Vol. 00, No. 0, Xxxxx 0000, pp. 000–000
ISSN 0000-0000 | EISSN 0000-0000 | 00 | 0000 | 0001

INFORMS
DOI 10.1287/xxxx.0000.0000

A Stochastic Representation for Nonlocal Parabolic PDEs with Applications

Min Dai
Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong
mindai@polyu.edu.hk

Steven Kou
Department of Finance, Questrom School of Business, Boston University, Boston, MA 02215, United States, kou@bu.edu

Chen Yang
Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Hong Kong SAR, China, cyang@se.cuhk.edu.hk

We establish a stochastic representation for a class of nonlocal parabolic terminal-boundary value problems, whose terminal and boundary conditions depend on the solution in the interior domain; in particular, the solution is represented as the expectation of functionals of a diffusion process with random jumps from boundaries. We discuss three applications of the representation, the first one on the pricing of dual-purpose funds, the second one on the connection to regenerative processes, and the third one on modeling the entropy on a one-dimensional non-rigid body.

Key words: Dual-purpose Funds; Feynman-Kac Representation; Linear Thermoelasticity; Nonlocal Problems; Parabolic PDEs
MSC2000 subject classification: 35C99, 60H30
OR/MS subject classification: Primary: Probability; secondary: Stochastic model applications

1. Introduction.

We give a stochastic representation for a class of nonlocal parabolic terminal-boundary value problems, whose terminal and boundary conditions depend on the solution in the interior domain; in particular, the solution is represented as the expectation of functionals of a diffusion process with random jumps from boundaries. We discuss three applications of the representation, one on the pricing of dual-purpose funds, for which we use the representation to get a partial differential equation (PDE) so that we can price these funds numerically, one on the connection of our results to regenerative processes, and the final example on modeling the entropy on a one-dimensional non-rigid body, for which we use the representation in the opposite direction, i.e. from the PDE we can find a stochastic representation which yields a different interpretation of the entropy.

1.1. Main Result.

We consider a Feynman-Kac type representation for a class of linear parabolic PDEs on a bounded smooth domain $\mathcal{O} \subset \mathbb{R}^d$, $d \geq 1$ with nonlocal boundary and terminal conditions in the following form

$$-\frac{\partial W}{\partial t}(t,x) - \mathcal{L}W(t,x) = f(t,x) \qquad \text{in } Q \tag{1}$$

$$W(t,x) = h(t,x) + \mathcal{B}W(t,x) \qquad \text{on } \partial_p Q, \tag{2}$$

where the interior domain is $Q := [0,T)\times\mathcal{O}$, the parabolic boundary is $\partial_p Q := (\{T\}\times\overline{\mathcal{O}})\cup([0,T)\times\partial\mathcal{O})$ taken as the union of terminal time and lateral boundary. Here the generator is

$$\mathcal{L}W = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\frac{\partial^2 W}{\partial x_i\partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial W}{\partial x_i} - rW.$$

The interesting feature is about the nonlocal boundary and terminal conditions:

$$\mathcal{B}W(t,x) = \iint_Q W(s,z)\nu_{t,x}(ds,dz), \tag{3}$$

where $\{\nu_{t,x}\}_{(t,x)\in\partial_p Q}$ is a collection of positive Borel measures on Q.

Two examples of the nonlocal condition (3) to be used in our later applications are: (1) By taking $\nu_{t,x}(ds,dz) = \delta_{0,g(x)}(ds,dz)$, i.e. the Dirac measure concentrated at $(0,g(x))$ for a function $g:\overline{\mathcal{O}}\to\mathcal{O}$, we get

$$\mathcal{B}W(t,x) = W(0,g(x)). \tag{4}$$

(2) By taking $\nu_{t,x}(ds,dz) = \delta_t(ds)K(x,z)dz$ for some function K on $\partial\mathcal{O}\times\overline{\mathcal{O}}$ and $x\in\partial\mathcal{O}$, we get

$$\mathcal{B}W(t,x) = \int_{\mathcal{O}} W(t,z)K(x,z)dz. \tag{5}$$

The main goal of this paper is to show that, under certain conditions to be given in Section 2, there is a unique solution to the above problem, which can be probabilistically represented as

$$\begin{aligned} W(t,x) = E_t^x\Bigg[&\int_t^\infty e^{-r(s-t)}\left(\prod_{t\le\theta_j<s}\overline{\nu}(v_{\theta_j-},X_{\theta_j-})\right)f(v_{s-},X_{s-})ds \\ &+\sum_{\theta_i\ge t} e^{-r(\theta_i-t)}\left(\prod_{t\le\theta_j<\theta_i}\overline{\nu}(v_{\theta_j-},X_{\theta_j-})\right)h(v_{\theta_i-},X_{\theta_i-})\Bigg] \end{aligned} \tag{6}$$

for all $(t,x)\in\overline{Q}$, where (v,X) is a càdlàg process satisfying

$$dX_s = b(v_s,X_s)ds + \sigma(v_s,X_s)dB_s, \qquad s\in(\theta_i,\theta_{i+1}), i\ge 0, \tag{7}$$

$$v_s = v_{\theta_i} + (s-\theta_i), \qquad s\in(\theta_i,\theta_{i+1}), i\ge 0.$$

Here $(\theta_i)_{i\ge1}$ is a sequence of passage times of (v,X) crossing $\partial_p Q$, at which (v,X) jumps into Q, and $\theta_0 = t$ for notational convenience; $\overline{\nu}(t,x) := \nu_{t,x}(Q)$; E_t^x means the expectation computed under the initial condition $v_{t-} = t, X_{t-} = x$; and the summation and products in (6) are over all $i\ge1$ and $j\ge1$ satisfying the constraints, respectively. Although our formulation is a terminal-boundary value problem, by a straightforward time-reverse we can also cover initial-boundary value problem. Figure 1 shows one sample path of (v,X).

The representation (6) has a financial interpretation as the expected present value of a future cash flow from a financial asset, consisting of a continuous payoff at rate f and a series of time-θ_i lump-sum payoffs $h(v_{\theta_i-},X_{\theta_i-})$, $i\ge1$. Starting from an initial position of one share in this asset, the nonincreasing process $\prod_{t\le\theta_j<s}\overline{\nu}(v_{\theta_j-},X_{\theta_j-})$, $s\ge t$ represents the quantity of the asset left immediately before time s.

We present three applications of the above representation. The first one, which uses the type of nonlocal condition in (4), is on the pricing of the dual-purpose funds (started in U.S. 1970's

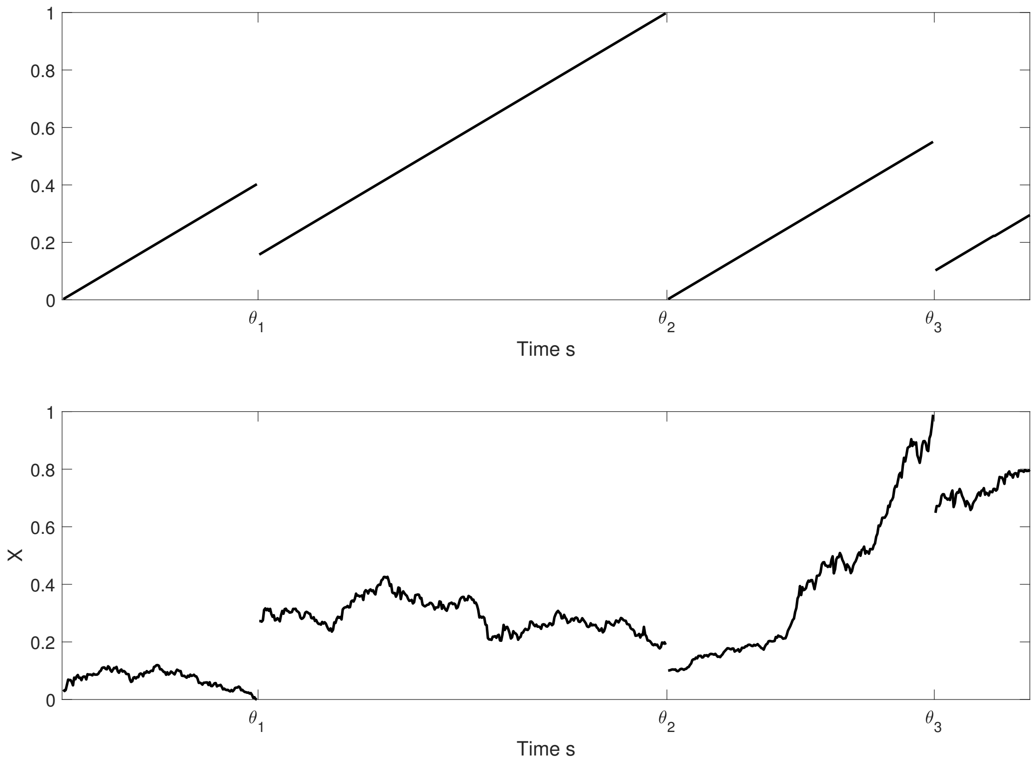

Figure 1. A Sample Path for (v, X).

Note. This figure illustrates one sample path of the process (v, X), where $\mathcal{O} = (0,1)$, $T = 1$. Therefore, the parabolic boundary $\partial_p Q = \{1\} \times \{0,1\}$, that is, (v, X) hits $\partial_p Q$ if and only if X hits 0 or 1, or v hits 1. Starting from time $s = 0$, (v, X) follows a space-time diffusion process. At θ_1 when X hits 0, X jumps back into $(0,1)$ and v jumps downwards, based on the jump measure $\tilde{\nu}_{v_{\theta_1-},0}$. The same kind of jumps of (v, X) occurs at θ_2 and θ_3, when v hits 1 and X hits 1, respectively.

and recently became very popular in China), where our representation result offers a rigorous characterization of the value of dual-purpose funds as well as an efficient numerical pricing method. We then discuss the connection between the process (v, X) and the regenerative process. We show that a special subclass of (v, X) is indeed regenerative, and present a stochastic representation result in terms of a regenerative process. As the third application, we discuss an example in thermodynamics theory modeling the entropy of a one-dimensional nonrigid body, which is based on the type of nonlocal condition in (5).

1.2. Literature Review. The Feynman-Kac representation result for classical solutions to linear parabolic terminal-boundary value problems with local terminal and boundary conditions has been well-documented in literature. For example, solutions to problems with Dirichlet type boundary condition can be represented by a diffusion process that is stopped on the boundary (c.f. Friedman (2012)); solutions to problems with Neumann type (or more generally, Robin type) boundary conditions can be represented via a diffusion process with reflection on boundary (c.f. Freidlin (1985)). For a class of Cauchy problems whose leading coefficient does not satisfy linear growth, Bayraktar and Xing (2010) derived the existence and uniqueness of classical solution as well as a stochastic representation (as the European option price).

A body of PDE literature discussed the existence, uniqueness and asymptotic behavior of solutions to parabolic initial-boundary value problems, with different locations of the nonlocalness

Table 1. Comparison between This Paper and Literature.

Panel I: Comparison on the Formulation of Nonlocal Initial/Boundary Conditions

	Nonlocalness Location	Type of ν	Discontinuity Allowed	Stochastic Representation
Chabrowski (1984a)	Initial Only	Discrete	No	No
Chabrowski (1984b)	Initial Only	Borel	No	No
Deng (1993) Olmstead and Roberts (1997)	Initial Only	Abs. Continuous	No	No
Friedman (1986)	Boundary Only	Abs. Continuous	No	No
Yin (1994), Pao (1995a)	Boundary Only	Abs. Continuous	No	No
Pao (1998)	Boundary Only	Abs. Continuous	No	No
Pao (1995b)	Initial & Boundary	Abs. Continuous	No	No
This Paper	Initial & Boundary	Borel	Yes	Yes

Panel II: Comparison on the Pricing of Dual-Purpose Funds

	Horizon	Complex Contract Clauses	Pricing Method
Ingersoll (1976)	Finite	No	PDE
Jarrow and O'Hara (1989)	Finite	No	PDE
Adams and Clunie (2006)	Finite	Yes	Monte Carlo
This Paper	Infinite	Yes	PDE

Panel III: Comparison with Regenerative Process Literature

	Main Focus
Sigman and Wolff (1993) Glasserman and Kou (1995)	Deriving approximations based on limiting theorem (e.g. limiting distribution)
This Paper	Showing that the expectation of functionals on exact solutions for some regenerative processes are the solutions of PDEs with nonlocal terminal and boundary conditions

Panel IV: Comparison with Inventory Management Literature

	Proving Existence of Solution to Nonlocal Problem	Showing Inventory Process is Regenerative
Cadenillas et al. (2010) Harrison and Sellke (1983)	Based on explicit solutions	No
This Paper	General proof not relying on finding explicit solutions	Yes

Note for Panel I: Nonlocalness location means whether the nonlocal condition appears in initial condition only, boundary condition only, or both initial and boundary conditions. The type of ν means whether ν is discrete, absolutely continuous (Abs. Continuous), or a general Borel measure. "Discontinuity Allowed" means whether $\nu_{t,x}$ can have discontinuous dependence on (t,x). "Stochastic Representation" means whether a stochastic representation result is obtained.

Note for Panel II: Horizon means whether the dual-purpose fund has an infinite life horizon or a finite life horizon; in China all dual-purpose funds have an infinite horizon. "Complex Contract Clauses" means whether the dual-purpose fund has a complex payoff scheme (e.g. from some protection mechanism).

(in initial/boundary data), types of measure $\nu_{t,x}(ds,dz)$ (discrete, absolutely continuous in s or z, or general Borel measures), and continuity of the measure $\nu_{t,x}$ with respect to its parameter. Table 1, Panel I summarizes the comparison between our work and the PDE literature on these specifications.

To our best knowledge, the current work is the first to explore the stochastic representation for parabolic problems with nonlocal terminal and boundary conditions, besides establishing the existence and uniqueness of solutions of nonlocal problems; in particular, we find that the solutions can be represented in terms of a space-time diffusion process with jumps from the parabolic boundary into the interior.

Furthermore, in addition to allowing nonlocalness in both initial (terminal) and boundary condition, and allowing ν to be Borel measure (instead of either discrete or absolutely continuous), different from the existing literature, we also allow $\nu_{t,x}$ to have discontinuous dependence on its parameter (t,x), not only for (t,x) in initial (terminal) time or lateral boundary, respectively, but also in their intersection. This kind of discontinuity is motivated and required by the application in pricing dual-purpose funds from finance, and with it, we look for solutions that are classical in the interior domain but can be discontinuous on the parabolic boundary. Here, $\nu_{t,x} = \delta_{0,g(x)}$ as in example (4). Specifically, on pre-specified payment dates $(t = T)$, the (transformed) underlying fund value jump from x to $g(x)$ as defined in (79). The discontinuity of g at $\{0,1\}$ then translates to the discontinuity of $\nu_{t,x}$ at $\{T\} \times \{0,1\}$. In addition, this kind of discontinuity also arises in our thermoelasticity example due to a technical reason. More precisely, although the original physical example has nonlocalness only in boundary and ν defined in boundary is continuous, when we transform it into our standard form (1), ν is discontinuous at the intersection between initial time and boundary.

