Strong Formulations for the Multistage Stochastic Self-Scheduling Unit Commitment

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Abstract

With the increasing penetration of renewable energy into the power grid system, the volatility of real-time electricity prices increases significantly. This brings challenges for independent power producers to provide optimal bidding strategies. The traditional approaches of only attending the day-ahead market might not be profitable enough without taking advantage of real-time price volatility. In this paper, we study the optimal bidding strategies for the independent power producers utilizing self-scheduling strategies to participate in the real-time market considering real-time electricity price volatility, with the objective of maximizing the total expected profit. Considering the correlations of renewable energy generation outputs among different time periods, the correlations of real-time prices are captured in our modeling framework, in which we explore a multistage stochastic scenario tree to formulate the price uncertainties. Accordingly, the derived multistage stochastic self-scheduling unit commitment problem is transformed as a deterministic equivalent mixed-integer linear programming formulation. To overcome the curse of dimensionality, we develop strong valid inequalities for the derived stochastic unit commitment polytope to speed up the algorithms to solve the problem. In particular, we derive strong valid inequalities that can provide the convex hull descriptions for the two-period case and a special class of the three-period cases with rigorous proofs provided. Furthermore, strong valid inequalities, including facet-defining proofs, for multistage cases are proposed to further strengthen the model. Finally, numerical experiments verify the effectiveness of our derived strong valid inequalities by incorporating them in a branch-and-cut framework.

Key words: stochastic integer programming; cutting planes; self-scheduling unit commitment

1 Introduction

The current deregulated wholesale electricity markets in U.S. mostly contain day-ahead and real-time markets for electricity trading, plus an auxiliary service market to help maintain system reliability [1]. For the day-ahead market, the independent power producers (IPPs) and consumers submit generation (selling) and load (buying) offers respectively to an Independent System Operator (ISO), which clears the market within a market-clearing procedure [12] to ensure power balance. During this procedure, locational marginal prices (LMPs) are taken as the basis of settlements [11] such that generation is compensated and demands are charged by LMPs [30]. Similarly, for the real-time market, the ISO takes real-time generation offers and clears the real-time market accordingly by accommodating the load discrepancy between day-ahead and real-time markets.

An IPP can participate in the day-ahead market, real-time market, and auxiliary service market independently or combinations of two or all three of them. Meanwhile, there are different ways to submit offers to the ISO to participate in the electricity wholesale markets. One common approach is through submitting a three-part offer (e.g., ERCOT and MISO) that includes start-up cost, minimum load cost, and energy bid. For this approach, after marketing clearings (day-ahead or real-time), the ISO notifies each IPP, whose offer is awarded, of the unit commitment status (for day-ahead) and generation amount (for real-time) at each time period. This approach could potentially lead to the inefficient commitment of conventional units as described in [31]. Considering this, the wholesale electricity market also allows IPPs to submit offers in the forms of "self-commitment" or "self-scheduling" [1]. For the self-commitment approach, IPPs decide the unit commitment and let the ISO decide the generation amount for each time period; for the self-scheduling approach, IPPs decide both the unit commitment and the generation amount for each time period. In practice, depending on the physical characteristics of each generator including minimum-up/down time, ramp rate, capacity limits, etc., offer submission strategies can be different. For instance, most coal-fired generators have larger minimum-up/down times and more restrictive start-up ramp rates and thus are not very flexible. It is common to submit a three-part offer to the ISO for these types of generators. On the other hand, most gas-fired generators are more flexible to start-up and shut-down due to their smaller minimum-up/down times and larger ramp rates. Meanwhile, due to increasing penetration of intermittent renewable energy generation following Obama's new energy plan [27], unexpected outage of generators and transmissions, and fluctuating electricity demand [29], the real-time electricity prices can be significantly volatile, which provides an option for gas-fired generators to participate in the real-time market utilizing the self-scheduling mode with the expectation to earn more profit.

In this paper, we study the optimal bidding strategy for an IPP (e.g., a gas-fired generator) to maximize its own profit by participating in the real-time market using the self-scheduling mode, under which the market prices are purely input. With this price-based decision making, our goal is to derive a best generation schedule to participate in the market subject to physical constraints of the generator. This is typically defined as the price-based unit commitment (PBUC) problem in the literature (see, e.g., [34, 20]).

Due to price uncertainty, stochastic optimization approaches have been utilized to enhance the self-scheduling process in the last decade. For instance, two-stage stochastic self-scheduling models have been developed in [3] and [32] by considering different market settings. In addition, a midterm two-stage stochastic hydrothermal scheduling model using the self-scheduling mode is developed in [39] and solved by using Monte Carlo scenario generation and reduction techniques described in [13]. Similar to [39], a two-stage stochastic self-scheduling unit commitment problem is studied in [21]. As compared to [3] and [32], [39] and [21] consider risk constraints that are reformulated by using auxiliary binary variables. Recently, a two-stage stochastic programming model to obtain the optimal bidding strategies for the day-ahead market in the Iberian Electricity Market is proposed in [17], and a two-stage stochastic PBUC model with chance constraints is proposed in [37], for which a sample average approximation method is introduced to solve the problem.

As compared to two-stage stochastic programming approaches, scenario-tree based multistage stochastic programming approaches allow incorporation of multistage forecasting information with varying accuracy, e.g., from one day to several hours ahead (see, e.g., [5], [4] and [35]). This provides more efficient decisions because the real-time electricity price forecast becomes more accurate as the time horizon shrinks. In addition, scenario-tree based multistage stochastic optimization approaches allow us to model the dependencies between consecutive time periods, reflecting current practices of dependencies between consecutive time periods of renewable energy outputs. Multistage stochastic unit commitment (MSUC) formulations were originally proposed for the power system operators in the 1990s [8, 36]. In these early studies, load uncertainty is considered and transmission constraints are ignored. For instance, in [36], an MSUC is introduced to model load uncertainty and is solved by using an augmented Lagrangian decomposition framework. Recently, an MSUC with transmission constraints is formulated and studied in [38]. In addition, in [10], a multistage stochastic self-scheduling model is proposed for the power producers participating in the day-ahead and auxiliary service markets.

For both two-stage and multistage stochastic self-scheduling unit commitment approaches mentioned above, an IPP needs to solve a stochastic unit commitment problem to obtain the expected profit, which leads to two-stage and multistage stochastic integer programs. Cutting plane approaches have been proven one of the most efficient approaches to speed up the branch-and-cut algorithms to solve the related problems [28], in particular, when a finite number of scenarios are taken and a deterministic equivalent formulation is obtained. There has been significant research progress on developing efficient cutting planes for two-stage stochastic integer programs. For instance, an early attempt utilizing cutting planes to solve two-stage variants appears in [19], for which an L-shaped method is proposed. In [7], lift-and-project cuts are applied to solve the deterministic equivalent of 0-1 stochastic mixed integer programs with possible integer variables in the first stage. In [2], a branch-and-bound algorithm for two-stage stochastic integer programs with mixed-integer first-stage variables and pure integer second-stage variables is proposed. In [33], a decomposition-based algorithm is developed for the two-stage stochastic mixed-integer linear programs (MILPs) emphasizing decomposition among the integer variables that appear in the first and second stages. Recently, cutting plane methods have also been developed to solve the stochastic programs in which probabilistic constraints are considered [24, 25]. The cutting planes and decomposition based algorithms for two-stage chance-constrained stochastic programs are further studied in [22, 40]. However, there has been only limited research on developing efficient algorithms to solve multistage stochastic integer programs. Along this direction, an efficient heuristic approach is proposed in [23], which combines the progressive hedging algorithm with a tabu search. A decomposition method based Lagrangian relaxation is described in [6] and a branch-and-price method is investigated in [26]. The value function approach is investigated in [18] to solve the multistage stochastic capacity planning problem. For the cutting plane approaches, in [15], strong valid inequalities are proposed to solve multistage stochastic uncapacitated lot-sizing problems, whereas a generalized procedure for generating cutting planes for multistage stochastic integer programs is explored in [14]. Among the inequalities derived for the chance constrained and multistage stochastic integer programs [15, 14, 25, 40], a part of them are cross-scenario inequalities, which can be derived following mixing procedure [16].

In this paper, we propose a scenario-tree based multistage stochastic self-scheduling unit commitment model for IPPs to participate in the real-time market, taking the self-scheduling mode with the consideration of price uncertainty. We explore efficient cutting plane algorithms to solve the corresponding large-sized deterministic equivalent formulations. Our contributions can be summarized as follows:

1. We introduced a multistage stochastic optimization model to help IPPs optimally submit an offer to the real-time market for a gas-fired generator so as to maximize the total expected profit, which can capture price uncertainty better than the deterministic PBUC models (cf., [34, 20]).

2. We derived strong valid inequalities that can describe the convex hulls for the two-period and three-period multistage stochastic self-scheduling problems. For each case, the number of inequalities is polynomial in terms of the number of scenarios.

3. We derived further cross-scenario strong valid inequalities for the general multistage formulation, so as to strengthen the original polytope. These cross-scenario inequalities are derived by utilizing the special physical structures of gas-fired generators, which are different from those in the literature. The number of this family of inequalities is also polynomial in terms of the number of scenarios.

4. Our proposed efficient cutting planes for the stochastic self-scheduling unit commitment polytope can be applied for other instances in which the corresponding polytope is embedded. For instance, the cutting planes can help speed up the branch-and-cut algorithms for the day-ahead and look-ahead reliability unit commitment runs. Meanwhile, our study will enrich the literature for solving multistage stochastic integer programs.

The remaining part of this paper is organized as follows. Section 2 provides the notation and formulation that describe the multistage stochastic self-scheduling problem. Then, in Sections 3 and 4, we derive the convex hull descriptions for the two-period case and a special case of the three-period problems, respectively. Furthermore, in Section 5, several families of cross-scenario facet-defining inequalities are derived. After that, in Section 6, we perform computational studies that verify the effectiveness of the proposed strong valid inequalities. Finally, in Section 7, we summarize our research.

2 Notation and Formulation

We assume electricity prices in the real-time market follow a discrete-time stochastic process evolving in a finite probability space. To describe the evolving process, a scenario tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ with T time periods is utilized to describe the possible realizations of the uncertain electricity prices, as shown in Figure 1. Each node $i \in \mathcal{V}$ at time t of the tree provides the state of the system that can be distinguished by information available up to time t (corresponding to a scenario realization from time 1 to time t). Accordingly, corresponding to each node $i \in \mathcal{V}$, we let $t(i)$ be its time period, $\mathcal{P}(i)$ be the set of nodes along the path from the root node (denoted as node 0) to node i, and p_i be the probability associated with the state represented by node i. In addition, each node i in the scenario tree, except the root node, has a unique parent i^-, and could have multiple children, denoted as set $\mathcal{C}(i)$. We let $\mathcal{V}(i)$ represent the set of all descendants of node i, including itself. Finally, we let $\mathcal{H}_r(i) = \Big\{k \in \mathcal{V}(i) : 0 \leq t(k) - t(i) \leq r - 1\Big\}$ be the set of nodes used to describe minimum-up and minimum-down time constraints (e.g., in Figure 1, $r = t(j) - t(i)$). The decisions corresponding to each node i are assumed to be made after observing the realizations of the problem parameters along the path from the root node to this node i, but are nonanticipative with respect to future realizations.

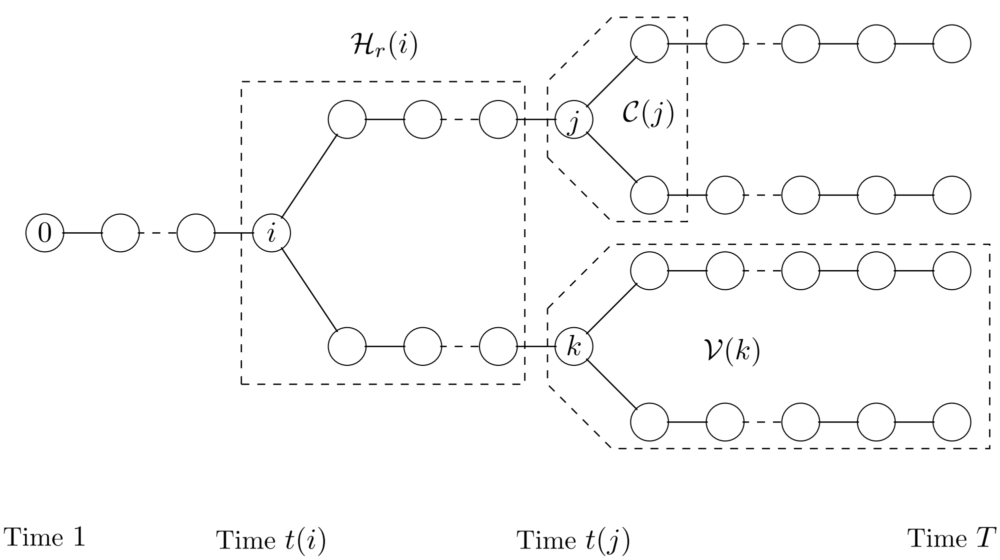

Figure 1: Multistage stochastic scenario tree

For the multistage stochastic self-scheduling problem, following the notation described above, we let q_i denote the electricity price at node i. Meanwhile, we should have the physical constraints for the gas-fired generator to be satisfied. The physical characteristics can be described as follows: we let L (ℓ) represent its minimum-up (down) time, $\overline{C}$ ($\underline{C}$) denote its upper (lower) generation limit if the generator is online, V^+ (V^-) denote its ramp-up (down) rate limit (indicating the maximum generation amount increment (decrement) between two consecutive time periods when the generator is online), $\bar{U}$ ($\underline{U}$) denote its start-up (shut-down) cost, and a nondecreasing convex function $f(\cdot)$ denote the fuel cost as a function of its electricity generation amount.

The decision variables of this problem include the "turn on" and "turn off" decisions, the "online" and "offline" statuses, and the generation amounts at different time periods corresponding to each node in the scenario tree. Accordingly, for each node i, we let binary variables (y_i, u_i, v_i) denote the unit commitment decisions: (1) y_i represents if the generator is online or offline at node i (i.e., $y_i = 1$ if yes; $y_i = 0$ otherwise), (2) u_i represents if the generator starts up or not at node i (i.e., $u_i = 1$ if yes; $u_i = 0$ otherwise), and (3) v_i represents if the generator shuts down or not at node i (i.e., $v_i = 1$ if yes; $v_i = 0$ otherwise). We also let continuous variable x_i denote the electricity generation amount at node i.

Based on the notation described above, the formulation for this problem can be described as follows:

$$
\begin{aligned}
\max \quad & \sum_{i\in\mathcal{V}} p_i\Big(q_i x_i - \big(\bar{U}u_i + \underline{U}v_i + f(x_i)\big)\Big) && \text{(1a)}\\
\text{s.t.} \quad & y_i - y_{i^-} \le y_k, \quad \forall i \in \mathcal{V}\setminus\{0\}, \forall k \in \mathcal{H}_L(i), && \text{(1b)}\\
& y_{i^-} - y_i \le 1 - y_k, \quad \forall i \in \mathcal{V}\setminus\{0\}, \forall k \in \mathcal{H}_\ell(i), && \text{(1c)}\\
& y_i - y_{i^-} \le u_i, \quad \forall i \in \mathcal{V}\setminus\{0\}, && \text{(1d)}\\
& v_i = y_{i^-} - y_i + u_i, \quad \forall i \in \mathcal{V}\setminus\{0\}, && \text{(1e)}\\
& \underline{C}y_i \le x_i \le \overline{C}y_i, \quad \forall i \in \mathcal{V}, && \text{(1f)}\\
& x_i - x_{i^-} \le V^+ y_{i^-} + \overline{C}(1 - y_{i^-}), \quad \forall i \in \mathcal{V}\setminus\{0\}, && \text{(1g)}\\
& x_{i^-} - x_i \le V^- y_i + \overline{C}(1 - y_i), \quad \forall i \in \mathcal{V}\setminus\{0\}, && \text{(1h)}\\
& y_i \in \{0,1\}, \quad \forall i \in \mathcal{V}; \quad u_i, v_i \in \{0,1\}, \quad \forall i \in \mathcal{V}\setminus\{0\}. && \text{(1i)}
\end{aligned}
$$

In the above formulation, the objective is to maximize the expected total profit, which is equal to the revenue minus the total cost, while the total cost includes start-up, shut-down, and fuel costs. Constraints (1b) represent the minimum-up time for the generator. That is, if the generator starts up at node i, then it should be kept online for all the nodes in $\mathcal{H}_L(i)$. Similarly, constraints (1c) represent the minimum-down time limits. If the generator shuts down at node i, then it should be kept offline for all the nodes in $\mathcal{H}_\ell(i)$. Constraints (1d) describe the turn on decision and constraints (1e) define the relationship among u, v, and y. Constraints (1f) describe the upper and lower bounds of electricity generation amount if the generator is online at node i. Constraints (1g) and (1h) describe the ramp-up rate and ramp-down rate limits, respectively. Typically the fuel cost function can be approximated by a piecewise linear function [9]. With this approximation, the deterministic equivalent formulation above can be reformulated as an MILP formulation.

Note here that v can be expressed by y and u, so we replace v_i by $y_{i^-} - y_i + u_i$ in the objective function and remove v variables and constraints (1e). Moreover, we can observe that constraints (1b)-(1d) allow $u_i = 1$ when $y_i = y_{i^-}$, or, $y_{i^-} = 1$ and $y_i = 0$. To eliminate these cases and keep consistent with the feasible region of the original problem, we add the following constraints into the formulation:

$$u_i \leq \min\{y_i,\ 1 - y_{i^-}\}, \quad \forall i \in \mathcal{V} \setminus \{0\}. \tag{2}$$

Thus, the final formulation for the problem, defined as MSS, can be expressed as

$$\max \left\{ \sum_{i \in \mathcal{V}} p_i (q_i x_i - (\bar{U} u_i + \underline{U}(y_{i^-} - y_i + u_i) + f(x_i))) : (x, y, u) \in P \right\},$$

where $P = \{(x, y, u) \in \mathbb{R}^{|\mathcal{V}|} \times \mathbb{B}^{|\mathcal{V}|} \times \mathbb{B}^{(|\mathcal{V}|-1)} : (1b) - (1d), (1f) - (1h),$ and $(2)\}$. In the following sections, we derive strong valid inequalities for this polyhedral structure.

3 Strengthening the Two-period Formulation

We start with deriving strong formulations for the two-period case of MSS. That is, we consider a case in which there is only one root node with several scenarios in the second period, each with a corresponding given probability. The corresponding figure is shown as below, which is a special case of the general structure as shown in Figure 1.

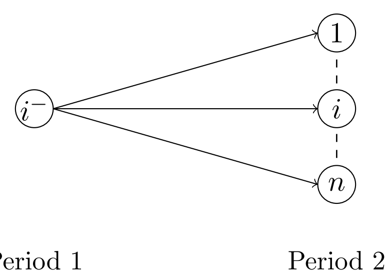

Figure 2: Two-period scenario tree

For Figure 2, we let $\mathcal{N} = \{1, 2, \cdots, n\}$ represent the set of scenario nodes in the second period, and they share the same parent node $i^- = 0$ ($\forall i \in \mathcal{N}$). Since there are only two periods, without loss of generality, we assume $L = \ell = 1$ and accordingly, the minimum-up/down time constraints (1b) and (1c) can be omitted here. Thus, the corresponding MSS can be described as follows:

$$P_2 := \Big\{ (x, y, u) \in \mathbb{R}^{n+1}_{+} \times \mathbb{B}^{n+1} \times \mathbb{B}^n : \quad y_i - y_{i^-} - u_i \leq 0, \quad \forall i \in \mathcal{N}, \tag{3a}$$

$$u_i - y_i \leq 0, \quad \forall i \in \mathcal{N}, \tag{3b}$$

$$u_i + y_{i^-} \leq 1, \quad \forall i \in \mathcal{N}, \tag{3c}$$

$$\underline{C} y_i \leq x_i \leq \overline{C} y_i, \quad \forall i \in \mathcal{N} \cup \{i^-\}, \tag{3d}$$

$$x_i - x_{i^-} \leq V^+ y_{i^-} + \overline{C}(1 - y_{i^-}), \quad \forall i \in \mathcal{N}, \tag{3e}$$

$$x_{i^-} - x_i \leq V^- y_i + \overline{C}(1 - y_i), \quad \forall i \in \mathcal{N} \Big\}. \tag{3f}$$

Note here that in P_2, there is no start-up decision (i.e., u variable) in the first period for the root node. Thus, the derived inequalities can be applied for each node in the scenario tree (e.g., as shown in Figure 1) and can be applied recursively. For notation brevity, we denote the convex hull of the set of feasible points in P_2 as conv(P_2). We let ϵ be an arbitrarily small positive real number and $[a, b]_{\mathbb{Z}}$ represent $[a, b] \cap \mathbb{Z}$ for integers a and b (i.e., $\{a, a+1, \cdots, b\}$). If $b < a$, then $[a, b]_{\mathbb{Z}} = \emptyset$.

3.1 Short Commitment Interval Case

We first study the case in which $\overline{C} - \underline{C} - V^+ - V^- > 0$, which corresponds to the industrial practices when the time interval of decision making is smaller than an hour (e.g., ERCOT real-time electricity prices are settled every 15 minutes[1], which can be used as the length of each time unit for the corresponding self-scheduling unit commitment). Observing the fact that start-up decisions in the second period affect the upper limit of the difference between generation amounts in two nodes, we derive the following proposition.

Proposition 1 *For each pair of nodes $(i, j) \in \mathcal{N}$ such that $i \neq j$, the following inequalities*

$$x_i - x_{i^-} \leq (\underline{C} + V^+) y_i - \underline{C} y_{i^-} + (\overline{C} - \underline{C} - V^+) u_i, \tag{4}$$

$$x_{i^-} - x_i \leq \overline{C} y_{i^-} - (\overline{C} - V^-) y_i + (\overline{C} - \underline{C} - V^-) u_i, \tag{5}$$

$$x_i - x_j \leq (\underline{C} + V^+ + V^-) y_i - (\overline{C} - V^+ - V^-) y_j + (\overline{C} - \underline{C} - V^+ - V^-)(y_{i^-} + u_i + u_j), \tag{6}$$

are valid for conv(P_2).

Proof: The detailed proofs are shown in E-companion A.1. ■

Now, utilizing inequalities (4), (5), and (6), we introduce the following linear programming formulation of conv(P_2) through adding trivial inequalities:

$$Q_2 := \Big\{ \quad (x, y, u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^n :$$

$$(3a), (3b), (3c), (3d), (4), (5), (6),$$

[1] ERCOT real-time market description available at http://www.ercot.com/mktinfo/rtm.

$$u_i \geq 0, \quad \forall i \in \mathcal{N} \Big\}, \tag{7}$$

for which we notice that $x_i \geq 0$ is guaranteed by (3b), (3d), and (7). Then we show that Q_2 describes the convex hull of P_2. That is, $Q_2 = \text{conv}(P_2)$. We first provide the following preliminary results.

Proposition 2 *Q_2 is full-dimensional.*

Proof: The detailed proofs are shown in E-companion A.2. ∎

Proposition 3 *Each inequality in Q_2 is facet-defining for conv(P_2).*

Proof: The detailed proofs are shown in E-companion A.3. ∎

Proposition 4 *The inequalities in Q_2 dominate those in P_2.*

Proof: The detailed proofs are shown in E-companion A.4. ∎

In the following part, we prove that all the extreme points of Q_2 are integral in y and u. To show this, we first provide the following Lemma.

Lemma 1 *For the following two-period MSS*

$$\begin{aligned} z^* = \max \quad & \sum_{i=0}^{n} a_i x_i + \sum_{i=0}^{n} b_i y_i + \sum_{i=1}^{n} c_i u_i \\ s.t. \quad & (x, y, u) \in P_2, \end{aligned} \tag{8}$$

where $(a, b, c) \in \mathbb{R}^{3n+2}$, there exists at least one optimal solution satisfying one of the following five conditions:

(1) $x_0 = 0$, $x_i \in \{0,\ \underline{C},\ \overline{C}\}$ for each $i = 1, \cdots, n$, and binary variables y and u are uniquely decided following the constraints in P_2;

(2) $x_0 = \underline{C}$, $x_i \in \{0,\ \underline{C},\ \underline{C} + V^+\}$ for each $i = 1, \cdots, n$, and binary variables y and u are uniquely decided following the constraints in P_2;

(3) $x_0 = \underline{C} + V^-$, $x_i \in \{0,\ \underline{C},\ \underline{C} + V^+ + V^-\}$ for each $i = 1, \cdots, n$, among which there must exist at least one scenario node k such that $x_k = \underline{C}$, and binary variables y and u are uniquely decided following the constraints in P_2;

(4) $x_0 = \overline{C} - V^+$, $x_i \in \{0,\ \overline{C},\ \overline{C} - V^+ - V^-\}$ *for each* $i = 1, \cdots, n$, *among which there must exist at least one scenario node* k *such that* $x_k = \overline{C}$, *and binary variables* y *and* u *are uniquely decided following the constraints in* P_2*;*

(5) $x_0 = \overline{C}$, $x_i \in \{0,\ \overline{C} - V^-,\ \overline{C}\}$ *for each* $i = 1, \cdots, n$, *and binary variables* y *and* u *are uniquely decided following the constraints in* P_2*.*

Proof: The detailed proofs are shown in E-companion A.5. ■

Proposition 5 *All the extreme points of* Q_2 *are integral in* y *and* u*.*

Proof: The detailed proofs are shown in E-companion A.6. ■

Theorem 1 $Q_2 = $ *conv*(P_2)*.*

Proof: First, we have both P_2 and Q_2 bounded from their formulation representations. Since all the inequalities in Q_2 are valid and facet-defining for conv(P_2) based on Propositions 1 and 3, we have $Q_2 \supseteq \text{conv}(P_2)$. Meanwhile, we have that the inequalities in Q_2 dominate those in P_2 based on Proposition 4 and any extreme point in Q_2 is binary in y and u based on Proposition 5. Thus $Q_2 = \text{conv}(P_2)$. ■

Example 1 *Considering a two-period case as shown in Figure 2, in which there are two nodes in the second period, with variables* (x_0, y_0) *corresponding to the root node and* (x_1, y_1, u_1) *and* (x_2, y_2, u_2) *corresponding to two scenario nodes respectively, plus the physical characteristics of the generator* $\overline{C} = 10$, $\underline{C} = 2$, $V^+ = 3$, $V^- = 4$, *and* $L = \ell = 1$, *we have*

$$
\begin{aligned}
\mathit{conv}(P_2) = Q_2 := \Big\{ & (x, y, u) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^2 : \\
& y_0 - y_1 + u_1 \geq 0;\quad y_0 - y_2 + u_2 \geq 0;\quad y_1 - u_1 \geq 0;\quad y_2 - u_2 \geq 0; \\
& y_0 + u_1 \leq 1;\quad y_0 + u_2 \leq 1;\quad u_1 \geq 0;\quad u_2 \geq 0;\quad 2y_i \leq x_i \leq 10y_i,\ i = 0, 1, 2; \\
& x_i - x_0 \leq 5y_i - 2y_0 + 5u_i,\ i = 1, 2;\quad x_0 - x_i \leq 10y_i - 6y_0 + 4u_i,\ i = 1, 2; \\
& x_1 - x_2 \leq y_0 + 9y_1 - 3y_2 + u_1 + u_2;\quad x_2 - x_1 \leq y_0 + 9y_2 - 3y_1 + u_1 + u_2 \Big\}.
\end{aligned}
$$

3.2 Hourly Commitment Interval Case

Our study can be extended to the cases in which $\overline{C} - \underline{C} - V^+ - V^- > 0$ does not hold, which corresponds to some hourly commitment interval cases. We describe the convex hull representations corresponding to different parameter settings as follows:

- For a two-period MSS with $\overline{C} - \underline{C} - V^+ - V^- \leq 0$, $\overline{C} - \underline{C} - V^+ > 0$, $\overline{C} - \underline{C} - V^- > 0$, the corresponding convex hull can be described as $Q_2^1 = \{(x, y, u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^n :$ (3a) − (3d), (4), (5), (7)$\}$.

- For a two-period MSS with $\overline{C} - \underline{C} - V^+ \leq 0$ and $\overline{C} - \underline{C} - V^- > 0$, the corresponding convex hull can be described as $Q_2^2 = \{(x, y, u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^n :$ (3a) − (3d), (5), (7)$\}$.

- For a two-period MSS with $\overline{C} - \underline{C} - V^+ > 0$ and $\overline{C} - \underline{C} - V^- \leq 0$, the corresponding convex hull can be described as $Q_2^3 = \{(x, y, u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^n :$ (3a) − (3d), (4), (7)$\}$.

- For a two-period MSS with $\overline{C} - \underline{C} - V^+ \leq 0$ and $\overline{C} - \underline{C} - V^- \leq 0$, the corresponding convex hull can be described as $Q_2^4 = \{(x, y, u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^n :$ (3a) − (3d), (7)$\}$.

The proofs are similar to those for the case in which $\overline{C} - \underline{C} - V^+ - V^- > 0$ and thus omitted here.

4 Strengthening the Three-period Formulations

We extend our study to the three-period case in this section. We derive the convex hull results for a special case and strong valid inequalities for the general three-period problems.

4.1 A Special Structure

We first consider a special structure in which the uncertainty is only explored in the third period. Note here that the inequalities derived for this substructure can also be applied for the whole problem. As shown in Figure 3, the problem parameters in the first two periods are realized. Meanwhile, several possible realizations of uncertain parameters, denoted as set $\mathcal{N} = \{1, 2, \cdots, n\}$, in the third period are explored. In addition, we let $i_0^- = i$, $i_1^- = i^-$, and i_k^- be the unique parent node of i_{k-1}^-, for $k \geq 2$. In other words, we define i_k^- be the k-fold parent of node i. To illustrate the main results and for notation brevity, we assume the ramp-up and ramp-down rates are the same (i.e., $V^+ = V^- = V$) for three and later on multi-period cases. The derived results can be easily extended to the general cases in which $V^+ \neq V^-$.

