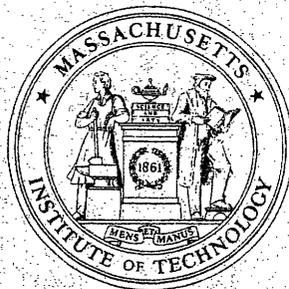


# OPERATIONS RESEARCH CENTER

working paper



**MASSACHUSETTS INSTITUTE  
OF TECHNOLOGY**

THE COMPENSATION METHOD APPLIED TO  
A ONE-PRODUCT PRODUCTION INVENTORY PROBLEM

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## ABSTRACT

This paper considers a one-product, one-machine production/inventory problem. Demand requests for the product are governed by a Poisson process with demand size being an exponential random variable. The production facility may be in production or idle; while in production, the facility produces continuously at a constant rate. The objective is to minimize system costs consisting of setup costs, inventory holding costs, and backorder costs. Given a two-critical-number policy, the problem is analyzed as a constrained Markov process using the compensation method. The policy space may then be searched to find the optimal policy.

## 1. Introduction

The intent of this paper is twofold:

- a) to analyze a one-product production/inventory problem assuming stochastic demand, and
- b) to illustrate the use of Green's function methods for the study of constrained Markov processes.

The production/inventory problem of concern has one product subject to Poisson demand requests with the size of each demand request being exponentially distributed. This product is produced on a single machine, which may be either setup and in production, or shutdown. System costs consist of setup costs, inventory holding costs, and backorder costs. This production/inventory problem differs from the pure inventory problem in that replenishment of inventory occurs continuously in the production context; for a pure inventory system, replenishment occurs in batches or lot sizes. Such problems are prevalent in continuous processing plants, such as in the chemical or glass industry, which are very capital-intensive and conducive to long production runs during which the product is continually added to inventory.

Heyman [4 ] and Bell [1] have considered a special case of this production/inventory problem in which positive inventory is now allowed, and a linear penalty or waiting cost is incurred for backorders. Both [4 ] and [1] find optimal policies by way of a square root formula analogous to the economic order quantity. Sobel [9 ] examines a more general production/inventory problem, and demonstrates that the optimal policy must be a two-critical-number policy; however, no computational method is given for finding this policy. Gavish and Graves [ 3 ] have analyzed the production/inventory problem where the demand is a Poisson process

with unit requests. Their analysis relies on concepts from queueing theory. The problem of the current paper is distinct from that in [ 3] in that the size of the demand requests is exponentially distributed; the methodology used to analyze the problem is also different.

The second purpose of this paper is to illustrate the use of Green's function methods. In particular, a compensation method is used to study a spatially-homogeneous Markov process which has been restricted by boundaries. This methodology has been developed by Keilson and is detailed in [ 5], [ 6]. These methods provide the means to study inhomogeneous Markov processes which are induced from a homogeneous Markov process by putting restrictions or modifications on the homogeneous process.

This paper is organized as follows: In Section 2, the compensation method using Green's functions is presented. Section 3 defines the problem of interest. In Section 4, the compensation method is applied to the production/inventory problem to find the ergodic probability density for the underlying inventory process. In Section 5, the results of Section 4 are used to determine an expected cost function which can be searched to find the minimum cost policy.

## 2. The Compensation Method

This section introduces the use of Green's functions and compensation densities for studying a special class of ergodic processes which arise from modifying spatially-homogeneous processes. The mathematical ideas have been developed more completely elsewhere (see below). The basic ideas will be presented in simplified form without full mathematical care to permit the presentation to be self-contained. The processes are assumed to be temporally homogeneous throughout; i.e., the laws governing changes are described by parameters and distributions unchanging in time. A spatially-homogeneous process is a process whose transition probability distribution depends only on the distance or space that is measured between the transition states. That is, the process  $\{X_H(t)\}$  is spatially homogeneous if the probability distribution

$P_H(x,y,t) = \Pr[X_H(t) \leq y \mid X_H(0) = x]$  is such that

$$(1) \quad P_H(x,y,t) = P_H(0,y-x,t) = G_H(y-x,t)$$

where  $G_H(w,t)$  is the probability distribution for the change in  $X_H$  in an interval of length  $t$ .

Consider a spatially-homogeneous process which is modified such that certain transitions are restricted. A typical restriction would be to limit the process by boundaries on the state space. For example, the queue length process for an M/M/1 queue with finite waiting room  $K$  may be thought of as the homogeneous random walk in continuous time on the lattice of all integers (negative and positive) restricted to the state space  $0,1,\dots,K$  by censoring transitions leaving this state space. This point of view was introduced by J. Keilson and developed in a series of papers [5], [6], [7], [8].

