{a},c] \setminus S^+$. There is a number $\ell \in (\tilde{a},c)$ such that $\sum_{s \in (\tilde{a},\ell] \setminus S^+} \min \left\{ C_s, \tilde{R}_{a,b} \right\} y_s \ge 2\tilde{R}_{a,b}$ and $\sum_{s \in (\ell,c] \setminus S^+} \min \left\{ C_s, \tilde{R}_{a,b} \right\} y_s \ge 2\tilde{R}_{a,b}$. Thus, $\tilde{R}'_{a,\ell} \ge \tilde{R}_{a,b}$ and $\tilde{R}'_{\ell,c} \ge \tilde{R}_{a,b}$, by the definition of \tilde{R}' in Equation (23). By the way we choose c in Statement 3 of Algorithm 2, we have that $\min \left\{ \tilde{R}'_{\tilde{a},a'}, \tilde{R}'_{a',c} \right\} \ge \min \left\{ \tilde{R}'_{\tilde{a},\ell}, \tilde{R}'_{\ell,c} \right\} \ge \tilde{R}_{a,b}$. In particular, $\tilde{R}'_{a',c} \ge \tilde{R}_{a,b}$. Let b' = c; then $(a',b'] = (a',c] \subseteq (a,c] \subseteq (a,b]$ and $\tilde{R}'_{a',b'} = \tilde{R}'_{a',c} \ge \tilde{R}_{a,b}$. This finishes the proof of the Lemma if Inequality (8) is satisfied for (a,b]. We now consider the second case in which Inequality (9) is satisfied for (a,b]. The proof for this case

We now consider the second case in which Inequality (9) is satisfied for (a,b]. The proof for this case is very similar to the previous case and thus we only give a sketch. Let $(\widehat{a},\widehat{b}]$ be the inclusive-minimal set in $\mathcal S$ that contains (a,b]. Similarly, we can prove $\widehat{b}-\widehat{a}\geq b-a\geq 6$ and $(\widehat{a},\widehat{b}]$ has two children $(\widehat{a},c]$ and $(c,\widehat{b}]$ in the laminar tree. By the way we choose $(\widehat{a},\widehat{b}]$, we have a< c< b. Inequality (9) implies $\sum_{s\in(a,b]\backslash S^+:C_s\geq\widetilde{R}_{a,b}}y_s\geq 6$. Thus, either $\sum_{s\in(a,c]\backslash S^+:C_s\geq\widetilde{R}_{a,b}}y_s\geq 3$ or $\sum_{s\in(c,b]\backslash S^+:C_s\geq\widetilde{R}_{a,b}}y_s\geq 3$. W.l.o.g, we assume the first case happens. We find a set $(\widetilde{a},c]\in\mathcal S$ and its right child $(a',c]\in\mathcal S$ as in the previous case. So, $(\widetilde{a},c]\supseteq(a,c]\supseteq(a',c]$. Since $\sum_{s\in(\widetilde{a},c]\backslash S^+:C_s\geq\widetilde{R}_{a,b}}y_s\geq 3$ and $y_s\leq 1$ for every $s\in(\widetilde{a},c]\setminus S^+$, there is an $\ell\in(\widetilde{a},c)$ such that $\sum_{s\in(\widetilde{a},\ell]\backslash S^+:C_s\geq\widetilde{R}_{a,b}}y_s\geq 1$ and $\sum_{s\in(\ell,c]\backslash S^+:C_s\geq\widetilde{R}_{a,b}}y_s\geq 1$. By the definition of \widetilde{R}' in Equation (23), we have $\widetilde{R}'_{a,\ell}\geq\widetilde{R}_{a,b}$ and $\widetilde{R}'_{\ell,c}\geq\widetilde{R}_{a,b}$. By the way we select c in Statement 3 of Algorithm 2, we have that min $\left\{\widetilde{R}'_{a,a'},\widetilde{R}'_{a',c}\right\}\geq\min\left\{\widetilde{R}'_{a,\ell},\widetilde{R}'_{\ell,c}\right\}\geq\widetilde{R}_{a,b}$. Let b'=c; then $(a',b']=(a',c]\subseteq(a,c]\subseteq(a,b]$ and $\widetilde{R}'_{a',b'}=\widetilde{R}'_{a',c}\geq\widetilde{R}_{a,b}$. This finishes the proof of the lemma for the second case.