4.1.1 Short Commitment Interval Case

For this special case, we first study the case in which $\overline{C} - \underline{C} - 2V > 0$. We consider different combinations of L and ℓ, including (1) $L = \ell = 1$, (2) $L = 2, \ell = 1$, (3) $L = 1, \ell = 2$, and (4) $L = \ell = 2$. For $L = \ell = 1$, the mathematical formulation of MSS can be described as follows:

$$P_3^1 := \Big\{ \quad (x, y, u) \in \mathbb{R}_+^{n+2} \times \mathbb{B}^{n+2} \times \mathbb{B}^{n+1} :$$

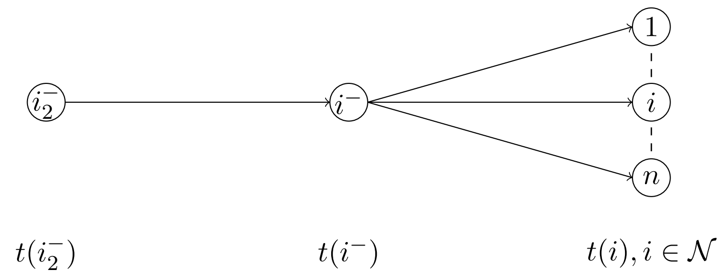

Figure 3: Three-period scenario tree: a special case

$$y_i - y_{i^-} - u_i \le 0, \quad \forall i \in \mathcal{N} \cup \{i^-\}, \tag{9a}$$

$$u_i - y_i \le 0, \quad \forall i \in \mathcal{N} \cup \{i^-\}, \tag{9b}$$

$$u_i + y_{i^-} \le 1, \quad \forall i \in \mathcal{N} \cup \{i^-\}, \tag{9c}$$

$$\underline{C} y_i \le x_i \le \overline{C} y_i, \quad \forall i \in \mathcal{N} \cup \{i_2^-, i^-\}, \tag{9d}$$

$$x_i - x_{i^-} \le V y_{i^-} + \overline{C}(1 - y_{i^-}), \quad \forall i \in \mathcal{N} \cup \{i^-\}, \tag{9e}$$

$$x_{i^-} - x_i \le V y_i + \overline{C}(1 - y_i), \quad \forall i \in \mathcal{N} \cup \{i^-\}\Big\}. \tag{9f}$$

We can observe that P_3^1 possesses the similar polyhedral structure as P_2 in Section 3. Accordingly, the linear description of conv(P_3^1), i.e., Q_3^1, can be described in a similar way. We describe the convex hull representation Q_3^1 as follows.

Theorem 2 *For a three-period MSS as shown in Figure 3 in which $L = \ell = 1$, the corresponding convex hull conv(P_3^1) can be described as follows:*

$$Q_3^1 := \Big\{(x, y, u) \in \mathbb{R}^{n+2} \times \mathbb{R}^{n+2} \times \mathbb{R}^{n+1} : (9a), (9b), (9c), (9d),$$

$$u_i \ge 0, \quad \forall i \in \mathcal{N} \cup \{i^-\}, \tag{10a}$$

$$x_i - x_{i^-} \le (\underline{C} + V) y_i - \underline{C} y_{i^-} + (\overline{C} - \underline{C} - V) u_i, \quad \forall i \in \mathcal{N} \cup \{i^-\}, \tag{10b}$$

$$x_{i^-} - x_i \le \overline{C} y_{i^-} - (\overline{C} - V) y_i + (\overline{C} - \underline{C} - V) u_i, \quad \forall i \in \mathcal{N} \cup \{i^-\}, \tag{10c}$$

$$x_i - x_{i_2^-} \le (\underline{C} + 2V) y_i - \underline{C} y_{i_2^-} + (\overline{C} - \underline{C} - 2V)(u_{i^-} + u_i), \quad \forall i \in \mathcal{N}, \tag{10d}$$

$$x_{i_2^-} - x_i \le \overline{C} y_{i_2^-} - (\overline{C} - 2V) y_i + (\overline{C} - \underline{C} - 2V)(u_{i^-} + u_i), \quad \forall i \in \mathcal{N}, \tag{10e}$$

$$x_i - x_j \le (\underline{C} + 2V) y_i - (\overline{C} - 2V) y_j + (\overline{C} - \underline{C} - 2V)(y_{i^-} + u_i + u_j), \forall i, j \in \mathcal{N}, j \ne i\Big\} \tag{10f}$$

Proof: The proofs are similar to those in Section 3 for Theorem 1 and thus omitted here. ■

Example 2 *Consider a three-period MSS as shown in Figure 3 with two scenario nodes in the third period. Let variables (x_0, y_0) correspond to the root node i_2^-, (x_1, y_1, u_1) correspond to node i^-, and*

(x_2, y_2, u_2) and (x_3, y_3, u_3) correspond to two scenario nodes i and j respectively. The generator data are $\overline{C} = 9$, $\underline{C} = 2$, $V^+ = V^- = V = 3$, and $L = \ell = 1$. Then we have

$$
\begin{aligned}
conv(P_3^1) = Q_3^1 := \Big\{&(x, y, u) \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^3 : y_1 - u_1 \geq 0; \quad y_2 - u_2 \geq 0; \quad y_3 - u_3 \geq 0;\\
& y_0 - y_1 + u_1 \geq 0; \quad y_1 - y_2 + u_2 \geq 0; \quad y_1 - y_3 + u_3 \geq 0;\\
& y_0 + u_1 \leq 1; \quad y_1 + u_2 \leq 1; \quad y_1 + u_3 \leq 1;\\
& 2y_i \leq x_i \leq 9y_i, \ i = 0, 1, 2, 3; \quad u_i \geq 0, \ i = 1, 2, 3;\\
& x_1 - x_0 \leq 5y_1 - 2y_0 + 4u_1; \quad x_i - x_1 \leq 5y_i - 2y_1 + 4u_i, \ i = 2, 3;\\
& x_0 - x_1 \leq 9y_0 - 6y_1 + 4u_1; \quad x_1 - x_i \leq 9y_1 - 6y_i + 4u_i, \ i = 2, 3;\\
& x_i - x_0 \leq 8y_i - 2y_0 + u_1 + u_i, \ i = 2, 3; \quad x_0 - x_i \leq 9y_0 - 3y_i + u_1 + u_i, \ i = 2, 3;\\
& x_2 - x_3 \leq y_0 + 8y_2 - 3y_3 + u_2 + u_3; \quad x_3 - x_2 \leq y_0 + 8y_3 - 3y_2 + u_2 + u_3\Big\}.
\end{aligned}
$$

Remark 1 *Note here that P_3^1, in which $L = \ell = 1$, is equivalent to the problem without minimum-up/down time constraints. It then can be considered as a relaxation for the cases in which the minimum-up/down times are larger than 1. Thus, the derived inequalities in Q_3^1 are valid for the cases in which the minimum-up/down times are larger than 1. This claim also holds for the general multi-period cases.*

Now we extend our study to three other combinations: (1) $L = \ell = 2$, (2) $L = 2$, $\ell = 1$, and (3) $L = 1$, $\ell = 2$, under the setting of Figure 3. We start with $L = \ell = 2$, and the corresponding mathematical formulation for the original polytope can be described as follows:

$$
\begin{aligned}
P_3^2 := \Big\{ \quad & (x, y, u) \in \mathbb{R}_+^{n+2} \times \mathbb{B}^{n+2} \times \mathbb{B}^{n+1} : \\
& u_{i^-} + u_i - y_i \leq 0, \quad \forall i \in \mathcal{N}, && \text{(11a)}\\
& y_{i_2^-} + u_{i^-} + u_i \leq 1, \quad \forall i \in \mathcal{N}, && \text{(11b)}\\
& y_i - y_{i^-} - u_i \leq 0, \quad \forall i \in \mathcal{N} \cup \{i^-\}, && \text{(11c)}\\
& \underline{C} y_i \leq x_i \leq \overline{C} y_i, \quad \forall i \in \mathcal{N} \cup \{i_2^-, i^-\}, && \text{(11d)}\\
& x_i - x_{i^-} \leq V y_{i^-} + \overline{C}(1 - y_{i^-}), \quad \forall i \in \mathcal{N} \cup \{i^-\}, && \text{(11e)}\\
& x_{i^-} - x_i \leq V y_i + \overline{C}(1 - y_i), \quad \forall i \in \mathcal{N} \cup \{i^-\}\Big\}. && \text{(11f)}
\end{aligned}
$$

Proposition 6 *For a three-period MSS as shown in Figure 3 in which $L = \ell = 2$, the following inequalities*

(10a), (10b), (10c), (10d), (10e), (10f),

$$x_i - x_j \le \overline{C} y_i - \underline{C} y_j - (\overline{C} - \underline{C} - 2V) u_{i^-}, \quad \forall i, j \in \mathcal{N}, j \ne i, \tag{12}$$

$$x_{i_2^-} - x_{i^-} + x_i \ge \underline{C} y_{i_2^-} - (\underline{C} + V) y_{i^-} + \underline{C} y_i, \quad \forall i \in \mathcal{N}, \tag{13}$$

$$x_{i_2^-} - x_{i^-} + x_i \le \overline{C} y_{i_2^-} - (\overline{C} - V) y_{i^-} + \overline{C} y_i, \quad \forall i \in \mathcal{N}, \tag{14}$$

$$x_{i_2^-} + x_i - x_j \ge \underline{C} y_{i_2^-} + \underline{C} y_i - (\underline{C} + 2V) y_j - (\overline{C} - \underline{C} - 2V) u_j, \quad \forall i, j \in \mathcal{N}, j \ne i, \tag{15}$$

$$x_{i_2^-} + x_i - x_j \le \overline{C} y_{i_2^-} + \overline{C} y_i - (\overline{C} - 2V) y_j + (\overline{C} - \underline{C} - 2V) u_j, \quad \forall i, j \in \mathcal{N}, j \ne i, \tag{16}$$

$$\begin{aligned} x_{i_2^-} - x_{i^-} + x_i - x_j \ge{}& \underline{C} y_{i_2^-} - (\overline{C} - V) y_{i^-} + (\overline{C} - 2V) y_i - (\underline{C} + 2V) y_j \\ & - (\overline{C} - \underline{C} - 2V)(u_{i^-} + u_i + u_j), \quad \forall i, j \in \mathcal{N}, j \ne i, \end{aligned} \tag{17}$$

$$\begin{aligned} x_{i_2^-} - x_{i^-} + x_i - x_j \le{}& \overline{C} y_{i_2^-} - (\underline{C} + V) y_{i^-} + (\underline{C} + 2V) y_i - (\overline{C} - 2V) y_j \\ & + (\overline{C} - \underline{C} - 2V)(u_{i^-} + u_i + u_j), \quad \forall i, j \in \mathcal{N}, j \ne i, \end{aligned} \tag{18}$$

are valid and facet-defining for conv(P_3^2).

Proof: The detailed proofs are shown in E-companion B.1. ■

Based on the above analysis, we can further obtain the convex hull description of P_3^2, i.e., Q_3^2, as follows.

Theorem 3 *For a three-period MSS as shown in Figure 3 in which $L = \ell = 2$, the corresponding convex hull conv(P_3^2) can be described as follows:*

$$Q_3^2 := \Big\{ (x, y, u) \in \mathbb{R}^{n+2} \times \mathbb{R}^{n+2} \times \mathbb{R}^{n+1} : \\ (10a) - (10f), (11a) - (11d), (12) - (18) \Big\}.$$

Proof: The proofs are similar to those in Section 3 for Theorem 1 and thus omitted here. ■

Example 3 *Consider the same case in Example 2 except $L = \ell = 2$. In the following, we only illustrate inequalities (11a), (11b), and (12) - (18) in Q_3^2 because other inequalities are the same as those described in Example 2.*

$$u_1 + u_i \le y_i, \quad i = 2, 3; \quad y_0 + u_1 + u_i \le 1, \quad i = 2, 3;$$

$$x_i - x_j \le 9 y_i - 2 y_j - u_1, \quad i, j = 2, 3, i \ne j;$$

$$x_0 - x_1 + x_i \ge 2 y_0 - 5 y_1 + 2 y_i, \quad i = 2, 3;$$

$$x_0 - x_1 + x_i \le 9 y_0 - 6 y_1 + 9 y_i, \quad i = 2, 3;$$

$$x_0 + x_i - x_j \ge 2 y_0 + 2 y_i - 8 y_j - u_j, \quad i, j = 2, 3, i \ne j;$$

$$x_0 + x_i - x_j \le 9y_0 + 9y_i - 3y_j + u_j, \quad i,j = 2,3, i \ne j;$$
$$x_0 - x_1 + x_i - x_j \ge 2y_0 - 6y_1 + 3y_i - 8y_j - u_1 - u_i - u_j, \quad i,j = 2,3, i \ne j;$$
$$x_0 - x_1 + x_i - x_j \le 9y_0 - 5y_1 + 8y_i - 3y_j + u_1 + u_i + u_j, \quad i,j = 2,3, i \ne j.$$

We can follow the similar procedure to derive the convex hull results for the rest two combinations: $L = 2,\ \ell = 1$ and $L = 1,\ \ell = 2$, as described below. The proofs are similar to those in Section 3 for Theorem 1 and thus omitted here.

Theorem 4 *For a three-period MSS as shown in Figure 3 in which $L = 1,\ \ell = 2$, the corresponding convex hull conv($P_3^{1,2}$) can be described as follows:*

$$Q_3^{1,2} := \Big\{(x,y,u) \in \mathbb{R}^{n+2} \times \mathbb{R}^{n+2} \times \mathbb{R}^{n+1} : \\ (9a) - (9b), (11b), (9d), (10a) - (10f)\Big\}.$$

Theorem 5 *For a three-period MSS as shown in Figure 3 in which $L = 2,\ \ell = 1$, the corresponding convex hull conv($P_3^{2,1}$) can be described as follows:*

$$Q_3^{2,1} := \Big\{(x,y,u) \in \mathbb{R}^{n+2} \times \mathbb{R}^{n+2} \times \mathbb{R}^{n+1} : \\ (10a) - (10f), (11a), (9c), (11c) - (11d), (12) - (18)\Big\}.$$

Remark 2 *We can observe that the convex hull descriptions of Theorems 2 and 4 are similar, with one inequality difference by replacing (9c) with (11b). It similarly happens between Theorems 3 and 5. This indicates that increasing ℓ does not increase the number of inequalities to describe the convex hull, while increasing L increases the number of inequalities required to define the convex hull dramatically, e.g., see comparison between Theorems 3 and 4.*

4.1.2 Hourly Commitment Interval Case

Our study can be simlarly extended to the cases in which $\overline{C} - \underline{C} - 2V > 0$ does not hold. For the cases in which $\overline{C} - \underline{C} - 2V \le 0$ and $\overline{C} - \underline{C} - V > 0$, we can obtain the convex hull representations corresponding to different values of L and ℓ as follows:

- For $L = \ell = 1$, the corresponding convex hull can be described as $\hat{Q}_3^1 = \{(x,y,u) \in \mathbb{R}^{n+2} \times \mathbb{R}^{n+2} \times \mathbb{R}^{n+1} : (9a) - (9d), (10a) - (10c)\}$.

- For $L = 1$ and $\ell = 2$, the corresponding convex hull can be described as $\hat{Q}_3^{1,2} = \{(x,y,u) \in \mathbb{R}^{n+2} \times \mathbb{R}^{n+2} \times \mathbb{R}^{n+1} : (9a) - (9b), (11b), (9d), (10a) - (10c)\}$.

- For $L=\ell=2$, the corresponding convex hull can be described as $\hat{Q}_3^2=\{(x,y,u)\in\mathbb{R}^{n+2}\times\mathbb{R}^{n+2}\times\mathbb{R}^{n+1}: (10a)-(10c),(11a)-(11d),(13)-(14)\}$.

- For $L=2$ and $\ell=1$, the corresponding convex hull can be described as $\hat{Q}_3^{2,1}=\{(x,y,u)\in\mathbb{R}^{n+2}\times\mathbb{R}^{n+2}\times\mathbb{R}^{n+1}: (10a)-(10c),(11a),(9c),(11c)-(11d),(13)-(14)\}$.

For the cases in which $\overline{C}-\underline{C}-V\leq 0$, we can obtain the convex hull representations corresponding to different values of L and ℓ as follows:

- For $L=\ell=1$, the corresponding convex hull can be described as $\bar{Q}_3^1=\{(x,y,u)\in\mathbb{R}^{n+2}\times\mathbb{R}^{n+2}\times\mathbb{R}^{n+1}: (9a)-(9d),(10a)\}$.

- For $L=1$ and $\ell=2$, the corresponding convex hull can be described as $\bar{Q}_3^{1,2}=\{(x,y,u)\in\mathbb{R}^{n+2}\times\mathbb{R}^{n+2}\times\mathbb{R}^{n+1}: (9a)-(9b),(11b),(9d),(10a)\}$.

- For $L=\ell=2$, the corresponding convex hull can be described as $\bar{Q}_3^2=\{(x,y,u)\in\mathbb{R}^{n+2}\times\mathbb{R}^{n+2}\times\mathbb{R}^{n+1}: (10a),(11a)-(11d)\}$.

- For $L=2$ and $\ell=1$, the corresponding convex hull can be described as $\bar{Q}_3^{2,1}=\{(x,y,u)\in\mathbb{R}^{n+2}\times\mathbb{R}^{n+2}\times\mathbb{R}^{n+1}: (10a),(11a),(9c),(11c)-(11d)\}$.

The proofs are similar to those in Section 3 for Theorem 1 and thus omitted here.

Remark 3 *From the representations of the convex hulls described so far, we can observe that there are no inequalities containing variables corresponding to three or more scenario nodes. It follows that the number of inequalities in the convex hull is a polynomial function (e.g., $\mathcal{O}(n^2)$) of the number of scenario nodes.*

4.2 The General Structure

Now, we consider a general three-period MSS problem, as shown in Figure 4. That being said, several scenarios are explored in the second period to form the second stage and each node in the second stage is followed by several scenarios to form the third stage. For instance, in Figure 4, node i in the third stage (the third period for this case) has a unique parent i^- in the second stage and share a two-fold parent i_2^- (i.e., the root node) with another node in the same third period, node j, whose parent node is node j^-. Without loss of generality, in this research, we assume the number of branches for each non-leaf node is fixed at n. Then, there are $1+n+n^2$ nodes in total in Figure 4, and we focus on the polyhedral results of the corresponding set $P_3:=\{(x,y,u)\in\mathbb{R}^{n^2+n+1}\times\mathbb{B}^{n^2+n+1}\times\mathbb{B}^{n^2+n}: (1b)-(1d),(1f)-(1h),\text{ and }(2)\}$.

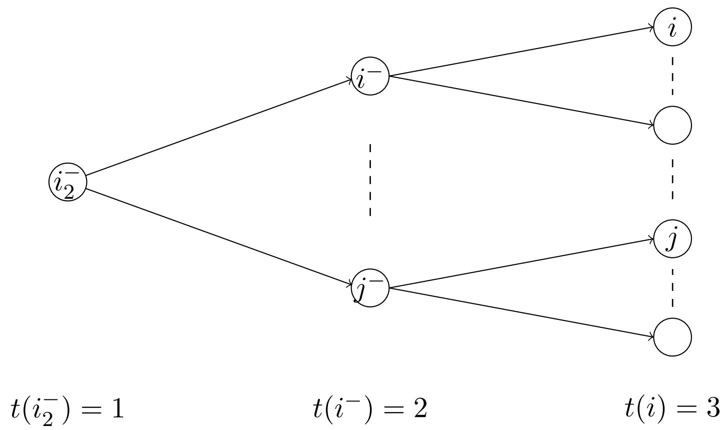

Figure 4: A generic three-period scenario tree

First, we consider the relationship among the generation amounts for three nodes linked through a cross-scenario path. We define a cross-scenario path as a path between two non-root nodes not belonging to the same scenario (i.e., these two non-root nodes are not in the same path from the root node to any non-root node). For instance, nodes i, i_2^-, and j^- in Figure 4 are linked through the cross-scenario path from node i to node j^- by passing through the root node i_2^-. The electricity generation amounts at these three nodes, i.e., x_i, $x_{i_2^-}$ and x_{j^-}, are correlated through ramp rate constraints (1g) - (1h). Through exploring the effect of unit commitment and start-up status of each node on this cross-scenario path, we can explore the relationships among x_i, $x_{i_2^-}$ and x_{j^-}, and two families of facet-defining inequalities (19) and (20) are derived as follows.

Proposition 7 *For a general three-period MSS with any pair of nodes $(i, j) \in \mathcal{V}$ such that $t(i) = t(j) = T$, $i^- \neq j^-$, and $\overline{C} - \underline{C} - 2V > 0$, the following inequalities*

$$x_i - x_{i_2^-} + x_{j^-} \leq (\underline{C} + 2V)y_i - \underline{C}y_{i_2^-} + (\overline{C} - \underline{C} - 2V)(u_{i^-} + u_i) + (\overline{C} - V)y_{j^-} + Vu_{j^-}, \tag{19}$$

$$x_{i_2^-} - x_i - x_{j^-} \leq \overline{C}y_{i_2^-} - (\overline{C} - 2V)y_i + (\overline{C} - \underline{C} - 2V)(u_{i^-} + u_i) - (\underline{C} + V)y_{j^-} + Vu_{j^-}, \tag{20}$$

are valid and facet-defining for conv(P_3).

Proof: The detailed proofs are shown in E-companion B.2. ■

Next, we investigate the difference of electricity generation amounts at any two nodes (e.g., i and j^- in Figure 4) that are not on the same path (scenario), but instead, these two nodes are linked through a cross-scenario path from node i to node j^- by passing through the root node i_2^-.

Proposition 8 *For a general three-period MSS with any pair of nodes $(i,j) \in \mathcal{V}$ such that $t(i) = t(j) = T$, $i^- \neq j^-$, and $\overline{C} - \underline{C} - 3V > 0$, the following inequalities*

$$x_i - x_{j^-} \leq (\underline{C} + 3V)y_i - (\overline{C} - 3V)y_{j^-} + (\overline{C} - \underline{C} - 3V)(y_{i_2^-} + u_{i^-} + u_i + u_{j^-}), \tag{21}$$

are valid and facet-defining for conv(P_3).

Proof: The detailed proofs are shown in E-companion B.3. ■

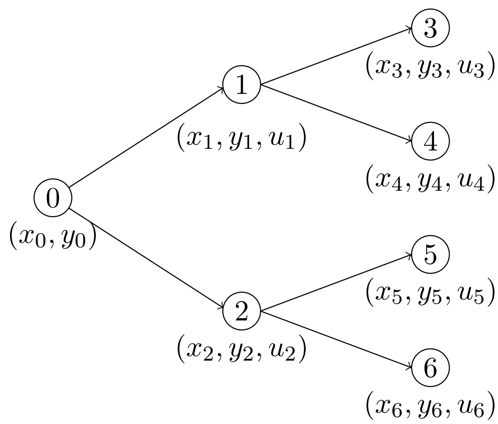

Figure 5: Example 4

Example 4 *Consider a three-period MSS in which each node contains two branches as shown in Figure 5. The generator data are $\overline{C} = 12$, $\underline{C} = 2$, and $V^+ = V^- = V = 3$. Then the following inequalities in the forms of (19) - (21) are valid and facet-defining:*

$$x_i - x_0 + x_2 \leq 8y_i - 2y_0 + 9y_2 + 4u_1 + 3u_2 + 4u_i, \quad i = 3, 4;$$

$$x_j - x_0 + x_1 \leq 8y_j - 2y_0 + 9y_1 + 3u_1 + 4u_2 + 4u_j, \quad j = 5, 6;$$

$$x_0 - x_i - x_2 \leq 12y_0 - 6y_i - 5y_2 + 4u_1 + 3u_2 + 4u_i, \quad i = 3, 4;$$

$$x_0 - x_j - x_1 \leq 12y_0 - 6y_j - 5y_1 + 3u_1 + 4u_2 + 4u_j, \quad j = 5, 6;$$

$$x_i - x_2 \leq y_0 + 11y_i - 3y_2 + u_1 + u_2 + u_i, \quad i = 3, 4;$$

$$x_2 - x_i \leq y_0 + 11y_2 - 3y_i + u_1 + u_2 + u_i, \quad i = 3, 4;$$

$$x_j - x_1 \leq y_0 + 11y_j - 3y_1 + u_1 + u_2 + u_j, \quad j = 5, 6;$$

$$x_1 - x_j \leq y_0 + 11y_1 - 3y_j + u_1 + u_2 + u_j, \quad j = 5, 6.$$

Remark 4 *In practice, the two-period and three-period cases explored in the previous and current sections can be utilized by IPPs as recourses when they submit self-scheduling offers for the real-time* *market. For instance, when an IPP submits a generation amount for a particular time period in real time, the two-period and three-period scenario trees generated based on the price forecast can be served as recourses to help make the real-time decision. This procedure can be applied recursively for making decisions for each time period.*

5 Strengthening the Multi-period Formulations

As described earlier, the inequalities derived in Sections 3 and 4 can be applied to solve the general multi-period problems, since we do not have a start-up decision variable for the root node. In this section, we further strengthen the general MSS formulation by exploring the inequalities covering multi-period nodes, which are additional to those described in Sections 3 and 4. We first consider the flower-structure scenario tree setting in Figure 6 in which branches in the last period are considered. We let $\mathcal{N} = \{1, \cdots, n\}$ denote the set of scenario nodes, and these nodes share the ancestors until to the root node of the whole scenario tree as shown in Figure 1. Therefore, we consider the original formulation as follows:

$$P_T^0 = \Big\{(x, y, u) \in \mathbb{R}_+^{n+T-1} \times \mathbb{B}^{n+T-1} \times \mathbb{B}^{n+T-2} : (1b) - (1d), (1f) - (1h), \text{ and } (2)\Big\}.$$

For this setting, we derive strong valid inequalities containing two and three continuous variables in the same and different paths (scenarios), respectively.

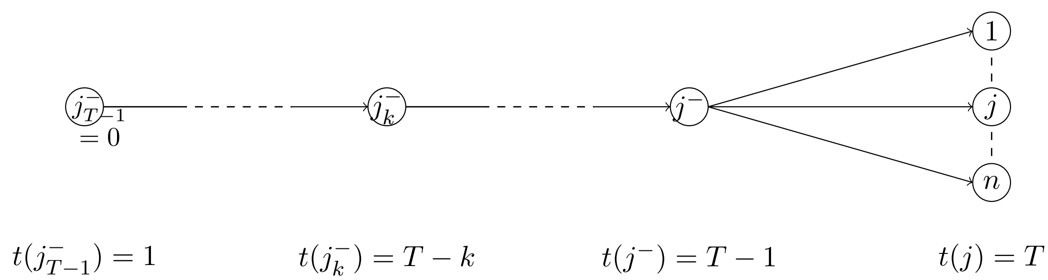

Figure 6: Multi-period scenario tree ($T \geq 3$)

First, we consider deriving strong valid inequalities to bound the difference of electricity generation amounts for any pair of two nodes in the tree. We consider the cases in which these two nodes are on the same path (scenario) and difference paths, respectively. For the case in which these two nodes are on the same path (scenario), e.g., i^-_k and i for any $i \in \mathcal{N}$, the difference is bounded from above and below by the combination of generation bound constraints (1f) and ramp rate constraints (1g) - (1h). Through additionally considering the start-up status of each node on this path between these two nodes, i.e., $u_{i^-_{k-1}}, \cdots, u_i$, we derive the explicit formulas of upper and lower bounds of $x_{i^-_k} - x_i$ in inequalities (22) and (23), respectively, both of which are facet-defining for conv(P^0_T).

Proposition 9 *For each $i \in \mathcal{N} \cup \{j^-, \cdots, j^-_{T-1-k}\}$, when $\overline{C} - \underline{C} - kV > 0$ and $k \in [1, T-1]_{\mathbb{Z}}$, the inequalities*

$$x_{i^-_k} - x_i \leq \overline{C} y_{i^-_k} - (\overline{C} - kV) y_i + (\overline{C} - \underline{C} - kV) \sum_{r=0}^{k-1} u_{i^-_r} \tag{22}$$

$$x_{i^-_k} - x_i \geq \underline{C} y_{i^-_k} - (\underline{C} + kV) y_i - (\overline{C} - \underline{C} - kV) \sum_{r=0}^{k-1} u_{i^-_r} \tag{23}$$

are valid and facet-defining for conv(P^0_T).

Proof: The detailed proofs are shown in E-companion C.1. ■

For the case in which these two nodes, e.g., nodes i and j, are on different paths (scenarios), the relationship between electricity generation amounts at these two nodes is linked through their common ancestors. Through investigating the start-up statuses of their ancestors and considering the minimum-up time restrictions, the difference of electricity generation amounts at these two nodes, i.e., $x_i - x_j$, can be bounded from above as shown in inequality (24), which is facet-defining for conv(P^0_T).

Proposition 10 *For each pair $i, j \in \mathcal{N}$ and $i \neq j$, when $L \geq 2$ and $\overline{C} - \underline{C} - 2V > 0$, the following inequality*

$$x_i - x_j \leq \overline{C} y_i - \underline{C} y_j - (\overline{C} - \underline{C} - 2V) \sum_{k=1}^{L-1} u_{i^-_k} \tag{24}$$

is valid and facet-defining for conv(P^0_T).

Proof: The detailed proofs are shown in E-companion C.2. ■

Note here that $L \geq 2$ ensures that x_i and x_j are correlated if there is a start-up in node i^-. For instance, if there is a start-up in node i^-, then all three nodes i, j, and i^- are online due to this $L \geq 2$ minimum-up time restriction, and accordingly $x_i - x_j$ is no larger than $2V$, which is smaller than $\overline{C} - \underline{C}$.