For such a modified process  $\{X(t)\}$ , the presentation of ideas is clarified by working with spatial densities rather than distribution functions even though the densities may sometimes exist only in a generalized sense. Thus let  $F(y,t) = \Pr[X(t) \leq y]$  and let  $f(y,t) = \frac{d}{dy} F(y,t)$  be its p.d.f. For  $X(t)$  we may write its transition density as  $p(x,y,t) = \frac{d}{dy} \Pr[X(t) \leq y \mid X(0) = x]$ . We may then relate  $p(x,y,t)$  to the homogeneous transition density  $p_H(x,y,t) = \frac{d}{dy} P_H(x,y,t)$  by

$$(2) \quad p(x,y,t) = p_H(x,y,t) + d(x,y,t)$$

where  $d(x,y,t)$  is defined by (2) to denote the transition restrictions. Let  $f_0(y,t) = P(0,y,t)$  and  $g_H(y,t) = \frac{d}{dy} G_H(y,t) = p_H(0,y,t)$ . Then for any  $\Delta$ ,  $0 < \Delta < t$ , we have

$$(3) \quad f_0(y,t) = \int f_0(x,t-\Delta)p(x,y,\Delta)dx.$$

Substituting (2) and (1) into (3) gives

$$(4) \quad \begin{aligned} f_0(y,t) &= \int f_0(x,t-\Delta)g_H(y-x,\Delta)dx + \int f_0(x,t-\Delta)d(x,y,\Delta)dx \\ &= f_0(y,t-\Delta) * g_H(y,\Delta) + c_0(\Delta,y,t) \end{aligned}$$

where  $c_0(\Delta,y,t) = \int f_0(x,t-\Delta)d(x,y,\Delta)dx$ . The asterisk  $*$  denotes convolution over the state variable, and  $c_0(\Delta,y,t)$  is a compensation density.

The compensation density, as defined in (4), represents the difference in probability flow over  $(t-\Delta,t)$  between the modified process and the homogeneous process. Note that the compensation density  $c_0(\Delta,y,t)$  is localized wherever restrictions have been made to the homogeneous process. We also note from (2) that  $\int d(x,y,t)dy = 0$ , and hence from (4) that the total compensation mass  $\int c_0(\Delta,y,t)dy$  is zero. By repeated substitution, when  $t = N\Delta$ , (4) may be rewritten as

$$(5) \quad f_o(y,t) = \sum_{i=0}^N c_o(\Delta,y,t-i\Delta) * g_H(y,i\Delta).$$

Here  $c_o(\Delta,y,0)$  is defined to be  $f_o(y,0) = \delta(y)$ , and  $g_H(y,0) = \delta(y)$ , the Dirac delta function. As  $\Delta \rightarrow 0$ , define  $c_o(y,t)$  such that a Taylor's expansion of the  $c_o(\Delta,y,t)$  about  $\Delta = 0$  gives  $c_o(\Delta,y,t) = \Delta c_o(y,t) + o(\Delta)$ . Substituting this into (5) and taking the limit as  $\Delta \rightarrow 0$ , we may replace the summation by integration:

$$(6) \quad f_o(y,t) = \int_0^t c_o(y,t-\tau) * g_H(y,\tau) d\tau.$$

When the modified process  $\{X(t)\}$  is ergodic, the density  $f_o(y,t)$  converges to the ergodic probability density  $f_\infty(y)$ ; furthermore, from (6), the compensation density  $c_o(y,t)$  must also converge to  $c_\infty(y)$ . Hence we have

$$(7) \quad f_\infty(y) = c_\infty(y) * g(y)$$

where  $g(y) = \int_0^\infty g_H(y,t) dt$  is the ergodic Green density for the homogeneous process  $\{X_H(t)\}$ .

Equation (7) can now be used to determine the ergodic distribution for the modified process. The value and justification of this formulation is the ease with which the Green density and compensation density may be found. The compensation density, and thence the ergodic density, is found by exploiting the simplicity in structure of the more simple underlying spatially homogeneous process. To identify the compensation density, (7) is used with the facts that the compensation density has zero total mass, and the modified process is restricted to a limited state space. In the following sections we present a model for which both the Green density and the compensation density can be identified.

### 3. The Production/Inventory Problem

The problem of interest is a single-machine, one-product production scheduling problem. Demand requests for the product are a Poisson process with the size of the demand requests distributed as an exponential random variable. Demand either is serviced from inventory or is backordered. The inventory holding cost rate and the backorder cost rate are assumed to be linear in the respective inventory and backorder levels, and these costs are accumulated continuously. A single machine or production facility is dedicated to the one product. The machine may be either setup and in production, or shutdown and idle. If the machine is setup, production into inventory is continuous at a constant production rate. It is assumed that variable production costs are constant, and that a fixed setup cost is incurred whenever the production facility is turned on. The production rate is greater than the average demand rate so to avoid the infinite backorder possibility. It is also assumed that setups and shutdowns do not consume any production capacity.

