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Antifoundation and Transitive Closure in the System of Zermelo

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Abstract The role of foundation with respect to transitive closure in the Zermelo system Z has been investigated by Boffa; our aim is to explore the role of antifoundation. We start by showing the consistency of "Z + antifoundation + transitive closure" relative to Z (by a technique well known for ZF). Further, we introduce a "weak replacement principle" (deductible from antifoundation and transitive closure) and study the relations among these three statements in Z via interpretations. Finally, we give some adaptations for ZF without infinity.

1 Definitions and prerequisites In this paper, by Z we mean the set theory of Zermelo without foundation. Recall that the axioms of Z are: extensionality, pairing, union set, power set, infinity in the original form of Zermelo: $\exists\xi\ \xi = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \dots\}$ and the local version AC of the axiom of choice: for every set of nonempty sets, there exists a choice-function. In Z, we shall use the notion of class as usual.

Remark 1.1 Although we have included the axiom of choice in Z for convenience, all our results remain true for Z $\setminus$ {AC} (except Proposition 2.3 and maybe Theorem 2.11 for which it is an open question). This uses the interpretability of Z in Z $\setminus${AC}.

Definition 1.2

1. The *ordered pair* of a and b is defined by $(a, b) = \{\{a\}, \{a, b\}\}$.
2. A *class-function* of domain A (class) is a class F of ordered pairs such that $(\forall a \in A)(\exists! b)((a, b) \in F)$; as usual, this unique b will be noted $F(a)$.
3. A *function* is a class-function which is a set.
4. An *ordinal* is a transitive set well-ordered by the relation $\in$. We denote by On the class of ordinals.
5. A class X is *transitive* if and only if $(\forall z \in X)(\forall t \in z)(t \in X)$.

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6. The *transitive closure* of a set x is defined by

$$\mathrm{TC}(x) = \bigcap\{y | x \subseteq y \;\&\; y \text{ is a transitive set}\}.$$

7. TC is the transitive closure axiom: $\forall x\, \exists t\; x \subseteq t \subseteq \mathcal{P}t$.
8. A *graph* is a structure of type $(G, \in_G)$ where G is a set and $\in_G \subseteq G \times G$. We will often allow ourselves just to write 'G' for '$(G, \in_G)$'. We also write '$(Y, \in_Y)$' for '$(Y, \in_G \cap (Y \times Y))$' to denote the graph obtained by restricting $\in_G$ to $Y \subseteq G$.
9. For a graph G and $g \in G$, we define $g_G = \{h \mid h \in_G g\}$.
10. A subset $X \subseteq G$ (graph) is *G-transitive* if and only if $(\forall z \in X)(\forall t \in_G z)(t \in X)$.
11. For a graph G, the *G-transitive closure* of $X \subseteq G$ is defined by

$$\mathrm{TC}_G(X) = \bigcap\{Y \mid X \subseteq Y \;\&\; Y \text{ is a set } G\text{-transitive}\}.$$

12. An *apg* (accessible pointed graph) is a structure of type $(G, \in_G, n(G))$ where G is a set, $\in_G \subseteq G \times G$, and $n(G)$ is a distinguished element of G, realizing $\mathrm{TC}_G(\{n(G)\}) = G$.
13. An *isomorphism of apg* is a graph isomorphism $f : G \to G'$ realizing

$$f(n(G)) = n(G').$$

14. A *decoration* of a graph G is a function d realizing

$$(\forall g \in G)(d(g) = \{d(z) \mid z \in_G g\}).$$

15. AFA is the well-known *antifoundation axiom*: each graph has exactly one decoration.
16. A structure of type $(X, \in_X)$ where X is a class and $\in_X \subseteq X \times X$ is called *well-founded* if and only if $\big(\forall a \subseteq X\big)\big(a \neq \emptyset \Longrightarrow (\exists z \in a)(\forall t \in_X z)\; t \notin a\big)$.
17. MOST is *Mostowski's collapsing principle*: each well-founded graph has a (necessarily unique) decoration.
18. WREP is the following *weak replacement principle* (it is a scheme in our first-order language). For any set a and F a class-function: if $\{F(x) \mid x \in a\}$ is a *transitive* class, then it is a set.

2 Relations among AFA*,* TC*, and* WREP

Proposition 2.1 (in Z) AFA + TC $\Longrightarrow$ WREP.

Remark 2.2 The axiom TC cannot be dropped as is shown by Theorem 2.12. Proposition 2.6 shows that, in Z, WREP + AFA does not imply TC.