Next, we consider deriving strong valid inequalities to explore the relationships of generation amounts among three nodes in the tree. Similarly, we discuss the cases in which the nodes are on the same path (scenario) and difference paths, respectively. For the case in which these three nodes are on the same path (scenario), e.g., i^-_{k+1}, i^-_k, and i for any $i \in \mathcal{N}$, from ramp rate constraints (1g) - (1h) and generation bound constraints (1f), we have the upper (lower) bounds of $x_{i^-_{k+1}} - x_{i^-_k}$ and x_i respectively. After considering the effect of start-up status of i^-_k, i.e., $u_{i^-_k}$, we can get a tighter upper (resp. lower) bound for $x_{i^-_{k+1}} - x_{i^-_k} + x_i$ in the following inequality (25) (resp. (26)) instead of simply summing up the upper (resp. lower) bounds of $x_{i^-_{k+1}} - x_{i^-_k}$ and x_i from (1h) (resp. (1g)) and (1f). That being said, (25) is tighter than $x_{i^-_{k+1}} - x_{i^-_k} + x_i \leq V y_{i^-_k} + \overline{C}(1 - y_{i^-_k}) + \overline{C} y_i$ since $V y_{i^-_k} + \overline{C}(1 - y_{i^-_k}) + \overline{C} y_i$ − RHS of (25) $= \overline{C} - \overline{C} y_{i^-_{k+1}} - (k-1)V u_{i^-_k} \geq \overline{C}(1 - y_{i^-_{k+1}} - u_{i^-_k}) \geq 0$, where the first inequality holds because $\overline{C} > (k-1)V$ from Proposition 11 and u is nonnegative, and the second inequality holds because $1 - y_{i^-_{k+1}} - u_{i^-_k} \geq 0$. The similar arguments hold for (26).

Proposition 11 *For each $i \in \mathcal{N} \cup \{j^-, \cdots, j^-_{T-k-2}\}$, when $\overline{C} - \underline{C} - kV > 0$, $k \in [1, \min\{L-1, T-2\}]_{\mathbb{Z}}$, and $L \geq 2$, the following inequalities*

$$x_{i^-_{k+1}} - x_{i^-_k} + x_i \leq \overline{C} y_{i^-_{k+1}} - (\overline{C} - V) y_{i^-_k} + \overline{C} y_i + (k-1)V u_{i^-_k} \tag{25}$$

$$x_{i^-_{k+1}} - x_{i^-_k} + x_i \geq \underline{C} y_{i^-_{k+1}} - (\underline{C} + V) y_{i^-_k} + \underline{C} y_i - (k-1)V u_{i^-_k} \tag{26}$$

are valid and facet-defining for conv(P^0_T).

Proof: The detailed proofs are shown in E-companion C.3. ■

The same logic can be applied to the three nodes, e.g., i^-_k, i^- and i, on the same path (scenario) by considering the start-up status of i (not i^-_k as described above). In this way, based on the upper (resp. lower) bounds of $x_{i^-_k}$ and $x_i - x_{i^-}$ for any $i \in \mathcal{N}$ obtained from (1f) and (1g) (resp. (1h)), we can get a tighter inequality to bound $x_{i^-_k} - x_{i^-} + x_i$ from above (resp. below). Thus, the following two families of inequalities can be derived.

Proposition 12 *For each $i \in \mathcal{N} \cup \{j^-, \cdots, j^-_{T-k-1}\}$ when $\overline{C} - \underline{C} - (k-1)V > 0$, $k \in [2, \min\{L, T-1\}]_{\mathbb{Z}}$, and $L \geq 2$, the following inequalities*

$$x_{i^-_k} - x_{i^-} + x_i \leq \overline{C} y_{i^-_k} - (\overline{C} - (k-1)V) y_{i^-} + (\overline{C} - (k-2)V) y_i + (k-2)V u_i \tag{27}$$

$$x_{i^-_k} - x_{i^-} + x_i \geq \underline{C} y_{i^-_k} - (\underline{C} + (k-1)V) y_{i^-} + (\underline{C} + (k-2)V) y_i - (k-2)V u_i \tag{28}$$

are valid and facet-defining for conv(P^0_T).

Proof: The detailed proofs are shown in E-companion C.4. ■

Note here that inequalities (27) and (28) are different from (25) and (26), except for the case in which $k = 2$ in Proposition 12 and $k = 1$ in Proposition 11.

For the case in which these three nodes are in different paths (scenarios), e.g., two scenario nodes in the last period plus their k-fold parent node, since the electricity generation amounts at any two scenario nodes in $\mathcal{N}$, e.g., i and j, are correlated through their ancestors while the generation amount at each scenario node is related to their ancestors through ramp rate constraints (1g) - (1h), we can derive facet-defining inequalities in Propositions 13 and 14 by incorporating these two relationships simultaneously.

Proposition 13 *For each pair $i, j \in \mathcal{N}, i \neq j$ when $\overline{C} - \underline{C} - kV > 0$, $k \in [2, \min\{L, T-1\}]_{\mathbb{Z}}$, and $L \geq 2$, the following inequalities*

$$x_{i_k^-} + x_i - x_j \leq \overline{C} y_{i_k^-} + \overline{C} y_i - (\overline{C} - kV) y_j - (k-2)V \sum_{r=1}^{k-1} u_{i_r^-} + (\overline{C} - \underline{C} - kV) u_j \tag{29}$$

$$x_{i_k^-} + x_i - x_j \geq \underline{C} y_{i_k^-} + \underline{C} y_i - (\underline{C} + kV) y_j + (k-2)V \sum_{r=1}^{k-1} u_{i_r^-} - (\overline{C} - \underline{C} - kV) u_j \tag{30}$$

are valid and facet-defining for conv(P_T^0).

Proof: The detailed proofs are shown in E-companion C.5. ■

Proposition 14 *For each pair $i, j \in \mathcal{N}, i \neq j$ when $\overline{C} - \underline{C} - kV > 0$, $k \in [2, \min\{L, T-1\}]_{\mathbb{Z}}$, and $L \geq 2$, the following inequalities*

$$x_{i_k^-} + x_i - x_j \leq \overline{C} y_{i_k^-} + (\overline{C} - (k-2)V) y_i - (\overline{C} - 2V) y_j + (k-2)V (y_{i^-} + u_i) + (\overline{C} - \underline{C} - 2V) u_j \tag{31}$$

$$x_{i_k^-} + x_i - x_j \geq \underline{C} y_{i_k^-} + (\underline{C} + (k-2)V) y_i - (\underline{C} + 2V) y_j - (k-2)V (y_{i^-} + u_i) - (\overline{C} - \underline{C} - 2V) u_j \tag{32}$$

are valid and facet-defining for conv(P_T^0).

Proof: The detailed proofs are shown in E-companion C.6. ■

Note here that inequalities (31) and (32) are different from (29) and (30), except for the case in which $k = 2$ in Propositions 13 and 14.

Finally, we extend the study to the complete scenario tree setting as described in Figure 7. For this setting, we only introduce the extra strong valid inequalities which are not described in the previous flower-structure scenario tree setting. For any two nodes i and j in Figure 7, we let $p = \text{argmax}\{t(k) : k \in \mathcal{P}(i) \cap \mathcal{P}(j)\}$. This indicates that the path from nodes i to j passing through node p as the node with the smallest time period is the shortest path from node i to node j in the scenario tree. We define $\mathcal{P}(i,p) = \mathcal{P}(i) \setminus \mathcal{P}(p)$ and the distance between node i and node j, $dist(i,j) = |\mathcal{P}(i,p)| + |\mathcal{P}(j,p)|$. Through investigating the relationships between x_i and x_j by incorporating the effects of node p, we have the following facet-defining inequality.

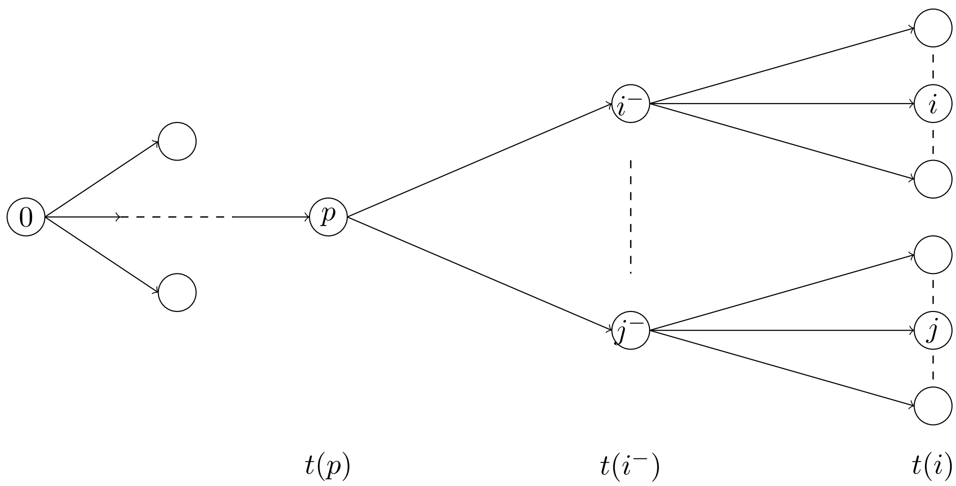

Figure 7: Scenario tree

Proposition 15 *For any pair of nodes $(i,j) \in \mathcal{V}$ such that $i \notin \mathcal{P}(j)$ and $j \notin \mathcal{P}(i)$, when $\overline{C} - \underline{C} - kV > 0$ with $k = dist(i,j)$, the following inequality*

$$x_i - x_j \leq (\underline{C} + kV)y_i - (\overline{C} - kV)y_j + (\overline{C} - \underline{C} - kV)(y_p + \sum_{s \in \mathcal{P}(i,p) \cup \mathcal{P}(j,p)} u_s) \tag{33}$$

is valid and facet-defining for conv(P).

Proof: The detailed proofs are described in E-companion C.7. ■

Note here that inequality (33) generalizes the results described in (10f) and (21) from two- and three-period cases to the general multi-period setting.

Remark 5 *Due to the structure of the linearly independent points constructed to prove the inequalities (22) - (32) to be facet-defining for conv(P_T^0), it can be observed that these inequalities (i.e., inequalities (22) - (32)) are also facet-defining for the general polytope conv(P). Meanwhile, the number of all facet-defining inequalities is a polynomial function of the input size of the scenario tree, e.g., in the order of $\mathcal{O}(|\mathcal{V}|^2)$.*

6 Computational Experiments

In this section, we report computational studies on testing the effectiveness of the strong valid inequalities, through implementing branch-and-cut algorithms on randomly generated instances. All the computational experiments were implemented on a computer with Intel Dual Core 2.60 GHz and 4 GB memory. Default CPLEX 12.5 in C++ via Concert Technology was applied to solve the instances.

6.1 Instance Generation for the Stochastic Self-Scheduling Problem

In our computational experiments, instances of gas-fired thermal generators for the MSS instances were generated based on the modified IEEE 118-bus system available online at motor.ece.iit.edu/data/SCUC_118. Three different generators were selected, and the detailed characteristics are shown in the following Table 1.

Table 1: Generator Data

G	$\underline{C}$ (MW)	$\overline{C}$ (MW)	$V^+ = V^-$ (MW/h)	$\bar{U}$ (\$)
1	35	160	38	80
2	61	300	58	100
3	130	620	150	300

For the electricity price, we assume it is uncertain and within the interval $[1, 31]$ at each node in the scenario tree. To test the variations of the proposed instances, we consider three types of minimum-up/down times, i.e., $L = \ell = 2, 3$, and 4. In addition, we test different structures of the scenario tree by considering variant numbers of branches for each non-leaf node. We let K represent the number of branches for each non-leaf node in the derived scenario tree. In our experiments, we consider (1) a binary tree (i.e., $K = 2$), (2) a ternary tree (i.e., $K = 3$), and (3) a quadtree (i.e., $K = 4$), with each non-leaf node containing two-four branches, each with the same probability. For $K = 2$, we let the number of time periods $T = 10, 11$, and 12 for the cases in which the minimum-up/down times $L = \ell = 2, 3$, and $T = 9, 10$, and 11 when $L = \ell = 4$. For $K = 3$, we let $T = 7$ and 8. For $K = 4$, we let $T = 6$ and 7. Thus, various instances are generated based on different combinations of K, T, and $L(\ell)$. For each combination of G, K, T, and $L(\ell)$, we test three instances and report the average value, with one hour time limit per run.

6.2 Results for the Stochastic Self-Scheduling Problem

We first test the performance of our cutting planes in tightening the LP relaxations for different MSS instances at the root node, and report the results in Table 2. In the table, the column labelled "LP Gap (%)" represents the LP relaxation gap of the original formulation MSS with respect to the best integer solution we obtained from default CPLEX and our branch-and-cut scheme. "LP Gap (%)" is defined as $(Z_{\text{LP}} - Z_{\text{MILP}})/Z_{\text{LP}}$, where Z_{LP} represents the objective value of the LP relaxation problem and Z_{MILP} represents the objective value of the best integer solution. Similarly, the column labelled "Cut (%)" represents the LP relaxation gap after adding our cutting planes. We can observe that the LP relaxation gap decreases dramatically after adding our developed strong valid inequalities as cutting planes in the root node to tighten the original formulation. The degree of such reduction is shown in the column labelled "Percentage (%)," which provides how much the gap is reduced based on the "LP Gap (%)." That is, Percentage(%)= (LP Gap(%) - Cut(%))/LP Gap(%). From the table, we can observe that approximately 70% reduction can be achieved, indicating the tremendous effect of our cutting planes.

Then, we test the performance of our cutting planes in speeding up the branch-and-cut algorithms to solve the instances. We test the instances for $L = \ell = 2$, $L = \ell = 3$, and $L = \ell = 4$, respectively, and the corresponding computational results are reported in Tables 3, 4, and 5. In our experiment, we compare the performance of default CPLEX MIP solver (e.g., "Default CPLEX") with the branch-and-cut algorithm we developed with the derived cutting planes embedded as User Cuts. In the tables, the column labelled "Gap (%)" reports the final optimality gap obtained within the given time limit. Meanwhile, the number in the square bracket indicates the number of instances not solved to default optimality (i.e., 0.01%) within the time limit. Accordingly, in the tables, we report the final optimality gap as the average value over those of the instances not solved to default optimality. The column labelled "CPU secs" represents the solution time taken to solve the problem. We report the average value over those of the instances solved to default optimality, whereas 3600 is reported for the cases when all three instances are not solved to default optimality. Besides the "Gap (%)" and "CPU secs" columns, the column labelled "# of Nodes" provides the number of branch-and-bound nodes that CPLEX explored, and the last column labelled "# of Cuts" represents the number of derived cutting planes utilized in the branch-and-cut algorithm to solve the instances.

From Tables 3, 4, and 5, we can observe that the performance of our branch-and-cut algorithm is much better than that of the default CPLEX. For the cases when both default CPLEX and

Table 2: Results for the Root Node

G	K	T	$L=\ell=2$ LP Gap (%)	Cut (%)	Percent -age (%)	$L=\ell=3$ LP Gap (%)	Cut (%)	Percent -age (%)	T	$L=\ell=4$ LP Gap (%)	Cut (%)	Percent -age (%)
1	2	10	8.76	2.52	71.23	15.16	4.74	68.72	9	17.79	5.87	67.01
		11	8.26	2.39	71.06	15.02	4.72	68.58	10	17.47	5.85	66.51
		12	8.06	2.34	71.01	14.76	4.54	69.24	11	17.67	6.09	65.54
	3	7	11.39	3.49	69.36	18.39	6.04	67.15	7	20.53	6.99	65.95
		8	10.58	3.25	69.29	18.59	6.38	65.69	8	19.92	7.17	64.01
	4	6	10.24	3.40	66.78	13.60	4.13	69.60	6	16.30	5.22	67.96
		7	10.09	3.27	67.59	14.20	4.65	67.26	7	15.50	4.58	70.47
2	2	10	10.53	3.01	71.45	10.89	3.48	68.04	9	22.91	7.51	67.21
		11	10.19	2.78	72.72	11.40	3.78	66.86	10	23.72	7.34	69.06
		12	9.95	2.79	71.94	11.67	3.79	67.51	11	23.88	7.52	68.51
	3	7	11.90	4.01	66.28	13.99	4.89	65.05	7	17.98	5.73	68.15
		8	11.33	3.86	65.93	15.08	5.46	63.79	8	16.87	4.94	70.72
	4	6	11.54	4.98	56.84	10.02	2.94	70.70	6	18.91	6.55	65.37
		7	11.24	4.73	57.89	11.15	3.25	70.82	7	13.54	2.26	83.27
3	2	10	7.91	2.15	72.85	15.44	4.91	68.19	9	17.35	5.88	66.11
		11	8.36	2.43	70.94	15.42	5.21	66.2	10	18.02	6.09	66.21
		12	7.66	1.85	75.85	13.08	4.85	62.92	11	15.78	5.74	63.62
	3	7	11.75	3.71	68.42	18.82	6.22	66.95	7	20.79	6.97	66.47
		8	11.01	3.55	67.77	16.12	5.28	67.25	8	16.75	6.06	63.83
	4	6	10.43	3.62	65.3	13.73	4.13	69.94	6	16.57	5.24	68.39
		7	10.36	2.61	74.81	13.16	4.81	63.44	7	16.22	5.46	66.32

our branch-and-cut scheme can solve the instances into default optimality, our approach takes a much shorter time and explores a much smaller number of branch-and-bound nodes than the default CPLEX does. For the cases when only the default CPLEX cannot solve the instances into default optimality, our branch-and-cut algorithm shows its advantage by solving the corresponding instances into default optimality. Meanwhile, for these instances, our approach does not take a long time to solve them and explores a much smaller number of branch-and-bound nodes. For the cases when both the default CPLEX and our branch-and-cut scheme cannot solve the instances into default optimality, our approach derives a much smaller optimality gap than the default CPLEX does.

6.3 Results for the Stochastic Network-Constrained Unit Commitment

We further report the performance of strong valid inequalities by testing the stochastic network-constrained unit commitment problem in which MSS is embedded. To differentiate generators, we

Table 3: Results for the Branch-and-cut Scheme ($L = \ell = 2$)

G	K	T	Default CPLEX Gap (%)	Default CPLEX # Nodes	Default CPLEX CPU secs	Branch-and-cut Gap (%)	Branch-and-cut # Nodes	Branch-and-cut CPU secs	Branch-and-cut # Cuts
1	2	10	0.00	1721	33.4	0.00	95	6.2	448
		11	0.00	11149	822.3	0.00	399	15.6	874
		12	0.17[2]	30195	2553	0.00	2337	115.5	1485
	3	7	0.00	5224	43.1	0.00	80	9.1	565
		8	0.05[3]	73172	3600	0.00	3006	151.5	1362
	4	6	0.00	8626	67.7	0.00	239	13.4	811
		7	0.24[2]	18084	348.6	0.00	3302	281.9	2625
2	2	10	0.00	3367	31.9	0.00	76	4.3	529
		11	0.07[1]	97208	139.9	0.00	422	10.7	1023
		12	0.15[3]	148936	3600	0.00	2357	79	1354
	3	7	0.00	11228	137.6	0.00	133	5.4	692
		8	0.25[3]	168255	3600	0.00	4203	131.5	1932
	4	6	0.00	4531	47.1	0.00	143	8.3	880
		7	0.30[3]	118651	3600	0.16[1]	41480	118	3260
3	2	10	0.00	2202	36.3	0.00	67	6.4	442
		11	0.03[1]	35512	109.1	0.00	380	14.3	881
		12	0.14[2]	59686	2424.2	0.00	1977	175.8	1548
	3	7	0.00	4109	51.6	0.00	107	11.3	570
		8	0.26[3]	59977	3600	0.00	1986	129.5	912
	4	6	0.00	8203	71.1	0.00	274	14.7	802
		7	0.37[3]	27958	3600	0.13[1]	9327	243.9	3009

add superscript k to each decision variable/parameter defined for generators. In addition, we let $\mathcal{B}$ and $\mathcal{A}$ represent the sets of buses and transmission lines, and $\mathcal{K}$ ($\mathcal{K}_b$) represent the set of generators (at bus b). We let C_{mn} and K^b_{mn} represent the capacity of the transmission line (m, n) and the line flow distribution factor for the flow on the transmission line (m, n) contributed by the net injection at bus b, respectively. We let D^b_i denote the load of bus b at node i. The formulation can be described as follows:

$$\min \quad \sum_{i\in\mathcal{V}} p_i \left(\sum_{k\in\mathcal{K}} \left(\bar{U}^k u^k_i + \underline{U}^k (y^k_{i^-} - y^k_i + u^k_i) + f^k(x^k_i) \right) \right) \tag{34}$$

s.t. (1b) – (1i) with superscript k added,

$$\sum_{k\in\mathcal{K}} x^k_i = \sum_{b\in\mathcal{B}} D^b_i, \quad \forall i \in \mathcal{V}, \tag{35}$$

$$-C_{mn} \leq \sum_{b\in\mathcal{B}} K^b_{mn} \left(\sum_{k\in\mathcal{K}_b} x^k_i - D^b_i \right) \leq C_{mn}, \quad \forall i \in \mathcal{V}, \forall (m, n) \in \mathcal{A}, \tag{36}$$

$$y^k_i \in \{0, 1\}, x^k_i \geq 0, \ \forall i \in \mathcal{V}, \ \forall k \in \mathcal{K} \text{ and } \ u^k_i \in \{0, 1\}, \ \forall i \in \mathcal{V} \setminus \{0\}, \ \forall k \in \mathcal{K}, \tag{37}$$

Table 4: Results for the Branch-and-cut Scheme ($L = \ell = 3$)

G	K	T	Default CPLEX			Branch-and-cut			
			Gap (%)	# Nodes	CPU secs	Gap (%)	# Nodes	CPU secs	# Cuts
1	2	10	0.00	5398	80.2	0.00	236	16	678
		11	0.21[3]	76132	3600	0.00	3660	121.3	958
		12	0.47[3]	32607	3600	0.00	10761	417.4	2050
	3	7	0.00	48355	889.6	0.00	1671	28.7	877
		8	1.14[3]	38510	3600	0.31[2]	28787	1610.7	2215
	4	6	0.00	36373	867.8	0.00	532	23.9	1030
		7	0.69[3]	24577	3600	0.12[1]	9947	1004	3361
2	2	10	0.00	13373	156.5	0.00	221	8.4	759
		11	0.11[3]	198128	3600	0.00	6433	160.4	1107
		12	0.52[3]	77355	3600	0.00	11780	529.7	2229
	3	7	0.00	20696	240.2	0.00	1229	19.7	815
		8	0.76[3]	6487	3600	0.28[1]	17333	189.3	2135
	4	6	0.00	1466	22.6	0.00	123	11	1194
		7	0.42[3]	36665	3600	0.00	2736	390.3	3439
3	2	10	0.00	10085	97.9	0.00	664	23.2	690
		11	0.16[3]	159498	3600	0.00	3742	154.2	991
		12	0.55[3]	26642	3600	0.00	9517	459.1	2136
	3	7	0.00	55416	984.8	0.00	2580	40.6	874
		8	0.77[3]	67590	3600	0.27[1]	19668	176.5	2430
	4	6	0.00	42823	904.2	0.00	364	20.8	1041
		7	0.80[3]	24550	3600	0.29[2]	17430	221.9	4103

where constraints (35) ensure the power balance and constraints (36) represent the capacity limit of each transmission line (m, n).

We test the instances randomly generated from a modified IEEE 118-bus system based on the one given online at `motor.ece.iit.edu/data/SCUC_118`. We let the number of generators $|\mathcal{K}| = 15, 20, 25,$ and 30 respectively, and the uncertain system load is within the interval $[0, 2\bar{D}]$ where $\bar{D}$ is proportional to the total generation capacity. The computational results are reported in Tables 6 and 7. From Table 6, we can observe that our proposed strong valid inequalities are effective because approximately 50% LP Gap reduction can be achieved by adding our proposed strong valid inequalities, as compared to default CPLEX.

From Table 7, we can observe that similar performance maintains for the branch-and-cut approach. For instance, our approach takes a shorter time when both default CPLEX and our branch-and-cut scheme can solve the instances into default optimality. There also exist instances for which the default CPLEX cannot obtain the optimal solution within the time limit while our

Table 5: Results for the Branch-and-cut Scheme ($L = \ell = 4$)

G	K	T	Default CPLEX			Branch-and-cut			
			Gap (%)	# Nodes	CPU secs	Gap (%)	# Nodes	CPU secs	# Cuts
1	2	9	0.00	2648	13.7	0.00	263	10.3	320
		10	0.00	73314	1429.2	0.00	2708	35	728
		11	0.26[3]	94806	3600	0.11[1]	31253	1087.2	1047
	3	7	0.09[1]	102170	34.4	0.00	984	34.5	762
		8	0.88[3]	23498	3600	0.21[2]	16822	230.3	2251
	4	6	0.00	182	28.4	0.00	6	29.8	1438
		7	0.47[3]	20869	3600	0.00	4274	783.1	3169
2	2	9	0.00	4232	29.2	0.00	178	5.7	382
		10	0.50[2]	177313	179	0.00	5857	142	577
		11	1.34[3]	55291	3600	0.60[2]	35652	70	1545
	3	7	0.07[1]	128965	33.7	0.00	8641	136.3	929
		8	0.79[3]	35275	3600	0.04[1]	26088	271.3	2226
	4	6	0.00	2181	40.1	0.00	604	31.6	1226
		7	0.75[2]	31866	3007.6	0.00	1459	387.5	3495
3	2	9	0.00	2562	14.4	0.00	327	9.6	321
		10	0.00	66079	984.6	0.00	2904	38.1	714
		11	0.56[3]	70682	3600	0.00	38442	1838.8	1060
	3	7	0.00	21832	589.9	0.00	813	29.8	765
		8	0.65[3]	29610	3600	0.18[2]	17072	206.4	2234
	4	6	0.00	236	32.5	0.00	4	29.1	1413
		7	0.53[3]	18942	3600	0.25[1]	3562	524.1	3091

Table 6: Results for the Root Node

K	T	$\lvert\mathcal{K}\rvert = 15$			$\lvert\mathcal{K}\rvert = 20$		
		LP Gap (%)	Cut (%)	Percentage (%)	LP Gap (%)	Cut (%)	Percentage (%)
2	9	0.7	0.38	45.7	0.66	0.31	53.07
	10	0.82	0.5	39.41	0.6	0.32	46.84
3	6	0.83	0.37	54.95	0.74	0.36	51.22
	7	1.09	0.53	51.75	1.74	1.04	40.48
K	T	$\lvert\mathcal{K}\rvert = 25$			$\lvert\mathcal{K}\rvert = 30$		
		LP Gap (%)	Cut (%)	Percentage (%)	LP Gap (%)	Cut (%)	Percentage (%)
2	9	0.54	0.28	47.83	0.89	0.53	40.59
	10	0.47	0.25	46.9	0.93	0.49	46.76
3	6	0.64	0.32	49.13	1.11	0.64	42.53
	7	0.86	0.49	43.45	1.08	0.67	37.56

approach can. For the cases when both approaches cannot solve the instances into default optimality, our approach obtains a smaller optimality gap than the default CPLEX does. We also notice that the improvement is less significant for the stochastic network-constrained unit commitment than the MSS problem due to the introduction of constraints (35) and (36).

Table 7: Results for the Branch-and-cut Scheme

$\lvert\mathcal{K}\rvert$	K	T	Default CPLEX			Branch-and-cut Scheme			
			Gap (%)	# of Nodes	CPU secs	Gap (%)	# of Nodes	CPU secs	# of Cuts
15	2	9	0.00	1203	465.5	0.00	777	260.1	97
		10	0.32 [2]	798	1075.5	0.07 [1]	906	1895.5	169
	3	6	0.07 [3]	91172	3600	0.02 [1]	86303	2905.9	117
		7	0.10 [1]	803	1427.3	0.04 [1]	1056	2064.5	129
20	2	9	0.05 [1]	6133	984.2	0.00	8899	816.4	106
		10	0.07 [3]	701	3600	0.03 [2]	1372	1521.3	135
	3	6	0.06 [3]	53070	3600	0.05 [1]	18616	3035.4	203
		7	0.72 [3]	1800	3600	0.25 [3]	1334	3600	168
25	2	9	0.03 [1]	4503	681.2	0.00	2717	575.3	82
		10	0.04 [3]	704	3600	0.00	920	2517.1	182
	3	6	0.04 [2]	37296	1660.7	0.00	17623	2021.8	205
		7	0.31 [2]	579	2578.6	0.02[1]	483	2963.5	241
30	2	9	0.00	703	801.6	0.00	657	521.6	352
		10	0.14 [3]	586	3594.5	0.08 [3]	140	3593.1	314
	3	6	0.09 [3]	24990	3606.9	0.07 [2]	14662	3609.5	381
		7	0.72 [3]	589	3607.7	0.63 [1]	166	2592.9	442

Finally, we report the performance of our approach by testing a 24 time period instance also based on the modified IEEE 118-bus system as described above. We generate a scenario tree with 24 time periods and 64 scenarios, with the number of branches at each node generated from $[1, 3]_{\mathbb{Z}}$. We let the time limit per run be two hours since we are solving larger instances. We also test $|\mathcal{K}| = 15, 20, 25$, and 30 cases and report the average value of three randomly generated instances for each case in Tables 8 and 9. We can also observe that around 50% LP Gap reduction at the root node and our branch-and-cut algorithm outperforms default CPLEX.