The state space for this system is characterized by the pair  $(I,k)$  where  $I$  is the inventory level (negative inventory being backorders), and  $k = 0,1$  denotes whether the facility is idle ( $k=0$ ) or producing ( $k=1$ ). A stationary decision policy is defined by the function  $d$  on the state space such that  $d(I,k) = 0$  is a decision to shutdown, while  $d(I,k) = 1$  is a decision to produce. In order for the policy to have finite cost, we must have  $d(I,0) \leq d(I,1)$ ; this rules out the paradoxical possibility of shutting down the facility at a given inventory level if the facility is setup, but setting up the facility if it is idle. Given that  $d(I,0) \leq d(I,1)$ , it is easy to see that any stationary policy, which results neither in infinite backorders nor infinite inventory, will revert

to a two-critical-number policy  $(I^*, I^{**})$  such that the facility is setup when inventory falls below  $I^*$  and is shutdown once inventory reaches  $I^{**}$  ( $I^{**} > I^*$ ). Consequently, any decision policy may be restated in terms of  $(I^*, I^{**})$  as

$$(8) \quad \begin{aligned} d(I,0) &= \begin{cases} 1 & \text{for } I < I^* \\ 0 & \text{for } I \geq I^* \end{cases} \\ d(I,1) &= \begin{cases} 1 & \text{for } I < I^{**} \\ 0 & \text{for } I \geq I^{**} \end{cases} \end{aligned}$$

Figure 1 gives a graphical representation of the state space and the possible transitions. While the machine is on, the inventory grows at a constant rate ( $\rightarrow$ ) subject to random demand requests ( $\curvearrowright$ ) which behave as an exponential jump process. Once the inventory reaches  $I^{**}$ , the machine is shutdown. Here, inventory is depleted by random demand requests, until the level drops below  $I^*$  and the machine is setup again.

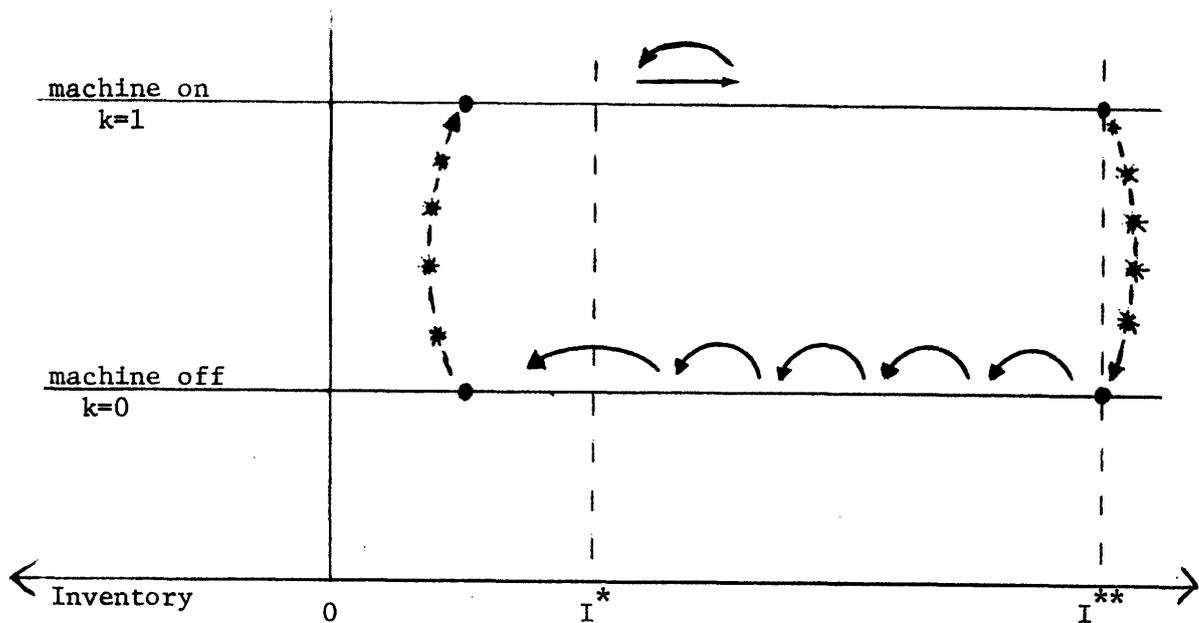


Figure 1: State Space Representation

In the framework of Figure 1, the process can be thought of as two spatially-homogeneous processes, corresponding to the machine on or off, which are modified and connected into one process by boundary restrictions and transitions. The theory of Green's functions and the compensation method can be applied to this process. Compensation densities are located around the boundaries ( $I^*$  and  $I^{**}$ ) so that the potential flow across these boundaries is annihilated and transferred. That is, when the machine is on, flow beyond  $I^{**}$  is annihilated and transferred to the machine being off; for the machine shutdown, the compensation mass redirects flows that will reduce inventory below  $I^*$ , to the corresponding state with the machine on. The next section details the development of this application.