With Lemma A.1, we can define our Laminar-KC instance. Let $R'_{a,b} = \widetilde{R}'_{a,b} + C((a,b] \cap S^+)$ for every $(a,b] \in \mathcal{S}$; thus $\widetilde{R}'_{a,b} = R'_{a,b} - C((a,b] \cap S^+)$. Then the Laminar-KC instance we focus on is (T,C,K,\mathcal{S},R') . By the definition of $\widetilde{R}'_{a,b}$, we have

either
$$\sum_{s \in (a,b] \setminus S^+} \min \left\{ C_s, \widetilde{R}'_{a,b} \right\} y_s \ge 2\widetilde{R}'_{a,b},$$
 or
$$\sum_{s \in (a,b] \setminus S^+: C_s \ge \widetilde{R}'_{a,b}} y_s \ge 1.$$

Thus, we can use Theorem 2.1 for the instance $(T, C, K, \mathcal{S}, R')$ and y to obtain a set $S^* \supseteq S^+$ such that $K(S^*) \leq \sum_{s \in [T]} y_s K_s$ and $C((a', b'] \cap S^*) \geq R'_{a', b'}$ for every set $(a', b'] \in \mathcal{S}$.

Now, focus on any interval (a,b] over [T]. If $\widetilde{R}_{a,b}=0$, then $C((a,b]\cap S^+)\geq R_{a,b}$ and the requirement for (a,b] is satisfied; thus we can assume $\widetilde{R}_{a,b}>0$. Then by Lemma A.1, we have a set $[a',b')\in\mathcal{S}$ such that $[a',b']\subseteq(a,b]$ and $\widetilde{R}'_{a',b'}\geq\widetilde{R}_{a,b}$. Thus, we have that $C(S^*\cap(a,b])-C(S^*\cap((a,a']\cup(b',b]))=C(S^*\cap(a',b'])\geq R'_{a',b'}=\widetilde{R}'_{a',b'}+C((a',b']\cap S^+)\geq \widetilde{R}_{a,b}+C((a',b']\cap S^+)$. So $C(S^*\cap(a,b])\geq \widetilde{R}_{a,b}+C((a',b']\cap S^+)+C(S^*\cap((a,a']\cup(b',b]))\geq \widetilde{R}_{a,b}+C((a,b]\cap S^+)=R_{a,b}$, where the last inequality used the fact that $S^*\supseteq S^+$. This finishes the proof of Theorem 2.2.

B Omitted Proofs

B.1 Proof of Lemma 3.1

Proof. For every $i \in [N]$, and for every s from 1 to r_i , let $x'_{s,i} = \min\left\{\frac{5}{2}x_{s,i}, 1 - x'_{[s-1],i}\right\}$. By this definition, we have that $x'_{[t],i} = \min\left\{\frac{5}{2}x_{[t],i}, 1\right\}$ for every $i \in [N], t \in [r_i]$. We shall first show that if $x^*_{[t],i} \leq x'_{[t],i} = \min\left\{\frac{5}{2}x_{[t],i}, 1\right\}$ for every $i \in [N]$ and $t \in [r_i]$, then $\mathsf{hcost}(x^*) \leq \frac{5}{2} \cdot \mathsf{hcost}(x)$.