Table 8: Results for the Root Node

$\lvert\mathcal{K}\rvert$	LP Gap (%)	Cut (%)	Percentage (%)	$\lvert\mathcal{K}\rvert$	LP Gap (%)	Cut (%)	Percentage (%)
15	0.55	0.23	57.13	25	0.44	0.22	50.73
20	0.74	0.44	39.96	30	0.53	0.31	41.88

7 Conclusions

In this paper, we proposed a multistage stochastic self-scheduling unit commitment model for an IPP to participate in the real-time market using the self-scheduling mode, so as to achieve the maximum expected total profit. The proposed model can help the IPP to optimally decide the

Table 9: Results for the Branch-and-cut Scheme

$\lvert\mathcal{K}\rvert$	Default CPLEX			Branch-and-cut Scheme			
	Gap (%)	# of Nodes	CPU secs	Gap (%)	# of Nodes	CPU secs	# of Cuts
15	0.00	166	3470.5	0.00	339	2259.4	441
20	0.00	280	4398.6	0.00	63	2706.6	466
25	0.02 [1]	1733	4659.6	0.00	958	5108.4	352
30	0.06 [3]	3767	7200	0.07 [1]	3838	5826.7	397

unit commitment status and economic dispatch amount at each time period, based on the probabilistic forecast of real-time prices. The proposed optimal decisions are dynamic, following the evaluation of real-time prices, which result in a higher total expected profit as compared to that derived by the two-stage stochastic optimization approach. By exploring the possible realizations of uncertain prices, a scenario-tree based multistage stochastic integer programming formulation was obtained and accordingly strong cutting planes were developed for the derived large-scale deterministic equivalent formulation. By exploring the scenario-tree structure and the unit commitment physical constraints characteristics, we developed strong valid inequalities to speed up the algorithms to solve the deterministic equivalent formulation. In particular, we derived strong formulations that can describe the convex hull for the two-period case. This study was also extended to provide the convex hull description for a special case of the three-period case. For the general multistage setting, our derived inequalities are cross-scenario and facet-defining for the whole scenario tree. The final numerical experiments on various data settings demonstrated the effectiveness of our proposed cutting planes, embedded in a branch-and-cut framework.

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Appendix A Two-period Convex Hull Proofs

A.1 Proof for Proposition 1

Proof: To prove the validity of inequality (4), we discuss the following four cases in terms of possible values of y_{i^-} and y_i:

(i) If $y_{i^-} = y_i = 0$, then $x_{i^-} = x_i = 0$ due to constraints (3d) and $u_i = 0$ due to constraints (3b). Thus, inequality (4) is valid.

(ii) If $y_{i^-} = 0$ and $y_i = 1$, then $u_i = 1$ following constraints (3a) and $x_{i^-} = 0$ due to constraints (3d). Then, inequality (4) converts to $x_i \leq \overline{C}$, which is valid because of (3d).

(iii) If $y_{i^-} = 1$ and $y_i = 0$, then $x_i = 0$ following constraints (3d) and $u_i = 0$ due to constraints (3b). Thus, inequality (4) converts to $x_{i^-} \geq \underline{C}$, which is valid because of (3d).

(iv) If $y_{i^-} = y_i = 1$, then $u_i = 0$ due to constraints (3c). Inequality (4) converts to $x_i - x_{i^-} \leq V^+$, which is valid because of (3e).

By symmetry, inequality (5) can be proven to be valid in a similar way.

To prove the validity of inequality (6), we discuss the following four cases in terms of possible values of y_i and y_j:

(i) If $y_i = y_j = 0$, then $x_i = x_j = 0$ due to constraints (3d) and $u_i = u_j = 0$ due to constraints (3b). Thus, inequality (6) is valid because of the assumption that $(\overline{C} - \underline{C} - V^+ - V^-)>0$ and the nonnegativity of y_{i^-}.

(ii) If $y_i = 0$ and $y_j = 1$, then $x_i = 0$ due to constraints (3d) and $u_i = 0$ due to constraints (3b). Since i^- is equivalent to j^-, inequality (6) converts to $x_j \geq \underline{C}y_j - (\overline{C} - \underline{C} - V^+ - V^-)(y_{j^-} + u_j - y_j)$, which is valid due to constraints (3d) and (3a).

(iii) If $y_i = 1$ and $y_j = 0$, then $x_j = 0$ due to constraints (3d) and $u_j = 0$ due to constraints (3b). Inequality (6) converts to $x_i \leq \overline{C}y_i + (\overline{C} - \underline{C} - V^+ - V^-)(y_{i^-} + u_i - y_i)$, which is valid due to constraints (3d) and (3a).

(iv) If $y_i = y_j = 1$, inequality (6) converts to $x_i - x_j \leq (V^+ + V^-) + (\overline{C} - \underline{C} - V^+ - V^-)(y_{i^-} + u_i + u_j - 1)$. If $y_{i^-} = 0$, then $u_i = u_j = 1$ because of constraints (3a). Hence (6) is further simplified to be $x_i - x_j \leq \overline{C} - \underline{C}$, which is valid due to constraints (3d). Otherwise, if $y_{i^-} = 1$, then $u_i = u_j = 0$ due to constraints (3c). Moreover, the difference between x_i and x_j is maximized when one of them increases by V^+ from x_{i^-} and another one decreases by V^- from x_{i^-} following the ramp-up and ramp-down rate limits. Thus $x_i - x_j \leq V^+ + V^-$ and it follows that (6) holds.

Therefore, inequalities (4), (5), and (6) are valid for $\text{conv}(P_2)$ as desired. ■

A.2 Proof for Proposition 2

Proof: We prove that $\dim(Q_2) = 3n + 2$, because there are $3n + 2$ decision variables in Q_2. We generate $3n + 3$ affinely independent points in Q_2. We sort the scenario nodes in the second period in the order as $1, 2, \cdots, n$ and label the node i^- as index 0 so that we have $n + 1$ nodes, i.e., $0, 1, 2, \cdots, n$. Since $0 \in Q_2$, we generate other $3n + 2$ linearly independent points in Q_2. First, we create $(\bar{x}^i, \bar{y}^i, \bar{u}^i) \in Q_2$ for each $i \in [0, n]_{\mathbb{Z}}$, where

$$\bar{x}^i_s = \begin{cases} \overline{C}, & s \in [0, i]_{\mathbb{Z}} \\ 0, & s \in [i + 1, n]_{\mathbb{Z}} \end{cases}, \quad \bar{y}^i_s = \begin{cases} 1, & s \in [0, i]_{\mathbb{Z}} \\ 0, & s \in [i + 1, n]_{\mathbb{Z}} \end{cases}, \text{ and } \bar{u}^i_s = 0, \quad s \in [1, n]_{\mathbb{Z}}.$$

Second, we create $(\hat{x}^i, \hat{y}^i, \hat{u}^i) \in Q_2$ for each $i \in [0, n]_{\mathbb{Z}}$, where

$$\hat{x}^i_s = \begin{cases} \underline{C}, & s \in [0, i]_{\mathbb{Z}} \\ 0, & s \in [i + 1, n]_{\mathbb{Z}} \end{cases}, \quad \hat{y}^i_s = \begin{cases} 1, & s \in [0, i]_{\mathbb{Z}} \\ 0, & s \in [i + 1, n]_{\mathbb{Z}} \end{cases}, \text{ and } \hat{u}^i_s = 0, \quad s \in [1, n]_{\mathbb{Z}}.$$

Third, we create $(\tilde{x}^i, \tilde{y}^i, \tilde{u}^i) \in Q_2$ for each $i \in [1, n]_{\mathbb{Z}}$, where

$$\tilde{x}^i_s = \begin{cases} 0, & s \in [0, i - 1]_{\mathbb{Z}} \\ \underline{C}, & s \in [i, n]_{\mathbb{Z}} \end{cases}, \quad \tilde{y}^i_s = \begin{cases} 0, & s \in [0, i - 1]_{\mathbb{Z}} \\ 1, & s \in [i, n]_{\mathbb{Z}} \end{cases}, \text{ and } \tilde{u}^i_s = \begin{cases} 0, & s \in [0, i - 1]_{\mathbb{Z}} \\ 1, & s \in [i, n]_{\mathbb{Z}} \end{cases}.$$

It is clear that $(\bar{x}^i, \bar{y}^i, \bar{u}^i)_{i=0}^n$, $(\hat{x}^i, \hat{y}^i, \hat{u}^i)_{i=0}^n$, and $(\tilde{x}^i, \tilde{y}^i, \tilde{u}^i)_{i=1}^n$ are linearly independent and therefore the statement is proved. ■

A.3 Proof for Proposition 3

Proof: For each inequality, we generate $3n + 2$ affinely independent points in $\text{conv}(P_2)$ that satisfy the inequality at equality. In the following proof, we follow the notation described in Proposition 2. Since $0 \in \text{conv}(P_2)$, we generate the remaining $3n + 1$ linearly independent points in $\text{conv}(P_2)$ for each inequality. In the following proofs, we use the superscript of (x, y, u), e.g., r in (x^r, y^r, u^r), to indicate the index of different points in $\text{conv}(P_2)$.

For inequalities (3a) $y_i - y_{i^-} - u_i \leq 0, \quad \forall i \in \mathcal{N}:$

We create five groups of points as follows:

(i) We create a point $(\acute{x}, \acute{y}, \acute{u}) \in \text{conv}(P_2)$ (totally one point) such that

$$\acute{x}_s = \begin{cases} \underline{C} + V^-, & s = 0 \\ \underline{C}, & s = i \\ 0, & \forall s \in [1,n]_{\mathbb{Z}} \setminus \{i\} \end{cases}, \quad \acute{y}_s = \begin{cases} 1, & \forall s \in \{0, i\} \\ 0, & \forall s \in [1,n]_{\mathbb{Z}} \setminus \{i\} \end{cases}, \text{ and } \acute{u}_s = 0, \ \forall s \in [1,n]_{\mathbb{Z}}.$$

(ii) For each $r \in [0,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are n points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P_2)$ such that

$$\bar{x}^r_s = \begin{cases} \underline{C}, & \forall s \in [0,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}^r_s = \begin{cases} 1, & \forall s \in [0,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \bar{u}^r_s = 0, \forall s.$$

(iii) For each $r \in [0,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are n points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P_2)$ such that

$$\hat{x}^r_s = \begin{cases} \overline{C}, & \forall s \in [0,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}^r_s = \begin{cases} 1, & \forall s \in [0,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}^r_s = 0, \forall s.$$

(iv) We create a point $(\dot{x}, \dot{y}, \dot{u}) \in \text{conv}(P_2)$ (totally one point) such that

$$\dot{x}_s = \begin{cases} 0, & s = 0 \\ \underline{C}, & \text{o.w.} \end{cases}, \quad \dot{y}_s = \begin{cases} 0, & s = 0 \\ 1, & \text{o.w.} \end{cases}, \text{ and } \dot{u}_s = 1, \forall s.$$

(v) For each $r \in [1,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are $n-1$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r) \in \text{conv}(P_2)$ such that

$$\tilde{x}^r_s = \begin{cases} \underline{C}, & \forall s \in [r,n]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{y}^r_s = \tilde{u}^r_s = \begin{cases} 1, & \forall s \in [r,n]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}.$$

Finally, these five groups of points are collected in Table 10, from which we can observe that $(\hat{x}, \hat{y}, \hat{u})$, $(\dot{x}, \dot{y}, \dot{u})$, and $(\tilde{x}, \tilde{y}, \tilde{u})$ are linearly independent because they can construct a lower-triangular matrix based on the values of y and u after Gaussian elimination on the u part. Moreover, $(\acute{x}, \acute{y}, \acute{u})$ and $(\bar{x}, \bar{y}, \bar{u})$ are further linearly independent with them because all of these five groups of points can construct a lower-triangular matrix after Gaussian elimination operation on the x (i.e., $\bar{x}$ and $\hat{x}$) part in the two groups of points $(\bar{x}, \bar{y}, \bar{u})$ and $(\hat{x}, \hat{y}, \hat{u})$ since $\overline{C} > \underline{C}$. Thus, we have created $1 + n + n + 1 + (n-1) = 3n + 1$ linearly independent points in $\text{conv}(P_2)$ as desired.

In the following proofs, we follow the similar way as described above to create linearly independent points in $\text{conv}(P_2)$ by firstly generating several groups of points that can construct a lower-triangular matrix in terms of the values of x and y (e.g., $(\acute{x}, \acute{y}, \acute{u})$, $(\bar{x}, \bar{y}, \bar{u})$, and $(\hat{x}, \hat{y}, \hat{u})$ for inequalities (3a) above) and then generating several groups of points that can construct an upper-triangular matrix in terms of the value of u (e.g., $(\dot{x}, \dot{y}, \dot{u})$ and $(\tilde{x}, \tilde{y}, \tilde{u})$ for inequalities (3a) above).

Group	x								y								u						
	x_{i-}	x_i	x_1	$\cdots$	x_{i-1}	x_{i+1}	$\cdots$	x_n	y_{i-}	y_i	y_1	$\cdots$	y_{i-1}	y_{i+1}	$\cdots$	y_n	u_i	u_1	$\cdots$	u_{i-1}	u_{i+1}	$\cdots$	u_n
$(\acute{x}, \acute{y}, \acute{u})$	$\underline{C}+V^-$	$\underline{C}$	0	$\cdots$	0	0	$\cdots$	0	1	1	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
$(\bar{x}, \bar{y}, \bar{u})$	$\underline{C}$	$\underline{C}$	0	$\cdots$	0	0	$\cdots$	0	1	1	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	0	0	$\cdots$	0	1	1	1	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	1	1	1	$\cdots$	1	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\cdots$	0	1	1	1	$\cdots$	1	1	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	1	1	1	$\cdots$	1	1	$\cdots$	1	0	0	$\cdots$	0	0	$\cdots$	0
$(\hat{x}, \hat{y}, \hat{u})$	$\overline{C}$	$\overline{C}$	0	$\cdots$	0	0	$\cdots$	0	1	1	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	0	0	$\cdots$	0	1	1	1	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	0	$\cdots$	0	1	1	1	$\cdots$	1	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\cdots$	0	1	1	1	$\cdots$	1	1	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	1	$\cdots$	1	1	$\cdots$	1	0	0	$\cdots$	0	0	$\cdots$	0
$(\dot{x}, \dot{y}, \dot{u})$	0	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	1	1	$\cdots$	1	1	$\cdots$	1	1	1	$\cdots$	1	1	$\cdots$	1
$(\tilde{x}, \tilde{y}, \tilde{u})$	0	0	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	0	1	$\cdots$	1	1	$\cdots$	1	0	1	$\cdots$	1	1	$\cdots$	1
	0	0	0	$\cdots$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	0	0	$\cdots$	1	1	$\cdots$	1	0	0	$\cdots$	1	1	$\cdots$	1
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
	0	0	0	$\cdots$	0	$\underline{C}$	$\cdots$	$\underline{C}$	0	0	0	$\cdots$	0	1	$\cdots$	1	0	0	$\cdots$	0	1	$\cdots$	1
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
	0	0	0	$\cdots$	0	0	$\cdots$	$\underline{C}$	0	0	0	$\cdots$	0	0	$\cdots$	1	0	0	$\cdots$	0	0	$\cdots$	1

Table 10: $3n+1$ linearly independent points for (3a)

For inequalities (3b) $u_i - y_i \leq 0, \quad \forall i \in \mathcal{N}$ **:**

We create $3n+1$ linearly independent points in $\mathrm{conv}(P_2)$ in the following five groups.

(i) For each $r \in [0,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are n points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \mathrm{conv}(P_2)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [0,r]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [0,r]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \in [1,n]_{\mathbb{Z}} \end{array}.$$

(ii) For each $r \in [0,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are n points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \mathrm{conv}(P_2)$ such that

$$\hat{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [0,r]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [0,r]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

(iii) For each $r \in [1,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are $n-1$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r) \in \text{conv}(P_2)$ such that

$$\tilde{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [r,n]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{y}_s^r = \tilde{u}_s^r = \begin{cases} 1, & \forall s \in [r,n]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}.$$

(iv) We create a point $(\acute{x}, \acute{y}, \acute{u}) \in \text{conv}(P_2)$ (totally one point) such that

$$\acute{x}_s = \begin{cases} \underline{C}, & s = i \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}_s = \begin{cases} 1, & s = i \\ 0, & \text{o.w.} \end{cases}, \text{ and } \acute{u}_s = \begin{cases} 1, & s = i \\ 0, & \text{o.w.} \end{cases}.$$

(v) We create a point $(\dot{x}, \dot{y}, \dot{u}) \in \text{conv}(P_2)$ (totally one point) such that

$$\dot{x}_s = \begin{cases} \overline{C}, & s = i \\ 0, & \text{o.w.} \end{cases}, \quad \dot{y}_s = \begin{cases} 1, & s = i \\ 0, & \text{o.w.} \end{cases}, \text{ and } \dot{u}_s = \begin{cases} 1, & s = i \\ 0, & \text{o.w.} \end{cases}.$$

We collect these $3n+1$ linearly independent points in Table 11.

Group	x	y	u
	$x_{i-}\ x_1 \cdots x_{i-1}\ x_{i+1} \cdots x_n\ x_i$	$y_{i-}\ y_1 \cdots y_{i-1}\ y_{i+1} \cdots y_n\ y_i$	$u_1 \cdots u_{i-1}\ u_{i+1} \cdots u_n\ u_i$
$(\bar{x}, \bar{y}, \bar{u})$	$\underline{C}\ 0 \cdots 0\ \ 0 \cdots 0\ 0$	$1\ 0 \cdots 0\ \ 0 \cdots 0\ 0$	$0 \cdots 0\ \ 0 \cdots 0\ 0$
	$\underline{C}\ \underline{C} \cdots 0\ \ 0 \cdots 0\ 0$	$1\ 1 \cdots 0\ \ 0 \cdots 0\ 0$	$0 \cdots 0\ \ 0 \cdots 0\ 0$
	$\vdots$	$\vdots$	$\vdots$
	$\underline{C}\ \underline{C} \cdots \underline{C}\ \ 0 \cdots 0\ 0$	$1\ 1 \cdots 1\ \ 0 \cdots 0\ 0$	$0 \cdots 0\ \ 0 \cdots 0\ 0$
	$\underline{C}\ \underline{C} \cdots \underline{C}\ \ \underline{C} \cdots 0\ 0$	$1\ 1 \cdots 1\ \ 1 \cdots 0\ 0$	$0 \cdots 0\ \ 0 \cdots 0\ 0$
	$\vdots$	$\vdots$	$\vdots$
	$\underline{C}\ \underline{C} \cdots \underline{C}\ \ \underline{C} \cdots \underline{C}\ 0$	$1\ 1 \cdots 1\ \ 1 \cdots 1\ 0$	$0 \cdots 0\ \ 0 \cdots 0\ 0$
$(\hat{x}, \hat{y}, \hat{u})$	$\overline{C}\ 0 \cdots 0\ \ 0 \cdots 0\ 0$	$1\ 0 \cdots 0\ \ 0 \cdots 0\ 0$	$0 \cdots 0\ \ 0 \cdots 0\ 0$
	$\overline{C}\ \overline{C} \cdots 0\ \ 0 \cdots 0\ 0$	$1\ 1 \cdots 0\ \ 0 \cdots 0\ 0$	$0 \cdots 0\ \ 0 \cdots 0\ 0$
	$\vdots$	$\vdots$	$\vdots$
	$\overline{C}\ \overline{C} \cdots \overline{C}\ \ 0 \cdots 0\ 0$	$1\ 1 \cdots 1\ \ 0 \cdots 0\ 0$	$0 \cdots 0\ \ 0 \cdots 0\ 0$
	$\overline{C}\ \overline{C} \cdots \overline{C}\ \ \overline{C} \cdots 0\ 0$	$1\ 1 \cdots 1\ \ 1 \cdots 0\ 0$	$0 \cdots 0\ \ 0 \cdots 0\ 0$
	$\vdots$	$\vdots$	$\vdots$
	$\overline{C}\ \overline{C} \cdots \overline{C}\ \ \overline{C} \cdots \overline{C}\ 0$	$1\ 1 \cdots 1\ \ 1 \cdots 1\ 0$	$0 \cdots 0\ \ 0 \cdots 0\ 0$
$(\tilde{x}, \tilde{y}, \tilde{u})$	$0\ \underline{C} \cdots \underline{C}\ \ \underline{C} \cdots \underline{C}\ \underline{C}$	$0\ 1 \cdots 1\ \ 1 \cdots 1\ 1$	$1 \cdots 1\ \ 1 \cdots 1\ 1$
	$\vdots$	$\vdots$	$\vdots$
	$0\ 0 \cdots \underline{C}\ \ \underline{C} \cdots \underline{C}\ \underline{C}$	$0\ 0 \cdots 1\ \ 1 \cdots 1\ 1$	$0 \cdots 1\ \ 1 \cdots 1\ 1$
	$0\ 0 \cdots 0\ \ \underline{C} \cdots \underline{C}\ \underline{C}$	$0\ 0 \cdots 0\ \ 1 \cdots 1\ 1$	$0 \cdots 0\ \ 1 \cdots 1\ 1$
	$\vdots$	$\vdots$	$\vdots$
	$0\ 0 \cdots 0\ \ 0 \cdots \underline{C}\ \underline{C}$	$0\ 0 \cdots 0\ \ 0 \cdots 1\ 1$	$0 \cdots 0\ \ 0 \cdots 1\ 1$
$(\acute{x}, \acute{y}, \acute{u})$	$0\ 0 \cdots 0\ \ 0 \cdots 0\ \underline{C}$	$0\ 0 \cdots 0\ \ 0 \cdots 0\ 1$	$0 \cdots 0\ \ 0 \cdots 0\ 1$
$(\dot{x}, \dot{y}, \dot{u})$	$0\ 0 \cdots 0\ \ 0 \cdots 0\ \overline{C}$	$0\ 0 \cdots 0\ \ 0 \cdots 0\ 1$	$0 \cdots 0\ \ 0 \cdots 0\ 1$

Table 11: $3n+1$ linearly independent points for (3b)

For inequalities (3c) $u_i + y_{i^-} \leq 1, \quad \forall i \in \mathcal{N}:$

These $3n+1$ linearly independent points for (3c) are the same as the linearly independent points for (3b).

For inequalities (3d) $\underline{C} y_i \leq x_i \leq \overline{C} y_i, \quad \forall i \in \mathcal{N} \cup \{i^-\}:$

Here we only prove the left side of (3d), i.e., $x_i \geq \underline{C} y_i$, as the proof for the right side follows the similar way. For the root node i^- indexed as 0, we create $3n+1$ linearly independent points in $\text{conv}(P_2)$ in the following three groups.

(i) For each $r \in [0,n]_{\mathbb{Z}}$ (totally there are $n+1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P_2)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [0,r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [0,r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \in [1,n]_{\mathbb{Z}} \end{array}.$$

(ii) For each $r \in [1,n]_{\mathbb{Z}}$ (totally there are n points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P_2)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [r,n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{y}_s^r = \hat{u}_s^r = \begin{cases} 1, & \forall s \in [r,n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}.$$

(iii) For each $r \in [1,n]_{\mathbb{Z}}$ (totally there are n points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r) \in \text{conv}(P_2)$ such that

$$\tilde{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [r,n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{y}_s^r = \tilde{u}_s^r = \begin{cases} 1, & \forall s \in [r,n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}.$$

We collect these $3n+1$ linearly independent points in Table 12.

For a fixed node $i \in \mathcal{N}$, we create $3n+1$ linearly independent points in $\text{conv}(P_2)$ in the following six groups.

(i) We create a point $(\acute{x}, \acute{y}, \acute{u}) \in \text{conv}(P_2)$ (totally one point) such that

$$\acute{x}_s = \begin{cases} \overline{C}, & s = 0 \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}_s = \begin{cases} 1, & s = 0 \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s = 0, \\ \forall s \in [1,n]_{\mathbb{Z}} \end{array}.$$

(ii) We create a point $(\dot{x}, \dot{y}, \dot{u}) \in \text{conv}(P_2)$ (totally one point) such that

$$\dot{x}_s = \begin{cases} \underline{C}, & s = 0 \\ 0, & \text{o.w.} \end{cases}, \quad \dot{y}_s = \begin{cases} 1, & s = 0 \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \dot{u}_s = 0, \\ \forall s \end{array}.$$

(iii) For each $r \in [0,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are n points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P_2)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [0,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [0,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s = 0, \\ \forall s \end{array}.$$

Group	x	y	u
	$x_{i^-}\ x_1 x_2 \cdots x_n$	$y_{i^-}\ y_1 y_2 \cdots y_n$	$u_1 u_2 \cdots u_n$
	$\underline{C}\ 0\ 0 \cdots 0$	$1\ 0\ 0 \cdots 0$	$0\ 0 \cdots 0$
	$\underline{C}\ \underline{C}\ 0 \cdots 0$	$1\ 1\ 0 \cdots 0$	$0\ 0 \cdots 0$
$(\bar{x}, \bar{y}, \bar{u})$	$\underline{C}\ \underline{C}\ \underline{C} \cdots 0$	$1\ 1\ 1 \cdots 0$	$0\ 0 \cdots 0$
	$\vdots\ \vdots\ \vdots\ \ \vdots$	$\vdots\ \vdots\ \vdots\ \ \vdots$	$\vdots\ \vdots\ \ \vdots$
	$\underline{C}\ \underline{C}\ \underline{C} \cdots \underline{C}$	$1\ 1\ 1 \cdots 1$	$0\ 0 \cdots 0$
	$0\ \underline{C}\ \underline{C} \cdots \underline{C}$	$0\ 1\ 1 \cdots 1$	$1\ 1 \cdots 1$
$(\hat{x}, \hat{y}, \hat{u})$	$0\ 0\ \underline{C} \cdots \underline{C}$	$0\ 0\ 1 \cdots 1$	$0\ 1 \cdots 1$
	$\vdots\ \vdots\ \vdots\ \ \vdots$	$\vdots\ \vdots\ \vdots\ \ \vdots$	$\vdots\ \vdots\ \ \vdots$
	$0\ 0\ 0 \cdots \underline{C}$	$0\ 0\ 0 \cdots 1$	$0\ 0 \cdots 1$
	$0\ \overline{C}\ \overline{C} \cdots \overline{C}$	$0\ 1\ 1 \cdots 1$	$1\ 1 \cdots 1$
$(\tilde{x}, \tilde{y}, \tilde{u})$	$0\ 0\ \overline{C} \cdots \overline{C}$	$0\ 0\ 1 \cdots 1$	$0\ 1 \cdots 1$
	$\vdots\ \vdots\ \vdots\ \ \vdots$	$\vdots\ \vdots\ \vdots\ \ \vdots$	$\vdots\ \vdots\ \ \vdots$
	$0\ 0\ 0 \cdots \overline{C}$	$0\ 0\ 0 \cdots 1$	$0\ 0 \cdots 1$

Table 12: $3n+1$ linearly independent points for $\underline{C}y_{i^-} \leq x_{i^-},\ i \in \mathcal{N}$

(iv) We create a point $(\grave{x}, \grave{y}, \grave{u}) \in \operatorname{conv}(P_2)$ (totally one point) such that

$$\grave{x}_s = \begin{cases} 0, & s = 0 \\ \underline{C}, & \text{o.w.} \end{cases}, \quad \grave{y}_s = \begin{cases} 0, & s = 0 \\ 1, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \grave{u}_s = 1, \\ \forall s \end{array}.$$

(v) For each $r \in [1, n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are $n-1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \operatorname{conv}(P_2)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [r, n]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{y}_s^r = \hat{u}_s^r = \begin{cases} 1, & \forall s \in [r, n]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}.$$

(vi) For each $r \in [1, n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are $n-1$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r) \in \operatorname{conv}(P_2)$ such that

$$\tilde{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [r, n]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{y}_s^r = \tilde{u}_s^r = \begin{cases} 1, & \forall s \in [r, n]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}.$$

We collect these $3n+1$ linearly independent points in Table 13.

For inequalities (4) $x_i - x_{i^-} \leq (\underline{C} + V^+)y_i - \underline{C}y_{i^-} + (\overline{C} - \underline{C} - V^+)u_i, \forall i \in \mathcal{N}$:

We create $3n+1$ linearly independent points in $\operatorname{conv}(P_2)$ in the following five groups.