#### 4. Ergodic Analysis of Production/Inventory Problem

This section applies the theory reviewed in Section 2 to the one-product production/inventory problem of Section 3. First, the Green's functions for the homogeneous processes corresponding to the machine on and off will be found. The compensation densities are then identified based on the Green's functions and on the problem structure as depicted in Figure 1. Equation (7) is then applied to find the ergodic probability density for the system.

Consider first the spatially-homogeneous process  $X_H^+(t)$  on the state space  $\{(I,1)\}$  when the machine is on and switching at  $I^{**}$  is ignored. At any time  $t$ , the state of the process is the net inventory level, that being the difference between total production and the accumulated demand requests. Note that no restrictions are placed on this inventory level. Assuming that  $X_H^+(0) = 0$ , and a constant production rate of one unit/time unit, we have

$$(9) \quad X_H^+(t) = t - S(t)$$

where  $S(t)$  is the accumulated demand at time  $t$ . If demand requests arrive at Poisson rate  $\lambda$ , and the size of the request is an exponential random variable with mean  $(1/\mu)$ , then  $S(t)$  has a probability density  $f_s(y,t)$  in the generalized sense, such that

$$(10) \quad f_s(y,t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^k}{k!} a^{(k)}(y)$$

where  $a(y) = 0$  for  $y < 0$ ,  $a(y) = \mu e^{-\mu y}$  for  $y \geq 0$ ,  $a^{(k)}(y)$  is the  $k$ -fold convolution of  $a(\cdot)$ , and  $a^{(0)}(y) = \delta(y)$ . The assumption that the demand rate is less than the production rate gives  $\lambda/\mu < 1$ .

The time-dependent probability density for the process  $X_H^+(t)$  given

that  $X_H^+(0) = 0$ , is

$$(11) \quad \begin{aligned} P_H^+(x=0, y, t) &= g_H^+(y, t) \\ &= f_s(t-y, t) \end{aligned}$$

Therefore the ergodic Green's density can be expressed as

$$(12) \quad \begin{aligned} g^+(y) &= \int_0^\infty g_H^+(y, t) dt \\ &= \int_0^\infty f_s(t-y, t) dt \end{aligned}$$

To solve (12), it is convenient to use the bilateral Laplace transformation.

Let

$$(13) \quad \begin{aligned} \gamma^+(s) &= \int_{-\infty}^\infty e^{-sy} g^+(y) dy \\ &= \frac{1}{\lambda[1-\alpha(-s)] + s} \quad \theta < s < \mu - \lambda \end{aligned}$$

where  $\alpha(s) = \int_{-\infty}^\infty e^{-sx} a(x) dx = \mu/(\mu+s)$ . Substituting into (13) we obtain

$$(14) \quad \begin{aligned} \gamma^+(s) &= \frac{\mu-s}{s(\mu-s-\lambda)} \\ &= \frac{(\mu/\mu-\lambda)}{s} + \frac{(\lambda/\mu-\lambda)}{\mu-s-\lambda} \end{aligned}$$

This transform can now be inverted with uniqueness when normalized, to give (see Widder [10])

$$(15) \quad \begin{aligned} g^+(y) &= \left(\frac{\lambda}{\mu-\lambda}\right) e^{(\mu-\lambda)y} && \text{for } y < 0, \\ &= \frac{\mu}{\mu-\lambda} && \text{for } y > 0. \end{aligned}$$

Similarly, the spatially-homogeneous process  $X_H^-(t)$  when the machine is off can be examined. Here,  $X_H^-(t)$  again corresponds to the net inventory level at time  $t$ , given no restrictions on the inventory range.