$$\begin{aligned} \mathsf{hcost}(x^*) &= \sum_{i \in [N]} d_i \sum_{s \in [r_i]} x_{s,i}^* h_i(s) = \sum_{i \in [N]} d_i \sum_{s \in [r_i]} x_{s,i}^* \sum_{t=s}^{r_i-1} (h_i(t) - h_i(t+1)) & \text{ since } h_i(r_i) = 0, \forall i \in [N] \\ &= \sum_{i \in [N]} d_i \sum_{t=1}^{r_i-1} (h_i(t) - h_i(t+1)) \sum_{s \in [t]} x_{s,i}^* \\ &\leq \frac{5}{2} \sum_{i \in [N]} d_i \sum_{t=1}^{r_i-1} (h_i(t) - h_i(t+1)) \sum_{s \in [t]} x_{s,i} = \frac{5}{2} \cdot \mathsf{hcost}(x). \end{aligned}$$

Thus, it suffices to find an $x^* \in [0,1]^{[T] \times [N]}$ such that $\sum_{s \in [T]} x_{s,i}^* = 1$ for every $i \in [N]$, $x_{s,i}^* = 0$ if $s \notin S^*$ or $s > r_i$, $\sum_{i \in [N]} x_{s,i} d_i \le C_s$ for every $s \in S^*$, and $x_{[t],i}^* \le x_{[t],i}'$ for every $i \in [N]$ and $t \in [r_i]$. This x^* can be found by solving the following fractional b-matching instance on the bipartite graph $(A \cup B, E)$, where $A = \{u_{s,i} : i \in [N], s \in [r_i]\}$ and $B = \{v_{s'} : s' \in S^*\}$. We assign the d_i units of demand for item i to vertices in $\{u_{s,i} : s \in [r_i]\}$ according to x': $u_{s,i}$ is assigned $x'_{s,i}d_i$ units of demand. In order to guarantee that $x_{[t],i}^* \le x_{[t],i}'$ for every $t \in [r_i]$, we guarantee that the $x'_{s,i}d_i$ units of demand assigned to $u_{s,i}$ can only be satisfied by orders in $[s, r_i]$. Thus, we define E as follows: for every $u_{s,i} \in A$ and $v_{s'} \in B$ such that $s' \in [s, r_i]$, we have an edge $(u_{s,i}, v_{s'}) \in E$. The goal of the fractional b-matching problem is to find a vector $z \in \mathbb{R}^E_{\geq 0}$ such that $\sum_{e \text{ incident to } u_{s,i}} z_e = x'_{s,i}d_i$ for every $u_{s,i} \in A$, and $\sum_{e \text{ incident to } v_{s'}} z_e \le C_{s'}$ for every $v_{s'} \in B$.

It is well-known that the above fractional b-matching instance is feasible if and only if for every $A'\subseteq A$, we have that $C(\{s':v_{s'}\in B \text{ is adjacent to some vertex in }A'\})\geq \sum_{u_{s,i}\in A'}x'_{s,i}d_i$. That is, $C\left(\bigcup_{u_{s,i}\in A'}[s,r_i]\right)\geq \sum_{u_{s,i}\in A'}x'_{s,i}d_i$. If $u_{s,i}\in A'$, then we can assume that for every $s'\in (s,r_i]$, we also have $u_{s',i}\in A'$; this does not change the left-side of the inequality but increases the right-side and makes the inequality harder to satisfy. So, we can assume there is a set $I\subseteq [N]$, $t\in [0,T)^I$ such that $t_i\in [0,r_i)$ for every $i\in I$ and $A'=\{u_{s,i}:i\in I,s\in (t_i,r_i]\}$. Then the neighbors of A' is $\{v_{s'}:s'\in \bigcup_{i\in I}(t_i,r_i)\}$.