We give three examples to illustrate the applications of our representation result. Our first example from finance is on the pricing of dual-purpose funds, originated in U.S. in 1970's, and has since been significantly modified and is currently very popular in China. This is a type of structured mutual funds that are split into lower risk/return shares and higher risk/return shares, each having a claim on the underlying asset pool. The perpetual interest and dividend payment cash flow of dual-purpose funds depends on the underlying fund value, which fluctuates in an interval and jumps back to the interior part once it hits the interval boundary according to some pre-specified rules. Our representation result provides a rigorous PDE characterization of dual-purpose funds value in terms of a parabolic PDE with both nonlocal boundary condition and terminal condition. This link with PDE also provides an iterative numerical algorithm for estimating the fund value, which is more efficient than the Monte Carlo simulation as suggested in Adams and Clunie (2006). Our result is also related with Ingersoll (1976), which focused on the pricing of dual-purpose funds in U.S. However, the model in Ingersoll (1976) is not applicable to the dual-purpose funds in China that have a more complicated payoff structure in an infinite time horizon. This calls for a more sophisticated valuation model, which partly motivated our study in this work.

As the second example, we discuss the relationship between (v, X) and regenerative processes. We find that a subclass of (v, X) with degenerate jump measure is indeed regenerative, but (v, X) is not regenerative in general. Our stochastic representation result then shows that the solution of PDEs with nonlocal terminal and boundary conditions can be represented in terms of regenerative processes with regeneration epochs being the hitting times of boundary of PDE's space-time domain. This result differs from the regenerative process literature in that, while the literature (e.g. Sigman and Wolff (1993); Glasserman and Kou (1995)) mainly focuses on approximation of regenerative processes based on limiting theorem (e.g. limiting distribution), we focus on the exact solutions for some regenerative processes, and show that the expectation of certain functionals on them is the solutions of PDEs with nonlocal terminal and boundary conditions. In the one-dimensional case, our representation for elliptic equations with nonlocal boundary condition corresponds to a regenerative underlying process which can be interpreted as the inventory level under a band control policy as discussed in Cadenillas et al. (2010) and Harrison and Sellke (1983). However, in contrast to these two papers in which explicit solutions are available, we derive the existence of solution in terms of a probabilistic representation that does not rely on the existence of explicit solutions.

Our third example from one-dimensional linear thermoelasticity theory models the change in the temperature distribution on a one-dimensional non-rigid body made from a homogeneous and isotropic material. Day (1982) found that, after a transformation, the entropy can be written as a solution to a parabolic equation with nonlocal boundary conditions. Our result gives the existence and uniqueness to the solution of this PDE, as well as a probabilistic representation of the unique solution. This probabilistic representation provides a new interpretation of the thermodynamics entropy in terms of particles.

This paper is organized as follows. Section 2 introduces notations and the basic formulation of the problem; Section 3 presents the main result of this paper (Theorem 3.1), a Feynman-Kac type representation for the nonlocal problem, establishing the existence and uniqueness of solutions to the PDE. Section 4 discusses the applications of our representation result. All proofs are deferred to Appendix.

2. Statement of the Problem. In this section we will first formulate the parabolic PDE with nonlocal terminal and boundary conditions, and then construct the underlying process for the stochastic representation. Based on this construction, we finally discuss the stochastic form of the value functions to be used in the representation in Section 3.

2.1. The Nonlocal Problem. Consider the parabolic terminal-boundary value problem (1) – (3) on $\overline{Q}$. We assume $b = (b_i)_{1\le i\le d}$ and $\sigma = (\sigma_{ij})_{1\le i,j\le d}$ are continuous functions satisfying the following conditions: $\forall (t,x), (\hat{t},\hat{x}) \in [0,T]\times\mathbb{R}^d$,

$$|b(t,x) - b(\hat{t},\hat{x})| + |\sigma(t,x) - \sigma(\hat{t},\hat{x})| \le \overline{K}(|x-\hat{x}| + |t-\hat{t}|), \tag{8}$$

$$|b(t,x)| + |\sigma(t,x)| \le \overline{K}(1+|x|), \tag{9}$$

and the uniform ellipticity condition: there exists $\Lambda > 0$ such that for all $\xi \in \mathbb{R}^d$ and $(t,x) \in Q$,

$$\sum_{i,j=1}^{d} a_{ij}(t,x)\xi_i\xi_j \ge \Lambda|\xi|^2, \tag{10}$$

where $a := \sigma\sigma^T$. For all $(t,x) \in \partial_p Q$, we further assume that $\nu_{t,x}$ is a positive Borel measure on Q such that

$$0 \le \overline{\nu}(t,x) := \nu_{t,x}(Q) \le 1$$

and $\nu_{t,x}((t,T)\times\mathcal{O}) = 0$; r is a positive constant; f is uniformly Hölder continuous (with exponent α) on $\overline{Q}$, and we denote $K_f = \max_{x\in\overline{Q}}|f(x)|$. We also assume the mapping $(t,x) \mapsto \nu_{t,x}$ and h defined on $\partial_p Q$ are Borel measurable (where the measurability of $\nu_{t,x}$ is with respect to the weak convergence topology), and satisfy the following continuity assumption.

Assumption 2.1. *The function h and the mapping $(t,x)\mapsto \nu_{t,x}$ are continuous on $\partial_p Q\backslash D$, where $D \subset \partial_p Q$ is a closed set satisfying the following property: there exist $\mathcal{T} = \{t_1,\cdots,t_N\} \subset [0,T]$ and $\lambda \in (0,d)$, such that for any $\varepsilon > 0$, there exist $s_1,\cdots,s_M \in \mathcal{T}$ and a finite collection of d-dimensional balls $B_1,\cdots,B_M$ such that $D \subset \cup_{i=1}^M(\{s_i\}\times B_i)\cap\partial_p Q$ and $\sum_{i=1}^M(\mathrm{diam}\ B_i)^\lambda < \varepsilon$, where* diam B *denotes the diameter of set B.*

Under Assumption 2.1, for any bounded function $u \in C(Q)$, we have that the terminal/boundary data

$$h(t,x) + \mathcal{B}u(t,x)$$

is also continuous on $\partial_p Q\backslash D$. The set D in Assumption 2.1 can include the lateral boundary at a finite collection of time points; in particular, it can include the intersection between the terminal time and lateral boundary, $\{T\}\times\partial\mathcal{O}$. In other words, Assumption 2.1 allows for inconsistencies between terminal and boundary conditions. The assumption on the covering balls together with the condition $\sum_{i=1}^M(\mathrm{diam}\ B_i)^\lambda < \varepsilon$ requires the set of discontinuity point at the terminal time T

to be a Lebesgue-null set (in fact, it has a λ-dimensional Hausdorff measure 0, and hence has a Hausdorff dimension no greater than λ) and nowhere dense in $\overline{\mathcal{O}}$.

EXAMPLES. Our formulation of the nonlocal boundary and terminal conditions can cover several cases in literature. (The literature mainly focuses on initial-boundary value problems, which are equivalent to the terminal-boundary value problem after a time-reverse $\tau = T - t$.) For instance, if we take

$$\begin{aligned} \nu_{t,x}(ds,dz) &= \begin{cases} \delta_t(ds)K(x,z)dz & \text{if } (t,x)\in[0,T)\times\partial\mathcal{O} \\ 0 & \text{otherwise} \end{cases} \\ h(t,x) &= \begin{cases} 0 & \text{if } (t,x)\in[0,T)\times\partial\mathcal{O} \\ \Psi(t,x) & \text{otherwise,} \end{cases} \end{aligned} \tag{11}$$

where $K\in C(\partial\mathcal{O}\times\overline{\mathcal{O}})$ nonnegative, $\int_{\mathcal{O}} K(x,z)dz \le 1,\ \forall x\in\partial\mathcal{O}$, and $\Psi\in C(\overline{\mathcal{O}})$, we get the type of nonlocal boundary condition and local initial condition in Friedman (1986); Pao (1995a, 1998):

$$W(t,x) = \int_{\mathcal{O}} K(x,z)W(t,z)dz \qquad \text{in } [0,T)\times\partial\mathcal{O} \tag{12}$$

$$W(T,x) = \Psi(x) \qquad \text{on } \overline{\mathcal{O}}. \tag{13}$$

On the other hand, if we take

$$\begin{aligned} \nu_{t,x}(ds,dz) &= \begin{cases} \sum_{i=1}^N \beta_i(x)\delta_{t_i,x}(ds,dz) & \text{if } (t,x)\in\{T\}\times\mathcal{O} \\ 0 & \text{otherwise} \end{cases} \\ h(t,x) &= \begin{cases} \Psi(x) & \text{if } (t,x)\in\{T\}\times\mathcal{O} \\ \phi(t,x) & \text{otherwise,} \end{cases} \end{aligned} \tag{14}$$

where for $i=1,\cdots,N$, $0\le t_i<T$, $\beta_i\ge 0$ and $\sum_{i=1}^N \beta_i(x)\le 1,\ \forall x\in\overline{\mathcal{O}}$, $\beta_i,\Psi\in C(\overline{\mathcal{O}})$, $\phi\in C(\overline{Q})$, we obtain the type of nonlocal initial condition and local boundary condition in Chabrowski (1984a):

$$\begin{aligned} W(t,x) &= \phi(t,x) && \text{on } [0,T]\times\partial\mathcal{O} \\ W(T,x) &= \Psi(x) + \sum_{i=1}^N \beta_i(x)W(t_i,x) && \text{in } \mathcal{O}. \end{aligned}$$

Note that $\nu_{t,x}$ and h as in (11) and (14) are continuous on $\partial_p Q\backslash D$ with $D=\{T\}\times\partial\mathcal{O}$. In particular, by taking $N=1$, $t_1=0$ and h such that $h(T,\cdot)=h(0,\cdot)$ in (14), we get the time-periodic parabolic problems as discussed in Lieberman (1999).

The specification of nonlocal operator in Chabrowski (1984b) is the closest to ours. Although Chabrowski (1984b) started with a general framework where the nonlocalness in the initial condition is in terms of a mapping F, the existence and uniqueness result for nonzero local boundary condition requires F to be defined in terms of a parameterized collection of positive Borel measures in a manner similar to (3), and the collection of measures needs to have continuous dependence on the parameter. Our formulation is more general than Chabrowski (1984b), as we allow this type of nonlocalness in both the terminal and boundary conditions, and the collection of measures can have discontinuous dependence on the parameter.

2.2. The Underlying Process. Next we give a construction of the underlying process for the representation. Take a filtered probability space $(\Omega,\mathbb{F},(\mathcal{F}_t)_{t\ge0},P)$ satisfying the usual assumptions, where a collection of independent d-dimensional Brownian motions $(B^i)_{i\ge1}$ is defined. Consider the state process (v,X,Y) valued in $[0,T]\times\overline{\mathcal{O}}\times[0,1]$ defined as follows. Starting from a point $(u,x,y)\in Q\times[0,1]$, v evolves as $dv_s=ds$, X evolves as (7) with B replaced by B^1, and Y is constant, until the first passage time

$$\theta_1 = \inf\{s>0: X_{s-}\in\partial\mathcal{O} \text{ or } v_{s-}=T\},$$

at which (v,X) exits Q by crossing $\partial_p Q$ at $\xi_1=(v_{\theta_1-},X_{\theta_1-})\in\partial_p Q$. At θ_1, (v,X) instantaneously jumps to a random point (v_1,x_1) in Q with distribution $\tilde\nu_{\xi_1}(ds,dz)$, where for $(v_0,x_0)\in\partial_p Q$, the probability measure $\tilde\nu_{v_0,x_0}(ds,dz)$ in Q is defined as

$$\tilde\nu_{v_0,x_0}(ds,dz)=\begin{cases}\nu_{v_0,x_0}(ds,dz)/\overline{\nu}(v_0,x_0) & \text{if } \overline{\nu}(v_0,x_0)>0\\ \delta_{v',x'}(ds,dz) & \text{if } \overline{\nu}(v_0,x_0)=0,\end{cases}\tag{15}$$

where we fix $v'=0$ and $x'\in\mathcal{O}$. Also, $Y_{\theta_1}=\overline{\nu}(\xi_1)\cdot Y_{\theta_1-}$. After the jump, v and X continue as $dv_s=ds$ and (7) with B replaced by B^2, respectively, and Y remains constant. And this same jump mechanism is repeated independently every time the process (v,X) hits $\partial_p Q$. Denote θ_i as the i-th passage time of $\partial_p Q$. Note that by construction,

$$Y_s=y\cdot\prod_{\theta_i\le s}\overline{\nu}(v_{\theta_i-},X_{\theta_i-}).\tag{16}$$

Also, due to the nature of v, the inter-jump times $\Delta\theta_i=\theta_{i+1}-\theta_i\le T$, a.s., therefore $\partial_p Q$ will be hit for an infinite number of times almost surely. Therefore, we have an infinite, increasing sequence of jump times $(\theta_i)_{i\ge1}$.

The state process (v,X,Y) has the following intuitive interpretation. X is the main underlying state process representing the location of a particle with mass Y, moving in $\mathcal{O}$ as a diffusion which depends on the clock v. At the time when the particle hits the domain boundary or the clock hits T, it jumps to a new point in $\mathcal{O}$ and the clock jumps backwards, all based on the probability measure $\tilde\nu$. As the particle jumps, it losses $1-\overline{\nu}(\xi)$ fraction of its mass, where ξ is the clock status and particle location right before the jump. In particular, at a point ξ' where $\overline{\nu}(\xi')=0$, the particle mass becomes 0 and remains 0 forever. After the jump, the particle and clock keep going until the next jump.

2.3. A Non-Explosive Condition. This type of processes has been studied in Grigorescu and Kang (2013). By definition (v,X,Y) is a time-homogeneous Markov process. To ensure that it is well-defined for all $t<\infty$, one needs to make sure that it is non-explosive, that is, $(\theta_i)_{i\ge1}$ tends to $+\infty$ a.s. The proof for non-explosiveness in Grigorescu and Kang (2013) requires the nondegeneracy of the transition probability. However, in our case, v is degenerate. To establish the non-explosiveness in our setting, we need the following assumption.

Assumption 2.2. *We assume that there exists $t_0\in[0,T)$, $\delta>0$ and $0<p_0\le1$, such that*

$$\tilde\nu_{v_0,x_0}([0,t_0]\times\mathcal{O})\ge p_0,\ \text{if } v_0=T\tag{17}$$

$$\tilde\nu_{v_0,x_0}([0,v_0]\times\mathcal{O}_\delta)\ge p_0,\ \text{if } v_0<T,\tag{18}$$

for all $(v_0,x_0)\in\partial_p Q$, where $\mathcal{O}_\delta=\{x\in\mathcal{O}:dist(x,\partial\mathcal{O})>\delta\}$. Note that since there exists $\delta>0$ such that $x'\in\mathcal{O}_\delta$, (17) and (18) are trivially satisfied at the point (v_0,x_0) where $\overline{\nu}(v_0,x_0)=0$.