Assuming  $X_H^-(0) = 0$ , we have

$$(16) \quad X_H^-(t) = -S(t).$$

Following the analysis of the process for the machine on, a similar development yields the following results:

$$(17) \quad \begin{aligned} p_H^-(x=0, y, t) &= g_H^-(y, t) \\ &= f_s^-(y, t) \end{aligned}$$

$$(18) \quad \begin{aligned} g^-(y) &= \int_0^\infty g_H^-(y, t) dt \\ &= \int_0^\infty f_s^-(y, t) dt \end{aligned}$$

$$(19) \quad \begin{aligned} \gamma^-(s) &= \int_{-\infty}^\infty e^{-sy} g^-(y) dy \\ &= \frac{1}{\lambda} - \frac{\mu}{\lambda s} \end{aligned} \quad \text{for } s < 0$$

$$(20) \quad \therefore \quad g^-(y) = \frac{1}{\lambda} \delta(y) + \frac{\mu}{\lambda} U(-y)$$

where  $U(-y) = 1$  for  $y \leq 0$ ,  $U(-y) = 0$  for  $y > 0$ .

Having found the Green's densities for the two homogeneous processes, we now must determine the appropriate compensation densities so to be able to apply equation (7). Here it will be useful to refer to Figure 2. The modified process is characterized by a two-critical-number policy ( $I^*$ ,  $I^{**}$ ). In Figure 2, the state space for the modified process is illustrated; it is convenient to translate the state space by  $I^{**}$ . That is,  $x=0$  corresponds to inventory  $I^{**}$ , while  $x = -\theta = -(I^{**} - I^*)$  is an inventory of  $I^*$ . For the modified process when the machine is on, the machine is shutdown when the inventory reaches  $I^{**}$  ( $x=0$ ); consequently, a negative compensation mass  $-c_A \delta(x)$  must be placed at  $x=0$  to offset the probability flow to the positive  $x$ -axis. A corresponding positive mass  $c_A \delta(x)$  is placed at  $x=0$  for the machine off, to represent the probability flow from shutting the machine down. When the machine is off, the machine is set up once inventory drops below  $I^*$  ( $x = -\theta$ );

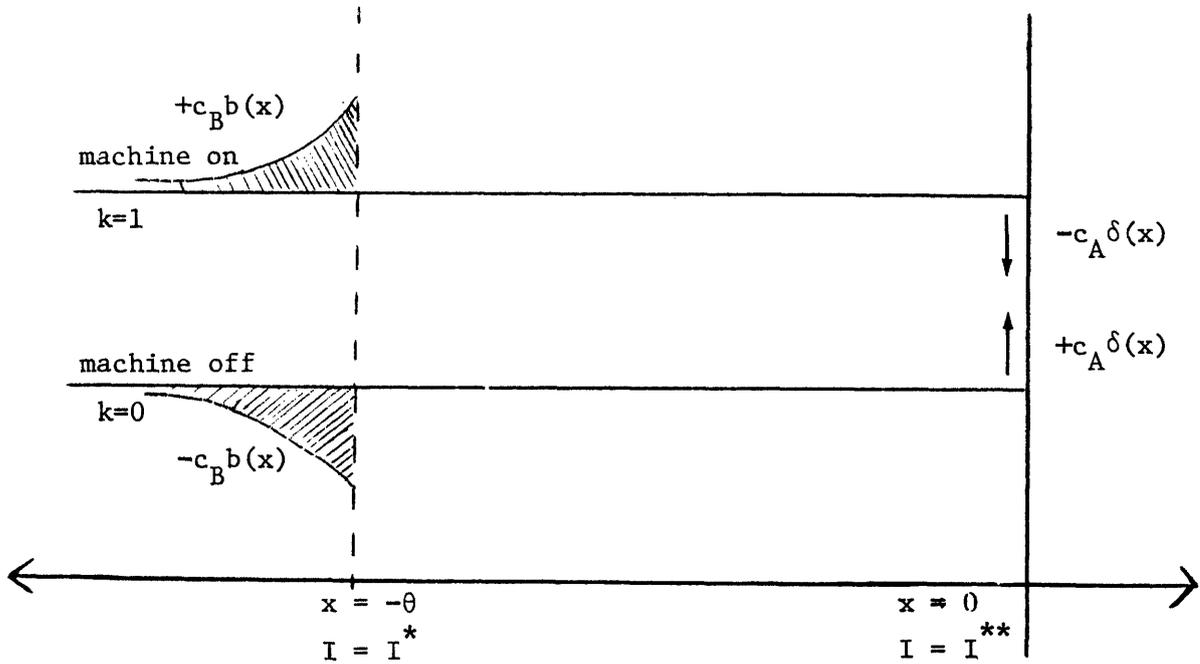


Figure 2: State Space Representation of Compensation Method

to absorb this flow, a negative compensation density  $[-c_B b(x)]$  is applied for  $x < -\theta$ . This compensation density appears as a positive probability flow  $[c_B b(x)]$  for the machine on to represent the transition from turning the machine on. Note that the compensation densities applied to the process when the machine is off are symmetric to those for the process when the machine is on. It will soon be seen that  $b(x) = \mu e^{\mu(x+\theta)}$  on  $(-\infty, -\theta)$  and  $c_A = c_B$ . The form of  $b(x)$  might have been expected since  $b(x)$  acts to annihilate the overshoot to  $(-\infty, -\theta)$  when the machine is off.