Thus, to guarantee the existence of x^* , it suffices to guarantee that for every such I and t, we have

$$\sum_{i \in I} x'_{(t_i, r_i], i} d_i \le \sum_{s' \in \bigcup_{i \in I} (t_i, r_i]} y_{s'}^* C_{s'}. \tag{24}$$

We can further assume $\bigcup_{i\in I}(t_i,r_i]$ is a time interval; otherwise, we can break I into two sets I' and I'' such that $\bigcup_{i\in I'}(t_i,r_i]$ and $\bigcup_{i\in I''}(t_i,r_i]$ are disjoint. Inequality (24) with I replaced with I', and the inequality with I replaced with I'', implies the Inequality (24). If $\bigcup_{i\in I}(t_i,r_i]=(a,b]$, then the right side of Inequality (24) is $\sum_{s\in(a,b]}y_s^*C_s=C(S^*\cap(a,b])$; we want to find the (I,t) pair with $\sum_{i\in I}(t_i,r_i]=(a,b]$ that maximize the left side. If some i has $r_i>b$ or $r_i\leq a$ then $i\notin I$. Otherwise, we can let $i\in I$; and $t_i=a$ will maximize $x'_{(t_i,r_i],i}$. So, the maximum possible value of the left side of Inequality (24) is

$$\sum_{i \in [N]: r_i \in (a,b]} x'_{(a,r_i],i} d_i = \sum_{i \in [N]: r_i \in (a,b]} \left(1 - x'_{[a],i}\right) d_i = \sum_{i \in [N]: r_i \in (a,b]} \max \left\{1 - \frac{5}{2} x_{[a],i}, 0\right\} d_i = R_{a,b}.$$

Thus, to guarantee the existence of x^* , it suffices to guarantee that for every interval (a, b], we have $C(S^* \cap (a, b]) \ge R_{a,b}$.

B.2 Proof of Lemma 4.2

Proof. We prove the lemma by induction. Suppose we are at the beginning of first iteration of the outer loop. By the initialization of S° and S^{*} in Statement 1, and the fact that $y_{s} = 1$ for every $s \in S^{+}$,

Constraints (20) to (22) are satisfied. The statement also sets the initial \widehat{R} to be \widetilde{R} . By Inequality (6) and Inequality (7), for every $(a,b] \in \mathcal{S}$ with $\widehat{R}_{a,b} > 0$, either Constraint (18) or Constraint (19) is satisfied. Thus, after Statement 3, Constraints (18) and (19) are satisfied. So, the lemma holds for the first iteration of the outer loop.

Now, we assume that y is a feasible solution to $\mathsf{LP}_{\mathsf{IterRound}}$ at the beginning of some iteration of the outer loop; we prove that it is also feasible at the beginning the next iteration, if it exists. We run Algorithm 1 from the beginning of this iteration. Since Statements 5 and 6 only remove sets from \mathcal{S}^1 and \mathcal{S}^2 , they do not destroy the feasibility of y. So, y is a feasible solution to $\mathsf{LP}_{\mathsf{IterRound}}$ before running Statement 7. Then, the statement will always find a feasible solution y to $\mathsf{LP}_{\mathsf{IterRound}}$. Statement 8 only adds knapsacks with $y_s = 0$ to S° and does not destroy the feasibility of y. Lemma 4.3 says that running an iteration of the middle loop does not destroy the feasibility of y. Thus, y is a feasible solution at the beginning of the next iteration of the outer loop. This finishes the proof of Lemma 4.2.

B.3 Proof of Lemma 4.4

Proof. We show that in each iteration of the outer loop, we either have added some new knapsack to S° in Statement 8, or have added some new knapsack to S^{*} in Statement 10. This proves that the algorithm will terminate in at most T iterations since there are only T knapsacks and $S^{\circ} \cap S^{*} = \emptyset$.

Let us run the algorithm from the beginning of an iteration of the outer loop, at which time we have $S^1 \cup S^2 \neq \emptyset$. Statements 5 and 6 remove a set from $S^1 \cup S^2$ only if $S^1 \cup S^2$ contains at least two sets. Thus after Statement 6 we still have $S^1 \cup S^2 \neq \emptyset$. So before Statement 7, there must be some constraint of form (18) or (19) in the LP. Thus, $S^0 \cup S^* \neq [T]$ since otherwise y can not satisfy the constraint due to Observations (4.1a) and (4.1c), contradicting Lemma 4.2.