Examples. Following the discussion in Section 2.1, Assumption 2.2 is weak enough to include all the examples in this paper under reasonable conditions. For the case (11), to satisfy Assumption 2.2, we need to assume that there exists $\delta>0$ and $0<p_0<1$ such that for all $x\in\partial\mathcal{O}$,

$$\frac{\int_{\mathcal{O}_\delta}K(x,z)dz}{\int_{\mathcal{O}}K(x,z)dz}\ge p_0.\tag{19}$$

In view of the continuity of K, a sufficient condition is $\int_{\mathcal{O}}K(x,z)dz>0$ for all $x\in\partial\mathcal{O}$. This way, the measure $\mu_x(A):=\frac{\int_A K(x,z)dz}{\int_{\mathcal{O}}K(x,z)dz}$ is continuous with respect to $x\in\partial\mathcal{O}$, and for any $0<p_0<1$, (19) must be satisfied for some $\delta > 0$, otherwise it will lead to the contradiction that $\mu_{x_0}(\mathcal{O}) < 1$ for some $x_0 \in \partial\mathcal{O}$. For the case (14), Assumption 2.2 is automatically satisfied by taking $t_0 = \max\{t_1, \cdots, t_n\}$.

Intuitively speaking, Assumption 2.2 imposes tightness on the jump measures: (17) says that the jump measures starting from the terminal time T assign a uniform probability for a set that is bounded away from the terminal time; (18) says that the jump measures starting from the boundary assign a uniform probability for a set that is bounded away from the boundary. These two conditions rule out the possibility of $\partial_p Q$ being hit for an infinite number of times in a short period, and therefore (v, X) is non-explosive, as is shown in the following lemma.

Lemma 2.1 **(Non-explosiveness)**. *Under Assumption 2.2, the increasing sequence $(\theta_i)_{i\geq 1}$ tends to $+\infty$ a.s. Furthermore, for any $r > 0$, we have*

$$\lim_{n\to\infty} \sup_{(u,x)\in Q} E^{u,x}\left[e^{-r\theta_n}\right] = 0, \tag{20}$$

where $E^{u,x}$ means the expectation calculated under the initial position (u, x) for (v, X).

2.4. The Value Functions. For the stochastic representation in Section 3, consider the following infinite-horizon problem for $(t, x) \in \overline{Q}$ and $y \in [0, 1]$:

$$\begin{aligned} V(t,x,y) = E_t^{t,x,y}\Bigg[&\int_t^\infty e^{-r(s-t)} Y_{s-} f(v_{s-}, X_{s-})ds \\ &+ \sum_{\theta_i \geq t} e^{-r(\theta_i - t)} Y_{\theta_i -} h(v_{\theta_i -}, X_{\theta_i -})\Bigg], \end{aligned} \tag{21}$$

where $E_t^{u,x,y}$ means the expectation computed under the initial condition $v_{t-} = u, X_{t-} = x, Y_{t-} = y$. That is, for $(u, x) \in Q$, the right hand side is equal to the expectation under the initial condition $v_t = u, X_t = x, Y_t = y$; for $(u, x) \in \partial_p Q$, however, by viewing t as a passage time of $\partial_p Q$, the right hand side expression equals $y \cdot h(u, x)$ plus the expectation computed under the initial distribution $\tilde{\nu}_{u,x}$ for (v, X) and $Y_t = \overline{\nu}(u, x) \cdot y$. To simplify notations, unless otherwise stated, in the following we always replace $E_t^{t,x,1}$ by E_t^x, omit the initial value of y when it is irrelevant to the expectation, and omit the subscript when $t = 0$.

In addition to the infinite-horizon problem, we also consider its accompanying finite-horizon problem: for $N = 1, 2, \cdots$, we define

$$\begin{aligned} \tilde{V}_N(t,u,x,y,m) = E_t^{u,x,y}\Bigg[&\int_t^{\gamma_N^{t,m}} e^{-r(s-t)} Y_{s-} f(v_{s-}, X_{s-})ds \\ &+ \sum_{t \leq \theta_i \leq \gamma_N^{t,m}} e^{-r(\theta_i - t)} Y_{\theta_i -} h(v_{\theta_i -}, X_{\theta_i -})\Bigg], \end{aligned} \tag{22}$$

for $t, m \in [0, NT]$, $u \in [0, T)$, $x \in \mathcal{O}$ and $y \in [0, 1]$, where

$$\begin{aligned} \gamma_N^{t,m} &= \inf\{s \geq t : NT - s - m_s^{t,m} \leq 0\}, \quad \gamma_N^t = \gamma_N^{t,0}, \\ m_s^{t,m} &= m + \sum_{t \leq \theta_i \leq s} (T - v_{\theta_i -} + v_{\theta_i}), \quad m_s^t = m_s^{t,0}. \end{aligned}$$

Note that the jumps of $m_s^{t,m}$ are bounded by T, since $v_{\theta_i} \leq v_{\theta_i -}$ by definition. We then denote

$$V_N(t,x,y) = \tilde{V}_N(t, t - \lfloor t \rfloor_T, x, y, 0),$$

where $\lfloor t \rfloor_T$ denotes the largest integer multiple of T that is no greater than t. Intuitively speaking, starting at $t \in [0, T)$, different from the infinite-horizon problem which goes on forever, the finite-horizon problem V_N ends at the N-th passage time θ_N. Therefore, one can expect that V_N converges to V as N goes to infinity. We will establish this in Lemma A.2.

Since the right hand side of (21) has a linear dependence on Y, and Y only changes at the collection of passage times, $(\theta_i)_{i\geq 1}$, at which its value jumps downward proportionally as indicated by (16), it is straightforward to see that $V(t,x,y)$ is homogeneous in y, that is,

$$V(t,x,y) = y \cdot V(t,x,1).$$

This is also true for V_N. So in the following, we will use $V(t,x)$ and $V_N(t,x)$ to denote $V(t,x,1)$ and $V_N(t,x,1)$ for simplicity.

To guarantee the finiteness of V and V_N, we assume that h has a linear growth with respect to s, that is,

$$|h(s,x)| \leq K_h \cdot s, \tag{23}$$

where K_h is a positive constant. It turns out that the boundedness of f, the decrease of Y at $(\theta_i)_{i\geq 1}$ and the growth condition on h together lead to the finiteness of value functions of both the infinite and finite horizon problems, as shown by the following proposition.

Proposition 2.1. *(The finiteness of value functions of both the infinite and finite horizon problems). Under Assumption 2.2 and assuming (23), $|V|$ and $|V_N|$ are bounded by $K_V := K_f/r + 4TMK_h/(1-p_2)$, where p_2 and M are as defined in (51).*

3. A Feynman-Kac Representation for Nonlocal Problems. All the assumptions in Section 2 are assumed throughout this section.

Theorem 3.1. *(A Feynman-Kac Representation). The terminal-boundary value problem (1) – (2) has a unique bounded solution $W \in C^{1,2}(Q) \cap C(\overline{Q} \backslash D)$, which can be represented as (6) for all $(t,x) \in \overline{Q}$.*

The right hand side expectation in (6) has an intuitive financial interpretation as the expected present value of a contingent future cash flow from a financial asset. The cash flow consists of continuous payments at the rate $f(v_{s-}, X_{s-})$ and a series of time-θ_i lump-sum payments $h(v_{\theta_i -}, X_{\theta_i -})$, both of which are dependent on the clock v and underlying asset value X. In addition, the quantity of the asset may decrease as the delivery of lump-sum payoffs. Starting from a unit position in this asset, this decrease is captured by the nonincreasing process $\prod_{t\leq\theta_j<s} \overline{\nu}(v_{\theta_j -}, X_{\theta_j -})$, $s \geq t$ representing the remaining quantity of the asset immediately before time s. Finally, the exponential factor $e^{-r(\cdot - t)}$ discounts the future values of the cash flow to present values. In Section 4, we will use this interpretation to connect the representation (6) with the value of dual-purpose funds.

There are three main difficulties in establishing Theorem 3.1: handling discontinuities (as in Assumption 2.1), nonlocalness of the terminal and boundary conditions, and deriving the stochastic representation. First, to handle the difficulty of discontinuities, we extend the classic Feynman-Kac representation (c.f. Theorem 5.2 in Friedman (2012)) by allowing the terminal and boundary data h to have discontinuities in a set $D \subset \partial_p Q$ (and bounded). By approximating h by a sequence of continuous functions, Proposition A.1 shows that the representation in the classic Feynman-Kac representation for Dirichlet problems still holds despite the discontinuity of h.

Secondly, the presence of nonlocalness in the terminal and boundary data creates difficulty for establishing both the existence and representation of the solution. The typical PDE argument for proving the existence of a classical solution to Dirichlet problems requires a priori regularity of the terminal and boundary data. However, this a priori regularity is typically not possessed by the nonlocal problems, since the regularity of the terminal and boundary data in turn depends on the regularity of solution that is not obtained yet. In view of this, we prove the existence of a bounded solution by an iterative construction in three steps:

(i) Starting from $W_0 = 0$, we iteratively define W_{i+1}, $i = 0, 1, \cdots$ as the unique bounded solution to the following equation

$$-\frac{\partial W}{\partial t}(t,x) - \mathcal{L}W(t,x) = f(t,x) \qquad \text{in } Q \tag{24}$$

$$W(t,x) = h(t,x) + \mathcal{B}W_i(t,x) \qquad \text{on } \partial_p Q. \tag{25}$$

The existence and uniqueness of a bounded solution to this problem is guaranteed by the extended Feynman-Kac representation for Dirichlet problems (Proposition A.1).

(ii) We prove that the sequence $(W_i)_{i\geq 1}$ constructed above converges uniformly on $\overline{Q}$ via a probabilistic argument, by showing that the solution W_i to (24) – (25) equals the finite-horizon value function V_i (as defined in Section 2.4) in Q for each i, and proving that $(V_i)_{\geq 1}$ converges in Q uniformly to the infinite-horizon value function V defined in (21).

(iii) Since W_i, $i \geq 1$ satisfy (24), a typical application of Schauder interior estimate and Arzelà-Ascoli theorem establishes that the limit W of $(W_i)_{i\geq 1}$ satisfies (1) in Q, and the uniform convergence gives W the desired continuity in $\overline{Q}\backslash D$.

Remark 3.1. The above construction and convergence of $(W_i)_{i\geq 1}$ also provide an iterative algorithm of numerically solving the parabolic PDE with nonlocal terminal and boundary data (1) – (2). Starting from the initial guess $\tilde{W}_0 = 0$, each time we plug in the value of $\tilde{W}_i$ in the last iteration step into the terminal and boundary condition and solve the local PDE (24) – (25) numerically to get $\tilde{W}_{i+1}$. Then, step (i) – (iii) above show that the sequence of functions $(\tilde{W}_i)_{i\geq 0}$ converges uniformly to the unique solution of the nonlocal problem.

Thirdly, as for the stochastic representation, from the (extended) Feynman-Kac representation for Dirichlet problems, any bounded solution $W \in C^{1,2}(Q) \cap C(\overline{Q}\backslash D)$ of (1) – (2) can be written as

$$W(t,x) = E_t^x \left[\int_t^\theta e^{-r(s-t)} f(v_s, \xi_s) ds + e^{-r(\theta-t)} \Big(h(\theta, \xi_\theta) + \mathcal{B}W(\theta, \xi_\theta) \Big) \right],$$

However, this is still an equation for W rather than a real stochastic representation, and it does not necessarily imply the uniqueness of solution. We solve this equation by iteratively substituting W (the whole right hand side expression) into $\mathcal{B}W(\theta, \xi_\theta)$, and showing that this procedure converges to the representation (6). Two essential ingredients for this convergence are the non-explosiveness established in Lemma 2.1 and the boundedness of value functions in Proposition 2.1.

3.1. Some Remarks. The idea of establishing the existence of solutions via constructing an approximating sequence of solutions was also used in Friedman (1986) and Pao (1995a,b). Friedman (1986) showed that the sequence is a uniform Cauchy sequence, thanks to the maximum principle and the assumption that $\overline{\nu}(t,x) = \rho < 1$. However, this assumption is not necessarily valid in our framework. On the other hand, assuming the existence of a pair of lower and upper solutions, Pao (1995a,b) proved that the sequence starting from an upper/lower solution is monotone with a corresponding lower/upper bound, and hence established the convergence. However, the construction of the upper/lower solution is not straightforward in general. In contrast, the sequence in our work is not necessarily monotone, as it starts from 0 that is not necessarily an upper/lower solution. Our construction has a clear probabilistic interpretation via the characterization of $(W_i)_{i\geq 1}$ using the value functions $(V_i)_{i\geq 1}$ of finite-horizon problems, which also leads to the uniform convergence of the sequence.

In the special case where only one of the boundary or terminal condition is nonlocal, the solution can be continuous at $\{T\} \times \partial\mathcal{O}$ even if $\nu_{t,x}$ and h are discontinuous there, provided that the boundary and terminal conditions are consistent. Take the case (12) studied in Friedman (1986) as an example. Recall that $K \in C(\partial\mathcal{O} \times \overline{\mathcal{O}})$, $\Psi \in C(\overline{\mathcal{O}})$, but $\nu_{t,x}$ and $h(t,x)$ as defined in (11) are not necessarily continuous on $\{T\} \times \partial\mathcal{O}$. From Theorem 3.1 we can claim that the solution $W \in C(\overline{Q} \backslash (\{T\} \times \partial\mathcal{O}))$. However, it can be shown that $W \in C(\overline{Q})$ if and only if the boundary condition and terminal condition are consistent in the following way

$$\Psi(x_0) = \int_{\mathcal{O}} K(x_0, z)\Psi(z)dz, \quad \forall x_0 \in \partial\mathcal{O}. \tag{26}$$

that is, the (local) terminal data Ψ satisfies the (nonlocal) boundary condition. This condition (26) seems to be missing in Friedman (1986); Yin (1994).

3.2. Three Extensions of Theorem 3.1. For the application in pricing dual-purpose funds, we need to consider PDE (1) together with the terminal and boundary condition

$$W(t,x) = \tilde{h}(t,x,\overline{\nu}(t,x)) + \mathcal{B}W(t,x) \quad \text{on } \partial_p Q \tag{27}$$

instead of (2), where $\tilde{h} : \partial_p Q \times [0,1] \to \mathbb{R}$ is measurable and has the additional dependence on $\overline{\nu}(t,x)$. Corollary 3.1 says that the stochastic representation result still holds if $\tilde{h}$ satisfies growth and continuity assumptions similar to those of h.

Corollary 3.1. *Assume $\tilde{h} \in C((\partial_p Q \backslash D) \times [0,1])$ and satisfies*

$$|\tilde{h}(s,x,u)| \leq K_{\tilde{h}}(1 - u + s), \tag{28}$$

then the terminal-boundary value problem (1) and (27) has a unique bounded solution $W \in C^{1,2}(Q) \cap C(\overline{Q} \backslash D)$, which can be represented as

$$\begin{aligned} W(t,x) = E_t^x \Bigg[&\int_t^\infty e^{-r(s-t)} \left(\prod_{t \leq \theta_j < s} \overline{\nu}(v_{\theta_j -}, X_{\theta_j -}) \right) f(v_{s-}, X_{s-}) ds \\ &+ \sum_{\theta_i \geq t} e^{-r(\theta_i - t)} \left(\prod_{t \leq \theta_j < \theta_i} \overline{\nu}(v_{\theta_j -}, X_{\theta_j -}) \right) \tilde{h}(v_{\theta_i -}, X_{\theta_i -}, \overline{\nu}(v_{\theta_i -}, X_{\theta_i -})) \Bigg], \end{aligned}$$

for $(t,x) \in \overline{Q}$.