We can now use equation (7) to find the ergodic probability densities for the modified process of interest. Define  $f^-(x)$  to be the probability density for the machine off; from (7) we have

$$(21) \quad f^-(x) = g^-(x) * [c_A \delta(x) - c_B b(x)]$$

$$(22) \quad \begin{aligned} \phi^-(s) &= \int_{-\infty}^{\infty} e^{-sx} f^-(x) dx \\ &= \gamma^-(s) \cdot [c_A - c_B \beta(s)] \end{aligned}$$

where  $\beta(s) = \int_{-\infty}^{\infty} e^{-sx} b(x) dx$ . Substituting (19) into (22) we have for  $s < 0$ :

$$(23) \quad \phi^-(s) = \frac{c_A}{\lambda} - \frac{c_A \mu}{\lambda s} - \frac{c_B \beta(s)}{\lambda} + \frac{c_B \mu \beta(s)}{s \lambda}$$

Inverting (23) we obtain

$$(24) \quad f^-(x) = \frac{c_A}{\lambda} \delta(x) + \frac{c_A \mu}{\lambda} [1-U(x)] - \frac{c_B}{\lambda} b(x) - \frac{c_B \mu}{\lambda} \bar{B}(x)$$

where  $U(x) = 0$  for  $x < 0$  and  $U(x) = 1$  for  $x \geq 0$ , and  $\bar{B}(x) = \int_x^{\infty} b(y) dy$ .

As  $x \rightarrow -\infty$ , we must have  $c_A = c_B \int_{-\infty}^{\infty} b(y) dy$  to ensure that  $f^-(x)$  vanishes;

if  $b(x)$  is normalized to have unit mass, then  $c_A = c_B = c$ . Now since  $f^-(x)$  has positive mass only on the interval  $[-\theta, 0]$ ,  $b(x)$  must be found such that  $f^-(x) = 0$  for  $x \leq -\theta$  and for  $x > 0$ ; that is knowledge of the underlying state space for the modified process can be used to identify the appropriate compensation density. These conditions translate to the following two equations:

$$(25) \quad b(x) + \mu \bar{B}(x) = 0 \quad \text{for } x > 0$$

$$(26) \quad \mu - b(x) - \mu \bar{B}(x) = 0 \quad \text{for } x \leq -\theta$$

The solution to (25), (26) is

$$(27) \quad \begin{aligned} b(x) &= \mu e^{\mu(\theta+x)} && \text{for } x \leq -\theta \\ &= 0 && \text{for } x > -\theta \end{aligned}$$

Therefore, we can rewrite (24) as

$$(28) \quad f^-(x) - \frac{c}{\lambda} \delta(x) = \begin{cases} 0 & \text{for } x \leq -\theta \\ \frac{c\mu}{\lambda} & \text{for } -\theta < x \leq 0 \\ 0 & \text{for } x > 0 \end{cases}$$

Thus the ergodic probability density when the machine is off is uniform over the interval  $[-\theta, 0]$  with a probability mass point at  $x = 0$ .

Having determined  $b(x)$ , we can now use equation (7) to find the ergodic probability density when the machine is on,  $f^+(x)$ .

$$(29) \quad f^+(x) = g^+(x) * [-c\delta(x) + cb(x)]$$

Substituting (15) and (27) into (29), we obtain

$$(30) \quad \begin{aligned} f^+(x) &= c\left(\frac{\mu}{\mu-\lambda}\right) e^{(\mu-\lambda)(\theta+x)} - c\left(\frac{\lambda}{\mu-\lambda}\right) e^{(\mu-\lambda)x} && \text{for } x \leq -\theta \\ &= c\left(\frac{\mu}{\mu-\lambda}\right) - c\left(\frac{\lambda}{\mu-\lambda}\right) e^{(\mu-\lambda)x} && \text{for } -\theta < x \leq 0 \\ &= 0 && \text{for } x > 0. \end{aligned}$$

The constant  $c$  can be found by setting the total probability mass for the modified process to one. That is, we have

$$(31) \quad \int f^+(x)dx + \int f^-(x)dx = 1$$

which results in

$$(32) \quad c = \frac{\lambda(\mu - \lambda)}{\mu(1 + \mu\theta)} .$$

This section has derived the ergodic probability density for the modified process which models the production/inventory system of interest. This ergodic density assumes a policy  $(I^*, I^{**})$  where  $\theta = I^{**} - I^*$ . To find the optimal values for  $(I^*, I^{**})$ , the results of this section can be used to write an expected cost function which is to be minimized over the parameter values. This is done in the next section.