Then we run Statement 7 to find a vertex-point solution y to $\mathsf{LP}_{\mathsf{IterRound}}$. We assume towards the contradiction that we have $S^\circ = \{s \in [T] : y_s = 0\}$ and $S^* = \{s \in [T] : y_s = 1\}$; otherwise we will add some knapsack to S° or S^* later. We choose a set of T linearly independent tight constraints that defines y. We require that this set contains all tight constraints of the form (20), (21) and (22) (these tight constraints are linearly independent). The number of tight constraints of form (18) and (19) in the independent set is exactly $T - |S^\circ| - |S^*| = |[T] \setminus (S^\circ \cup S^*)| \ge 1$. Let $S' \subseteq S^1 \cup S^2$ be the family of intervals that corresponds to these tight constraints; so $|S'| = |[T] \setminus (S^\circ \cup S^*)| \ge 1$.

If we have two sets $(a,b], (a',b'] \in \mathcal{S}' \subseteq \mathcal{S}^1 \cup \mathcal{S}^2$ such that $(a',b'] \subseteq (a,b]$, then Statements 5 and 6 guaranteed that $(a',b'] \setminus (S^{\circ} \cup S^{*}) \subseteq (a',b'] \setminus (S^{\circ} \cup S^{*})$. Otherwise $(a',b'] \setminus (S^{\circ} \cup S^{*}) = (a',b'] \setminus (S^{\circ} \cup S^{*})$ and Statements 5 and 6 must have removed either (a,b] or (a',b'] from $\mathcal{S}^1 \cup \mathcal{S}^2$. Thus, there must be a knapsack in $(a,b] \setminus (a',b']$ that is not in $S^{\circ} \cup S^{*}$.

Now we focus on the laminar forest defined by the set \mathcal{S}' (the forest is not empty). We assign each knapsack $s \in [T] \setminus (S^{\circ} \cup S^{*})$ to the minimal set $(a,b] \in \mathcal{S}'$ that contains s. If some $(a,b] \in \mathcal{S}'$ has exactly one child (a',b'] in the laminar forest, then $((a,b] \setminus (a',b']) \setminus (S^{\circ} \cup S^{*}) \neq \emptyset$ and some knapsack must be assigned to (a,b]. The number of leaves in the laminar forest is strictly more than the number of non-leaves that have at least two children. Since the number of nodes in the laminar forest is $|\mathcal{S}'| = |[T] \setminus (S^{\circ} \cup S^{*})|$, and each inner node with exactly one child is assigned at least one knapsack in $[T] \setminus (S^{\circ} \cup S^{*})$, there must be a leaf (a,b] in the forest such that (a,b] is assigned at most one knapsack in $[T] \setminus (S^{\circ} \cup S^{*})$. For this (a,b], we have $\sum_{s \in (a,b] \setminus S^{*}} y_{s} = \sum_{s \in (a,b] \setminus (S^{\circ} \cup S^{*})} y_{s} + \sum_{s \in (a,b] \cap S^{\circ}} y_{s} < 1$, since $|(a,b] \setminus (S^{\circ} \cup S^{*})| \leq 1$, every $s \notin S^{*}$ has $y_{s} < 1$, and every $s \in (a,b] \cap S^{\circ}$ has $y_{s} = 0$. Recall that $\widehat{R}_{a,b} > 0$ by Observation (4.1c). If $(a,b] \in \mathcal{S}^{1}$, then $\sum_{s \in (a,b] \setminus S^{*}} \min \left\{ C_{s}, \widehat{R}_{a,b} \right\} y_{s} < \sum_{s \in (a,b] \setminus S^{*}} \widehat{R}_{a,b} y_{s} < \widehat{R}_{a,b} < 2\widehat{R}_{a,b}$, contradicting Constraint (18) for (a,b]. If $(a,b] \in \mathcal{S}^{2}$, then $\sum_{s \in (a,b] \setminus S^{*}} y_{s} < 1$ contradicts Constraint (19) for (a,b]. This finishes the proof of the lemma.