For the application in linear thermodynamics, we need to modify the definition of the nonlocal operator $\mathcal{B}$ in (3) to

$$\tilde{\mathcal{B}}W(t,x) = -\iint_Q W(s,z)\nu_{t,x}(ds,dz), \tag{29}$$

and consider PDE (1) together with the terminal and boundary condition

$$W(t,x) = h(t,x) + \tilde{\mathcal{B}}W(t,x) \quad \text{on } \partial_p Q. \tag{30}$$

Note that the only difference of (29) from (3) is the minus sign before the integral. Similar to Theorem 3.1, we have the following result.

COROLLARY 3.2. *The terminal-boundary value problem* (1) *and* (30) *has a unique bounded solution* $W \in C^{1,2}(Q) \cap C(\overline{Q} \backslash D)$*, which can be represented as*

$$W(t,x) = E_t^x \Bigg[\int_t^\infty e^{-r(s-t)} \Bigg(\prod_{t \le \theta_j < s} -\overline{\nu}(v_{\theta_j -}, X_{\theta_j -}) \Bigg) f(v_{s-}, X_{s-}) ds + \sum_{\theta_i \ge t} e^{-r(\theta_i - t)} \Bigg(\prod_{t \le \theta_j < \theta_i} -\overline{\nu}(v_{\theta_j -}, X_{\theta_j -}) \Bigg) h(v_{\theta_i -}, X_{\theta_i -}) \Bigg],$$

for $(t,x) \in \overline{Q}$.

Finally, we also consider the following stochastic representation for elliptic equations. We assume that $b(t,x)$, $\sigma(t,x)$, $f(t,x)$ and $h(t,x)$ are constant in t, and denote them as $b(x)$, $\sigma(x)$, $f(x)$ and $h(x)$ for simplicity, and

$$\mathcal{L}'W = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 W}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial W}{\partial x_i} - rW.$$

Also assume that $h \in C(\overline{\mathcal{O}})$. Let $(\mu_x)_{x \in \partial \mathcal{O}}$ be a collection of Borel measures on $\mathcal{O}$, such that $x \mapsto \mu_x$ is continuous, and $0 < \overline{\mu}(x) := \mu_x(\mathcal{O}) \le 1, \forall x \in \partial \mathcal{O}$. Finally, we define (v, X) as in Section 2.2, with

$$\tilde{\nu}_{v_0, x_0}(ds, dz) = \begin{cases} \delta_{v_0}(ds) \cdot \mu_{x_0}(dz) / \overline{\mu}(x_0) & \text{if } v_0 < T \\ \delta_0(ds) \cdot \mu_{x_0}(dz) / \overline{\mu}(x_0) & \text{if } v_0 = T \text{ and } x_0 \in \partial \mathcal{O} \\ \delta_0(ds) \cdot \delta_{x_0}(dz) & \text{if } v_0 = T \text{ and } x_0 \in \mathcal{O} \end{cases} \tag{31}$$

Note that ν satisfies Assumption 2.2. Furthermore, X is a time-homogeneous Markov process independent of v. Indeed, the drift and diffusion coefficients b and σ are independent of time; also, X jumps if and only if it hits $\partial \mathcal{O}$, with a distribution $\mu_{x_0}(dz)/\overline{\mu}(x_0)$ depending only on the hitting point $x_0 \in \partial \mathcal{O}$. On the other hand, under initial condition $v_t = t$, we have $v_s = s - \lfloor s \rfloor$.

COROLLARY 3.3. *The elliptic boundary value problem*

$$\mathcal{L}'w(x) = -f(x) \qquad \text{in } \mathcal{O} \tag{32}$$

$$w(x) = h(x) + \int_{\mathcal{O}} w(z) \mu_x(dz) \qquad \text{on } \partial \mathcal{O} \tag{33}$$

has a unique solution $w \in C^2(\mathcal{O}) \cap C(\overline{\mathcal{O}})$*, which can be represented as*

$$w(x) = E^x \Bigg[\int_0^\infty e^{-rs} \Bigg(\prod_{0 \le \theta_j < s} \overline{\mu}(X_{\theta_j -}) \Bigg) f(X_{s-}) ds + \sum_{\theta_i \ge 0} e^{-r\theta_i} \Bigg(\prod_{0 \le \theta_j < \theta_i} \overline{\mu}(X_{\theta_j -}) \Bigg) h(X_{\theta_i -}) \Bigg], \tag{34}$$

where X *is defined above.*

4. Applications of the Stochastic Representation for Nonlocal Problems.

As applications of the above theoretical results of stochastic representation for nonlocal problems, in this section we discuss three examples, one in finance for the pricing of dual-purpose funds, one for the connection with regenerative processes, and the other in linear thermoelasticity for modeling the temperature distribution on a one-dimensional non-rigid body. For the first application, one uses the representation to get a partial differential equation (PDE) so that one can price these funds numerically. For the third application, one uses the representation in the opposite direction, i.e. from the PDE one can find a stochastic representation which yields a different interpretation of the entropy.

4.1. Pricing of Dual-Purpose Funds. The dual-purpose funds are a type of structured mutual funds that can be split into lower risk/return shares (A shares) and higher risk/return shares (B shares), each having a claim on the underlying fund (here and in the following, unless otherwise mentioned, by dual-purpose funds we mean the type of funds in China, c.f. Panel II of Table 1). The underlying funds are typically index-tracking open-end funds, while the A and B shares are closed-end funds listed in the exchange. The A shares behave like a bond, while the B shares behave like a leveraged ETF. Thanks to a two-way conversion clause, the fund holders can "combine" their positions in A and B shares with a ratio in number of shares $\ell : 1-\ell$ into positions of the underlying fund ($0<\ell<1$), or do a split vice versa. Therefore, an absence-of-arbitrage argument implies that the market value of A shares (V_A), B shares (V_B) and the net asset value of the underlying fund (S) must satisfy the relationship

$$\ell V_A + (1-\ell)V_B = S. \tag{35}$$

In the following discussion we mainly focus on the pricing of A shares, since the value of B shares can be derived directly via (35).

We formulate the pricing model under the Black-Scholes framework. We assume that, during normal trading days when no interest/dividend payment occurs, S follows the SDE under the risk-neutral measure $\mathbb{Q}$

$$dS_s = (r-c)S_s ds + \sigma S_s dB_s, \tag{36}$$

where c is the management fee rate, and B is a one-dimensional Brownian motion. The net asset value of A shares grows linearly at the rate R (equals the one-year benchmark deposit rate plus a premium), i.e.

$$NAV_s^A = 1 + Rv_s, \tag{37}$$

where v_s is the time from the last interest payment date (either a pre-specified payment date or a contingent reset date) before s, satisfying $dv_s = ds$. The net asset value of B shares is then defined by contract as

$$NAV_s^B = (S_s - \ell NAV_s^A)/(1-\ell).$$

The payments of dual-purpose funds are delivered at three types of dates: upward reset dates τ_i, downward reset dates η_i and pre-specified payment dates ζ_i, defined as

$$\begin{aligned}
&\tau_i = \inf\{s > \tau_{i-1} | S_{s-} \ge H_u\}, \quad \eta_i = \inf\{s > \eta_{i-1} | NAV_{s-}^B \le H_d\},\\
&\zeta_i = \inf\{s > \zeta_{i-1} | v_{s-} = 1, S_{s-} < H_u \text{ and } NAV_{s-}^B > H_d\}.
\end{aligned}$$

At τ_i, A shares receive a payment $Rv_{\tau_i -}$, after which $NAV_{\tau_i}^A = S_{\tau_i} = 1$. At η_i, A shares receive a payment $Rv_{\eta_i -} + 1 - H_d$, after which $NAV_{\eta_i}^A = S_{\eta_i} = 1$ while each one share of A shrinks to H_d share. At ζ_i, A shares receive a payment R, after which $NAV_{\zeta_i} = 1$ and $S_{\zeta_i} = S_{\zeta_i -} - \ell R$.

Therefore, under the risk-neutral pricing framework, the value of A shares (with current underlying fund value S and time from last interest payment date t) is equal to

$$\begin{aligned}
V_A(t,S) = E_t^*\Bigg[&\sum_{\zeta_i \ge t} e^{-r(\zeta_i - t)} Y_{\zeta_i -} R + \sum_{\tau_i \ge t} e^{-r(\tau_i - t)} Y_{\tau_i -} R v_{\tau_i -}\\
&+ \sum_{\eta_i \ge t} e^{-r(\eta_i - t)} Y_{\eta_i -}(Rv_{\eta_i -} + 1 - H_d)\Bigg].
\end{aligned} \tag{38}$$

where E_t^* means the expectation calculated under the risk-neutral measure with initial conditions $S_{t-} = S$, $v_{t-} = t$ and $Y_{t-} = 1$, and Y is a pure jump process which jumps at η_i from $Y_{\eta_i -}$ to $Y_{\eta_i} = H_d Y_{\eta_i -}$ and remains constant otherwise.

Denote $H(t) = \ell(1+Rt) + (1-\ell)H_d,\ Q' = \{(t,S): 0 \le t < 1,\ \ H(t) < S < H_u\},\ D' = \{1\} \times \{H(1), H_u\}$, and

$$\mathcal{L}_{BS}W = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} + (r-c)S\frac{\partial W}{\partial S} - rW.$$

To make the problem well-defined, we assume that $H_u - H(1) > 0$. This assumption is satisfied by all the contracts in the real market.

By transforming the equation from Q' to a rectangular domain, Corollary 3.1 leads to the following proposition, whose proof is given in Appendix B.

Proposition 4.1. *V_A is the unique bounded solution in $C^{1,2}(Q') \cap C(\overline{Q'} \backslash D')$ to the following nonlocal parabolic terminal-boundary value problem:*

$$\begin{aligned}
\frac{\partial W}{\partial t} + \mathcal{L}_{BS}W &= 0 && (t,S) \in Q' && (39)\\
W(1,S) &= R + W(0, S-\ell R) && S \in (H(1), H_u) && (40)\\
W(t,H_u) &= Rt + W(0,1) && t \in [0,1] && (41)\\
W(t,H(t)) &= Rt + 1 - H_d + H_d W(0,1) && t \in [0,1]. && (42)
\end{aligned}$$

Remark 4.1. The pricing of dual-purpose funds in U.S., which has a much simpler future cash flow (c.f. Panel II of Table 1), was considered in Ingersoll (1976) under the Black-Scholes framework by viewing the higher risk/return shares as European options with a finite maturity. However, his method and result are not applicable to dual-purpose funds considered in this paper, since the cash flow of the latter is much more complicated, featuring an infinite time horizon and a periodic payoff structure. In contrast, our theoretical framework for the representation of nonlocal problems can be applied to price the type of dual-purpose funds considered in this section. On the other hand, our representation result also provides an iterative numerical algorithm for solving the one-dimensional PDE (39) – (42) (see the discussion in Remark 3.1), which is perhaps less time-consuming than the Monte Carlo simulation suggested in Adams and Clunie (2006).

4.2. Connection with Regenerative Processes. One main feature of the process (v, X) defined in Section 2 is that once it hits the parabolic boundary $\partial_p Q$, it "regenerates" and starts off from Q with an initial distribution. This suggests a possible relationship between (v, X) and the wide class of regenerative processes, which is the main focus of this subsection. It turns out that a certain subset of (v, X) is indeed regenerative.

We first briefly introduce the concept of regenerative process (c.f. Sigman and Wolff (1993); Glasserman and Kou (1995)). A process Z is called (wide-sense) regenerative if there exists a random variable $R_1 > 0$ such that

1. $\{Z_{t+R_1}, t \ge 0\}$ is independent of R_1;
2. $\{Z_{t+R_1}, t \ge 0\}$ and $\{Z_t, t \ge 0\}$ are equal in distribution.

Here R_1 is referred to as a regeneration epoch, since Z "regenerates" at time R_1. And one can show that there exists $(R_j)_{j\ge 2}$ such that $\left(\sum_{j=1}^k R_j\right)_{k\ge 1}$ is also a sequence of regeneration epochs. $\{Z_t : 0 \le t < R_1\}$, $\{Z_t : R_1 \le t < R_2\}$, ... are called cycles. In some cases the second requirement on Z only holds starting from the second cycle, and we refer to such process Z as delayed regenerative.

The next proposition shows that (v, X) with a certain class of jump measures is regenerative.

Proposition 4.2. *If the jump measure $\tilde{\nu}_{t,x}(ds, dz)$ does not depend on (t, x), i.e.,*

$$\tilde{\nu}_{t,x}(ds, dz) = \delta_0(ds)\mu(dz), \tag{43}$$

where μ is a probability measure on $\mathcal{O}$, the process (v,X) defined in Section 2 is wide-sense delayed regenerative with regeneration epoch $(\theta_i)_{i\geq 1}$, where θ_i is the i-th hitting time of (v,X) on the parabolic boundary $\partial_p Q$. Furthermore, the parabolic equation

$$-\frac{\partial W}{\partial t}(t,x) - \mathcal{L}W(t,x) = f(t,x) \quad \text{in } Q$$

with nonlocal terminal and boundary conditions

$$W(t,x) = h(t,x) + \int_\Omega W(0,y)\mu(dy) \quad \text{on } \partial_p Q$$

has a unique solution $W \in C^{1,2}(Q)\cap C(\overline{Q}\backslash D)$, which can be represented in terms of (v,X) as

$$W(t,x) = E_t^x\left[\int_t^\infty e^{-r(s-t)} f(v_{s-},X_{s-})ds + \sum_{\theta_i\geq t} e^{-r(\theta_i-t)}h(v_{\theta_i -},X_{\theta_i -})\right].$$

Remark 4.2. The process (v,X) and function W in Proposition 4.2 have the following interpretation as inventory management under a specific type of adjustment strategies. X is the inventory diffusion process, and v is the time from last inventory adjustment. The inventory is adjusted based on the following strategy: at time θ_1 when X hits the boundary of $\mathcal{O}$, or the time from last adjustment reaches T, an adjustment takes place. Immediately after the adjustment, X is valued in $\mathcal{O}$ randomly with the distribution μ, independent of the inventory before the adjustment, the time from last adjustment resets to 0, and X continues as a diffusion process until the next adjustment. On the other hand, the function W can be viewed as the expected accumulated cost of this inventory, consisting of running costs at the rate $f(v_{s-},X_{s-})$ at time s, as well as adjustment cost $h(v_{\theta_i -},X_{\theta_i -})$ at each adjustment.

The following example shows that (v,X) is not necessarily regenerative with $(\theta_i)_{i\geq 1}$ as epochs if (43) is not satisfied.

Example. Consider $d=1$, $\Omega=(-1,1)$, and (v,X) defined as in Section 2 with $b=0$, $\sigma=1$, and $\tilde{\nu}_{t,x}(ds,dz)=\delta_0(ds)\delta_{g(x)}(dz)$, where $g:[-1,1]\to(-1,1)$ such that $g(x)=x/2$. (v,X) is not regenerative since $X(\theta_1)$ is not independent of θ_1, which violates the first requirement in the definition of regenerative process. Indeed, under initial distribution $(v,X)=(0,x)$, $X_{\theta_1}\in(-1/2,1/2)$ implies $\theta_1=1$. The dependency between θ_1 and X_{θ_1} is caused by the relationship $X_{\theta_1}=g(X_{\theta_1 -})$ as well as the fact that θ_1 depends on $X_{\theta_1 -}$. If $g(x)$ is constant, this kind of dependency breaks, which is the case in Proposition 4.2.