### 5. Policy Determination for Production/Inventory Problem

In this section, an expected cost expression is formulated, and a discussion is given for its optimization. In the previous section, the ergodic densities for the modified process were found for when the machine is off and is on. Here it will be useful to combine these two densities into one density, and to transform this density back to an inventory scale. For a given policy  $(I^*, I^{**})$ , if we let  $\pi(x)$  be the ergodic probability density of having an inventory level of  $x$ , then we have

$$(33) \quad \pi(x) = f^-(x-I^{**}) + f^+(x-I^{**}) .$$

That is, using the fact that  $\theta = I^{**} - I^*$ ,

$$(34) \quad \begin{aligned} \pi(x) &= c\left(\frac{\mu}{\mu-\lambda}\right) e^{-(\mu-\lambda)(I^*-x)} - c\left(\frac{\lambda}{\mu-\lambda}\right) e^{-(\mu-\lambda)(I^{**}-x)} && \text{for } x \leq I^* \\ &= \frac{c}{\lambda} \delta(x-I^{**}) + \frac{c\mu^2}{\lambda(\mu-\lambda)} - c\left(\frac{\lambda}{\mu-\lambda}\right) e^{-(\mu-\lambda)(I^{**}-x)} && \text{for } I^* < x \leq I^{**} \\ &= 0 && \text{for } x > I^{**} \end{aligned}$$

for  $c$  given by (32).

Assuming a linear inventory holding cost rate  $h$  per time unit, a linear backorder cost rate  $b$  per time unit, and a fixed setup cost  $K$ , the expected cost rate per time unit for a policy  $(I^*, I^{**})$  is given by

$$(35) \quad C(I^*, I^{**}) = cK + b \int_{-\infty}^0 (-x) \pi(x) dx + h \int_0^{I^{**}} x \pi(x) dx$$

where both the constant  $c$  and the density function  $\pi$  are functions of the policy parameters  $(I^*, I^{**})$ . In (35) it is assumed that  $I^{**} \geq 0$ ; provided that  $b > 0$ , the optimal policy must have  $I^{**} \geq 0$ . Setups occur at rate  $c$  since  $c$  reflects the rate of probability flow from the machine being off to being on. Instead of assuming that the backorder cost depends on the

length of time an item is backordered, it may be that the backorder cost is a linear function of the expected number of items backordered; in this case, the backorder cost term in (35) is replaced by  $b \int_0^\infty \pi(x) dx$ .

For  $I^* \geq 0$  ( $\theta \leq I^{**}$ ), expression (35) may be written as

$$(36) \quad C(I^*, I^{**}) = cK + ch \left[ -\frac{1}{(\mu-\lambda)^2} + \frac{\mu I^{**}}{\lambda(\mu-\lambda)} - \frac{\mu(I^{**}-I^*)}{(\mu-\lambda)^2} + \frac{1}{2} \frac{\mu^2(I^{**}-I^*)^2}{\lambda(\mu-\lambda)} \right] \\ + \frac{c(h+b)}{(\mu-\lambda)^3} \left[ \mu e^{-(\mu-\lambda)I^*} - \lambda e^{-(\mu-\lambda)I^{**}} \right]$$

Alternatively, equation (36) can be expressed as a function of  $I^*$  and  $\theta = I^{**} - I^*$ :

$$(37) \quad C(I^*, \theta) = cK + ch \left[ \frac{-1}{(\mu-\lambda)^2} + \frac{\mu(I^*+\theta)}{\lambda(\mu-\lambda)} - \frac{\mu\theta}{(\mu-\lambda)^2} + \frac{1}{2} \frac{\mu^2\theta(\theta+2I^*)}{\lambda(\mu-\lambda)} \right] \\ + \frac{c(h+b)}{(\mu-\lambda)^3} \left\{ [e^{-(\mu-\lambda)I^*}] [\mu - \lambda e^{-(\mu-\lambda)\theta}] \right\}$$

For a given  $\theta$ ,  $C(I^*, \theta)$  is convex with respect to  $I^*$ . Consequently, given  $\theta$ , the optimal value of  $I^*$ , ignoring integer restrictions, satisfies the following:

$$(38) \quad I^* = \max(0, \hat{I}), \text{ where}$$

$$(39) \quad e^{-(\mu-\lambda)\hat{I}} = \left( \frac{h}{h+b} \right) \left\{ \frac{(\mu-\lambda)(\mu+\mu^2\theta)}{\lambda[\mu-\lambda e^{-(\mu-\lambda)\theta}]} \right\}$$

Using (38), (39), the cost expression (37) may be easily searched over values for  $\theta$  to find the best policy  $(I^*, I^{**})$  where  $\theta = I^{**} - I^*$  and  $I^* \geq 0$ . If the integer restrictions on the policy parameters cannot be ignored, then (38), (39) finds the best continuous value for  $I^*$ , which must be rounded either up or down to give the best integer value.