(i) We create a point $(\acute{x}, \acute{y}, \acute{u}) \in \operatorname{conv}(P_2)$ (totally one point) such that

$$\acute{x}_s = \begin{cases} \underline{C}, & s = 0 \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}_s = \begin{cases} 1, & s = 0 \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s = 0, \\ \forall s \in [1, n]_{\mathbb{Z}} \end{array}.$$

Group	x						y						u				
	x_{i-}	x_i	$x_1\cdots$	x_{i-1}	$x_{i+1}\cdots$	x_n	y_{i-}	y_i	$y_1\cdots$	y_{i-1}	$y_{i+1}\cdots$	y_n	u_i	$u_1\cdots$	u_{i-1}	$u_{i+1}\cdots$	u_n
$(\acute{x},\acute{y},\acute{u})$	$\overline{C}$	0	$0\cdots$	0	0	$\cdots 0$	1	0	$0\cdots$	0	0	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
$(\dot{x},\dot{y},\dot{u})$	$\underline{C}$	0	$0\cdots$	0	0	$\cdots 0$	1	0	$0\cdots$	0	0	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
$(\bar{x},\bar{y},\bar{u})$	$\underline{C}$	$\underline{C}$	$0\cdots$	0	0	$\cdots 0$	1	1	$0\cdots$	0	0	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
	$\underline{C}$	$\underline{C}$	$\underline{C}\cdots$	0	0	$\cdots 0$	1	1	$1\cdots$	0	0	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\underline{C}$	$\underline{C}$	$\underline{C}\cdots$	$\underline{C}$	0	$\cdots 0$	1	1	$1\cdots$	1	0	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
	$\underline{C}$	$\underline{C}$	$\underline{C}\cdots$	$\underline{C}$	$\underline{C}$	$\cdots 0$	1	1	$1\cdots$	1	1	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\underline{C}$	$\underline{C}$	$\underline{C}\cdots$	$\underline{C}$	$\underline{C}$	$\cdots \underline{C}$	1	1	$1\cdots$	1	1	$\cdots 1$	0	$0\cdots$	0	0	$\cdots 0$
$(\grave{x},\grave{y},\grave{u})$	0	$\underline{C}$	$\underline{C}\cdots$	$\underline{C}$	$\underline{C}$	$\cdots \underline{C}$	0	1	$1\cdots$	1	1	$\cdots 1$	1	$1\cdots$	1	1	$\cdots 1$
$(\hat{x},\hat{y},\hat{u})$	0	0	$\underline{C}\cdots$	$\underline{C}$	$\underline{C}$	$\cdots \underline{C}$	0	0	$1\cdots$	1	1	$\cdots 1$	0	$1\cdots$	1	1	$\cdots 1$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	0	0	$0\cdots$	$\underline{C}$	$\underline{C}$	$\cdots \underline{C}$	0	0	$0\cdots$	1	1	$\cdots 1$	0	$0\cdots$	1	1	$\cdots 1$
	0	0	$0\cdots$	0	$\underline{C}$	$\cdots \underline{C}$	0	0	$0\cdots$	0	1	$\cdots 1$	0	$0\cdots$	0	1	$\cdots 1$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	0	0	$0\cdots$	0	0	$\cdots \underline{C}$	0	0	$0\cdots$	0	0	$\cdots 1$	0	$0\cdots$	0	0	$\cdots 1$
$(\tilde{x},\tilde{y},\tilde{u})$	0	0	$\overline{C}\cdots$	$\overline{C}$	$\overline{C}$	$\cdots \overline{C}$	0	0	$1\cdots$	1	1	$\cdots 1$	0	$1\cdots$	1	1	$\cdots 1$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	0	0	$0\cdots$	$\overline{C}$	$\overline{C}$	$\cdots \overline{C}$	0	0	$0\cdots$	1	1	$\cdots 1$	0	$0\cdots$	1	1	$\cdots 1$
	0	0	$0\cdots$	0	$\overline{C}$	$\cdots \overline{C}$	0	0	$0\cdots$	0	1	$\cdots 1$	0	$0\cdots$	0	1	$\cdots 1$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	0	0	$0\cdots$	0	0	$\cdots \overline{C}$	0	0	$0\cdots$	0	0	$\cdots 1$	0	$0\cdots$	0	0	$\cdots 1$

Table 13: $3n+1$ linearly independent points for $\underline{C}y_i \le x_i, i \in \mathcal{N}$

(ii) For each $r \in [0,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are n points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \mathrm{conv}(P_2)$ such that

$$\bar{x}^r_s = \begin{cases} \underline{C}, & \forall s \in [0,r]_{\mathbb{Z}} \setminus \{i\} \\ \underline{C} + V^+, & s = i \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}^r_s = \begin{cases} 1, & \forall s \in [0,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}^r_s = 0, \\ \forall s \end{array}.$$

(iii) For each $r \in [0,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are n points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \mathrm{conv}(P_2)$ such that

$$\hat{x}^r_s = \begin{cases} \overline{C} - V^+, & s = 0 \\ \overline{C}, & \forall s \in [1,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}^r_s = \begin{cases} 1, & \forall s \in [0,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \hat{u}^r_s = 0, \\ \forall s \end{array}.$$

(iv) We create a point $(\dot{x}, \dot{y}, \dot{u}) \in \mathrm{conv}(P_2)$ (totally one point) such that

$$\dot{x}_s = \begin{cases} 0, & s = 0 \\ \overline{C}, & \text{o.w.} \end{cases}, \quad \dot{y}_s = \begin{cases} 0, & s = 0 \\ 1, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \dot{u}_s = 1, \\ \forall s \end{array}.$$

(v) For each $r \in [1, n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are $n-1$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r) \in \text{conv}(P_2)$ such that

$$\tilde{x}^r_s = \begin{cases} \overline{C}, & s \in [r, n]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{y}^r_s = \tilde{u}^r_s = \begin{cases} 1, & s \in [r, n]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}.$$

We collect these $3n+1$ linearly independent points in Table 14.

Group	x						y						u				
	x_{i^-}	x_i	$x_1\cdots$	x_{i-1}	x_{i+1}	$\cdots x_n$	y_{i^-}	y_i	$y_1\cdots$	y_{i-1}	y_{i+1}	$\cdots y_n$	u_i	$u_1\cdots$	u_{i-1}	u_{i+1}	$\cdots u_n$
$(\acute{x}, \acute{y}, \acute{u})$	$\underline{C}$	0	$0\cdots$	0	0	$\cdots 0$	1	0	$0\cdots$	0	0	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
$(\bar{x}, \bar{y}, \bar{u})$	$\underline{C}$	$\underline{C}+V^+$	$0\cdots$	0	0	$\cdots 0$	1	1	$0\cdots$	0	0	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
	$\underline{C}$	$\underline{C}+V^+$	$\underline{C}\cdots$	0	0	$\cdots 0$	1	1	$1\cdots$	0	0	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\underline{C}$	$\underline{C}+V^+$	$\underline{C}\cdots$	$\underline{C}$	0	$\cdots 0$	1	1	$1\cdots$	1	0	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
	$\underline{C}$	$\underline{C}+V^+$	$\underline{C}\cdots$	$\underline{C}$	$\underline{C}$	$\cdots 0$	1	1	$1\cdots$	1	1	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\underline{C}$	$\underline{C}+V^+$	$\underline{C}\cdots$	$\underline{C}$	$\underline{C}$	$\cdots \underline{C}$	1	1	$1\cdots$	1	1	$\cdots 1$	0	$0\cdots$	0	0	$\cdots 0$
$(\hat{x}, \hat{y}, \hat{u})$	$\overline{C}-V^+$	$\overline{C}$	$0\cdots$	0	0	$\cdots 0$	1	1	$0\cdots$	0	0	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
	$\overline{C}-V^+$	$\overline{C}$	$\overline{C}\cdots$	0	0	$\cdots 0$	1	1	$1\cdots$	0	0	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\overline{C}-V^+$	$\overline{C}$	$\overline{C}\cdots$	$\overline{C}$	0	$\cdots 0$	1	1	$1\cdots$	1	0	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
	$\overline{C}-V^+$	$\overline{C}$	$\overline{C}\cdots$	$\overline{C}$	$\overline{C}$	$\cdots 0$	1	1	$1\cdots$	1	1	$\cdots 0$	0	$0\cdots$	0	0	$\cdots 0$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\overline{C}-V^+$	$\overline{C}$	$\overline{C}\cdots$	$\overline{C}$	$\overline{C}$	$\cdots \overline{C}$	1	1	$1\cdots$	1	1	$\cdots 1$	0	$0\cdots$	0	0	$\cdots 0$
$(\dot{x}, \dot{y}, \dot{u})$	0	$\overline{C}$	$\overline{C}\cdots$	$\overline{C}$	$\overline{C}$	$\cdots \overline{C}$	0	1	$1\cdots$	1	1	$\cdots 1$	1	$1\cdots$	1	1	$\cdots 1$
$(\tilde{x}, \tilde{y}, \tilde{u})$	0	0	$\overline{C}\cdots$	$\overline{C}$	$\overline{C}$	$\cdots \overline{C}$	0	0	$1\cdots$	1	1	$\cdots 1$	0	$1\cdots$	1	1	$\cdots 1$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	0	0	$0\cdots$	$\overline{C}$	$\overline{C}$	$\cdots \overline{C}$	0	0	$0\cdots$	1	1	$\cdots 1$	0	$0\cdots$	1	1	$\cdots 1$
	0	0	$0\cdots$	0	$\overline{C}$	$\cdots \overline{C}$	0	0	$0\cdots$	0	1	$\cdots 1$	0	$0\cdots$	0	1	$\cdots 1$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	0	0	$0\cdots$	0	0	$\cdots \overline{C}$	0	0	$0\cdots$	0	0	$\cdots 1$	0	$0\cdots$	0	0	$\cdots 1$

Table 14: $3n+1$ linearly independent points for (4)

For inequalities (5) $x_{i^-} - x_i \leq \overline{C} y_{i^-} - (\overline{C} - V^-) y_i + (\overline{C} - \underline{C} - V^-) u_i, \forall i \in \mathcal{N}$:

We create $3n+1$ linearly independent points in conv(P_2) in the following five groups.

(i) We create a point $(\acute{x}, \acute{y}, \acute{u}) \in \text{conv}(P_2)$ (totally one point) such that

$$\acute{x}_s = \begin{cases} \overline{C}, & s = 0 \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}_s = \begin{cases} 1, & s = 0 \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s = 0, \\ \forall s \in [1, n]_{\mathbb{Z}} \end{array}.$$

(ii) For each $r \in [0,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are n points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P_2)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [0,r]_{\mathbb{Z}} \setminus \{i\} \\ \overline{C} - V^-, & s = i \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [0,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(iii) For each $r \in [0,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are n points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P_2)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C} + V^-, & s = 0 \\ \underline{C}, & \forall s \in [1,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [0,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

(iv) We create a point $(\dot{x}, \dot{y}, \dot{u}) \in \text{conv}(P_2)$ (totally one point) such that

$$\dot{x}_s = \begin{cases} 0, & s = 0 \\ \underline{C}, & \text{o.w.} \end{cases}, \quad \dot{y}_s = \begin{cases} 0, & s = 0 \\ 1, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \dot{u}_s = 1, \\ \forall s \end{array}.$$

(v) For each $r \in [1,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are $n-1$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r) \in \text{conv}(P_2)$ such that

$$\tilde{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [r,n]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{y}_s^r = \tilde{u}_s^r = \begin{cases} 1, & \forall s \in [r,n]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}.$$

We collect these $3n+1$ linearly independent points in Table 15.

For inequalities (6) $x_i - x_j \leq (\underline{C} + V^+ + V^-)y_i - (\overline{C} - V^+ - V^-)y_j + (\overline{C} - \underline{C} - V^+ - V^-)(y_{i^-} + u_i + u_j), \forall i, j \in \mathcal{N}, j \neq i:$

Without loss of generality, we let $i < j$ and create $3n+1$ linearly independent points in $\text{conv}(P_2)$ in the following eight groups.

(i) We create a point $(\dot{x}, \dot{y}, \dot{u}) \in \text{conv}(P_2)$ (totally one point) such that

$$\dot{x}_s = \begin{cases} \overline{C}, & \forall s \in \{0,i\} \\ 0, & \text{o.w.} \end{cases}, \quad \dot{y}_s = \begin{cases} 1, & \forall s \in \{0,i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \dot{u}_s = 0, \\ \forall s \in [1,n]_{\mathbb{Z}} \end{array}.$$

(ii) We create a point $(\grave{x}, \grave{y}, \grave{u}) \in \text{conv}(P_2)$ (totally one point) such that

$$\grave{x}_s = \begin{cases} \overline{C} - V^+, & \forall s \in \{0,i\} \\ 0, & \text{o.w.} \end{cases}, \quad \grave{y}_s = \begin{cases} 1, & \forall s \in \{0,i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \grave{u}_s = 0, \\ \forall s \end{array}.$$

(iii) We create a point $(\acute{x}, \acute{y}, \acute{u}) \in \text{conv}(P_2)$ (totally one point) such that

$$\acute{x}_s = \begin{cases} \underline{C}, & \forall s \in \{0,j\} \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}_s = \begin{cases} 1, & \forall s \in \{0,j\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s = 0, \\ \forall s \end{array}.$$

Group	x_{i-}	x_i	$x_1 \cdots x_{i-1}$	$x_{i+1} \cdots x_n$	y_{i-}	y_i	$y_1 \cdots y_{i-1}$	$y_{i+1} \cdots y_n$	u_i	$u_1 \cdots u_{i-1}$	$u_{i+1} \cdots u_n$
$(\acute{x}, \acute{y}, \acute{u})$	$\overline{C}$	0	$0 \cdots 0$	$0 \cdots 0$	1	0	$0 \cdots 0$	$0 \cdots 0$	0	$0 \cdots 0$	$0 \cdots 0$
$(\bar{x}, \bar{y}, \bar{u})$	$\overline{C}$	$\overline{C} - V^-$	$0 \cdots 0$	$0 \cdots 0$	1	1	$0 \cdots 0$	$0 \cdots 0$	0	$0 \cdots 0$	$0 \cdots 0$
	$\overline{C}$	$\overline{C} - V^-$	$\overline{C} \cdots 0$	$0 \cdots 0$	1	1	$1 \cdots 0$	$0 \cdots 0$	0	$0 \cdots 0$	$0 \cdots 0$
	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$
	$\overline{C}$	$\overline{C} - V^-$	$\overline{C} \cdots \overline{C}$	$0 \cdots 0$	1	1	$1 \cdots 1$	$0 \cdots 0$	0	$0 \cdots 0$	$0 \cdots 0$
	$\overline{C}$	$\overline{C} - V^-$	$\overline{C} \cdots \overline{C}$	$\overline{C} \cdots 0$	1	1	$1 \cdots 1$	$1 \cdots 0$	0	$0 \cdots 0$	$0 \cdots 0$
	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$
	$\overline{C}$	$\overline{C} - V^-$	$\overline{C} \cdots \overline{C}$	$\overline{C} \cdots \overline{C}$	1	1	$1 \cdots 1$	$1 \cdots 1$	0	$0 \cdots 0$	$0 \cdots 0$
$(\hat{x}, \hat{y}, \hat{u})$	$\underline{C} + V^-$	$\underline{C}$	$0 \cdots 0$	$0 \cdots 0$	1	1	$0 \cdots 0$	$0 \cdots 0$	0	$0 \cdots 0$	$0 \cdots 0$
	$\underline{C} + V^-$	$\underline{C}$	$\underline{C} \cdots 0$	$0 \cdots 0$	1	1	$1 \cdots 0$	$0 \cdots 0$	0	$0 \cdots 0$	$0 \cdots 0$
	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$
	$\underline{C} + V^-$	$\underline{C}$	$\underline{C} \cdots \underline{C}$	$0 \cdots 0$	1	1	$1 \cdots 1$	$0 \cdots 0$	0	$0 \cdots 0$	$0 \cdots 0$
	$\underline{C} + V^-$	$\underline{C}$	$\underline{C} \cdots \underline{C}$	$\underline{C} \cdots 0$	1	1	$1 \cdots 1$	$1 \cdots 0$	0	$0 \cdots 0$	$0 \cdots 0$
	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$
	$\underline{C} + V^-$	$\underline{C}$	$\underline{C} \cdots \underline{C}$	$\underline{C} \cdots \underline{C}$	1	1	$1 \cdots 1$	$1 \cdots 1$	0	$0 \cdots 0$	$0 \cdots 0$
$(\dot{x}, \dot{y}, \dot{u})$	0	$\underline{C}$	$\underline{C} \cdots \underline{C}$	$\underline{C} \cdots \underline{C}$	0	1	$1 \cdots 1$	$1 \cdots 1$	1	$1 \cdots 1$	$1 \cdots 1$
$(\tilde{x}, \tilde{y}, \tilde{u})$	0	0	$\underline{C} \cdots \underline{C}$	$\underline{C} \cdots \underline{C}$	0	0	$1 \cdots 1$	$1 \cdots 1$	0	$1 \cdots 1$	$1 \cdots 1$
	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$
	0	0	$0 \cdots \underline{C}$	$\underline{C} \cdots \underline{C}$	0	0	$0 \cdots 1$	$1 \cdots 1$	0	$0 \cdots 1$	$1 \cdots 1$
	0	0	$0 \cdots 0$	$\underline{C} \cdots \underline{C}$	0	0	$0 \cdots 0$	$1 \cdots 1$	0	$0 \cdots 0$	$1 \cdots 1$
	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$
	0	0	$0 \cdots 0$	$0 \cdots \underline{C}$	0	0	$0 \cdots 0$	$0 \cdots 1$	0	$0 \cdots 0$	$0 \cdots 1$

Table 15: $3n + 1$ linearly independent points for (5)

(iv) For each $r \in [0, n]_{\mathbb{Z}} \setminus \{i, j\}$ (totally there are $n - 1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \operatorname{conv}(P_2)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [0, r]_{\mathbb{Z}} \cup \{j\} \setminus \{i\} \\ \underline{C} + V^+ + V^-, & s = i \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [0, r]_{\mathbb{Z}} \cup \{i, j\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(v) For each $r \in [0, n]_{\mathbb{Z}} \setminus \{i, j\}$ (totally there are $n - 1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \operatorname{conv}(P_2)$ such that

$$\hat{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [0, r]_{\mathbb{Z}} \cup \{i\} \setminus \{j\} \\ \overline{C} - V^+ - V^-, & s = j \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [0, r]_{\mathbb{Z}} \cup \{i, j\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

(vi) We create a point $(\ddot{x}, \ddot{y}, \ddot{u}) \in \operatorname{conv}(P_2)$ (totally one point) such that

$$\ddot{x}_s = \begin{cases} 0, & s = 0 \\ \overline{C}, & s = i \\ \underline{C}, & \text{o.w.} \end{cases}, \quad \ddot{y}_s = \begin{cases} 0, & s = 0 \\ 1, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \ddot{u}_s = 1, \\ \forall s \end{array}.$$

(vii) We create a point $(\breve{x}, \breve{y}, \breve{u}) \in \text{conv}(P_2)$ (totally one point) such that

$$\breve{x}_s = \begin{cases} 0, & \forall s \in \{0, i\} \\ \underline{C}, & \text{o.w.} \end{cases}, \quad \breve{y}_s = \begin{cases} 0, & \forall s \in \{0, i\} \\ 1, & \text{o.w.} \end{cases}, \text{ and } \breve{u}_s = \begin{cases} 0, & s = i \\ 1, & \text{o.w.} \end{cases}.$$

(viii) For each $r \in [1, n]_{\mathbb{Z}} \setminus \{i, j\}$ (totally there are $n - 2$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r) \in \text{conv}(P_2)$ such that

$$\tilde{x}^r_s = \begin{cases} \underline{C}, & \forall s \in [r, n]_{\mathbb{Z}} \setminus \{i, j\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{y}^r_s = \tilde{u}^r_s = \begin{cases} 1, & \forall s \in [r, n]_{\mathbb{Z}} \setminus \{i, j\} \\ 0, & \text{o.w.} \end{cases}.$$

We collect these $3n + 1$ linearly independent points in Table 16.

Group	x				y	u
	x_{i-}	x_i	x_j	$x_1 x_2 \cdots x_n$	$y_{i-}\, y_i y_j y_1 y_2 \cdots y_n$	$u_i u_j u_1 u_2 \cdots u_n$
$(\dot{x}, \dot{y}, \dot{u})$	$\overline{C}$	$\overline{C}$	0	$0\ 0 \cdots 0$	$1\ \ 1\ 0\ 0\ 0 \cdots 0$	$0\ 0\ 0\ 0 \cdots 0$
$(\grave{x}, \grave{y}, \grave{u})$	$\overline{C} - V^+$	$\overline{C}$	0	$0\ 0 \cdots 0$	$1\ \ 1\ 0\ 0\ 0 \cdots 0$	$0\ 0\ 0\ 0 \cdots 0$
$(\acute{x}, \acute{y}, \acute{u})$	$\underline{C}$	0	$\underline{C}$	$0\ 0 \cdots 0$	$1\ \ 0\ 1\ 0\ 0 \cdots 0$	$0\ 0\ 0\ 0 \cdots 0$
$(\bar{x}, \bar{y}, \bar{u})$	$\underline{C}$	$\underline{C} + V^+ + V^-$	$\underline{C}$	$0\ 0 \cdots 0$	$1\ \ 1\ 1\ 0\ 0 \cdots 0$	$0\ 0\ 0\ 0 \cdots 0$
	$\underline{C}$	$\underline{C} + V^+ + V^-$	$\underline{C}$	$\underline{C}\ 0 \cdots 0$	$1\ \ 1\ 1\ 1\ 0 \cdots 0$	$0\ 0\ 0\ 0 \cdots 0$
	$\underline{C}$	$\underline{C} + V^+ + V^-$	$\underline{C}$	$\underline{C}\ \underline{C} \cdots 0$	$1\ \ 1\ 1\ 1\ 1 \cdots 0$	$0\ 0\ 0\ 0 \cdots 0$
	$\vdots$	$\vdots$	$\vdots$	$\vdots\ \vdots\ \ \vdots$	$\vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ \vdots$	$\vdots\ \vdots\ \vdots\ \vdots\ \ \vdots$
	$\underline{C}$	$\underline{C} + V^+ + V^-$	$\underline{C}$	$\underline{C}\ \underline{C} \cdots \underline{C}$	$1\ \ 1\ 1\ 1\ 1 \cdots 1$	$0\ 0\ 0\ 0 \cdots 0$
$(\hat{x}, \hat{y}, \hat{u})$	$\overline{C}$	$\overline{C}$	$\overline{C} - V^+ - V^-$	$0\ 0 \cdots 0$	$1\ \ 1\ 1\ 0\ 0 \cdots 0$	$0\ 0\ 0\ 0 \cdots 0$
	$\overline{C}$	$\overline{C}$	$\overline{C} - V^+ - V^-$	$\overline{C}\ 0 \cdots 0$	$1\ \ 1\ 1\ 1\ 0 \cdots 0$	$0\ 0\ 0\ 0 \cdots 0$
	$\overline{C}$	$\overline{C}$	$\overline{C} - V^+ - V^-$	$\overline{C}\ \overline{C} \cdots 0$	$1\ \ 1\ 1\ 1\ 1 \cdots 0$	$0\ 0\ 0\ 0 \cdots 0$
	$\vdots$	$\vdots$	$\vdots$	$\vdots\ \vdots\ \ \vdots$	$\vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ \vdots$	$\vdots\ \vdots\ \vdots\ \vdots\ \ \vdots$
	$\overline{C}$	$\overline{C}$	$\overline{C} - V^+ - V^-$	$\overline{C}\ \overline{C} \cdots \overline{C}$	$1\ \ 1\ 1\ 1\ 1 \cdots 1$	$0\ 0\ 0\ 0 \cdots 0$
$(\ddot{x}, \ddot{y}, \ddot{u})$	0	$\overline{C}$	$\underline{C}$	$\underline{C}\ \underline{C} \cdots \underline{C}$	$0\ \ 1\ 1\ 1\ 1 \cdots 1$	$1\ 1\ 1\ 1 \cdots 1$
$(\breve{x}, \breve{y}, \breve{u})$	0	0	$\underline{C}$	$\underline{C}\ \underline{C} \cdots \underline{C}$	$0\ \ 0\ 1\ 1\ 1 \cdots 1$	$0\ 1\ 1\ 1 \cdots 1$
$(\tilde{x}, \tilde{y}, \tilde{u})$	0	0	0	$\underline{C}\ \underline{C} \cdots \underline{C}$	$0\ \ 0\ 0\ 1\ 1 \cdots 1$	$0\ 0\ 1\ 1 \cdots 1$
	0	0	0	$0\ \underline{C} \cdots \underline{C}$	$0\ \ 0\ 0\ 0\ 1 \cdots 1$	$0\ 0\ 0\ 1 \cdots 1$
	$\vdots$	$\vdots$	$\vdots$	$\vdots\ \vdots\ \ \vdots$	$\vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ \vdots$	$\vdots\ \vdots\ \vdots\ \vdots\ \ \vdots$
	0	0	0	$0\ 0 \cdots \underline{C}$	$0\ \ 0\ 0\ 0\ 0 \cdots 1$	$0\ 0\ 0\ 0 \cdots 1$

Table 16: $3n + 1$ linearly independent points for (6)

For inequalities (7) $u_i \geq 0, \quad \forall i \in \mathcal{N}$:

We create $3n + 1$ linearly independent points in $\text{conv}(P_2)$ in the following five groups.

(i) We create a point $(\acute{x}, \acute{y}, \acute{u}) \in \text{conv}(P_2)$ (totally one point) such that

$$\acute{x}_s = \begin{cases} \underline{C}, & s = 0 \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}_s = \begin{cases} 1, & s = 0 \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \acute{u}^r_s = 0, \\ \forall s \in [1, n]_{\mathbb{Z}} \end{array}.$$

(ii) For each $r \in [0,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are n points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \mathrm{conv}(P_2)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [0,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [0,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(iii) We create a point $(\dot{x}, \dot{y}, \dot{u}) \in \mathrm{conv}(P_2)$ (totally one point) such that

$$\dot{x}_s = \begin{cases} \overline{C}, & s = 0 \\ 0, & \text{o.w.} \end{cases}, \quad \dot{y}_s = \begin{cases} 1, & s = 0 \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \dot{u}_s^r = 0, \\ \forall s \end{array}.$$

(iv) For each $r \in [0,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are n points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \mathrm{conv}(P_2)$ such that

$$\hat{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [0,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [0,r]_{\mathbb{Z}} \cup \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

(v) For each $r \in [1,n]_{\mathbb{Z}} \setminus \{i\}$ (totally there are $n-1$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r) \in \mathrm{conv}(P_2)$ such that

$$\tilde{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [r,n]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{y}_s^r = \tilde{u}_s^r = \begin{cases} 1, & \forall s \in [r,n]_{\mathbb{Z}} \setminus \{i\} \\ 0, & \text{o.w.} \end{cases}.$$

We collect these $3n+1$ linearly independent points in Table 17.

In summary, all the inequalities in Q_2 are facet-defining for $\mathrm{conv}(P_2)$. ■

A.4 Proof for Proposition 4

Proof: Considering the fact that (3a) - (3d) are duplicated in P_2 and Q_2, we only need to consider the dominance relationships between ((3e), (3f)) in P_2 and ((4), (5), (6), (7)) in Q_2.

(i) Inequality (3e) for each $i \in \mathcal{N}$ is dominated by a linear combination of (3a), (3c), and (4) since RHS (the right hand side) of (4) - RHS of (3e) $= \Big((\overline{C} - V^+)y_{i^-} - \overline{C}\Big) - \Big(\underline{C}y_{i^-} - (\underline{C} + V^+)y_i - (\overline{C} - \underline{C} - V^+)u_i\Big) = \overline{C}(u_i + y_{i^-} - 1) + (\underline{C} + V^+)(y_i - y_{i^-} - u_i) \leq 0$, where the last inequality follows from (3a) and (3c).

(ii) Inequality (3f) for each $i \in \mathcal{N}$ is dominated by a linear combination of (3c), (7), and (5) since RHS of (5) - RHS of (3f) $= \Big(\overline{C}y_{i^-} - (\overline{C} - V^-)y_i + (\overline{C} - \underline{C} - V^-)u_i\Big) - \Big(-(\overline{C} - V^-)y_i + \overline{C}\Big)$ $= \overline{C}(y_{i^-} + u_i - 1) - (\underline{C} + V^-)u_i \leq 0$, where the last inequality follows from (3c) and the non-negativity of u.

Thus each inequality in P_2 is dominated by one inequality or a combination of inequalities in Q_2 and the proposition holds. ■

Group	x						y						u				
	x_{i-}	x_i	$x_1 \cdots$	x_{i-1}	x_{i+1}	$\cdots x_n$	y_{i-}	y_i	$y_1 \cdots$	y_{i-1}	y_{i+1}	$\cdots y_n$	u_i	$u_1 \cdots$	u_{i-1}	u_{i+1}	$\cdots u_n$
$(\acute{x},\acute{y},\acute{u})$	$\underline{C}$	0	$0 \cdots$	0	0	$\cdots 0$	1	0	$0 \cdots$	0	0	$\cdots 0$	0	$0 \cdots$	0	0	$\cdots 0$
$(\bar{x},\bar{y},\bar{u})$	$\underline{C}$	$\underline{C}$	$0 \cdots$	0	0	$\cdots 0$	1	1	$0 \cdots$	0	0	$\cdots 0$	0	$0 \cdots$	0	0	$\cdots 0$
	$\underline{C}$	$\underline{C}$	$\underline{C} \cdots$	0	0	$\cdots 0$	1	1	$1 \cdots$	0	0	$\cdots 0$	0	$0 \cdots$	0	0	$\cdots 0$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\underline{C}$	$\underline{C}$	$\underline{C} \cdots$	$\underline{C}$	0	$\cdots 0$	1	1	$1 \cdots$	1	0	$\cdots 0$	0	$0 \cdots$	0	0	$\cdots 0$
	$\underline{C}$	$\underline{C}$	$\underline{C} \cdots$	$\underline{C}$	$\underline{C}$	$\cdots 0$	1	1	$1 \cdots$	1	1	$\cdots 0$	0	$0 \cdots$	0	0	$\cdots 0$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\underline{C}$	$\underline{C}$	$\underline{C} \cdots$	$\underline{C}$	$\underline{C}$	$\cdots \underline{C}$	1	1	$1 \cdots$	1	1	$\cdots 1$	0	$0 \cdots$	0	0	$\cdots 0$
$(\dot{x},\dot{y},\dot{u})$	$\overline{C}$	0	$0 \cdots$	0	0	$\cdots 0$	1	0	$0 \cdots$	0	0	$\cdots 0$	0	$0 \cdots$	0	0	$\cdots 0$
$(\hat{x},\hat{y},\hat{u})$	$\overline{C}$	$\overline{C}$	$0 \cdots$	0	0	$\cdots 0$	1	1	$0 \cdots$	0	0	$\cdots 0$	0	$0 \cdots$	0	0	$\cdots 0$
	$\overline{C}$	$\overline{C}$	$\overline{C} \cdots$	0	0	$\cdots 0$	1	1	$1 \cdots$	0	0	$\cdots 0$	0	$0 \cdots$	0	0	$\cdots 0$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\overline{C}$	$\overline{C}$	$\overline{C} \cdots$	$\overline{C}$	0	$\cdots 0$	1	1	$1 \cdots$	1	0	$\cdots 0$	0	$0 \cdots$	0	0	$\cdots 0$
	$\overline{C}$	$\overline{C}$	$\overline{C} \cdots$	$\overline{C}$	$\overline{C}$	$\cdots 0$	1	1	$1 \cdots$	1	1	$\cdots 0$	0	$0 \cdots$	0	0	$\cdots 0$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\overline{C}$	$\overline{C}$	$\overline{C} \cdots$	$\overline{C}$	$\overline{C}$	$\cdots \overline{C}$	1	1	$1 \cdots$	1	1	$\cdots 1$	0	$0 \cdots$	0	0	$\cdots 0$
$(\tilde{x},\tilde{y},\tilde{u})$	0	0	$\underline{C} \cdots$	$\underline{C}$	$\underline{C}$	$\cdots \underline{C}$	0	0	$1 \cdots$	1	1	$\cdots 1$	0	$1 \cdots$	1	1	$\cdots 1$
	0	0	$0 \cdots$	$\underline{C}$	$\underline{C}$	$\cdots \underline{C}$	0	0	$0 \cdots$	1	1	$\cdots 1$	0	$0 \cdots$	1	1	$\cdots 1$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	0	0	$0 \cdots$	0	$\underline{C}$	$\cdots \underline{C}$	0	0	$0 \cdots$	0	1	$\cdots 1$	0	$0 \cdots$	0	1	$\cdots 1$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	0	0	$0 \cdots$	0	0	$\cdots \underline{C}$	0	0	$0 \cdots$	0	0	$\cdots 1$	0	$0 \cdots$	0	0	$\cdots 1$

Table 17: $3n+1$ linearly independent points for (7)

A.5 Proof for Lemma 1

Proof: Let $A^+ = \{i \in [1,n]_{\mathbb{Z}} : a_i \geq 0\}$, $A^- = \{i \in [1,n]_{\mathbb{Z}} : a_i < 0\}$. Based on the structure of Figure 2, we discuss two different cases based on the unit commitment status ("online" or "offline") at the root node.