The next proposition shows a stochastic representation for elliptic equation in terms of a regenerative process. Different from Proposition 4.2, here the jump measure is allowed to have dependence on the hitting point on the domain boundary, and the regeneration epochs are chosen as the hitting times of a subset of the boundary.

Proposition 4.3. *Let μ_{a_1} and μ_{a_2} be probability measures on $\mathcal{O}=(a_1,a_2)$. Then X as in Corollary 3.3 is wide-sense delayed regenerative with either $(\theta_i^1)_{i\geq 1}$ or $(\theta_i^2)_{i\geq 1}$ as epochs, where θ^j denotes the hitting times of a_j, $j=1,2$. Furthermore, the problem*

$$\mathcal{L}'w(x) = -f(x) \quad \text{in } \mathcal{O} \tag{44}$$

$$w(a_1) = h(a_1) + \int_{a_1}^{a_2} w(z)\mu_{a_1}(dz) \tag{45}$$

$$w(a_2) = h(a_2) + \int_{a_1}^{a_2} w(z)\mu_{a_2}(dz) \tag{46}$$

has a unique solution $w \in C^2(\mathcal{O}) \cap C(\overline{\mathcal{O}})$*, which can be represented as*

$$w(x) = E^x\left[\int_0^\infty e^{-rs} f(X_{s-})ds + \sum_{\theta_i \geq 0} e^{-r\theta_i} h(X_{\theta_i -})\right]. \tag{47}$$

Remark 4.3. This proposition gives a PDE characterization for the value function in a class of infinite-horizon stochastic cash management problem with control band policy (c.f. Harrison and Sellke (1983), Cadenillas et al. (2010)). Cadenillas et al. (2010) studied a cash inventory management problem with mean-reverting inventory, and found that the optimal strategy is a control band policy (c.f. Theorem 4.1 in Cadenillas et al. (2010)), that is, there exist $a_1 < b_1 < b_2 < a_2$ such that whenever the inventory level reaches a_1 or a_2, it jumps to b_1 or b_2, respectively, due to the intervention. Since the inventory level follows an Ornstein-Uhlenbeck process without intervention, it is exactly the process X as described in the above proposition with $\sigma(x) = \sigma$, $b(x) = \kappa(\rho - x)$, $\mu_{a_1} = \delta_{b_1}$, and $\mu_{a_2} = \delta_{b_2}$. Therefore, the inventory level is regenerative by choosing the epochs as the hitting times of a_1 (or hitting times of a_2). Furthermore, under this optimal control band policy, the value function in Cadenillas et al. (2010) is equal to (47) with $f(x) = (x-\rho)^2$, $h(a_1) = C + c(b_1 - a_1)$, and $h(a_2) = D + d(b_2 - a_2)$, for some $\rho \in \mathbb{R}$ and $C, c, D, d > 0$. Therefore, Proposition 4.3 shows that the value function V in Cadenillas et al. (2010) restricted on $[a_1, a_2]$ is the unique classical solution to

$$\begin{aligned}
&\mathcal{L}' w(x) = -f(x) \quad \text{in } (a_1, a_2) \\
&w(a_1) = C + c(b_1 - a_1) + w(b_1) \\
&w(a_2) = D + d(b_2 - a_2) + w(b_2).
\end{aligned}$$

A similar result holds for Harrison and Sellke (1983), which assumes that the inventory process follows a Brownian motion instead of a mean-reverting process, and that the inventory has no holding cost. Therefore, $\sigma(x) = \sigma$, $b(x) = \mu$, $f(x) = 0$, and $\mu_{a_i}, i = 1, 2$ are defined above, and V satisfies the same equation. Note that in these two papers, the existence of solution to the equation with nonlocal boundary condition (i.e. the equation for cost function under a given strategy) is shown explicitly by finding the general solution. In contrast, our result does not rely on the existence of explicit solutions, hence it is applicable to a wider class of problems that are analytically intractable.

4.3. One-dimensional Linear Thermoelasticity. The nonlocal problem can also be used in studying linear thermoelasticity theory in Day (1982). Consider a one-dimensional slab made from a homogeneous and isotropic material. Its two ends have fixed position and are maintained at the reference temperature, but its interior part undergoes temperature change and displacement over time. Day (1982) showed that, after a transformation, the entropy $\eta(t, x)$ at time t and location x satisfies the equation

$$\frac{\partial \eta}{\partial t} = \frac{1}{1+\delta^2}\frac{\partial^2 \eta}{\partial x^2} \qquad \text{in } (0, T] \times (0, 1) \tag{48}$$

$$\eta(0, x) = \eta_0(x) \qquad \text{on } [0, 1] \tag{49}$$

$$\eta(t, x) = -\delta^2 \int_0^1 \eta(t, z)dz \qquad \text{on } (0, T] \times \{0, 1\}, \tag{50}$$

which is an initial-boundary value problem with local initial data and nonlocal boundary data. In this equation, $\delta \in (0, 1]$ is a constant quantity related to the elastic moduli, the length of the slab, the reference temperature and the specific heat per volume. One can refer to Day (1982) for the physics background and the detailed derivation of this equation.

Here we assume that $\eta(0,x) = \eta_0(x)$ satisfies Assumption 2.1, i.e. it can have a "nice" set of discontinuities D'. For instance, the initial entropy can have a jump discontinuity. Also, (49) and (50) can be inconsistent. This is more general than the assumption in Day (1982), which implicitly assumed a continuous initial entropy by assuming the smoothness of temperature in the whole closure of parabolic domain.

Based on the above PDE with nonlocal boundary data, our result can provide a stochastic representation for the entropy η. Specifically, we define X as a diffusion process satisfying

$$dX_t = \frac{\sqrt{2}}{\sqrt{1+\delta^2}} dW_t$$

in $(0,1)$. When X hits the boundary of the interval at 0 or 1, it jumps to a point in $(0,1)$ according to the uniform probability measure on $(0,1)$. We denote $(\theta_i)_{i\geq 1}$ as the sequence of hitting times of X at 0 or 1. We take Y as a piecewise constant process such that

$$Y_{t-} = 1, \quad Y_{\theta_i} = -\delta^2 \cdot Y_{\theta_i -} \text{ if } \theta_i < T, i \geq 1, \text{ and } Y_s = 0 \text{ for } s \geq T.$$

From Corollary 3.2, we have the following representation result. The details of the proof are given in Appendix B.

PROPOSITION 4.4. *The transformed entropy η described above can be represented as*

$$\eta(T-t,x) = E_t^x\left[Y_{T-} \cdot \eta_0(X_{T-})\right], \quad \forall (t,x) \in [0,T] \times [0,1],$$

where X and Y are as defined above.

REMARK 4.4. The above probabilistic representation leads to the following new interpretation. Consider a particle with unit mass located at x at the current time t that diffuses in the interval $(0,1)$. Each time it hits the boundary 0 or 1, it jumps back to a random location in $(0,1)$ with a uniform distribution, while losing $1-\delta^2$ fraction of its mass. At the terminal time T, the output (or impact) of this system per unit mass is $(-1)^N \eta_0(x_T)$, which depends on the time-T location of the particle x_T, the initial entropy distribution η_0, and the random variable N denoting the boundary-hitting times of X in $[0,T)$. The right hand side expectation in the above proposition can then be understood as the time-t expected output of the whole system. This new probabilistic representation yields a new estimation method for η via simulation; one can simulate the system (X,Y) as in the above construction, and calculate the mean payoff as an estimation.

We end this section by a numerical illustration on the above thermoelasticity problem. We set $\delta = 0.5, T = 0.1$, and take the initial data $\eta_0(x) = \left(-x - \frac{1}{2}\right)\mathbf{1}_{x\leq\frac{1}{2}} + \left(\frac{3}{2} - x\right)\mathbf{1}_{x>\frac{1}{2}}$, which has a jump discontinuity at $x = 0.5$, and is inconsistent with the nonlocal boundary condition at $x = 0,1$. The PDE is solved using the numerical algorithm described in Remark 3.1, by setting the initial guess $\tilde{\eta}_0 = 0$, and plugging $\tilde{\eta}_i$ into (50) to solve $\tilde{\eta}_{i+1}$ in the $(i+1)$-th iteration, for $i \geq 0$.

Figure 2 shows that the discontinuity in the initial data is immediately smoothed out after $t = 0$, as shown by the dotted line for $t = 0.0001$. As time increases, the entropy redistributes so that η becomes more and more flat with respect to x. Also note that η equals 0 at $x = 0,1$ for all curves except for the initial data, and therefore satisfies the nonlocal boundary condition (50) since the average entropy with respect to x is 0 for those three curves.

To demonstrate Proposition 4.4, we use Monte Carlo simulation to estimate the right hand side expectation. To this end, we simulate the dynamics of (X,Y) starting from $t = 0$ and $X_0 = x$ using the Euler scheme. Figure 3 shows two sample paths starting from $x = 0.8$ illustrated by two different colors, whose thickness represents $|Y|$, or the mass of the particle as in the interpretation in Remark 4.4. For instance, the black curve starts with unit mass, hits the upper boundary around $t = 0.035$, and jumps back to 0.3181 with its mass reduced to 0.25. Then, it continues the diffusion

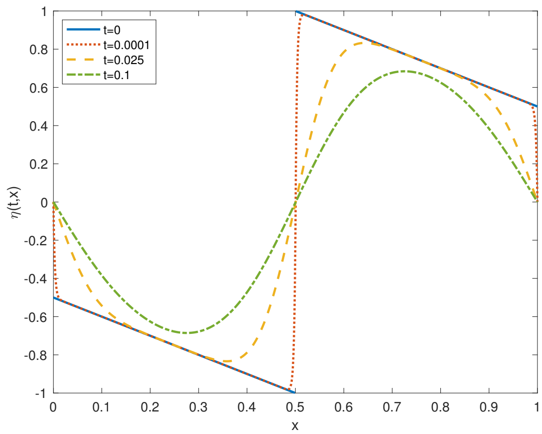

FIGURE 2. Numerical Solution to the PDE.

Note. This figure shows the solution to equation (48) – (50) for $x \in [0,1]$ and $t = 0$ (solid curve, which is the initial entropy η_0), $t = 0.0001$ (dotted curve), $t = 0.025$ (dashed curve) and $t = 0.05$ (dash-dotted curve). Default parameter: $\delta = 0.5$ and $T = 0.1$. Initial data: $\eta_0(x) = \left(-x - \frac{1}{2}\right)\mathbf{1}_{x\le\frac{1}{2}} + \left(\frac{3}{2} - x\right)\mathbf{1}_{x>\frac{1}{2}}$. Note that the initial data η_0 has a jump discontinuity at $x = 0.5$, and is inconsistent with the boundary condition. The discontinuity is smoothed out immediately after the initial time, as illustrated by the dotted curve. The solution is calculated by numerically solving the PDE using the algorithm outlined in Remark 3.1.

until it hits the lower boundary around $t = 0.095$, and jumps back with its mass further reduced to 0.0625. The simulation estimates for $\eta(T,x)$ at $x = 0.2, 0.4, 0.6, 0.8$ and the corresponding standard errors based on the Monte Carlo estimator with 10,000 sample paths are reported in Table 2. The simulation produces estimates that are very close to the numerical solution to the PDE (48) – (50), and costs about 300 seconds for each point, which is significantly higher than the running time of the PDE numerical solution, 2.11 seconds.

Note that (48) – (50) is one dimensional in space. For a similar equation with a higher space dimension, the PDE-based numerical solution will be much slower, if feasible at all, than the simulation-based estimation, due to the curse of dimensionality. Furthermore, it is much easier to do computer programming for the simulation-based estimation than for the PDE-based method.

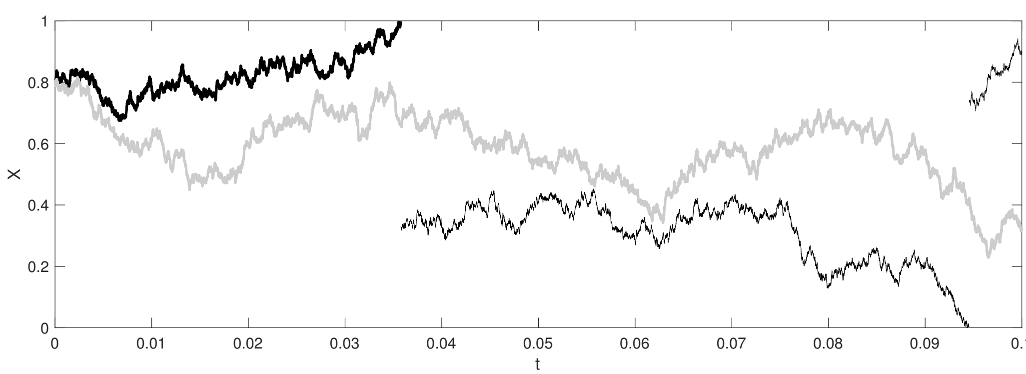

FIGURE 3. Simulated Sample Paths.

Note. This figure shows two sample paths starting from $x = 0.8$ in the simulation, marked with different colors. Default parameters: $T = 0.1, \delta = 0.5$. The thickness of the curves represents the absolute value of Y, or the mass of the particle in the interpretation in Remark 4.4. For instance, the black curve starts with mass 1, and hits the upper boundary around $t = 0.035$. Its mass is then reduced to 0.25, until it hits the lower boundary around $t = 0.095$, after which its mass is further reduced to 0.0625. On the other hand, the path depicted by the gray curve starts from $x = 0.8$ and reaches the terminal time T without hitting the boundaries once.

Table 2. Comparison of Simulation-based Estimation and PDE Numerical Solution.

x	PDE Numerical Solution	Simulation Estimates (Standard Error)	Simulation Running Time (Seconds)
0.2	-0.038617	-0.038610 (0.000047)	300.82
0.4	-0.023867	-0.023894 (0.000071)	298.28
0.6	0.023867	0.023837 (0.000060)	293.20
0.8	0.038617	0.038527 (0.000046)	319.68

This table shows a comparison between simulation-based estimation and PDE numerical solution for $\eta(T,x)$ at $x = 0.2, 0.4, 0.6, 0.8$. Default parameters: $T = 0.1, \delta = 0.5$. The PDE numerical solution is calculated using the iterative algorithm outlined in Remark 3.1. Simulation-based estimation is calculated as a Monte Carlo simulation estimator with 10,000 sample path, and the standard error is estimated using 100 realizations of the estimator. In comparison, the running time of the PDE numerical solution is 2.11 seconds.

Finally, we estimate the distribution of N, the number of jumps in $[0,T)$. Table 3 shows that, starting from $x = 0.5$ that is away from the boundary, the probability of $N = n$ jumps decreases as n increases, and the probability of having more than 6 jumps is small. However, starting from $x = 0.9$ that is close to the boundary, the probability of having no jump is lower than that of having one jump.

Table 3. Distribution of the Number of Jumps N.