For  $I^* \leq 0$  ( $\theta > I^{**}$ ), the evaluation of (35) gives

$$(40) \quad C(I^*, I^{**}) = cK + ch \left[ \frac{-1}{(\mu-\lambda)^2} + \frac{\mu I^{**}}{\lambda(\mu-\lambda)} - \frac{\mu(I^{**}-I^*)}{(\mu-\lambda)^2} + \frac{1}{2} \frac{\mu^2(I^{**}-I^*)^2}{\lambda(\mu-\lambda)} \right] \\ + \frac{c(h+b)}{(\mu-\lambda)} \left\{ \left( \frac{\mu}{\mu-\lambda} \right) \left[ \frac{1}{(\mu-\lambda)} - I^* \right] + \left( \frac{\mu}{2\lambda} I^* \right)^2 - \frac{\lambda e^{-(\mu-\lambda)I^{**}}}{(\mu-\lambda)^2} \right\},$$

or equivalently,

$$(41) \quad C(I^*, \theta) = cK + ch \left[ \frac{-1}{(\mu-\lambda)^2} + \frac{\mu(\theta+I^*)}{\lambda(\mu-\lambda)} - \frac{\mu\theta}{(\mu-\lambda)^2} + \frac{\mu^2\theta(\theta+2I^*)}{2\lambda(\mu-\lambda)} \right] \\ + \frac{c(h+b)}{(\mu-\lambda)} \left[ \left( \frac{\mu}{\mu-\lambda} \right) \left( \frac{1}{\mu-\lambda} - I^* \right) + \frac{(\mu I^*)}{2\lambda} - \frac{\lambda e^{-(\mu-\lambda)(I^*+\theta)}}{(\mu-\lambda)^2} \right]$$

Unfortunately this cost expression is not convex, and a two-dimensional search must be used to find the best policy assuming  $I^* \leq 0$ . Note that to ensure that  $I^{**} \geq 0$ , for a given value of  $\theta$  we need search only over  $-\theta \leq I^* \leq 0$ .

A series of test problems have been solved, and their solutions are reported in Table 1. Here the policy parameters have been restricted to be integer. The behavior of the optimal policy is as expected from standard inventory theory.  $I^*$ , corresponding to a reorder point, increases as the backorder cost rate increases.  $\theta$ , corresponding to an order quantity, increases as the setup cost increases. Finally, both parameters  $(I^*, \theta)$  increase as the machine utilization, equal to  $\lambda/\mu$ , is raised.

Table 1: Computation Results

$\mu$	$\lambda$	$h$	$b$	$K$	$I^*$	$\theta$	expected cost rate
1	.5	1	2	25	-1	4	3.62
1	.5	1	2	50	-1	6	4.68
1	.5	1	6	25	0	5	5.20
1	.5	1	6	50	0	6	6.16
1	.5	1	10	25	1	5	6.09
1	.5	1	10	50	1	6	7.07
1	.9	1	2	25	-1	5	11.44
1	.9	1	2	50	6	7	11.74
1	.9	1	6	25	15	6	19.91
1	.9	1	6	50	15	7	20.22
1	.9	1	10	25	20	5	24.43
1	.9	1	10	50	19	7	24.74

## 6. Conclusion

This paper has a dual purpose to analyze a specific production/inventory problem and to illustrate the use of the compensation method. The problem of interest is a simple single-machine, single-product production/inventory problem. This problem is very similar to the standard stochastic demand inventory problem, except for stock replenishment occurring continually rather than in discrete batches. Whereas the inventory problem has been well-studied, there has been little work on its production/inventory analog.

Through the introduction of Green's functions and the compensation method, the single-machine, single-product problem assuming continuous review and Poisson demand arrivals with exponential demand requests has been completely characterized on a continuous state space. A similar analysis can be done on the lattice assuming a continuous review policy with Poisson demand arrivals but with demand requests distributed as a geometric random variable. The compensation method was vital to this analysis; indeed it is a quite powerful methodology for making constrained homogeneous processes tractable. It allowed the analysis to take advantage of the structural simplicity of the underlying homogeneous processes in characterizing the modified process.

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