1) The generator is offline at the root node, i.e., $x_0 = y_0 = 0$. For this case, we further discuss the following two situations based on if $i \in A^+$ or $i \in A^-$ for each $i \in [1,n]_{\mathbb{Z}}$:

    (i) If $i \in A^+$, to maximize the objective function (8), the generator at node i should be scheduled online at its maximum generation amount $\overline{C}$ following constraints (3d) and (3e) if $a_i\overline{C}+b_i+c_i \geq 0$ or offline otherwise. It follows that $(x_i, y_i, u_i) = (\overline{C}, 1, 1)$ if the generator is online at node i or $(x_i, y_i, u_i) = (0, 0, 0)$ otherwise.

(ii) If $i \in A^-$, to maximize the objective function (8), the generator at node i should be scheduled online at its minimum generation amount $\underline{C}$ if $a_i\underline{C} + b_i + c_i \geq 0$ or offline otherwise. It follows that $(x_i, y_i, u_i) = (\underline{C}, 1, 1)$ if the generator is online at node i or $(x_i, y_i, u_i) = (0, 0, 0)$ otherwise.

From the above (i) and (ii), we verified Claim (1).

2) The generator is scheduled online at the root node, i.e., $y_0 = 1$. It follows that $u_i = 0$ for all $i = 1, \cdots, n$. For notation brevity, we let $\bar{A}^+(x_0) = \{i \in [1, n]_{\mathbb{Z}} : a_i \geq 0, \quad a_i \min\{\overline{C}, x_0 + V^+\} + b_i \geq 0\}$ and $\bar{A}^-(x_0) = \{i \in [1, n]_{\mathbb{Z}} : a_i < 0, \quad a_i \max\{\underline{C}, x_0 - V^-\} + b_i \geq 0\}$. Similar to 1), we further discuss the following two situations based on if $i \in A^+$ or $i \in A^-$ for each $i \in [1, n]_{\mathbb{Z}}$:

(i) If $i \in A^+$, to maximize the objective function (8), the generator at node i should be scheduled online at $\min\{\overline{C}, x_0 + V^+\}$ following constraints (3d) and (3e) if $i \in \bar{A}^+(x_0)$ or offline otherwise. It follows that $(x_i, y_i, u_i) = (\min\{\overline{C}, x_0 + V^+\}, 1, 0)$ if the generator is online at node i or $(x_i, y_i, u_i) = (0, 0, 0)$ otherwise.

(ii) If $i \in A^-$, to maximize the objective function (8), the generator at node i should be scheduled online at $\max\{\underline{C}, x_0 - V^-\}$ following constraints (3d) and (3f) if $i \in \bar{A}^-(x_0)$ or offline otherwise. It follows that $(x_i, y_i, u_i) = (\max\{\underline{C}, x_0 - V^-\}, 1, 0)$ if the generator is online at node i or $(x_i, y_i, u_i) = (0, 0, 0)$ otherwise.

Based on the above (i) and (ii), we can write the optimal objective value of (8) for a given set of $(a_0, b_0, a_i, b_i, c_i), i = 1, \cdots, n$, as a function of x_0. Denote it as $g(x_0) = (a_0x_0 + b_0) + \sum_{i \in \bar{A}^+(x_0)}(a_i \min\{\overline{C}, x_0 + V^+\} + b_i) + \sum_{i \in \bar{A}^-(x_0)}(a_i \max\{\underline{C}, x_0 - V^-\} + b_i)$, which is a continuous function with respect to x_0 on $[\underline{C}, \overline{C}]$. Thus, $z^* = \max\{g(x_0) : \underline{C} \leq x_0 \leq \overline{C}\}$.

To obtain an explicit formula of z^*, we continue considering the following three situations:

(1) If $\underline{C} \leq x_0 \leq \underline{C} + V^-$, it follows that $\min\{\overline{C}, x_0 + V^+\} = x_0 + V^+$ and $\max\{\underline{C}, x_0 - V^-\} = \underline{C}$. Then we have $g(x_0) = (a_0x_0 + b_0) + \sum_{i \in A^+}[a_i(x_0 + V^+) + b_i]^+ + \sum_{i \in A^-}[a_i\underline{C} + b_i]^+$, where we define $[t]^+ = \max\{0, t\}$ for $\forall t \in \mathbb{R}$. Thus, $g(x_0)$ is a convex function with respect to x_0 on $[\underline{C}, \underline{C} + V^-]$. It follows that the optimal solutions happen at the points where $x_0 = \underline{C}$ or $\underline{C} + V^-$. Now we discuss these two scenarios as follows.

- When $x_0 = \underline{C}$, $x_i, i = 1, \cdots, n$, can be obtained based on (i) and (ii) right above. Thus, Claim (2) is verified.

- When $x_0 = \underline{C} + V^-$, there exists at least one x_k for some $k \in \{1, \cdots, n\}$ such that $x_k = \underline{C}$. This can be proven by contradiction argument. If no such x_k exists, then x_i can only be either 0 or $\underline{C} + V^+ + V^-$ based on the calculation (i) and (ii) right above. Without loss of generality, we let $x_i = \underline{C} + V^+ + V^-$ for each $i \in \mathcal{N}_1 \subseteq \mathcal{N}$ and $x_i = 0$ for each $i \in \mathcal{N} \setminus \mathcal{N}_1$. It is easy to observe that this solution (denoted as $(x; y; u)$) can be written as a linear combination of two solutions $(\hat{x}; \hat{y}; \hat{u})$ and $(\tilde{x}; \tilde{y}; \tilde{u})$, i.e., $(x; y; u) = \frac{1}{2}(\hat{x}; \hat{y}; \hat{u}) + \frac{1}{2}(\tilde{x}; \tilde{y}; \tilde{u})$, where $\hat{y} = \tilde{y} = y, \hat{u} = \tilde{u} = u, \hat{x}_0 = x_0 + \epsilon, \tilde{x}_0 = x_0 - \epsilon, \hat{x}_i = x_i + \epsilon$ and $\tilde{x}_i = x_i - \epsilon$ for each $i \in \mathcal{N}_1$, and $\hat{x}_i = \tilde{x}_i = x_i$ for each $i \in \mathcal{N} \setminus \mathcal{N}_1$. This is a contradiction since $(x; y; u)$ should be an extreme point of $\text{conv}(P_2)$ if there is only one optimal solution for (8). Thus, Claim (3) is verified.

(2) If $\underline{C} + V^- \leq x_0 \leq \overline{C} - V^+$, it follows that $\min\{\overline{C}, x_0 + V^+\} = x_0 + V^+$ and $\max\{\underline{C}, x_0 - V^-\} = x_0 - V^-$. Then we have $g(x_0) = (a_0 x_0 + b_0) + \sum_{i \in A^+}[a_i(x_0 + V^+) + b_i]^+ + \sum_{i \in A^-}[a_i(x_0 - V^-) + b_i]^+$, which is a convex function with respect to x_0 on $[\underline{C} + V^-, \overline{C} - V^+]$. Thus, the optimal solutions happen at the points where $x_0 = \underline{C} + V^-$ or $\overline{C} - V^+$. Here we only need to discuss the case when $x_0 = \overline{C} - V^+$. Similarly, we can follow the similar argument in (1) right above to verify Claim (4).

(3) If $\overline{C} - V^+ \leq x_0 \leq \overline{C}$, it follows that $\min\{\overline{C}, x_0 + V^+\} = \overline{C}$ and $\max\{\underline{C}, x_0 - V^-\} = x_0 - V^-$. Then we have $g(x_0) = (a_0 x_0 + b_0) + \sum_{i \in A^+}(a_i \overline{C} + b_i)^+ + \sum_{i \in A^-}[a_i(x_0 - V^-) + b_i]^+$, which is a convex function with respect to x_0 on $[\overline{C} - V^+, \overline{C}]$. Thus, the optimal solutions happen at the points where $x_0 = \overline{C} - V^+$ or $\overline{C}$. We only need to consider the case when $x_0 = \overline{C}$, because the other case has been covered in (2) above. For this case, $x_i, i = 1, \cdots, n$, can be defined based on (i) and (ii) right above in 2). Hence, Claim (5) is verified.

This completes the proof. ■

A.6 Proof for Proposition 5

Proof: First, we prove a claim that every point in the five groups of points described in Lemma 1 satisfies $3n + 2$ linearly independent inequalities in Q_2 at equality, which indicates that they are extreme points of Q_2. We prove this claim in five situations:

1) For Group (1) points, we have $x_0 = y_0 = 0$ and let $x_i = 0$ for $i \in [1, r]_{\mathbb{Z}}$, $x_i = \underline{C}$ for $i \in [r+1, s]_{\mathbb{Z}}$, and $x_i = \overline{C}$ for $i \in [s+1, n]_{\mathbb{Z}}$ for some given r and s. It follows that $y_i = u_i = 0$ for $i \in [1, r]_{\mathbb{Z}}$, $y_i = u_i = 1$ for $i \in [r+1, n]_{\mathbb{Z}}$. Without loss of generality, we only consider the case in which

$r \geq 1$, $s \geq r+1$, and $n \geq s+1$. That is, there exists at least one scenario corresponding to each possible generation amount x_i of 0, $\underline{C}$, or $\overline{C}$. Then, the following $3n+2$ linearly independent inequalities, $x_i - \underline{C}y_i \geq 0$ (for each $i = 0, 1, \cdots, s$), $x_i - \overline{C}y_i \leq 0$ (for each $i = s+1, \cdots, n$), (3a) ($i = 1$), (3b) (for each $i = 1, \cdots, n$), (7) (for each $i = 1, \cdots, r$), and (3c) (for each $i = r+1, \cdots, n$) are tight.

2) For Group (2) points, we have $x_0 = \underline{C}$, $y_0 = 1$ and let $x_i = 0$ for $i \in [1, r]_{\mathbb{Z}}$, $x_i = \underline{C}$ for $i \in [r+1, s]_{\mathbb{Z}}$, and $x_i = \underline{C} + V^+$ for $i \in [s+1, n]_{\mathbb{Z}}$ for some given r and s. It follows that $u_i = 0$ for $i \in [1, n]_{\mathbb{Z}}$, $y_i = 0$ for $i \in [1, r]_{\mathbb{Z}}$, and $y_i = 1$ for $i \in [r+1, n]_{\mathbb{Z}}$. Without loss of generality, we only consider the case in which $r \geq 1$, $s \geq r+1$, and $n \geq s+1$. That is, there exists at least one scenario corresponding to each possible generation amount x_i of 0, $\underline{C}$, or $\underline{C} + V^+$. The following $3n+2$ linearly independent inequalities, $x_i - \underline{C}y_i \geq 0$ (for each $i = 0, 1, \cdots, s$), (4) (for each $i = s+1, \cdots, n$), (3c) ($i = 1$), (3b) (for each $i = 1, \cdots, r$), (3a) (for each $i = r+1, \cdots, n$), and (7) (for each $i = 1, \cdots, n$) are tight.

3) For Group (3) points, we have $x_0 = \underline{C} + V^-$, $y_0 = 1$ and let $x_i = 0$ for $i \in [1, r]_{\mathbb{Z}}$, $x_i = \underline{C}$ for $i \in [r+1, s]_{\mathbb{Z}}$, and $x_i = \underline{C} + V^+ + V^-$ for $i \in [s+1, n]_{\mathbb{Z}}$ for some given r and s. Without loss of generality, we only consider the case in which $r \geq 1$, $s \geq r+1$, and $n \geq s+1$. That is, there exists at least one scenario corresponding to each possible generation amount x_i of 0, $\underline{C}$, or $\underline{C} + V^+ + V^-$. The following $3n+2$ linearly independent inequalities, (5) ($i = r+1$), $x_i - \underline{C}y_i \geq 0$ (for each $i = 1, \cdots, s$), (4) (for each $i = s+1, \cdots, n$), (3c) ($i = 1$), (3b) (for each $i = 1, \cdots, r$), (3a) (for each $i = r+1, \cdots, n$), and (7) (for each $i = 1, \cdots, n$) are tight.

4) For Group (4) points, we have $x_0 = \overline{C} - V^+$, $y_0 = 1$ and let $x_i = 0$ for $i \in [1, r]_{\mathbb{Z}}$, $x_i = \overline{C}$ for $i \in [r+1, s]_{\mathbb{Z}}$, and $x_i = \overline{C} - V^+ - V^-$ for $i \in [s+1, n]_{\mathbb{Z}}$ for some given r and s. Without loss of generality, we only consider the case in which $r \geq 1$, $s \geq r+1$, and $n \geq s+1$. That is, there exists at least one scenario corresponding to each possible generation amount x_i of 0, $\overline{C}$, or $\overline{C} - V^+ - V^-$. The following $3n+2$ linearly independent inequalities, (4) ($i = r+1$), $x_i - \overline{C}y_i \leq 0$ (for each $i = 1, \cdots, s$), (6) (for each $i = s+1, \cdots, n$ and $j = s$), (3c) ($i = 1$), (3b) (for each $i = 1, \cdots, r$), (3a) (for each $i = r+1, \cdots, n$), and (7) (for each $i = 1, \cdots, n$) are tight.

5) For Group (5) points, we have $x_0 = \overline{C}$, $y_0 = 1$ and let $x_i = 0$ for $i \in [1, r]_{\mathbb{Z}}$, $x_i = \overline{C}$ for $i \in [r+1, s]_{\mathbb{Z}}$, and $x_i = \overline{C} - V^-$ for $i \in [s+1, n]_{\mathbb{Z}}$ for some given r and s. It follows that $u_i = 0$ for $i \in [1, n]_{\mathbb{Z}}$, $y_i = 0$ for $i \in [1, r]_{\mathbb{Z}}$, and $y_i = 1$ for $i \in [r+1, n]_{\mathbb{Z}}$. Without loss of generality, we only consider the case in which $r \geq 1$, $s \geq r+1$, and $n \geq s+1$. That is, there exists at least one

scenario corresponding to each possible generation amount x_i of 0, $\overline{C}$, or $\overline{C} - V^-$. The following $3n+2$ linearly independent inequalities, $x_i - \overline{C}y_i \leq 0$ (for each $i = 0, 1, \cdots, s$), (5) (for each $i = s+1, \cdots, n$), (3c) ($i = 1$), (3b) (for each $i = 1, \cdots, r$), (3a) (for each $i = r+1, \cdots, n$), and (7) (for each $i = 1, \cdots, n$) are tight.

Lemma 1 and the above claim indicate that

$$\text{every extreme point of } \mathrm{conv}(P_2) \text{ is also an extreme point of } Q_2. \tag{38}$$

Combining this with the fact that $0 \leq y, u \leq 1$ and every inequality in Q_2 is facet-defining for $\mathrm{conv}(P_2)$ as shown in Proposition 3, we can claim that all the extreme points of Q_2 are integral in y and u. This can be proved by using a contradiction method. First of all, based on inequalities (3a)-(3d) and Propositions 1 and 4, we have both $\mathrm{conv}(P_2)$ and Q_2 bounded and

$$\mathrm{conv}(P_2) \subseteq Q_2 \subseteq \hat{P}_2, \text{ where } \hat{P}_2 = \{(x, y, u) \in \mathbb{R}_+^{n+1} \times [0,1]^{n+1} \times [0,1]^n : (3a) - (3f)\}. \tag{39}$$

Now the argument is as follows: if the claim is not true, i.e., there exists at least one fractional extreme point v in Q_2, then following $\mathrm{conv}(P_2) \subseteq Q_2$ as stated in (39), we have $v \notin \mathrm{conv}(P_2)$, which means there exists a facet in $\mathrm{conv}(P_2)$, denoted as $\mathcal{H} = \{(x, y, u) \in P_2 : \pi(x, y, u) = \pi_0\}$, with its induced hyperplane $\mathcal{H}' = \{(x, y, u) \in \mathbb{R}^{3n+2} : \pi(x, y, u) = \pi_0\}$ separating $\mathrm{conv}(P_2)$ and v. Accordingly, there are at least $3n+2$ (linearly independent) extreme points of $\mathrm{conv}(P_2)$ (also extreme points of Q_2 following (38)) to construct $\mathcal{H}$. Meanwhile, since $\mathrm{conv}(P_2)$ is bounded, facet $\mathcal{H}$ intersects at least $3n+2$ facets of $\mathrm{conv}(P_2)$ (note here that each extreme point among the $3n+2$ extreme points is the intersection of $\mathcal{H}$ and other at least $3n+1$ facets. Therefore, at least $3n+2$ facets are required to generate more than one extreme point on $\mathcal{H}$). Now we select these $3n+2$ facets (denoted as set Λ) of $\mathrm{conv}(P_2)$, and the hyperplanes induced by them (denoted as set Λ') should intersect at v. Otherwise, if they cannot intersect at v, then there must exist a facet in Q_2, which is not a facet in $\mathrm{conv}(P_2)$, intersecting with other facets at v. This contradicts with Proposition 3.

In the following, we find the contradiction by arguing that v should not be an intersection of the hyperplanes in Λ' if v is factional. To approach this, we extend v backwards along each hyperplane $\Lambda'_k \in \Lambda'$ to the corresponding facet $\Lambda_k \in \Lambda, k = 1, \cdots, 3n+2$. In this way, along each Λ_k, there should exist at least one extreme point $v'_k \in \Lambda'_k \cap$ $\mathrm{conv}(P_2)$ and $v'_k \notin \mathbb{W}$, where $\mathbb{W}$ is defined as the set of extreme points on the intersection boundaries between $\mathcal{H}$ and each facet in Λ. However, we can claim that there exists at least one $\Lambda'_m \in \Lambda'$ in which no corresponding v'_m exists, which leads to a contradiction. In the remaining part of this proof, we prove this claim.

In general, the easiest case to obtain the most possible v'_ks is that v contains one fractional entry and conv(P_2) is generated by the intersection of $\mathcal{H}$ and hypercube $\mathcal{X} = \{(x, y, u) \in \mathbb{R}^{n+1} \times [0, 1]^{2n+1} : \underline{C} \leq x \leq \overline{C}\}$ in which only one vertex in $\mathcal{X}$ is cut off. It is due to the following reasons:

(i) When v contains a smaller number of fractional entries, e.g., one fractional entry instead of two, v lies in more $(3n + 1)$-faces in $\mathcal{X}$, which makes it easier to extend to the facets in conv(P_2) to obtain the corresponding v'.

(ii) If v has only one fractional entry, then when more vertices in $\mathcal{X}$ are removed to construct conv(P_2), less vertices are left in conv(P_2) and thus it is less likely to find the corresponding v' along $\Lambda'_k, 1 \leq k \leq 3n + 2$.

Now we consider the easiest case in which $v = (x, y, u)$ has only one fractional entry, e.g., denoted as y_r with $0 < y_r < 1$, and conv(P_2) is constructed by removing one vertex, denoted as $\hat{v} = (\hat{x}, \hat{y}, \hat{u})$, from $\mathcal{X}$. It is easy to observe that $\hat{v}$ is also separated by $\mathcal{H}$ from conv(P_2). For this case, if $\hat{y}_r = 0$, then we cannot find the corresponding v' along Λ'_r with $y_r = 0$. Note here that for this case $3n + 2$ hyperplanes in Λ' represent all the $3n + 2$ $(3n + 1)$-faces constructing $\hat{v}$ in which the hyperplane with the expression $y_r = 0$ is included. Similarly, if $\hat{y}_r = 1$, then we cannot find the corresponding v' along Λ'_r with $y_r = 1$. Therefore, it is a contradiction and the original conclusion holds.

In summary, we have shown that all the extreme points of Q_2 are integral in y and u. ■

Appendix B Proofs for Three-period Formulations

B.1 Proof for Proposition 6

Proof: The proof for (10a) is trivial and thus omitted here. The validity proofs for (10b) - (10f) and (12) are similar to those for inequalities (4) - (6) in Q_2, because all of them have two continuous variables and the validity proof arguments for different combinations are similar. In the following, we first provide the validity proofs for (13) - (16).

For (13), we prove the claim by discussing different cases in terms of the values of $y_{i_2^-}$ and y_{i^-}:

(i) If $y_{i_2^-} = y_{i^-} = 0$, then $x_{i_2^-} = x_{i^-} = 0$ due to (11d). Then (13) converts to $x_i \geq \underline{C} y_i$, which is valid because of (11d).

(ii) If $y_{i_2^-} = 0$ and $y_{i^-} = 1$, then $u_{i^-} = 1$ due to (11c) and further $y_i = 1$ due to (11a) and nonnegativity of u_i. Then (13) converts to $x_{i^-} - x_i \leq V$, which is valid because of (11f).

(iii) If $y_{i_2^-} = y_{i^-} = 1$, then (13) converts to $x_{i_2^-} - x_{i^-} + x_i \geq -V + \underline{C}y_i$, which is valid because $x_{i_2^-} - x_{i^-} \geq -V$ due to (11e) and $x_i \geq \underline{C}y_i$ due to (11d).

(iv) If $y_{i_2^-} = 1$ and $y_{i^-} = 0$, then $u_{i^-} = u_i = 0$ due to (11b) and nonnegativity of u_{i^-} and u_i and further $y_i = 0$ due to (11c). In addition, $y_{i^-} = y_i = 0$ leads to $x_{i^-} = x_i = 0$ due to (11d). It follows that (13) converts to $x_{i_2^-} \geq \underline{C}y_{i_2^-}$, which is valid because of (11d).

Due to symmetry, the proof for (14) is similar to that for (13) and thus omitted here.

For (15), we prove the claim by discussing different cases in terms of the values of $y_{i_2^-}$ and y_j:

(i) If $y_{i_2^-} = y_j = 0$, then $x_{i_2^-} = x_j = 0$ due to (11d) and $u_j = 0$ due to (11a). Then (13) converts to $x_i \geq \underline{C}y_i$, which is valid because of (11d).

(ii) If $y_{i_2^-} = 0$ and $y_j = 1$, then $x_{i_2^-} = 0$ due to (11d). We further discuss two situations: 1) if $y_{i^-} = 0$, then $y_{j^-} = y_{i^-} = 0$ since $j^- = i^-$ and thus $u_j = 1$ because of (11c). Then (15) converts to $x_i - x_j \geq \underline{C}y_i - \overline{C}$, which is valid because $x_i \geq \underline{C}y_i$ and $x_j \leq \overline{C}$; 2) if $y_{i^-} = 1$, then $u_{i^-} = 1$ due to (11c) and further $u_i = u_j = 0$ because of (11b). Also, we have $y_i = 1$ because of (11a). It follows that (15) converts to $x_j - x_i \leq 2V$, which is valid since $x_j - x_i$ is maximized when x_j is increased by V and x_i is decreased by V from x_{i^-}.

(iii) If $y_{i_2^-} = 1$ and $y_j = 0$, then $x_j = 0$ due to (11d) and $u_j = 0$ due to (11a) and nonnegativity of u_{i^-}. Then, (15) converts to $x_{i_2^-} + x_i \geq \underline{C} + \underline{C}y_i$, which is valid because of (11d).

(iv) If $y_{i_2^-} = y_j = 1$, we have $u_{i^-} = u_i = u_j = 0$ due to (11b). Then $y_{i^-} = y_{j^-} = 1$ because of (11c) and $i^- = j^-$, and (15) converts to $x_{i_2^-} + x_i - x_j \geq -2V + \underline{C}y_i$, which is valid because $x_{i_2^-} - x_j \geq -2V$ due to the ramp-up constraints (11e) since $y_{i_2^-} = y_{i^-} = y_j = 1$ and $x_i \geq \underline{C}y_i$.

The proof for (16) is similar to that for (15) due to symmetry and thus omitted here.

Finally, inequalities (12) - (16) are facet-defining for $\text{conv}(P_3^2)$, which will be provided in Section 5 in the proofs for Propositions 10 - 11 and Proposition 13.

Now we prove the validity of (17). We discuss the following four cases in terms of possible values of $y_{i_2^-}$ and y_j:

(i) If $y_{i_2^-} = y_j = 0$, then $x_{i_2^-} = x_j = 0$ due to constraints (11d), $u_{i^-} = u_j = 0$ due to constraints (11a), and $y_{i^-} = 0$ due to constraints (11c). It follows that $y_i = u_i$ and (17) converts to $x_i \geq \underline{C}y_i$, which is valid because of (11d).

(ii) If $y_{i_2^-} = 0$ and $y_j = 1$, we further discuss the following two cases in terms of possible values of y_{i^-}:

1) If $y_{i^-} = 0$, we have $u_j = y_j = 1$ and $u_i = y_i$ due to constraints (11c) and (11a). It follows that (17) converts to $x_i - x_j \geq \underline{C} y_i - \overline{C}$, which is valid because of (11d).

2) If $y_{i^-} = 1$, we have $u_{i^-} = y_i = 1,\ u_i = u_j = 0$ following (11a) - (11c). It follows that (17) converts to $-x_{i^-} + x_i - x_j \geq -\overline{C} - V$, which is valid since $-x_{i^-} + x_i \geq -V$ following (11f) and $-x_j \geq -\overline{C}$ following (11d).

(iii) If $y_{i_2^-} = 1$ and $y_j = 0$, then $x_j = 0$ due to constraints (11d), and $u_{i^-} = u_i = u_j = 0, y_{i^-} \geq y_i$ following (11a) - (11c). We further discuss the following three cases in terms of possible values of y_{i^-} and y_i:

1) If $y_{i^-} = y_i = 0$, (17) converts to $x_{i_2^-} \geq \underline{C}$, which is valid due to (11d).

2) If $y_{i^-} = 1$ and $y_i = 0$, (17) converts to $x_{i_2^-} - x_{i^-} \geq \underline{C} - \overline{C} + V$, which is valid since $x_{i_2^-} - x_{i^-} \geq -V$ following (11e) and $\overline{C} - \underline{C} - 2V > 0$.

3) If $y_{i^-} = y_i = 1$, (17) converts to $x_{i_2^-} - x_{i^-} + x_i \geq \underline{C} - V$, which is valid since $x_{i_2^-} \geq \underline{C}$ following (11d) and $-x_{i^-} + x_i \geq -V$ following (11f).

(iv) If $y_{i_2^-} = y_j = 1$, then $y_{i^-} = 1,\ u_{i^-} = u_i = u_j = 0$ following (11a) - (11c). It follows that (17) converts to $x_{i_2^-} - x_{i^-} + x_i - x_j \geq (\overline{C} - 2V) y_i - \overline{C} - V$. We further discuss the following two cases in terms of possible values of y_i:

1) If $y_i = 0$, (17) converts to $x_{i_2^-} - x_{i^-} - x_j \geq -\overline{C} - V$, which is valid since $x_{i_2^-} - x_{i^-} \geq -V$ following (11e) and $-x_j \geq -\overline{C}$ following (11d).

2) If $y_i = 1$, (17) converts to $x_{i_2^-} - x_{i^-} + x_i - x_j \geq -3V$, which is valid since $x_{i_2^-} - x_{i^-} \geq -V$ following (11e) and $x_j - x_i \leq 2V$ because $x_j - x_i$ is maximized when x_j increases by V and x_i decreases by V based on x_{i^-}.

By symmetry, the validity proof for (18) is similar to that for (17) and therefore is omitted here. As a result, we also only provide the facet-defining proof for (17) as follows.