N	0	1	2	3	4	5	≥ 6
$x = 0.5$	0.5794	0.2608	0.1068	0.0371	0.0116	0.0032	0.0011
$x = 0.9$	0.1806	0.3618	0.2495	0.1261	0.0532	0.0195	0.0093

This table shows the distribution of N, i.e. the number of jumps of X in $t \in [0,T)$. Default parameters: $T = 0.1, \delta = 0.5$. For $x = 0.5$, i.e. X starting from a point away from the boundary, the probability decreases as the realization of number of jumps increases, and jumps at the hitting time θ_i according to $Y_{\theta_i} = -\delta^2 \cdot Y_{\theta_i -}$ if $\theta_i < T$, and $Y_s = 0$ for $s \geq T$. However, for $x = 0.9$, i.e. X starting from a point close to the boundary, it is less likely to have no jump than to have one jump.

Appendix A: Proofs of the Main Results.

Proof of Lemma 2.1. Denote the sequence of hitting times of $Q_T := \{T\} \times \overline{\mathcal{O}}$ and $[0,T) \times \partial\mathcal{O}$ as (τ_i) and (η_i), respectively. First note that by construction, starting from $[0,t_0] \times \mathcal{O}$, it takes at least $\tilde{t}_0 = T - t_0$ amount of time for (v,X) to hit Q_T. Therefore, $P^{u,x}(\Delta\tau_i \geq \tilde{t}_0) \geq p_0$ for all $(u,x) \in Q$ and $i \geq 1$ thanks to Assumption 2.2.

On the other hand, taking $p_1 \in (1-p_0, 1)$, there exists $t_1 > 0$ small enough, such that starting from any $(u,x) \in Q$, the probability of the event of (v,X) hitting $[0,T) \times \partial\mathcal{O}$ consecutively twice within t_1 amount of time without hitting Q_T in-between is uniformly bounded by p_1. Indeed, on the first hit at $(u_0,x_0) \in [0,T) \times \partial\mathcal{O}$, either (1) (v,X) jumps to $[0,u_0] \times (\mathcal{O} \backslash \mathcal{O}_\delta)$, or (2) (v,X) jumps to $[0,u_0] \times \mathcal{O}_\delta$, where δ is as in Assumption 2.2. The probability of this event under case (1) is bounded by $1-p_0$ thanks to Assumption 2.2. The probability of this event under case (2) is bounded by the probability of a diffusion process ξ defined as (54) starting at time $\tilde{t} \in [0,T)$ from $\tilde{x} \in \mathcal{O}_\delta$ and hitting $\partial\mathcal{O}$ within $t_1 \wedge (T - \tilde{t})$ amount of time, which is bounded by

$$P\left[\sup_{s \in [\tilde{t}, (\tilde{t}+t_1) \wedge T]} |\xi_s^{\tilde{t},\tilde{x}} - \tilde{x}| \geq \delta\right] \leq C(1+|\tilde{x}|^2)t_1/\delta^2,$$

thanks to Chebyshev inequality and the estimation of the expectation of sup-norm of solutions to SDE (c.f. Theorem 5.2.3 in Friedman (2012)). So to make the probability of this event under case (2) smaller than $p_1 + p_0 - 1$, it suffices to take t_1 small enough such that $C(1+|\tilde{x}|^2)t_1/\delta^2 \leq p_1 + p_0 - 1$ for all $\tilde{x} \in \mathcal{O}_\delta$, by the boundedness of $\mathcal{O}_\delta$. Aggregating these two sub-events gives the bound p_1.

Let M be a positive integer large enough such that $(1-p_0)^M + p_1 < 1$. We claim that for any $n \geq 0$, $P(\theta_{n+4M} - \theta_{n+1} < \tilde{t}_0 \wedge t_1) < (1-p_0)^M + p_1$, uniformly in the starting point. To see this, a key observation is that the $4M$ consecutive hits at θ_{n+1} to θ_{n+4M} must contain at least $2M$ τ's (denoted as $\tau_1, \cdots, \tau_{2M}$) or two consecutive η's (denoted as η_1, η_2), therefore $\theta_{n+4M} - \theta_{n+1} < \tilde{t}_0 \wedge t_1$ implies $\tau_{2M} - \tau_1 < \tilde{t}_0 \wedge t_1$ or $\eta_2 - \eta_1 < \tilde{t}_0 \wedge t_1$. Starting from any $(u,x) \in Q$, the probability of the second event is bounded by p_1. Next we claim that the probability of the first event is bounded by $(1-p_0)^M$, which establishes the bound $(1-p_0)^M + p_1$. To see this, denoting $A_M = \{\tau_2 - \tau_1 < \tilde{t}_0 \wedge t_1, \cdots, \tau_{2M} - \tau_{2M-1} < \tilde{t}_0 \wedge t_1\}$, we have

$$\begin{aligned}
&P^{u,x}(\tau_{2M} - \tau_1 < \tilde{t}_0 \wedge t_1) \leq P^{x,u}(A_M) \\
&\quad = P^{u,x}(A_{M-1}) \cdot P^{u,x}(\tau_{2M} - \tau_{2M-1} < \tilde{t}_0 \wedge t_1 | A_{M-1}) \\
&\quad = P^{u,x}(A_{M-1}) \cdot E^{u,x}\left[P^{v_{\tau_{2M-2}}, X_{\tau_{2M-2}}}(\tau_2 - \tau_1 < \tilde{t}_0 \wedge t_1) | A_{M-1}\right] \\
&\quad \leq P^{u,x}(A_{M-1}) \cdot (1-p_0) \leq (1-p_0)^M,
\end{aligned}$$

where the last inequality is from mathematical induction.

Finally, since for all $(u,x) \in Q$,

$$\begin{aligned}
E^{u,x}\left[e^{-r\theta_{4M}}\right] &\leq e^{-r(\tilde{t}_0 \wedge t_1)} P^{u,x}(\theta_{4M} \geq \tilde{t}_0 \wedge t_1) + P^{u,x}(\theta_{4M} < \tilde{t}_0 \wedge t_1) \\
&\leq e^{-r(\tilde{t}_0 \wedge t_1)} + \left(1 - e^{-r(\tilde{t}_0 \wedge t_1)}\right)\left((1-p_0)^M + p_1\right) \\
&:= p_2 < 1,
\end{aligned} \tag{51}$$

we claim, by induction and the Markov property,

$$\sup_{(u,x) \in Q} E^{u,x}\left[e^{-r\theta_{4nM}}\right] \leq p_2^n. \tag{52}$$

This gives (20) and concludes the proof by sending n to ∞.

To see (52), (51) proves the $n=1$ case. Assume that the claim is true for $n=k$, we have

$$\begin{aligned}\sup_{(u,x)\in Q} E^{u,x}\left[e^{-r\theta_{4(k+1)M}}\right] &= \sup_{(u,x)\in Q} E^{u,x}\left[e^{-r\theta_{4kM}} E^{v_{\theta_{4kM}},X_{\theta_{4kM}}}\left[e^{-r\theta_{4M}}\right]\right]\\ &\le \sup_{(u,x)\in Q} E^{u,x}\left[e^{-r\theta_{4kM}} \sup_{(u,x)\in Q} E^{u,x}\left[e^{-r\theta_{4M}}\right]\right]\\ &\le p_2\cdot \sup_{(u,x)\in Q} E^{u,x}\left[e^{-r\theta_{4kM}}\right]\\ &\le p_2^{k+1}.\end{aligned}$$

This proves (52).

Proof of Proposition 2.1. It is straightforward that the first integral term in the expectation in (21) is uniformly bounded by K_f/r, thanks to the boundedness of f. For the second summation term, from the growth condition (23), we know that

$$\sum_{\theta_i\ge t} e^{-r(\theta_i-t)} Y_{\theta_i-}|h(v_{\theta_i-},X_{\theta_i-})| \le K_h\cdot \sum_{\theta_i\ge t} e^{-r(\theta_i-t)} v_{\theta_i-}.$$

Since $0\le v_{\theta_i-}\le T$, the expectation of the right hand side term is bounded by $4TMK_h/(1-p_2)$ thanks to (52). These give the bound on V. The bound on V_N can be derived similarly.

Proposition A.1 **(Feynman-Kac Representation for Local Terminal and Boundary Data).** *Assume that h is bounded on $\partial_p Q$ and continuous on $\partial_p Q\backslash D$, where $D\subset \partial_p Q$ is the set of discontinuity points of h satisfying Assumption 2.1.*

Then the unique bounded solution $w\in C^{1,2}(Q)\cap C(\overline{Q}\backslash D)$ to the parabolic terminal-boundary value problem

$$\begin{aligned}-\frac{\partial w}{\partial t} - \mathcal{L}w &= f(t,x) && \text{in } Q\\ w(t,x) &= h(t,x) && \text{on } \partial_p Q\end{aligned}$$

can be represented as

$$w(t,x) = E_t^x\left[\int_t^{\tau\wedge T} e^{-r(s-t)} f(s,\xi_s)ds + e^{-r((\tau\wedge T)-t)} h(\tau\wedge T,\xi_{\tau\wedge T})\right], \tag{53}$$

for all $(t,x)\in\overline{Q}$, where ξ is a diffusion that starts from x at time t and follows

$$d\xi_t = b(t,\xi_t)dt + \sigma(t,\xi_t)dB_t, \tag{54}$$

and τ is the first time that ξ hits $\partial\mathcal{O}$,

Proof of Proposition A.1. The existence of a bounded solution can be proved by smoothing the boundary and terminal data. Specifically, we approximate the data h by a sequence of continuous functions h_n having the bound $\sup_{(t,x)\in\partial_p Q}|h_n(t,x)| \le \sup_{(t,x)\in\partial_p Q}|h(t,x)| < \infty$, and h_n is different from h only in a set D_n. D_n is the union of the collection of covering sets $\left((s_i^{(n)}-1/n, s_i^{(n)}+1/n)\times B_i^{(n)}\right)\cap\partial_p Q$ that intersects with D, where $\{s_i^{(n)}\}$ and $\{B_i^{(n)}\}$ are as in Assumption 2.1 with $\varepsilon=1/n$. Note that by construction, $D\subset D_n$ and $\lim_{n\to\infty} D_n = D$. For each problem with data h_n, from the classic Feynman-Kac representation result (c.f. Theorem 5.2 in Friedman (2012)), there exists a unique solution $w_n\in C^{1,2}(Q)\cap C(\overline{Q})$ that can be represented as

$$w_n(t,x) = E_t^x\left[\int_t^{\tau\wedge T} e^{-r(s-t)} f(s,\xi_s)ds\right] + E_t^x\left[e^{-r((\tau\wedge T)-t)} h_n(\tau\wedge T,\xi_{\tau\wedge T})\right].$$

Note that the difference between $w_n(t,x)$ and $w(t,x)$ defined in (53) is dominated by

$$E_t^x\left[|h_n(\tau\wedge T,\xi_{\tau\wedge T})-h(\tau\wedge T,\xi_{\tau\wedge T})|\right]\le C\cdot P_t^x\left[(\tau\wedge T,\xi_{\tau\wedge T})\in D_n\right],$$

where $C=2\sup_{(t,x)\in\partial_p Q}|h(t,x)|$. Therefore, w_n converges uniformly to w on any compact subset in $\overline{Q}\backslash D$, since σ is uniformly elliptic. By going through all these compact sets, we see that w_n also converges pointwise on $\overline{Q}\backslash D$ to w and $w\in C(\overline{Q}\backslash D)$. Also by definition, w satisfies the terminal and boundary conditions. Finally, since h_n is uniformly bounded, w_n is also uniformly bounded by some M, thanks to the maximum principle. Therefore the Schauder interior estimate gives

$$||w_n||_{2+\alpha}\le K\cdot(||f||_\alpha+||w_n||_0)\le K\cdot(||f||_\alpha+M)<\infty,$$

and so from Arzelà-Ascoli theorem, by extracting subsequences on each compact set in Q, we have $w\in C^{1,2}(Q)$, is bounded, and satisfies the equation.

For the proof of the representation part, take $(t,x)\in Q$, let $Q_n=\{(t,x)\in Q: dist((t,x),\partial_p Q)>1/n\}$, and $\eta_n=\inf\{s\ge t:(s,\xi_s)\notin Q_n\}$. Then $(\eta_i,\xi_{\eta_i})\in int(Q)$, and by using Ito's formula on w, we have

$$E_t^x\left[e^{-r(\eta_i-t)}w(\eta_i,\xi_{\eta_i})\right]=w(t,x)-E_t^x\left[\int_t^{\eta_i}e^{-r(s-t)}f(s,\xi_s)ds\right].$$

Since $\eta_i\nearrow\tau\wedge T, a.s.$ and f is bounded, the right hand side expectation converges to

$$E_t^x\left[\int_t^{\tau\wedge T}e^{-r(s-t)}f(s,\xi_s)ds\right].$$

For the left hand side, since ξ has continuous sample paths, $(\eta_i,\xi_{\eta_i})\to(\tau\wedge T,\xi_{\tau\wedge T}), a.s.$ as $i\to\infty$. Also, since ξ is a non-degenerate diffusion, $P_t^x((\tau\wedge T,\xi_{\tau\wedge T})\in D)=0$. By the continuous mapping theorem, $w(\eta_i,\xi_{\eta_i})\to w(\tau\wedge T,\xi_{\tau\wedge T}), a.s.$. Thanks to the boundedness of w, the left hand side converges to

$$E_t^x\left[e^{-r((\tau\wedge T)-t)}w(\tau\wedge T,\xi_{\tau\wedge T})\right],$$

and the representation result follows.

LEMMA A.1. *For $(t,x)\in Q$, $i\ge 1$, the function W_i defined in Section 3 can be represented as*

$$W_i(t,x)=V_i(t,x), \tag{55}$$

where V_i is the finite-horizon value function as defined in Section 2.4.

Proof of Lemma A.1. For $i=1$, the result follows from Proposition A.1. Assume that we have proved

$$W_k(t,x)=V_k(t,x),\quad \forall(t,x)\in Q.$$

Denote $\theta=\inf\{s\ge t|X_{s-}\in\partial\mathcal{O}\text{ or }v_{s-}=T\}$. We first show that for $(t,x)\in Q$, we have

$$\begin{aligned}V_{k+1}(t,x)=E_t^x\Bigg[&\int_t^\theta e^{-r(s-t)}f(v_{s-},X_{s-})ds\\&+e^{-r(\theta-t)}Y_\theta V_k(v_\theta,X_\theta)+e^{-r(\theta-t)}h(\theta,X_{\theta-})\Bigg].\end{aligned} \tag{56}$$

To see this, at θ, v jumps to $(v_\theta, X_\theta) \in Q$ according to the jump measure $\tilde{\nu}$, Y jumps from 1 to $Y_\theta = \overline{\nu}(\theta, X_{\theta-})$ and m jumps from 0 to $T - v_{\theta-} + v_\theta = T - \theta + v_\theta$ by definition. Therefore we have

$$\begin{aligned} V_{k+1}(t,x) &= \tilde{V}_{k+1}(t, t - \lfloor t \rfloor_T, x, 1, 0) \\ &= E_t^x \bigg[\int_t^\theta e^{-r(s-t)} f(v_{s-}, X_{s-}) ds \\ &\quad + e^{-r(\theta-t)} Y_\theta \tilde{V}_{k+1}(\theta, v_\theta, X_\theta, 1, T - \theta + v_\theta) \\ &\quad + e^{-r(\theta-t)} h(\theta, X_{\theta-}) \bigg]. \end{aligned}$$

Comparing with (56), it remains to show that

$$\tilde{V}_{k+1}(\theta, v_\theta, X_\theta, 1, T - \theta + v_\theta) = V_k(v_\theta, X_\theta). \tag{57}$$

Indeed, by the Markov property and time-homogeneity of (v, X, Y), for any $u \in [0,T]$, $w \in [0,T)$ and $x \in \mathcal{O}$,

$$\begin{aligned} &\tilde{V}_{k+1}(u, w, x, 1, T - u + w) \\ &= E_u^{w,x,1} \bigg[\int_u^{\gamma_{k+1}^{u,w+T-u}} e^{-r(s-u)} f(v_{s-}, X_{s-}) ds \\ &\quad + \sum_{u \le \theta_i \le \gamma_{k+1}^{u,w+T-u}} e^{-r(\theta_i - u)} h(v_{\theta_i -}, X_{\theta_i -}) \bigg] \\ &= E_{w+T}^{w,x,1} \bigg[\int_{w+T}^{\gamma_{k+1}^{w+T,0}} e^{-r(s-w-T)} f(v_{s-}, X_{s-}) ds \\ &\quad + \sum_{w+T \le \theta_i \le \gamma_{k+1}^{w+T,0}} e^{-r(\theta_i - w - T)} h(v_{\theta_i -}, X_{\theta_i -}) \bigg] \\ &= \tilde{V}_{k+1}(w+T, w, x, 1, 0) \\ &= V_{k+1}(w+T, x) \\ &= V_k(w, x). \end{aligned}$$

Now it remains to show that V_{k+1} given as in (56) agrees with W_{k+1} in Q. To see this, by using Proposition A.1 on the equation for W_{k+1}, we have that

$$\begin{aligned} W_{k+1}(t,x) &= E_t^x \bigg[\int_t^\theta e^{-r(s-t)} f(v_{s-}, X_{s-}) ds \\ &\quad + e^{-r(\theta-t)} \Big(\mathcal{B} W_k(v_{\theta-}, X_{\theta-}) + h(v_{\theta-}, X_{\theta-}) \Big) \bigg] \\ &= E_t^x \bigg[\int_t^\theta e^{-r(s-t)} f(v_{s-}, X_{s-}) ds \\ &\quad + e^{-r(\theta-t)} \Big(Y_\theta W_k(v_\theta, X_\theta) + h(v_{\theta-}, X_{\theta-}) \Big) \bigg]. \end{aligned} \tag{58}$$

But since $W_k = V_k$ in Q from the induction hypothesis, we see that (58) agrees with V_{k+1} in (56). And the lemma follows from mathematical induction.

Lemma A.2. *The sequence of functions (W_i) converges uniformly in $\overline{Q}$.*

Proof of Lemma A.2. Based on Lemma A.1, it suffices to show that $V_i(t,x)$ converges uniformly to $V(t,x)$ in Q. Indeed, once we show this, then for any $\varepsilon > 0$, there exists $N > 0$ such that for any $m, n \geq N$,

$$\sup_{(t,x)\in Q} |W_m(t,x) - W_n(t,x)| < \varepsilon.$$

Then, due to the definition of (W_i) given in Section 3, for any $m, n \geq N+1$ and $(t,x) \in \partial_p Q$,

$$|W_m(t,x) - W_n(t,x)| \leq \iint_Q |W_{m-1}(s,z) - W_{n-1}(s,z)| \nu_{t,x}(ds, dz) < \varepsilon \cdot \nu_{t,x}(Q) \leq \varepsilon.$$

As a result, we can include the boundary data into the previous inequality and get

$$\sup_{(t,x)\in \overline{Q}} |W_m(t,x) - W_n(t,x)| < \varepsilon,$$

which concludes the proof.

To prove the uniform convergence of $V_i(t,x)$ in Q, first observe that for $s \in [t, NT]$,

$$\begin{aligned}
\{\gamma_N^t \leq s\} &= \{NT - s - m_s^t \leq 0\} \\
&= \left\{ NT - s - \sum_{i=1}^{N-1} (T - v_{\theta_i -} + v_{\theta_i}) - (T - v_{\theta_N -}) \leq 0 \right\} \\
&= \{\theta_N \leq s\},
\end{aligned}$$

where the last equality is due to $v_{\theta_{i+1}-} - v_{\theta_i} = \theta_{i+1} - \theta_i$. The difference between $V(t,x,y)$ and $V_N(t,x,y)$ is given by

$$\int_{\gamma_N^t}^{\infty} e^{-r(s-t)} Y_{s-} f(v_{s-}, X_{s-}) ds + \sum_{\theta_i > \gamma_N^t} e^{-r(\theta_i - t)} Y_{\theta_i -} h(v_{\theta_i -}, X_{\theta_i -}).$$

The expectation of this expression is dominated by

$$K_V e^{rt} \cdot E_t^x \left[e^{-r\gamma_N^t} \right] = K_V e^{rt} \cdot E_t^x \left[e^{-r\theta_N} \right],$$

where K_V is as in Proposition 2.1. The result then follows from Lemma 2.1 by sending N to infinity.

Proposition A.2. *The uniform limit of $(W_i)_{i\geq 1}$ on $\overline{Q}$ is a bounded solution in $C^{1,2}(Q) \cap C(\overline{Q}\backslash D)$ to the nonlocal problem (1) - (2).*

Proof of Proposition A.2. Lemma A.2 together with the continuity of W_i in $\overline{Q}\backslash D$ shows that W_i converges uniformly to a function W that is continuous in $\overline{Q}\backslash D$. From Lemma A.1 and Proposition 2.1, $\sup_i ||W_i||_0 \leq K_V < \infty$, and so W is bounded. To show that W satisfies the equation in the interior region Q, by the Schauder interior estimates of parabolic problem (c.f. Theorem 3.5 in Friedman (2013)), there exists K independent of W_i, such that

$$||W_i||_{2+\alpha} \leq K\left(||W_i||_0 + ||f||_\alpha\right) \leq K\left(K_V + ||f||_\alpha\right). \tag{59}$$

Since the right hand side does not depend on i, from Arzelà-Ascoli theorem, by extracting subsequences (W_{i_k}) on each compact subset of Q, we have $W \in C^{1,2}(Q)$, and W satisfies the equation in Q.

Finally, to show that W also satisfies the boundary and terminal conditions, it suffices to note that

$$W_{i+1}(s,x) = h(s,x) + \mathcal{B}W_i(s,x), \quad \forall i \geq 0 \tag{60}$$

on the parabolic boundary, and then send i to infinity.

Proof of Theorem 3.1. The existence of solutions is established in Proposition A.2. It remains to show that any bounded solution $W \in C^{1,2}(Q) \cap C(\overline{Q} \backslash D)$ can be represented by (6).

Step 1: For this solution W, the terminal data and boundary data are continuous except in D. So according to Proposition A.1,

$$W(t,x) = E_t^x \Bigg[\int_t^\theta e^{-r(s-t)} f(v_s, \xi_s) ds + e^{-r(\theta-t)} \Big(h(\theta, \xi_\theta) + \mathcal{B}W(\theta, \xi_\theta) \Big) \Bigg], \quad \forall (t,x) \in \overline{Q}, \tag{61}$$

where ξ and θ satisfy

$$d\xi_s = b(s, \xi_s) ds + \sigma(s, \xi_s) dB_s, \tag{62}$$
$$\theta = \inf\{ s \geq t : \xi_s \in \partial \mathcal{O} \} \wedge T. \tag{63}$$

Recall the definition of the jump of (v, X) from the boundary, and note that

$$X_s = \xi_s, \forall s < \theta, \quad X_{\theta-} = \xi_\theta \tag{64}$$

Equation (61) becomes

$$W(t,x) = E_t^x \Bigg[e^{-r(\theta-t)} Y_{\theta-} h(\theta, X_{\theta-}) + \int_t^\theta e^{-r(s-t)} Y_{s-} f(v_{s-}, X_{s-}) ds + e^{-r(\theta-t)} Y_\theta W(v_\theta, X_\theta) \Bigg]. \tag{65}$$

Step 2: Take $\theta_1 = \theta$, $\theta_{i+1} = \inf\{ s > \theta_i : X_{s-} \in \partial \mathcal{O} \text{ or } v_{s-} = T \}$, $i \geq 0$. We claim that for all $N \geq 1$, $W(t,x)$ equals

$$u_N(t,x) = E_t^x \Bigg[\sum_{i=1}^N e^{-r(\theta_i - t)} Y_{\theta_i -} h(v_{\theta_i -}, X_{\theta_i -}) + \int_t^{\theta_N} e^{-r(s-t)} Y_{s-} f(v_{s-}, X_{s-}) ds + e^{-r(\theta_N - t)} Y_{\theta_N} W(v_{\theta_N}, X_{\theta_N}) \Bigg]. \tag{66}$$

Indeed, when $N = 1$, this is true from (65). Now assume that this has been proved for $N = k$. Then for $N = k+1$,

$$\begin{aligned} u_{k+1}(t,x) = E_t^x \Bigg\{ & \sum_{i=1}^k e^{-r(\theta_i - t)} Y_{\theta_i -} h(v_{\theta_i -}, X_{\theta_i -}) + \int_t^{\theta_k} e^{-r(s-t)} Y_{s-} f(v_{s-}, X_{s-}) ds \\ & + E_{\theta_k}^{v_{\theta_k}, X_{\theta_k}, Y_{\theta_k}} \Big[e^{-r(\theta_{k+1} - t)} Y_{\theta_{k+1}-} h(v_{\theta_{k+1}-}, X_{\theta_{k+1}-}) \\ & + \int_{\theta_k}^{\theta_{k+1}} e^{-r(s-t)} Y_{s-} f(v_{s-}, X_{s-}) ds + e^{-r(\theta_{k+1} - t)} Y_{\theta_{k+1}} W(v_{\theta_{k+1}}, X_{\theta_{k+1}}) \Big] \Bigg\}. \end{aligned} \tag{67}$$

Due to the time-homogeneity and Markov property of (v, X, Y),

$$\begin{aligned} & E_{\theta_k}^{v_{\theta_k}, X_{\theta_k}, Y_{\theta_k}} \left[e^{-r(\theta_{k+1} - t)} Y_{\theta_{k+1}-} h(v_{\theta_{k+1}-}, X_{\theta_{k+1}-}) \right] \\ & = e^{-r(\theta_k - t)} E_{\theta_k}^{v_{\theta_k}, X_{\theta_k}, Y_{\theta_k}} \left[e^{-r(\theta_{k+1} - \theta_k)} Y_{\theta_{k+1}-} h(v_{\theta_{k+1}-}, X_{\theta_{k+1}-}) \right] \\ & = e^{-r(\theta_k - t)} Y_{\theta_k} E_{v_{\theta_k}}^{X_{\theta_k}} \left[e^{-r(\theta_1 - v_{\theta_k})} Y_{\theta_1 -} h(v_{\theta_1 -}, X_{\theta_1 -}) \right]. \end{aligned} \tag{68}$$

Similarly,

$$
\begin{aligned}
&E_{\theta_k}^{v_{\theta_k},X_{\theta_k},Y_{\theta_k}}\left[\int_{\theta_k}^{\theta_{k+1}} e^{-r(s-t)}Y_{s-}f(v_{s-},X_{s-})ds + e^{-r(\theta_{k+1}-t)}Y_{\theta_{k+1}}W(v_{\theta_{k+1}},X_{\theta_{k+1}})\right] \\
&= e^{-r(\theta_k-t)}Y_{\theta_k}E_{v_{\theta_k}}^{X_{\theta_k}}\left[\int_{v_{\theta_k}}^{\theta_1} e^{-r(s-v_{\theta_k})}Y_{s-}f(v_{s-},X_{s-})ds + e^{-r(\theta_1-v_{\theta_k})}Y_{\theta_1}W(v_{\theta_1},X_{\theta_1})\right].
\end{aligned}
\tag{69}
$$

Using (65) and the fact that $v_{\theta_1-}=\theta_1=\theta$, the summation of (68) and (69) equals $e^{-r(\theta_k-t)}Y_{\theta_k}W(v_{\theta_k},X_{\theta_k})$. So by plugging (68) and (69) into (67), $u_{k+1}(t,x)=W(t,x)$ is established.

Step 3: Finally, we show that

$$
\begin{aligned}
W(t,x) = E_t^x\Bigg[&\sum_{\theta_i\ge t} e^{-r(\theta_i-t)}Y_{\theta_i-}h(v_{\theta_i-},X_{\theta_i-}) \\
&+\int_t^{\infty} e^{-r(s-t)}Y_{s-}f(v_{s-},X_{s-})ds\Bigg].
\end{aligned}
\tag{70}
$$

Indeed, based on the claim in Step 2, $\forall N>0$, the absolute difference between $W(t,x)$ and the RHS is dominated by

$$
\begin{aligned}
E_t^x\Bigg[e^{-r(\theta_N-t)}\cdot\Bigg|&Y_{\theta_N}W(v_{\theta_N},X_{\theta_N}) - \int_{\theta_N}^{\infty} e^{-r(s-\theta_N)}Y_{s-}f(v_{s-},X_{s-})ds \\
&-\sum_{i=N+1}^{\infty} e^{-r(\theta_i-\theta_N)}Y_{\theta_i-}h(v_{\theta_i-},X_{\theta_i-})\Bigg|\Bigg].
\end{aligned}
\tag{71}
$$

Since $|W|$ is bounded by some $K_W>0$ and $Y_u\le Y_{t-}=1$, $\forall u\ge t$, this difference is bounded by (which can be proved similarly to Proposition 2.1)

$$
(K_V+K_W)e^{rt}E_t^x\left[e^{-r\theta_N}\right],
\tag{72}
$$

which goes to 0 as N goes to infinity thanks to Lemma 2.1. The representation (6) is obtained by recalling $Y_{u-}=y\cdot\prod_{t\le\theta_i<u}\overline{\nu}(v_{\theta_i-},X_{\theta_i-})$.

Proof of Corollary 3.1. Note that $\tilde{h}\in C((\partial_pQ\backslash D)\times[0,1])$ means

$$
\tilde{h}(t,x,\overline{\nu}(t,x))+\mathcal{B}u(t,x)
$$

is still continuous for $(t,x)\in\partial_pQ\backslash D$ for $u\in C(Q)$. Also, from the growth condition (28), we can still show the boundedness of V and V_N (replacing $h(v_{\theta_i-},X_{\theta_i-})$ by $\tilde{h}(v_{\theta_i-},X_{\theta_i-},\overline{\nu}(v_{\theta_i-},X_{\theta_i-}))$ in (21) and (22)) as in Proposition 2.1, by taking $K_V=K_f/r+K_h(1+4TM/(1-p_2))$. The existence and representation of solution can then be established by following the strategy in the proof of Theorem 3.1.

Proof of Corollary 3.2. It suffices to modify the jumps of the piecewise constant process Y to $Y_{\theta_i}=-\delta^2\cdot Y_{\theta_i-}$. The corollary then follows from the strategy in the proof of Theorem 3.1.