Similar to the proof for Proposition 2, we can prove that $\text{conv}(P_3^2)$ is full-dimensional with $\dim(\text{conv}(P_3^2)) = 3n + 5$. Thus we prove that inequality (17) is facet-defining for $\text{conv}(P_3^2)$ by creating $3n + 5$ affinely independent points in $\text{conv}(P_3^2)$ that satisfy (17) at equality. For notation brevity, for Figure 3, we sort all the nodes in the order of $i_2^-,\ i^-, 1, 2, \cdots, n$ with node i_2^- labelled as -1 and node i^- labelled as 0. In addition, due to the symmetry of scenario nodes in $\mathcal{N}$, without loss of generality, we only consider the case in which $i = 1$ and $j = 2$. Since $0 \in \text{conv}(P_3^2)$, we provide the remaining $3n + 4$ linearly independent points in the following eight groups:

(i) For $r = -1$ (totally there is one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r)$ in $\text{conv}(P_3^2)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}, & s = -1 \\ 0, & \forall s \in [0, n]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s = -1 \\ 0, & \forall s \in [0, n]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{c} \bar{u}_s^r = 0, \\ \forall s \in [0, n]_{\mathbb{Z}} \end{array}.$$

(ii) For $r = 0$ (totally there is one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r)$ in $\text{conv}(P_3^2)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}, & \forall s \in \{-1, 1\} \\ \underline{C} + V, & s = 0 \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [-1, 1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{c} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(iii) For $r = 1$ (totally there is one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r)$ in $\text{conv}(P_3^2)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, & \forall s \in \{0, 2\} \\ \overline{C} - V, & s = -1 \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [-1, 2]_{\mathbb{Z}} \setminus \{1\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{c} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(iv) For each $r \in [2, n]_{\mathbb{Z}}$ (totally there are $n - 1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r)$ in $\text{conv}(P_3^2)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}, & \forall s \in \{-1, 1\} \\ \underline{C} + V, & s = 0 \\ \underline{C} + 2V, & \forall s \in [2, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [-1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{c} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

Note here that Groups (i) to (iv) (totally $n + 2$ points) construct a lower-triangular matrix in terms of y, which are sorted horizontally in the order of $y_{i_2^-}, y_{i^-}, y_1, y_2, \cdots, y_n$. Next, we construct n linearly independent points through providing different values on x corresponding to each point (except the first and second ones, i.e., $r = -1$ and 0) in Groups (i) to (iv) through keeping y the same, making the constructed points linearly independent with Groups (i) to (iv).

(v) For $r = 1$ (totally there is one point), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r)$ in $\text{conv}(P_3^2)$ such that

$$\hat{x}_s^r = \begin{cases} \overline{C} - V - \epsilon, & s = -1 \\ \overline{C} - \epsilon, & s = 0 \\ \overline{C}, & s = 2 \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [-1, 2]_{\mathbb{Z}} \setminus \{1\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{c} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

(vi) For each $r \in [2,n]_{\mathbb{Z}}$ (totally there are $n-1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r)$ in $\text{conv}(P_3^2)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C} + \epsilon, & \forall s \in \{-1, 1\} \\ \underline{C} + V + \epsilon, & s = 0 \\ \underline{C} + 2V + \epsilon, & \forall s \in [2, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [-1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

Note here that the variable $u = 0$ for all the points in the six groups of points above. Next we construct the remaining $n+2$ linearly independent points through creating an upper-triangular matrix in terms of the variable u.

(vii) For each $r \in [0,n]_{\mathbb{Z}}$ (totally there are $n+1$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r)$ in $\text{conv}(P_3^2)$ such that

$$\tilde{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [r, n]_{\mathbb{Z}} \setminus \{1\} \\ \overline{C} - V, & s = 1, \text{ if } r = 0 \\ \underline{C}, & s = 1, \text{ if } r = 1 \\ 0, & \text{o.w.} \end{cases}, \quad \tilde{y}_s^r = \begin{cases} 1, & \forall s \in [r, n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{u}_s^r = \begin{cases} 1, & s = r, \text{ if } r \leq 0 \\ 1, & \forall s \in [r, n]_{\mathbb{Z}}, \text{ if } r \geq 1 \\ 0, & \text{o.w.} \end{cases}.$$

(viii) We create a point $(\dot{x}, \dot{y}, \dot{u})$ in $\text{conv}(P_3^2)$ (totally one point) such that

$$\dot{x}_s = \begin{cases} \overline{C} - \epsilon, & s = 0 \\ \overline{C} - V - \epsilon, & s = 1 \\ \overline{C}, & \forall s \in [2, n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \dot{y}_s = \begin{cases} 1, & \forall s \in [0, n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \dot{u}_s = \begin{cases} 1, & s = 0, \\ 0, & \text{o.w.} \end{cases}.$$

In summary, we create $3n+4$ points in $\text{conv}(P_3^2)$, with a similar structure in Table 16. It follows that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=-1}^n$, $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=1}^n$, $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r)_{r=0}^n$, and $(\dot{x}, \dot{y}, \dot{u})$ are linearly independent. Therefore, the statement is proved. ∎

For the following proofs, we omit the validity proofs due to the similarities with those provided in Proposition 1 and Proposition 6, and only provide the facet-defining proofs.

B.2 Proof for Proposition 7

Proof: We prove that (19) is facet-defining and omit the proof for (20) due to symmetry. Following the breadth-first search rule, we sort the n^2+n+1 scenario nodes (from Figure 4) in the following Figure 8, where the root node is the 0th one and the last one is the (n^2+n)th one. Due to the symmetry of scenario nodes following their parent node, without loss of generality, we assume scenario node i^- is the 1st child and j^- is the 2nd child following the root node i_2^-, scenario node i is the first child following node i^-, and node j is the first child following node j^-, i.e., node i is the

$(n+1)$st one and node j is the $(2n+1)$st one in the whole tree, as shown in Figure 8. Thus we only need to prove that $x_{n+1} - x_0 + x_2 \leq (\underline{C}+2V)y_{n+1} - \underline{C}y_0 + (\overline{C} - \underline{C} - 2V)(u_1 + u_{n+1}) + (\overline{C} - V)y_2 + Vu_2$ is facet-defining for $\mathrm{conv}(P_3)$.

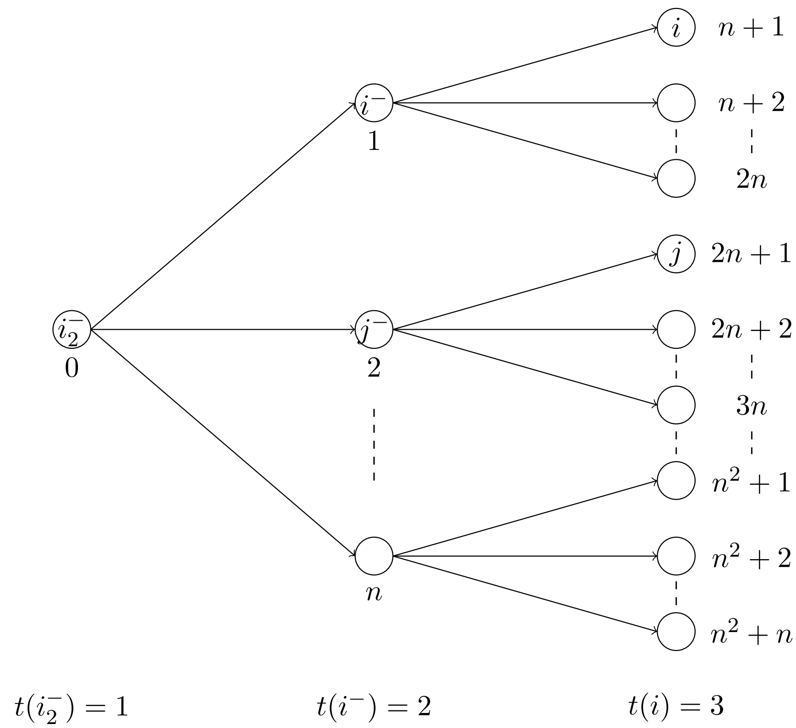

Figure 8: A generic three-period scenario tree

Now we generate $3n^2 + 3n + 2$ affinely independent points in $\mathrm{conv}(P_3)$ that satisfy (19) at equality. Since $0 \in \mathrm{conv}(P_3)$, we generate other $3n^2 + 3n + 1$ linearly independent points in the following groups.

(i) For each $r \in [0, 1]_{\mathbb{Z}}$ (totally there are two points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \mathrm{conv}(P_3)$ such that

$$\acute{x}^r_s = \begin{cases} \underline{C}, & \forall s \in [0, r]_{\mathbb{Z}} \\ 0, & \forall s \in [r+1, n^2+n]_{\mathbb{Z}} \end{cases}, \quad \acute{y}^r_s = \begin{cases} 1, & \forall s \in [0, r]_{\mathbb{Z}} \\ 0, & \forall s \in [r+1, n^2+n]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \acute{u}^r_s = 0, \\ \forall s \in [1, n^2+n]_{\mathbb{Z}} \end{array}.$$

(ii) For each $r \in [n+1, 2n]_{\mathbb{Z}}$ (totally there are n points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \mathrm{conv}(P_3)$ such that

$$\acute{x}^r_s = \begin{cases} \underline{C}, & s = 0 \\ \underline{C} + V, & s = 1 \\ \underline{C} + 2V, & \forall s \in [n+1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}^r_s = \begin{cases} 1, & \forall s \in [0, 1]_{\mathbb{Z}} \cup [n+1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \acute{u}^r_s = 0, \\ \forall s \end{array}.$$

(iii) For each $r \in [2, n]_{\mathbb{Z}}$ (totally there are $n - 1$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \mathrm{conv}(P_3)$ such that

$$\acute{x}_s^r = \begin{cases} \overline{C} - V, & s = 0 \\ \overline{C}, & \forall s \in [1, r]_{\mathbb{Z}} \cup [n+1, 2n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & \forall s \in [0, r]_{\mathbb{Z}} \cup [n+1, 2n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s^r = 0, \\ \forall s \end{array}.$$

(iv) For each $r \in [2n+1, n^2+n]_{\mathbb{Z}}$ (totally there are $n^2 - n$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P_3)$ such that

$$\acute{x}_s^r = \begin{cases} \overline{C} - V, & s = 0 \\ \overline{C}, & \forall s \in [1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & \forall s \in [0, 1]_{\mathbb{Z}} \cup [n+1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s^r = 0, \\ \forall s \end{array}.$$

Note here that Groups (i) to (iv) ($n^2 + n + 1$ points) construct a lower-triangular matrix in terms of y, which are sorted horizontally in the order of $y_0, y_1, y_{n+1}, \cdots, y_{2n}, y_2, \cdots, y_n, y_{2n+1}, \cdots, y_{3n}, \cdots, y_{n^2+1}, \cdots, y_{n^2+n}$. Next, we construct $n^2 + n$ linearly independent points through providing different values on x corresponding to each point (except the first one, i.e., $r = 0$) in Groups (i) to (iv) through keeping y the same, making the constructed points linearly independent with Groups (i) to (iv).

(v) For $r = 1$ (totally there is one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P_3)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}, & s = 0 \\ \underline{C} + V, & s = 1 \\ 0, & \forall s \in [2, n^2+n]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [0, r]_{\mathbb{Z}} \\ 0, & \forall s \in [r, n^2+n]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \in [1, n^2+n]_{\mathbb{Z}} \end{array}.$$

(vi) For each $r \in [n+1, 2n]_{\mathbb{Z}}$ (totally there are n points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P_3)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C} - 2V, & s = 0 \\ \overline{C} - V, & s = 1 \\ \overline{C}, & \forall s \in [n+1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [0, 1]_{\mathbb{Z}} \cup [n+1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(vii) For each $r \in [2, n]_{\mathbb{Z}}$ (totally there are $n - 1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P_3)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C} - 2V, & s = 0 \\ \overline{C} - V, & \forall s \in [1, r]_{\mathbb{Z}} \\ \overline{C}, & \forall s \in [n+1, 2n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [0, r]_{\mathbb{Z}} \cup [n+1, 2n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(viii) For each $r \in [2n+1, n^2+n]_{\mathbb{Z}}$ (totally there are $n^2 - n$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P_3)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C} - 2V, & s = 0 \\ \overline{C} - V, & \forall s \in [1, n]_{\mathbb{Z}} \cup [2n+1, r]_{\mathbb{Z}} \\ \overline{C}, & \forall s \in [n+1, 2n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [0, 1]_{\mathbb{Z}} \\ & \quad \cup [n+1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

Note here that the variable $u = 0$ for all the points in the eight groups of points above. Next we construct the remaining $n^2 + n$ linearly independent points through creating an upper-triangular matrix in terms of the variable u.

(ix) For each $r \in [1, n]_{\mathbb{Z}}$ (totally there are n points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P_3)$ such that

$$\hat{x}^r_s = \begin{cases} \overline{C}, & \forall s \in [r, n^2+n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}^r_s = \begin{cases} 1, & \forall s \in [r, n^2+n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}^r_s = \begin{cases} 1, & \forall s \in [r, n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}.$$

(x) For each $r \in [n+1, n^2+n]_{\mathbb{Z}}$ (totally there are n^2 points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P_3)$ such that

$$\hat{x}^r_s = \begin{cases} \underline{C}, & \forall s \in [n+1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{y}^r_s = \hat{u}^r_s = \begin{cases} 1, & \forall s \in [n+1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}.$$

In summary, we create $3n^2 + 3n + 1$ points in $\text{conv}(P_3)$, with a similar structure of Table 16. It follows that $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=0}^{n^2+n}$, $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^{n^2+n}$, and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=1}^{n^2+n}$ are linearly independent. Therefore, the statement is proved. ∎

B.3 Proof for Proposition 8

Proof: Similar to the proof in E-companion B.2 and following the notation in Figure 8 above, we only need to prove that $x_1 - x_{2n+1} \le (\underline{C}+3V)y_1 - (\overline{C}-3V)y_{2n+1} + (\overline{C}-\underline{C}-3V)(y_0+u_1+u_2+u_{2n+1})$ is facet-defining for $\text{conv}(P_3)$. In the following we create $3n^2+3n+1$ linearly independent points in $\text{conv}(P_3)$ that satisfy this inequality at equality. First, we create $2n^2+2n+1$ linearly independent points through constructing a lower-triangular matrix in terms of the values x and y.

(i) For each $r \in [1, 2n]_{\mathbb{Z}}$ (totally there are $2n$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P_3)$ such that

$$\acute{x}^r_s = \begin{cases} \overline{C}, & \forall s \in [0, r]_{\mathbb{Z}} \\ 0, & \forall s \in [r+1, n^2+n]_{\mathbb{Z}} \end{cases}, \quad \acute{y}^r_s = \begin{cases} 1, & \forall s \in [0, r]_{\mathbb{Z}} \\ 0, & \forall s \in [r+1, n^2+n]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \acute{u}^r_s = 0, \\ \forall s \in [1, n^2+n]_{\mathbb{Z}} \end{array}.$$

(ii) For each $r \in [2n+1, n^2+n]_{\mathbb{Z}}$ (totally there are $n^2 - n$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P_3)$ such that

$$\acute{x}^r_s = \begin{cases} \overline{C} - V, & s = 0 \\ \overline{C}, & \forall s \in \{1\} \cup [n+1, 2n]_{\mathbb{Z}} \\ \overline{C} - 2V, & \forall s \in [2, n]_{\mathbb{Z}} \\ \overline{C} - 3V, & \forall s \in [2n+1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}^r_s = \begin{cases} 1, & \forall s \in [0, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \acute{u}^r_s = 0, \\ \forall s \end{array}.$$

(iii) For each $r \in [1, 2n]_{\mathbb{Z}}$ (totally there are $2n$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P_3)$ such that

$$\bar{x}^r_s = \begin{cases} \overline{C} - V, & \forall s \in [0, r]_{\mathbb{Z}} \setminus \{1\} \\ \overline{C}, & s = 1 \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}^r_s = \begin{cases} 1, & \forall s \in [0, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}^r_s = 0, \\ \forall s \end{array}.$$

(iv) For each $r \in [2n+1, n^2+n]_{\mathbb{Z}}$ (totally there are n^2-n points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P_3)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}+2V, & s=0 \\ \underline{C}+3V, & \forall s \in \{1\} \cup [n+1,2n]_{\mathbb{Z}} \\ \underline{C}+V, & \forall s \in [2,n]_{\mathbb{Z}} \\ \underline{C}, & \forall s \in [2n+1,r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [0,r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(v) We create a point $(\dot{x}, \dot{y}, \dot{u}) \in \text{conv}(P_3)$ (totally one point) such that

$$\dot{x}_s = \begin{cases} \underline{C}, & \forall s \in [2,n]_{\mathbb{Z}} \cup \{0, 2n+1\} \\ 0, & \text{o.w.} \end{cases}, \quad \dot{y}_s = \begin{cases} 1, & \forall s \in [2,n]_{\mathbb{Z}} \cup \{0, 2n+1\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \dot{u}_s = 0, \\ \forall s \end{array}.$$

Next, we create the remaining n^2+n linearly independent points through constructing an upper-triangular matrix in terms of the value u. It is similar to the proof in E-companion B.2.

(vi) For $r = 1$ (totally there is one point), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P_3)$ such that

$$\hat{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [n+1,2n]_{\mathbb{Z}} \cup \{1\} \\ 0, & s=0 \\ \underline{C}, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 0, & s=0 \\ 1, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & \forall s \in [r,n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}$$

(vii) For each $r \in [2,n]_{\mathbb{Z}}$ (totally there are $n-1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P_3)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [r,n]_{\mathbb{Z}} \\ & \cup [2n+1, n^2+n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [r,n]_{\mathbb{Z}} \\ & \cup [2n+1, n^2+n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & \forall s \in [r,n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}$$

(viii) For each $r \in [n+1, n^2+n]_{\mathbb{Z}}$ (totally there are n^2 points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P_3)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [n+1,r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [n+1,r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & \forall s \in [n+1,r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}$$

It is clear that $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1}^{n^2+n}$, $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^{n^2+n}$, $(\dot{x}, \dot{y}, \dot{u})$, and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=1}^{n^2+n}$ are linearly independent. Therefore, the statement is proved. ■

Appendix C Proofs for Multi-period Formulations

To simplify the process of generating linearly independent points, we sort the nodes in Figure 6 with ordered integer numbers and label $j_{T-1}^-, j_{T-2}^-, \cdots, j^-$ with integers $-(T-1), -(T-2), \cdots, -1$ respectively. Thus, $x_{-(T-1)}, x_{-(T-2)}, \cdots, x_{-1}$ correspond to $x_{j_{T-1}^-}, x_{j_{T-2}^-}, \cdots, x_{j^-}$ respectively. Similarly, the scenario nodes in $\mathcal{N}$ are ordered and labelled with integers $0, 1, 2, \cdots, n-1$, where node

0 is the first scenario node and node $n-1$ is the last scenario node. Therefore, all the $n+T-1$ nodes in Figure 6 are labelled in an order with integers from $-(T-1)$ to $n-1$.

For the facet-defining proof for each proposition, i.e., Propositions 9 - 14, we generate $3n+3T-4$ affinely independent points in $\text{conv}(P_T^0)$ that satisfy the inequality at equality. Since $0 \in \text{conv}(P_T^0)$, we generate other $3n+3T-5$ linearly independent points. Similar to the proofs in Appendices B.2 and B.3, we construct a lower-triangular matrix in terms of the value y and an upper-triangular matrix in terms of the value u. In the following proofs, we use the superscript of (x, y, u), e.g., r in (x^r, y^r, u^r), to indicate the index of different points in $\text{conv}(P_T^0)$.

C.1 Proof for Proposition 9

Proof: We only prove that inequality (22) is facet-defining for $\text{conv}(P_T^0)$, because the facet-defining proof for inequality (23) is similar due to symmetry and thus omitted here. Due to the symmetry of the scenario nodes in $\mathcal{N}$, without loss of generality, we assume node i to be the last scenario node in $\mathcal{N}$, i.e., $i = n-1$. We create $3n+3T-5$ linearly independent points in $\text{conv}(P_T^0)$ in the following groups.

(i) For each $r \in [-(T-1), -(k+1)]_{\mathbb{Z}}$ (totally there are $T-k-1$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P_T^0)$ such that

$$\acute{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \forall s \in [r+1, n-1]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s^r = 0, \\ s \in [-(T-2), n-1]_{\mathbb{Z}} \end{array}.$$

(ii) For each $r \in [-(T-1), n-2]_{\mathbb{Z}}$ (totally there are $T+n-2$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P_T^0)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(iii) For $r = n-1$ (totally there is one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P_T^0)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [-(T-1), -(k+1)]_{\mathbb{Z}} \\ \overline{C} - (k+s)V, & \forall s \in [-k, -1]_{\mathbb{Z}} \\ \overline{C} - kV, & \text{o.w.} \end{cases}, \quad \begin{array}{l} \bar{y}_s^r = 1, \\ \forall s \end{array}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(iv) We create a point $(\dot{x}, \dot{y}, \dot{u}) \in \text{conv}(P_T^0)$ (totally one point) such that

$$\dot{x}_s = \begin{cases} \underline{C} + kV, & s \in [-(T-1), -(k+1)]_{\mathbb{Z}} \\ \underline{C} - sV, & s \in [-k, -1]_{\mathbb{Z}} \\ \underline{C}, & \text{o.w.} \end{cases}, \quad \begin{array}{l} \dot{y}_s = 1, \\ \forall s \end{array}, \text{ and } \begin{array}{l} \dot{u}_s = 0, \\ \forall s \end{array}.$$

(v) For each $r \in [-(T-2), -k]_{\mathbb{Z}}$ (totally there are $T-k-1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P_T^0)$ such that

$$\hat{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [r, -(k+1)]_{\mathbb{Z}} \\ \overline{C} - (k+s)V, & \forall s \in [-k, -1]_{\mathbb{Z}} \\ \overline{C} - kV, & \forall s \in [0, n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [r, n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

(vi) For each $r \in [-(k-1), n-1]_{\mathbb{Z}}$ (totally there are $n+k-1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P_T^0)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [r, n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [r, n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases},$$

$$\text{and } \hat{u}_s^r = \begin{cases} 1, & s = r, \text{ if } r \leq -1 \\ 1, & \forall s \in [r, n-1]_{\mathbb{Z}}, \text{ if } r \geq 0 \\ 0, & \text{o.w.} \end{cases}.$$

(vii) For each $r \in [-(k-1), n-2]_{\mathbb{Z}}$ (totally there are $n+k-2$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r) \in \text{conv}(P_T^0)$ such that

$$\tilde{x}_s^r = \begin{cases} \underline{C} + \epsilon, & \forall s \in [r, n-2]_{\mathbb{Z}} \\ \underline{C}, & s = n-1 \\ 0, & \text{o.w.} \end{cases}, \quad \tilde{y}_s^r = \begin{cases} 1, & \forall s \in [r, n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases},$$

$$\text{and } \tilde{u}_s^r = \begin{cases} 1, & s = r, \text{ if } r \leq -1 \\ 1, & \forall s \in [r, n-1]_{\mathbb{Z}}, \text{ if } r \geq 0 \\ 0, & \text{o.w.} \end{cases}.$$

From the description above, we can observe that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=-(T-1)}^{n-1}$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=-(T-2)}^{n-1}$ are linearly independent because they can construct a lower-triangular matrix based on the values of y and u after Gaussian elimination on the u part. Moreover, $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=-(T-1)}^{-(k+1)}$, $(\dot{x}, \dot{y}, \dot{u})$, and $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r)_{r=-(k-1)}^{n-2}$ are further linearly independent with them because all of these five groups of points can construct a lower-triangular matrix after Gaussian elimination on the x part. Thus, we have created $(T+n-1) + (T+n-2) + (T-k-1) + (n+k-2) + 1 = 3n + 3T - 5$ linearly independent points in $\text{conv}(P_T^0)$ as desired. ■

C.2 Proof for Proposition 10

Proof: Due to the symmetry of the scenario nodes in $\mathcal{N}$, without loss of generality, we assume nodes i and j to be the last two scenario nodes in $\mathcal{N}$, i.e., $i = n-2$, $j = n-1$. We create $3n+3T-5$ linearly independent points in $\text{conv}(P_T^0)$ in the following groups.

(i) For each $r \in [-(T-1), n-2]_{\mathbb{Z}}$ (totally there are $T+n-2$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r)$ in $\operatorname{conv}(P_T^0)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{matrix} \bar{u}_s^r = 0, \\ \forall s \end{matrix}.$$

(ii) For $r = n-1$ (totally there is one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r)$ in $\operatorname{conv}(P_T^0)$ such that

$$\bar{x}_s^r = \begin{cases} 0, & s = n-2 \\ \underline{C}, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 0, & s = n-2 \\ 1, & \text{o.w.} \end{cases}, \text{ and } \begin{matrix} \bar{u}_s^r = 0, \\ \forall s \end{matrix}.$$

(iii) For each $r \in [-(T-1), n-3]_{\mathbb{Z}}$ (totally there are $T+n-3$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r)$ in $\operatorname{conv}(P_T^0)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{matrix} \hat{u}_s^r = 0, \\ \forall s \end{matrix}.$$

(iv) For each $r \in [-(T-2), -L]_{\mathbb{Z}}$ (totally there are $T-L-1$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r)$ in $\operatorname{conv}(P_T^0)$ such that

$$\tilde{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [r, n-3]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \tilde{y}_s^r = \begin{cases} 1, & \forall s \in [r, n-3]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

(v) For each $r \in [-(L-1), -1]_{\mathbb{Z}}$ (totally there are $L-1$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r)$ in $\operatorname{conv}(P_T^0)$ such that

$$\tilde{x}_s^r = \begin{cases} \underline{C} + V, & \forall s \in [r, n-3]_{\mathbb{Z}} \\ \underline{C} + 2V, & s = n-2 \\ \underline{C}, & s = n-1 \\ 0, & \text{o.w.} \end{cases}, \quad \tilde{y}_s^r = \begin{cases} 1, & \forall s \in [r, n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

(vi) For each $r \in [0, n-1]_{\mathbb{Z}}$ (totally there are n points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r)$ in $\operatorname{conv}(P_T^0)$ such that

$$\tilde{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [r, n-2]_{\mathbb{Z}} \\ \underline{C}, & s = n-1 \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{y}_s^r = \tilde{u}_s^r = \begin{cases} 1, & \forall s \in [r, n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}.$$

(vii) We create a point $(\dot{x}, \dot{y}, \dot{u}) \in \operatorname{conv}(P_T^0)$ (totally one point) such that

$$\dot{x}_s = \begin{cases} \overline{C} - V, & \forall s \in [-(L-1), \\ & \quad n-3]_{\mathbb{Z}} \\ \overline{C}, & s = n-2 \\ \overline{C} - 2V, & s = n-1 \\ 0, & \text{o.w.} \end{cases}, \quad \dot{y}_s = \begin{cases} 1, & \forall s \in [-(L-1), \\ & \quad n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \dot{u}_s = \begin{cases} 1, & s = -(L-1) \\ 0, & \text{o.w.} \end{cases}.$$

It is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=-(T-1)}^{n-1}$, $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=-(T-1)}^{n-3}$, $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r)_{r=-(T-2)}^{n-1}$, and $(\dot{x}, \dot{y}, \dot{u})$ are linearly independent and therefore the statement is proved. ∎

C.3 Proof for Proposition 11

Proof: We only prove that inequality (25) is facet-defining for $\text{conv}(P_T^0)$, because the facet-defining proof for inequality (26) is similar due to symmetry and thus omitted here. Due to the symmetry of the scenario nodes in $\mathcal{N}$, without loss of generality, we assume node i to be the last scenario node in $\mathcal{N}$, i.e., $i = n-1$. We create $3n + 3T - 5$ linearly independent points in $\text{conv}(P_T^0)$ in the following groups.

(i) For each $r \in [-(T-1), -(k+1)]_{\mathbb{Z}}$ (totally there are $T-k-1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(ii) For each $r \in [-k, n-1]_{\mathbb{Z}}$ (totally there are $n+k$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [-(T-1), -(k+1)]_{\mathbb{Z}} \\ & \quad \cup[-(k-1), r]_{\mathbb{Z}} \\ \overline{C} - V, & s = -k \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(iii) For each $r \in [-(T-1), -(k+2)]_{\mathbb{Z}}$ (totally there are $T-k-2$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

(iv) For each $r \in [-k, n-2]_{\mathbb{Z}}$ (totally there are $n+k-1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C} + V, & \forall s \in [-(T-1), -(k+1)]_{\mathbb{Z}} \\ & \quad \cup[-(k-1), r]_{\mathbb{Z}} \\ \underline{C}, & s = -k \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

(v) For each $r \in [-(T-2), -(k+1)]_{\mathbb{Z}}$ (totally there are $T-k-2$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\tilde{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [r, -(k+1)]_{\mathbb{Z}} \\ & \cup[-(k-1), n-1]_{\mathbb{Z}} \\ \overline{C} - V, & s = -k \\ 0, & \text{o.w.} \end{cases}, \quad \tilde{y}_s^r = \begin{cases} 0, & \forall s \in [-(T-1), r-1]_{\mathbb{Z}} \\ 1, & \text{o.w.} \end{cases}, \text{ and } \tilde{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

(vi) For $r=-k$ (totally there is one point), we create $(\tilde{x}^r,\tilde{y}^r,\tilde{u}^r)$ in $\operatorname{conv}(P_T^0)$ such that

$$\tilde{x}_s^r = \begin{cases} \underline{C}+(k+s)V, & \forall s\in[r,-1]_{\mathbb{Z}} \\ \underline{C}+kV, & \forall s\in[0,n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \tilde{y}_s^r = \begin{cases} 0, & \forall s\in[-(T-1), \\ & \quad r-1]_{\mathbb{Z}} \\ 1, & \text{o.w.} \end{cases}, \text{ and } \tilde{u}_s^r = \begin{cases} 1, & s=r \\ 0, & \text{o.w.} \end{cases}.$$

(vii) For each $r\in[-(k-1),n-1]_{\mathbb{Z}}$ (totally there are $n+k-1$ points), we create $(\tilde{x}^r,\tilde{y}^r,\tilde{u}^r)$ in $\operatorname{conv}(P_T^0)$ such that

$$\tilde{x}_s^r = \begin{cases} \overline{C}, & \forall s\in[r,n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \tilde{y}_s^r = \begin{cases} 0, & \forall s\in[-(T-1),r-1]_{\mathbb{Z}} \\ 1, & \text{o.w.} \end{cases},$$

$$\text{and } \tilde{u}_s^r = \begin{cases} 1, & s=r, \text{ if } r\le -1 \\ 1, & \forall s\in[r,n-1]_{\mathbb{Z}}, \text{ if } r\ge 0 \\ 0, & \text{o.w.} \end{cases}.$$

(viii) We create a point $(\dot{x},\dot{y},\dot{u})\in\operatorname{conv}(P_T^0)$ (totally one point) such that

$$\dot{x}_s = \begin{cases} \overline{C}+sV, & \forall s\in[-k,-1]_{\mathbb{Z}} \\ \overline{C}, & \forall s\in[0,n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \dot{y}_s = \begin{cases} 0, & \forall s\in[-(T-1), \\ & \quad -(k+1)]_{\mathbb{Z}} \\ 1, & \text{o.w.} \end{cases}, \text{ and } \dot{u}_s = \begin{cases} 1, & s=-k \\ 0, & \text{o.w.} \end{cases}.$$

It is clear that $(\bar{x}^r,\bar{y}^r,\bar{u}^r)_{r=-(T-1)}^{n-1}$, $(\hat{x}^r,\hat{y}^r,\hat{u}^r)_{r=-(T-1),r\neq-(k+1)}^{n-2}$, $(\tilde{x}^r,\tilde{y}^r,\tilde{u}^r)_{r=-(T-2)}^{n-1}$, and $(\dot{x},\dot{y},\dot{u})$ are linearly independent and therefore the statement is proved. ■

C.4 Proof for Proposition 12

Proof: We only prove that inequality (27) is facet-defining for $\operatorname{conv}(P_T^0)$, because the facet-defining proof for inequality (28) is similar due to symmetry and thus omitted here. Due to the symmetry of the scenario nodes in $\mathcal{N}$, without loss of generality, we assume node i to be the last scenario node in $\mathcal{N}$, i.e., $i=n-1$. We create $3n+3T-5$ linearly independent points in $\operatorname{conv}(P_T^0)$ in the following groups.