Proof of Corollary 3.3. We consider a parallel parabolic terminal boundary value problem for $W(t,x)$:

$$
-\frac{\partial W}{\partial t}(t,x)-\mathcal{L}W(t,x)=f(x) \qquad \text{in } Q \tag{73}
$$

$$
W(t,x)=h(t,x)+\int_Q W(s,z)\nu_{t,x}(ds,dz) \qquad \text{on } \partial_pQ, \tag{74}
$$

where

$$\nu_{t,x}(ds,dz) = \begin{cases} \delta_0(ds)\delta_x(dz) \text{ if } (t,x)\in\{T\}\times\mathcal{O} \\ \delta_0(ds)\mu_x(dz) \text{ if } (t,x)\in\{T\}\times\partial\mathcal{O} \\ \delta_t(ds)\mu_x(dz) \text{ otherwise} \end{cases}$$
$$h(t,x) = \begin{cases} 0 \quad \text{if } (t,x)\in\{T\}\times\mathcal{O} \\ h(x) \text{ otherwise.} \end{cases} \tag{75}$$

It is straightforward to verify that h and ν satisfy the assumptions in Section 2 and Section 3 (with the discontinuity set $D:=\{T\}\times\partial\mathcal{O}$), except for (28). However, since $h(s,x)$ is bounded, Proposition 2.1 still holds. Therefore, by Theorem 3.1, we see that this equation has a unique solution $W\in C^{1,2}(Q)\cap C(\overline{Q}\backslash D)$ that can be written as (6). However, the fact that X does not depend on v means that the right hand side of (6) is equal to (34) which is independent of t. Therefore, $w(x):=W(0,x)\in C^2(\mathcal{O})\cap C(\overline{\mathcal{O}})$ satisfies (32) – (33), which gives the existence of solution. On the other hand, for any solution w to (32) – (33), define $W(t,x)=w(x)$ for $t\in[0,T]$, we can see that W is a solution to (73) – (74), which, again, can be represented as (34). This shows the uniqueness of solution and the representation.

Appendix B: Proofs for the Applications.

Proof of Proposition 4.1. First we transform S_t to a process $X_t\in[0,1]$:

$$X_t=\Gamma(v_t,S_t)=\frac{S_t-H(v_t)}{H_u-H(v_t)}. \tag{76}$$

So for X, the upper and lower threshold is 1 and 0, respectively. X can be understood as the relative distance from S_t to the lower threshold $H(v_t)$ in the range $[H(v_t),H_u]$. Under this transform, a direct application of Itô's formula gives the dynamics of X in-between payment dates

$$dX_s=b(v_s,X_s)dt+\sigma(v_s,X_s)dB_s,$$

where

$$b(v,x)=\left(\frac{\ell R}{H_u-H(v)}+r-c\right)(x-1)+\frac{(r-c)H_u}{H_u-H(v)} \tag{77}$$
$$\sigma(v,x)=\sigma\left(x+\frac{H(v)}{H_u-H(v)}\right). \tag{78}$$

Also, after the transform, the definitions of τ_i, η_i and ζ_i become

$$\tau_i=\inf\{s>\tau_{i-1}|X_{s-}\ge 1\},\quad \eta_i=\inf\{s>\eta_{i-1}|X_{s-}\le 0\},$$
$$\zeta_i=\inf\{s>\zeta_{i-1}|v_{s-}=1,X_{s-}\in(0,1)\}.$$

On these dates, the state process X jumps to a new level:

$$X_{\zeta_i}=\left(1-\frac{\ell R}{H_u-H(0)}\right)X_{\zeta_i-},\quad X_{\tau_i}=X_{\eta_i}=\frac{1-H(0)}{H_u-H(0)},$$

and on downward reset dates η_i's the number of A shares reduces to a fraction of the number immediately before reset (due to a shrink of the shares), i.e. $Y_{\eta_i}=H_dY_{\eta_i-}$.

Comparing the definitions of X, Y, v with the settings in Section 2, we have $\mathcal{O}=(0,1)$, $T=1$, and

$$\begin{aligned}
g(x) &= \left(\frac{1-H_u}{H_u-H(0)}+1\right)\cdot 1_{x=0,1}(x) \\
&\quad + \left(1-\frac{aR}{H_u-H(0)}\right)x\cdot 1_{0<x<1}(x) \\
\tilde{\nu}_{t,x} &= \delta_{0,g(x)}(ds,dz) \\
\overline{\nu}(t,x) &= H_d\cdot 1_{x=0}(x)+1_{0<x\le 1}(x) \\
\theta_i &= \inf\{s>\theta_{i-1}|X_{s-}=0 \text{ or } X_{s-}=1 \text{ or } v_{s-}=1\}.
\end{aligned} \tag{79}$$

Note that since $\overline{\nu}(t,x)$ is independent of t, we denote it by $\overline{\nu}(x)$ for simplicity. According to the contract, there is no continuous payment, therefore $f=0$. Also, we can see that the payments of A shares at pre-specified payment dates and upward and downward reset dates are given by $\tilde{h}(v_{\theta_i-},X_{\theta_i-},\overline{\nu}(X_{\theta_i-}))$ where

$$\tilde{h}(v,x,u)=1-u+Rv. \tag{80}$$

As a result, the risk-neutral pricing formula (38) becomes

$$V(t,x)=E_t^x\left[\sum_{\theta_i\ge t}e^{-r(\theta_i-t)}Y_{\theta_i-}\tilde{h}(v_{\theta_i-},X_{\theta_i-},\overline{\nu}(X_{\theta_i-}))\right]. \tag{81}$$

It is straightforward to check that the coefficient b, σ, $\tilde{h}$ and the measure ν satisfy the conditions in Section 2. Specifically, note that g has jump discontinuities at $x=0,1$, which is translated into the discontinuities of $\tilde{\nu}_{t,x}$ at $t=T$, $x=0,1$, and can be covered by Assumption 2.1. Therefore, from Corollary 3.1 we have that V defined in (81) is the unique solution in $C^{1,2}([0,1)\times(0,1))\cap C([0,1]\times[0,1]\backslash\{1\}\times\{0,1\})$ to the following nonlocal parabolic terminal-boundary value problem:

$$-\frac{\partial W}{\partial t}(t,x)-\mathcal{L}W(t,x)=0 \qquad \text{in } [0,1)\times(0,1) \tag{82}$$

$$W(1,x)=R+W(0,g(x)) \qquad \text{in } (0,1) \tag{83}$$

$$W(t,0)=1-\overline{\nu}(0)+Rt+\overline{\nu}(0)W(0,g(0)) \qquad \text{on } [0,1] \tag{84}$$

$$W(t,1)=Rt+W(0,g(1)) \qquad \text{on } [0,1], \tag{85}$$

Proposition 4.1 is then established by reversing the transform $x=(S-H(t))/(H_u-H(t))$.

Proof of Proposition 4.2 Recall that $(v_{\theta_1},X_{\theta_1})$ follows the distribution $\tilde{\nu}_{v_{\theta_1-},X_{\theta_1-}}$. The condition on $\tilde{\nu}$ implies that $v_{\theta_1}=0$ and X_{θ_1} follows the distribution μ that does not depend on θ_1. Therefore, $(v_{\theta_1},X_{\theta_1})$ is independent of θ_1. Furthermore, since (v,X) starts over from $(v_{\theta_1},X_{\theta_1})$ as (7) with an independent Brownian motion, this also implies that $\{(v_{\theta_1+t},X_{\theta_1+t}),t>0\}$ is independent of θ_1. On the other hand, starting from the second cycle, $\{(v_{t+\theta_{i+1}},X_{t+\theta_{i+1}}),t\ge 0\}$ and $\{(v_{t+\theta_i},X_{t+\theta_i}),t\ge 0\}$ are both time-homogeneous Markov process satisfying the same dynamics and having the same initial distribution $\delta_0(ds)\mu(dz)$, therefore they are equal in distribution.

Finally, to establish stochastic representation, it remains to check that $\nu_{t,x}$ satisfies the assumptions. From the PDE, $\nu_{t,x}=\delta_0(ds)\mu(dz)$ which is already a probability measure, hence we have $\nu_{t,x}=\tilde{\nu}_{t,x}$. Since $\tilde{\nu}_{t,x}$ does not depend on (t,x), we infer that $(t,x)\mapsto\tilde{\nu}_{t,x}$ is continuous. Furthermore, (17) is trivially satisfied since $\tilde{\nu}$ is a Dirac measure concentrated at 0 in the time direction. In the space direction, since μ is a probability measure on the open set $\mathcal{O}$, for any $0<p_0\le 1$, there exists $\delta>0$ such that $\mu(\mathcal{O}_\delta)>p_0$, and this δ satisfies (18).

Proof of Proposition 4.4. Denote $\tilde{\eta}(t,x) = e^{-r(T-t)}\eta(T-t,x)$, then $\tilde{\eta}$ satisfies

$$\begin{aligned} 0 &= -\frac{\partial\tilde{\eta}}{\partial t} - \frac{1}{1+\delta^2}\frac{\partial^2\tilde{\eta}}{\partial x^2} + r\tilde{\eta} && \text{in } [0,T)\times(0,1) \\ \tilde{\eta}(T,x) &= \eta_0(x) && \text{on } [0,1] \\ \tilde{\eta}(t,x) &= -\delta^2\int_0^1 \tilde{\eta}(t,z)dz && \text{on } [0,T)\times\{0,1\}. \end{aligned}$$

Comparing this problem to (29), we see that $\nu_{t,x}(ds,dz) = \delta_t(ds)\cdot\delta^2 dz$ if $(t,x)\in[0,T)\times\{0,1\}$ and 0 otherwise. As a result, we have $\tilde{\nu}_{t,x}(ds,dz) = \delta_t(ds)dz$ on $[0,T)\times(0,1)$ if $x=0,1$ and $t\in[0,T)$, and $\delta_{0,x'}(ds,dz)$ otherwise. Also the equation (48) suggests that X follows $dX_s = \frac{\sqrt{2}}{\sqrt{1+\delta^2}}dW_s$ between the jump times, $f=0$, and the boundary and initial conditions suggest that $h(t,x) = 1_{\{t=T\}}\eta_0(x)$. Therefore, applying Corollary 3.2 we have that the above problem has a unique solution $\tilde{\eta}\in C^{1,2}([0,T)\times(0,1))\cap C([0,T]\times[0,1]\backslash\{T\}\times(D'\cup\{0,1\}))$ that can be represented as

$$\tilde{\eta}(t,x) = E_t^x\left[e^{-r(T-t)}Y_{T-}\cdot\eta_0(X_{T-})\right],$$

where $Y_s = \prod_{t\le\theta_j\le s} -\overline{\nu}(v_{\theta_j-}, X_{\theta_j-})$. Note that the expression inside the expectation does not depend on the state of system (v,X,Y) on or after T because $\overline{\nu}(T,\cdot)=0$, which means that $Y_s=0$ for all $s\ge T$. Also it is straightforward that the piecewise constant process Y jumps according to $Y_{\theta_i} = -\delta^2\cdot Y_{\theta_i-}$ for $\theta_i<T$. Switching from $\tilde{\eta}$ back to η, Proposition 4.4 follows.

Proof of Proposition 4.3. The existence, uniqueness, and representation of solution follow from Corollary 3.3, since $\partial\mathcal{O} = \{a_1,a_2\}$ and $x\mapsto\mu_x$ is continuous. It remains to show X is regenerative. Without loss of generality, we only prove this by choosing θ^1 as epoch. By construction, $\{X_{\theta_1^1+t}, t\ge 0\}$ is independent of θ_1^1 by the same reasoning as in the proof of Proposition 4.2. On the other hand, starting from the second cycle, $\{X_{\theta_i^1+t}, t\ge 0\}$ and $\{X_{\theta_{i+1}^1+t}, t\ge 0\}$ are both time-homogeneous Markov process satisfying the same dynamics and having the same initial distribution $\delta_{a_1}(dz)$, therefore they are equal in distribution.

Acknowledgements The research of Dai is supported in part by NSFC grant 11671292, and Singapore MOE AcRF grants R-146-000-243-114 and R-703-000-032-112. The research of Kou was supported in part by NSFC grant 71571129 and Singapore MOE grant MOE2017-T2-1-101. The research of Yang is supported in part by a start-up fund and the Direct Allocation Grant 4055132 at the Chinese University of Hong Kong.

References

Adams AT, Clunie JB (2006) Risk assessment techniques for split capital investment trusts. *Annals of Actuarial Science* 1(1):7–36.

Bayraktar E, Xing H (2010) On the uniqueness of classical solutions of Cauchy problems. *Proceedings of the American Mathematical Society*, volume 138, 2061–2064.

Cadenillas A, Lakner P, Pinedo M (2010) Optimal control of a mean-reverting inventory. *Operations research* 58(6):1697–1710.

Chabrowski J (1984a) On nonlocal problems for parabolic equations. *Nagoya Math J* 93:109–131.

Chabrowski J (1984b) On the nonlocal problem with a functional for parabolic equation. *Funkcial Ekvac Ser Int* 27(1):101–123.

Day WA (1982) Extensions of a property of the heat equation to linear thermoelasticity and other theories. *Quarterly of Applied Mathematics* 40(3):319–330.

Deng K (1993) Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions. *Journal of Mathematical Analysis and Applications* (179):630–637.

Freidlin MI (1985) *Functional Integration and Partial Differential Equations* (Princeton University Press).

Friedman A (1986) Monotonic decay of solutions of parabolic equations with nonlocal boundary conditions. *Quarterly of Applied Mathematics* 44(3):401–407.

Friedman A (2012) *Stochastic Differential Equations and Applications* (Courier Corporation).

Friedman A (2013) *Partial Differential Equations of Parabolic Type* (Courier Dover Publications).

Glasserman P, Kou SG (1995) Limits of first passage times to rare sets in regenerative processes. *The Annals of Applied Probability* 5(2):424—445.

Grigorescu I, Kang M (2013) Markov processes with redistribution. *Markov Processes and Related Fields* 19(3):497–520.

Harrison JM, Sellke TM (1983) Impulse control of Brownian motion. *Mathematics of Operations Research* 8(3):454–466.

Ingersoll JE (1976) A theoretical and empirical investigation of the dual purpose funds: An application of contingent-claims analysis. *Journal of Financial Economics* 3(1):83–123.

Jarrow RA, O'Hara M (1989) Primes and scores: An essay on market imperfections. *The Journal of Finance* 44(5):1263–1287.

Lieberman GM (1999) Time-periodic solutions of linear parabolic differential equations. *Communications in partial differential equations* 24(3-4):631–663.

Olmstead WE, Roberts CA (1997) The one-dimensional heat equation with a nonlocal initial condition. *Applied Mathematics Letters* 10(3):89–94.

Pao CV (1995a) Dynamics of reaction-diffusion equations with nonlocal boundary conditions. *Quarterly of Applied Mathematics* 3(1):173–186.

Pao CV (1995b) Reaction diffusion equations with nonlocal boundary and nonlocal initial conditions. *Journal of Mathematical Analysis and Applications* 195(3):702–718.

Pao CV (1998) Asymptotic behavior of solutions of reaction-diffusion equations with nonlocal boundary conditions. *Journal of computational and applied mathematics* 88(1):225–238.

Sigman K, Wolff RW (1993) A review of regenerative processes. *SIAM review* 35(2):269–288.

Yin YF (1994) On nonlinear parabolic equations with nonlocal boundary condition. *Journal of Mathematical Analysis and Applications* 185(1):161–174.