(i) For each $r\in[-(T-1),-2]_{\mathbb{Z}}$ (totally there are $T-2$ points), we create $(\bar{x}^r,\bar{y}^r,\bar{u}^r)$ in $\operatorname{conv}(P_T^0)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, & \forall s\in[-(T-1),r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s\in[-(T-1),r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{matrix} \bar{u}_s^r=0, \\ \forall s \end{matrix}.$$

(ii) For each $r\in[-1,n-2]_{\mathbb{Z}}$ (totally there are n points), we create $(\bar{x}^r,\bar{y}^r,\bar{u}^r)$ in $\operatorname{conv}(P_T^0)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, & \forall s\in[-(T-1),-(k+1)]_{\mathbb{Z}} \\ \overline{C}-(k+s)V, & \forall s\in[-k,-1]_{\mathbb{Z}} \\ \overline{C}-(k-1)V, & \forall s\in[0,r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s\in[-(T-1),r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{matrix} \bar{u}_s^r=0, \\ \forall s \end{matrix}.$$

(iii) For $r = n-1$ (totally there is one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C} - V, & s = -1 \\ \overline{C}, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{y}_s^r = 1, \ \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(iv) For each $r \in [-(T-1), -(k+1)]_{\mathbb{Z}}$ (totally there are $T-k-1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \ \hat{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

(v) For each $r \in [-(k-1), -2]_{\mathbb{Z}}$ (totally there are $k-2$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\hat{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [-(T-1), r-1]_{\mathbb{Z}} \\ \overline{C} - \epsilon, & s = r \\ 0, & \text{o.w.} \end{cases}, \ \hat{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

(vi) For each $r \in [-1, n-2]_{\mathbb{Z}}$ (totally there are n points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C} + (k-1)V, & \forall s \in [-(T-1), -(k+1)]_{\mathbb{Z}} \\ \underline{C} - (s+1)V, & \forall s \in [-k, -1]_{\mathbb{Z}} \\ \underline{C}, & \forall s \in [0, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \ \hat{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

(vii) For each $r \in [-(T-2), n-1]_{\mathbb{Z}}$ (totally there are $T+n-2$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\tilde{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [r, n-1]_{\mathbb{Z}} \setminus \{-1\} \\ \overline{C} - V, & \forall s \in [r, n-1]_{\mathbb{Z}} \cap \{-1\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{y}_s^r = \begin{cases} 0, & \forall s \in [-(T-1), r-1]_{\mathbb{Z}} \\ 1, & \text{o.w.} \end{cases},$$

$$\text{and } \tilde{u}_s^r = \begin{cases} 1, & s = r, \text{ if } r \leq -1 \\ 1, & \forall s \in [r, n-1]_{\mathbb{Z}}, \text{ if } r \geq 0 \\ 0, & \text{o.w.} \end{cases}.$$

(viii) We create a point $(\dot{x}, \dot{y}, \dot{u}) \in \text{conv}(P_T^0)$ (totally one point) such that

$$\dot{x}_s = \begin{cases} \underline{C}, & s = -1 \\ \underline{C} + V, & \forall s \in [0, n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \ \dot{y}_s = \begin{cases} 0, & \forall s \in [-(T-1), -2]_{\mathbb{Z}} \\ 1, & \text{o.w.} \end{cases}, \text{ and } \dot{u}_s = \begin{cases} 1, & s = -1 \\ 0, & \text{o.w.} \end{cases}.$$

It is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=-(T-1)}^{n-1}$, $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=-(T-1), r \neq -k}^{n-2}$, $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r)_{r=-(T-2)}^{n-1}$, and $(\dot{x}, \dot{y}, \dot{u})$ are linearly independent and therefore the statement is proved. ∎

C.5 Proof for Proposition 13

Proof: We only prove that inequality (29) is facet-defining for $\text{conv}(P_T^0)$, because the facet-defining proof for inequality (30) is similar due to symmetry and thus omitted here. Due to the symmetry of the scenario nodes in $\mathcal{N}$, without loss of generality, we assume nodes i and j to be the last two scenario nodes in $\mathcal{N}$, i.e., $i = n-2$, $j = n-1$. We create $3n + 3T - 5$ linearly independent points in $\text{conv}(P_T^0)$ in the following groups.

(i) For each $r \in [-(T-1), n-2]_{\mathbb{Z}}$ (totally there are $T+n-2$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(ii) For $r = n-1$ (totally there is one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [-(T-1), -k]_{\mathbb{Z}} \\ \overline{C} - (k+s)V, & \forall s \in [-(k-1), -1]_{\mathbb{Z}} \\ \overline{C} - kV, & \forall s \in [0, n-3]_{\mathbb{Z}} \cap \{n-1\} \\ 0, & s = n-2 \end{cases}, \quad \bar{y}_s^r = \begin{cases} 0, & s = n-2 \\ 1, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

(iii) For each $r \in [-(T-1), -(k+1)]_{\mathbb{Z}}$ (totally there are $T-k-1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

(iv) For each $r \in [-(k-1), n-3]_{\mathbb{Z}}$ (totally there are $n+k-3$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\hat{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [-(T-1), -k]_{\mathbb{Z}} \\ \overline{C} - \epsilon, & \forall s \in [-(k-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

(v) For $r = n-1$ (totally there is one point), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C} + kV, & \forall s \in [-(T-1), -k]_{\mathbb{Z}} \\ \underline{C} - sV, & \forall s \in [-(k-1), -1]_{\mathbb{Z}} \\ \underline{C}, & \forall s \in [0, n-3]_{\mathbb{Z}} \cap \{n-1\} \\ 0, & s = n-2 \end{cases}, \quad \hat{y}_s^r = \begin{cases} 0, & s = n-2 \\ 1, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

(vi) For each $r \in [-(T-2), -k]_{\mathbb{Z}}$ (totally there are $T-k-1$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\tilde{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [r, n-3]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \tilde{y}_s^r = \begin{cases} 1, & \forall s \in [r, n-3]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

(vii) For each $r \in [-(k-1), -1]_{\mathbb{Z}}$ (totally there are $k-1$ points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\tilde{x}_s^r = \begin{cases} \overline{C} - V, & \forall s \in [r, -1]_{\mathbb{Z}} \\ \overline{C}, & \forall s \in [0, n-2]_{\mathbb{Z}} \\ \overline{C} - 2V, & s = n-1 \\ 0, & \text{o.w.} \end{cases}, \quad \tilde{y}_s^r = \begin{cases} 0, & \forall s \in [-(T-1), r-1]_{\mathbb{Z}} \\ 1, & \text{o.w.} \end{cases}, \text{ and } \tilde{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

(viii) For each $r \in [0, n-1]_{\mathbb{Z}}$ (totally there are n points), we create $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\tilde{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [r, n-2]_{\mathbb{Z}} \\ \underline{C}, & s = n-1 \\ 0, & \text{o.w.} \end{cases}, \quad \tilde{y}_s^r = \begin{cases} 1, & \forall s \in [r, n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{u}_s^r = \begin{cases} 1, & \forall s \in [r, n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}.$$

(ix) We create a point $(\dot{x}, \dot{y}, \dot{u}) \in \text{conv}(P_T^0)$ (totally one point) such that

$$\dot{x}_s = \begin{cases} \underline{C} + V, & s = -1 \\ \underline{C} + 2V, & \forall s \in [0, n-2]_{\mathbb{Z}} \\ \underline{C}, & s = n-1 \\ 0, & \text{o.w.} \end{cases}, \quad \dot{y}_s = \begin{cases} 1, & \forall s \in [-1, n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \dot{u}_s = \begin{cases} 1, & s = -1 \\ 0, & \text{o.w.} \end{cases}.$$

It is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=-(T-1)}^{n-1}$, $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=-(T-1), r\neq -k, \neq n-2}^{n-1}$, $(\tilde{x}^r, \tilde{y}^r, \tilde{u}^r)_{r=-(T-2)}^{n-1}$, and $(\dot{x}, \dot{y}, \dot{u})$ are linearly independent and therefore the statement is proved. ■

C.6 Proof for Proposition 14

Proof: We only prove that inequality (31) is facet-defining for $\text{conv}(P_T^0)$, because the facet-defining proof for inequality (32) is similar due to symmetry and thus omitted here. Due to the symmetry of the scenario nodes in $\mathcal{N}$, without loss of generality, we assume nodes i and j to be the first two scenario nodes in $\mathcal{N}$, i.e., $i = 0$, $j = 1$. We create $3n + 3T - 5$ linearly independent points in $\text{conv}(P_T^0)$ in the following groups.

(i) For each $r \in [-(T-1), -2]_{\mathbb{Z}} \cup \{0\}$ (totally there are $T-1$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\acute{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \acute{u}_s^r = 0, \ \forall s.$$

(ii) For each $r \in [1, n-1]_{\mathbb{Z}}$ (totally there are $n-1$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\acute{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [-(T-1), -2]_{\mathbb{Z}} \cup \{0\} \\ \overline{C} - V, & s = -1 \\ \overline{C} - 2V, & \forall s \in [1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \acute{u}_s^r = 0, \ \forall s.$$

(iii) For each $r \in [-(T-1), -(k+1)]_{\mathbb{Z}}$ (totally there are $T-k-1$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\grave{x}_s^r = \begin{cases} \underline{C}, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \grave{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \grave{u}_s^r = 0, \\ \forall s \end{array}.$$

(iv) For each $r \in [-(k-1), -2]_{\mathbb{Z}}$ (totally there are $k-2$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\grave{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [-(T-1), r-1]_{\mathbb{Z}} \\ \overline{C} - \epsilon, & s = r \\ 0, & \text{o.w.} \end{cases}, \quad \grave{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \grave{u}_s^r = 0, \\ \forall s \end{array}.$$

(v) For $r = 0$ (totally there is one point), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\grave{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [-(T-1), -2]_{\mathbb{Z}} \cup \{0\} \\ \overline{C} - \epsilon, & s = -1 \\ 0, & \text{o.w.} \end{cases}, \quad \grave{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), 0]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \grave{u}_s^r = 0, \\ \forall s \end{array}.$$

(vi) For each $r \in [1, n-1]_{\mathbb{Z}}$ (totally there are $n-1$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\grave{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [-(T-1), -k]_{\mathbb{Z}} \\ \overline{C} - (k+s)V, & \forall s \in [-(k-1), -1]_{\mathbb{Z}} \\ \overline{C} - kV, & \forall s \in [1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \grave{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), \\ & \quad r]_{\mathbb{Z}} \setminus \{0\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \grave{u}_s^r = 0, \\ \forall s \end{array}.$$

(vii) We create a point $(\dot{x}, \dot{y}, \dot{u}) \in \text{conv}(P_T^0)$ (totally one point) such that

$$\dot{x}_s^r = \begin{cases} \underline{C} + kV, & \forall s \in [-(T-1), -k]_{\mathbb{Z}} \\ \underline{C} - sV, & \forall s \in [-(k-1), -1]_{\mathbb{Z}} \\ \underline{C}, & s = 1 \\ 0, & \text{o.w.} \end{cases}, \quad \dot{y}_s^r = \begin{cases} 1, & \forall s \in [-(T-1), \\ & \quad -1]_{\mathbb{Z}} \cup \{1\} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \begin{array}{l} \dot{u}_s^r = 0, \\ \forall s \end{array}.$$

(viii) For each $r \in [-(T-2), -1]_{\mathbb{Z}}$ (totally there are $T-2$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [r, -2]_{\mathbb{Z}} \cup \{0\} \\ \overline{C} - V, & s = -1 \\ \overline{C} - 2V, & \forall s \in [1, n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 0, & \forall s \in [-(T-1), r-1]_{\mathbb{Z}} \\ 1, & \text{o.w.} \end{cases}, \text{ and } \bar{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

(ix) For each $r \in [0, n-1]_{\mathbb{Z}}$ (totally there are n points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r)$ in $\text{conv}(P_T^0)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, & \forall s \in [r, n-1]_{\mathbb{Z}} \cap \{0\} \\ \underline{C}, & \forall s \in [r, n-1]_{\mathbb{Z}} \setminus \{0\} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & \forall s \in [r, n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \bar{u}_s^r = \begin{cases} 1, & \forall s \in [r, n]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}.$$

(x) We create a point $(\ddot{x}, \ddot{y}, \ddot{u}) \in \mathrm{conv}(P_T^0)$ (totally one point) such that

$$\ddot{x}_s^r = \begin{cases} \underline{C} + V, & s = -1 \\ \underline{C} + 2V, & s = 0 \\ \underline{C}, & \forall s \in [1, n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \ddot{y}_s^r = \begin{cases} 1, & \forall s \in [-1, n-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \ddot{u}_s^r = \begin{cases} 1, & s = -1 \\ 0, & \text{o.w.} \end{cases}.$$

It is clear that $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=-(T-1), r \neq -1}^{n-1}$, $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=-(T-1), r \neq -k, r \neq -1}^{n-1}$, $(\dot{x}, \dot{y}, \dot{u})$, $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=-(T-2)}^{n-1}$, and $(\ddot{x}, \ddot{y}, \ddot{u})$ are linearly independent and therefore the statement is proved. ■

C.7 Proof for Proposition 15

Proof: To prove the validity, we discuss the following four cases considering different values of y_i and y_j:

(i) If $y_i = y_j = 0$, it follows that $x_i = x_j = 0$ due to constraints (1f). Then inequality (33) is valid for $\mathrm{conv}(P)$ since $\overline{C} - \underline{C} - kV > 0$ and $y, u \geq 0$.

(ii) If $y_i = 1$ and $y_j = 0$, it follows that $x_j = 0$ due to constraints (1f). Since $y_i = 1$, then $y_p + \sum_{s \in \mathcal{P}(i,p)} u_s \geq 1$ so that the generator can be online at node i. It follows that the right hand side of (33), $(\underline{C} + kV) + (\overline{C} - \underline{C} - kV)(y_p + \sum_{s \in \mathcal{P}(i,p)} u_s)$ is greater than $\overline{C}$. Then (33) is valid because $x_i \leq \overline{C}$ due to (1f).

(iii) If $y_i = 0$ and $y_j = 1$, it follows that $x_i = 0$ due to (1f). Since $y_j = 1$, then $y_p + \sum_{s \in \mathcal{P}(j,p)} u_s \geq 1$ so that the generator can be online at node j. It follows that the right hand side of (33), $-(\overline{C} - kV) + (\overline{C} - \underline{C} - kV)(y_p + \sum_{s \in \mathcal{P}(i,p)} u_s)$ is greater than $-\underline{C}$. Then (33) is valid because $-x_j \leq -\underline{C}$ due to (1f).

(iv) If $y_i = y_j = 1$, we discus the following two cases:

1) If $y_p = 0$, it follows that $\sum_{s \in \mathcal{P}(i,p) \cup \mathcal{P}(j,p)} u_s = \sum_{s \in \mathcal{P}(i,p)} u_s + \sum_{s \in \mathcal{P}(j,p)} u_s \geq 2$ so that the generator can be online at both nodes i and j. Then the right hand side of (33) is greater than $\overline{C} - \underline{C}$. It follows that (33) is valid because $x_i \geq \overline{C}$ and $x_j \leq \underline{C}$.

2) If $y_p = 1$, we discuss the following two situations:

- If $\sum_{s \in \mathcal{P}(i,p) \cup \mathcal{P}(j,p)} u_s \geq 1$, this case reduces to the above case.
- If $\sum_{s \in \mathcal{P}(i,p) \cup \mathcal{P}(j,p)} u_s = 0$, then the generator keeps online in both paths $\mathcal{P}(i,p)$ and $\mathcal{P}(j,p)$. Inequality (33) converts to $x_i - x_j \leq kV$, which is valid because $|\mathcal{P}(i,p)| + |\mathcal{P}(j,p)| = k$ and thus the difference between x_i and x_j is maximized when x_i increases by $|\mathcal{P}(i,p)|V$ from x_p and x_j decreases by $|\mathcal{P}(j,p)|V$ from x_p.

Furthermore, similar to the proofs described for Proposition 8, we can easily construct linearly independent points in conv(P) to prove that (33) is facet-defining for conv(P) as follows.

We create $3|\mathcal{V}|-1$ affinely independent points in conv(P) that satisfy inequality (33) at equality. Since $0 \in$ conv(P), we create the remaining $3|\mathcal{V}|-2$ linearly independent points.

For the convenience of generating points, we label the nodes in the tree as follows. As shown in Figure 9, due to the symmetry of the scenario tree, we can first reorganize and label the nodes in the tree such that $t(i) \leq t(j)$. Next, we label the nodes in $\mathcal{V}_1$ (in Figure 9) following the breadth-first search rule as we did for the nodes in Figure 8, with root node 0 as the 0th one. Then, we can reorganize and label the nodes along the path $\mathcal{P}(i,j) = \mathcal{P}(i,p) \cup \mathcal{P}(j,p) \cup \{p\}$ as shown in Figure 9 in the order of $p, i^-_{k_1-1}, i^-_{k_1-2}, \cdots, i^-, i, j^-_{k_2-1}, j^-_{k_2-2}, \cdots, j^-, j$, where $k_1 = dist(i,p) = |\mathcal{P}(i,p)|$ and $k_2 = dist(j,p) = |\mathcal{P}(j,p)|$ as described before. For the remaining nodes in $\hat{\mathcal{V}} = \mathcal{V} \setminus \{\mathcal{V}_1 \cup \mathcal{P}(i,j)\}$, we continue labelling them following the breath-first rule as we did for $\mathcal{V}_1$.

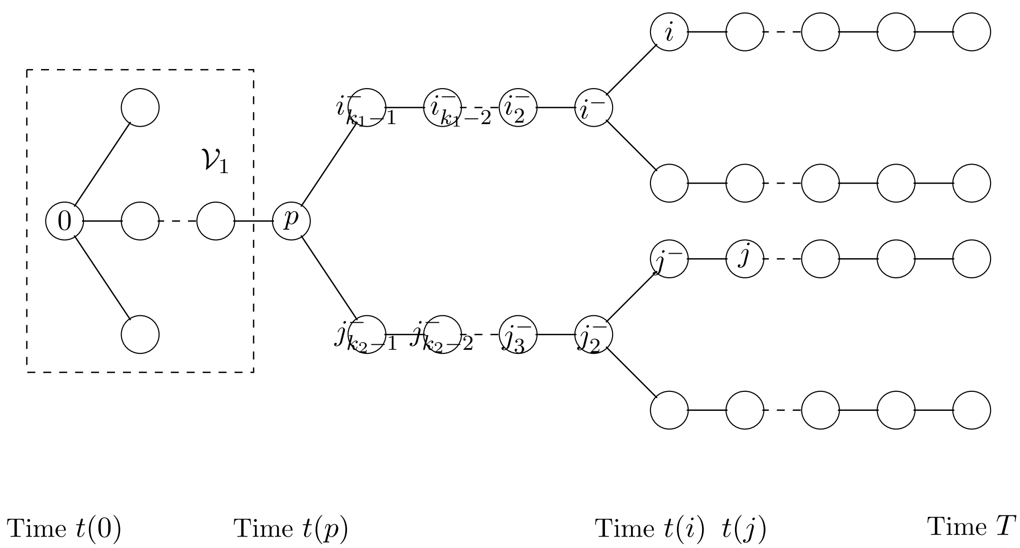

Figure 9: Complete scenario tree

In general, we create the points in two main steps. First, similar to the proof in E-companion B.3 for Proposition 8, we create two groups of points, G_1 (e.g., points $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1}^{n^2+n}$ in the proof of Proposition 8) and G_2 (e.g., points $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^{n^2+n}$ in the proof of Proposition 8). Second, we create the remaining points which form an upper-triangular matrix in terms of the value u. Finally, all the generated points can form a similar structure as the ones described in Table 16.

Now we explain the details to construct the points in G_1 and G_2 for the first main step. First, noticing that following the construction as described in E-companion B.3 for Proposition 8, we can

Table 18: Matrix in terms of y

row	$y_n, \forall n \in \mathcal{V}_1$	y_p	$y_n, \forall n \in \mathcal{P}(i,p)$	$y_n, \forall n \in \mathcal{P}(j,p)$	$y_n, \forall n \in \hat{\mathcal{V}}$
0	1 0 $\cdots$ 0				
1	1 1 $\cdots$ 0	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\vdots$	$\vdots$ $\vdots$ $\vdots$ $\vdots$				
$\lvert\mathcal{V}_1\rvert - 1$	1 1 1 1				
$\lvert\mathcal{V}_1\rvert$		1		0 0 $\cdots$ 0 0	
$\lvert\mathcal{V}_1\rvert + 1$		1		1 0 $\cdots$ 0 0	
$\lvert\mathcal{V}_1\rvert + 2$	$\mathbf{I}$	1	$\mathbf{I}$	1 1 $\cdots$ 0 0	$\mathbf{0}$
$\vdots$		1		$\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$	
$\lvert\mathcal{V}_1\rvert + \lvert\mathcal{P}(j,p)\rvert$		1		1 1 1 1 0	
$\lvert\mathcal{V}_1\rvert + \lvert\mathcal{P}(j,p)\rvert + 1$		*1*	*0 0 $\cdots$ 0*		
$\lvert\mathcal{V}_1\rvert + \lvert\mathcal{P}(j,p)\rvert + 2$		1	1 0 $\cdots$ 0		
$\lvert\mathcal{V}_1\rvert + \lvert\mathcal{P}(j,p)\rvert + 3$	$\mathbf{I}$	1	1 1 $\cdots$ 0	$\mathbf{I}$	$\mathbf{0}$
$\vdots$		1	$\vdots$ $\vdots$ $\vdots$ $\vdots$		
$\lvert\mathcal{V}_1\rvert + \lvert\mathcal{P}(i,j)\rvert$		1	1 1 1 1		
$\lvert\mathcal{V}_1\rvert + \lvert\mathcal{P}(i,j)\rvert + 1$		1			1 0 $\cdots$ 0
$\lvert\mathcal{V}_1\rvert + \lvert\mathcal{P}(i,j)\rvert + 2$	$\mathbf{I}$	1	$\mathbf{I}$	$\mathbf{I}$	1 1 $\cdots$ 0
$\vdots$		1			$\vdots$ $\vdots$ $\vdots$ $\vdots$
$\lvert\mathcal{V}\rvert - 1$		1			1 1 1 1

observe that 1) the y part for G_1 and G_2 can be easily transformed to a lower-triangular matrix as shown in Table 18, with each row corresponding to one point, and 2) $u_n = 0$ for $\forall n \in \mathcal{V} \setminus \{0\}$. Then, we assign the value x for each row in Table 18. For each row (except the row whose value y is *italicized*), two groups of values are assigned to x corresponding to each point listed below to make the given inequality (33) tight (since $x_n = 0$ when the corresponding $y_n = 0$, we only assign the value to x_n when the corresponding $y_n = 1$).

(i) For each row $r \in [0, |\mathcal{V}_1| - 1]$, let $x_n = \overline{C}$ ($\forall n : y_n = 1$) and assign this point to G_1; let $x_n = \underline{C}$ ($\forall n : y_n = 1$) and assign this point to G_2.

(ii) For each row $r \in [|\mathcal{V}_1|, |\mathcal{V}_1| + |\mathcal{P}(j,p)|]$, let $x_n = \overline{C}$ ($\forall n : y_n = 1$) and assign this point to G_1; let $x_i = \overline{C}, x_n = \overline{C} - V(\forall n \neq i : y_n = 1)$ and assign this point to G_2.

(iii) For each row $r \in [|\mathcal{V}_1| + |\mathcal{P}(j,p)| + 1, |\mathcal{V}_1| + |\mathcal{P}(i,j)| - 1]$, let $x_n = \underline{C}$ ($\forall n : y_n = 1$) and assign this point to G_1; For each row $r \in [|\mathcal{V}_1| + |\mathcal{P}(j,p)| + 2, |\mathcal{V}_1| + |\mathcal{P}(i,j)| - 1]$, let $x_j = \underline{C},\ x_n = \underline{C} + V(\forall n \neq j : y_n = 1)$ and assign this point to G_2.

(iv) For row $r = |\mathcal{V}_1| + |\mathcal{P}(i,j)|$, we only consider assigning the value x corresponding to each node in $\mathcal{P}(i,j)$, because the value x corresponding to the remaining nodes can be assigned easily as long as the feasibility is guaranteed. We assign the value x as follows: for the point in G_1, $x_p = x_{i^-_{k_1}} = \overline{C} - k_1 V$, $x_{i^-_{k_1-1}} = \overline{C} - (k_1-1)V$, $x_{i^-_{k_1-2}} = \overline{C} - (k_1-2)V$, $\cdots$, $x_{i^-} = \overline{C} - V$, $x_i = \overline{C}$, $x_{j^-_{k_2-1}} = \overline{C} - (k_1+1)V$, $x_{j^-_{k_2-2}} = \overline{C} - (k_1+2)V$, $\cdots$, and $x_j = \overline{C} - (k_1+k_2)V = \overline{C} - kV$; for the point in G_2, $x_p = x_{j^-_{k_2}} = \underline{C} + k_2 V$, $x_{j^-_{k_2-1}} = \underline{C} + (k_2-1)V$, $x_{j^-_{k_2-2}} = \underline{C} + (k_2-2)V$, $\cdots$, $x_j = \underline{C}$, $x_{i^-_{k_1-1}} = \underline{C} + (k_2+1)V$, $x_{i^-_{k_1-2}} = \underline{C} + (k_2+2)V$, $\cdots$, $x_{i^-} = \underline{C} + (k_1+k_2-1)V$, and $x_i = \underline{C} + (k_1+k_2)V = \underline{C} + kV$.

(v) For each row $r \in [|\mathcal{V}_1| + |\mathcal{P}(i,j)| + 1, |\mathcal{V}| - 1]$, for each $n \in \mathcal{V}_1 \cup \mathcal{P}(i,j)$, we use the same approach as we did in the above Case (iv). For each $n \in \hat{\mathcal{V}}$, the value x_n can be assigned easily, as long as the feasibility is guaranteed.

In this way, we obtain in total $2|\mathcal{V}| - 1$ linearly independent points in $\text{conv}(P)$.

Table 19: Upper-triangular matrix in terms of u

y_0	y_1	y_2	$\cdots$	y_n	$\cdots$	$y_{\|\mathcal{V}\|-1}$	u_1	u_2	$\cdots$	u_n	$\cdots$	$u_{\|\mathcal{V}\|-1}$
0	1	1		$\cdots$		1	1	1		$\cdots$		0
0	0	1			$\cdots\cdots$		0	1			$\cdots\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	0	$\cdots$	1	$y_{n'} = 1, \forall n' \geq n$		0	0	$\cdots$	1	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	0	0	0	0	1	0	0	0	0	0	1

For the second main step, we create the remaining $|\mathcal{V}| - 1$ linearly independent points. We first follow the breadth-first search labelling scheme to label the nodes in the tree as described in Figure 8. The generated points are shown in Table 19 in which each row corresponds to a created point. Meanwhile, we let each row correspond to a non-root node (i.e., nodes with labels $n = 1, 2, \ldots, |\mathcal{V}| - 1$ in the tree). For the row corresponding to node n, we assign the values for the y part by setting $y_{n'} = 1$ for each $n' \geq n$. In this way, the u values can be uniquely decided as shown in the right half of Table 19. We can observe that the u part forms an upper-triangular matrix based on our labelling scheme, which immediately implies these $|\mathcal{V}| - 1$ points are linearly independent by themselves, as well as linearly independent with the points created above in G_1 and G_2. The remaining task is to create x values for each point to make it feasible and satisfy inequality (33) at equality. To generate these values, when $y_p = y_i = y_j = 1$, the value x can be assigned in the same way as we did in Case (iv) above. For other situations, the x values can be similarly assigned as described in Cases (i) to (iii) and (iv) above, and the construction is omitted here. ■