Proof of Proposition 2.1: It suffices to prove that if a class-function J injects a transitive class X into a set b then X is a set. Define a graph G by

$$G = \{J(t) \mid t \in X\} \;\&\; J(z) \in_G J(t) \quad \text{iff} \quad z \in t.$$

The axiom AFA guarantees that $(G, \in_G)$ has a decoration d. It is easy to check that both $d \circ J$ and the identity function are decorations for the graph $(\mathrm{TC}(\{a\}), \in)$ (for each a fixed in X), so that (by AFA): $(\forall z \in X)(d(J(z) = z))$. So X is exactly the image of the set G under the function d and is itself a set. □

Proposition 2.3 (in Z + MOST) *Every set is in bijection with an ordinal.*

Proof: Let a be a set and let $u \notin a$. Let ch be a choice-function on $\mathcal{P}a \setminus \{\varnothing\}$. Define a function f by induction on $\alpha \in \mathsf{On}$:

$$f(\alpha) = \begin{cases} ch(a \setminus \{ f(\beta) \mid \beta < \alpha\}) & \text{if this set is not empty,} \\ u & \text{otherwise.} \end{cases}$$

There exists an ordinal α such that $f(\alpha) = u$. Otherwise On could be injected in a by a class-function F. Define a graph $(G, \in_G)$ in the following way: $G = \mathrm{im}(f)$, $\in_G = \{(x, y) \mid F^{-1}(x) \in F^{-1}(y)\}$. $(G, \in_G)$ is a set since $G \subset a$ and this graph is obviously wellfounded, so has a decoration d. One check by induction on $\alpha \in \mathsf{On}$ that, for $x \in G, d(x) = F^{-1}(x)$. This gives $\mathrm{im}(d) = \mathsf{On}$ and On would be a set, a contradiction. Now the first ordinal α with $f(\alpha) = u$ is in bijection with a. □

Definition 2.4 We define in Z, $\#a$ as being the least ordinal in bijection with a if it exists. By Proposition 2.3, $\#a$ always exists in Z + MOST.

The aim is now to show the interpretability of antifoundation in Zermelo. The proof is similar to the one in ZF (see, e.g., Aczel [1]).

Proposition 2.5 *There is an interpretation of* Z + AFA + TC *in* Z.

Proof: We use the well-known "trick" of *graph-models*; the reader can find earlier variants of this in Hinnion [7], Forti and Honsell [5], and [1].

Consider as *universe* the class M of the strongly extensional (in the sense of [1]) apg's. Let us recall that a bisimulation on a graph G is an equivalence $\sim$ on G such that "$x \in_G y \sim y' \Longrightarrow (\exists x' \in_G y')(x' \sim x)$" holds in G; that any graph G admits a maximum bisimulation which is exactly the union of all bisimulations on G; that G is strongly extensional if and only if (definition) its maximum bisimulation coincides with equality (on G). It is also useful to keep in mind that any strongly extensional graph G is necessarily *Finsler-(strongly) extensional*, that is, it satisfies

$$\forall x, y \in G \, \big(((\mathrm{TC}_G(\{x\}), \in_G, x) \overset{\text{as apg}}{\cong} (\mathrm{TC}_G(\{y\}), \in_G, y)) \Longrightarrow x = y \big).$$

The reader can find more about bisimulations and strong forms of extensionality in Hinnion [8] and [1]. The $\in$-relation of our interpretation is defined for $G, G' \in M$ by

$$G' \in_M G \iff \big(\exists x \in_G n(G)\big)\big((\mathrm{TC}_G(\{x\}), \in_G, x) \overset{\text{as apg}}{\cong} G'\big).$$

At last, interpret equality on M as apg-isomorphism, that is, $G =_M G'$ if and only if $G \cong G'$. It is a simple routine task to check that the structure $(M, \in_M, =_M)$ is an interpretation for Z + AFA +TC. □

Proposition 2.6 *There is a supertransitive interpretation of* Z + AFA + ¬TC + WREP *in* Z + AFA +TC +$\exists\beth_\omega$.

Comment 2.7 The axiom $\exists\beth_\omega$ is the following: the *beth number* of level ω (the first limit ordinal) does exist as a set (actually $\beth_\omega$ is the cardinal

$$\bigcup_{n\in\omega}\beth_n \text{ with } \beth_0 = \aleph_0 \text{ and } \beth_{n+1} = 2^{\beth_n}).$$

Comment 2.8 An interpretation is supertransitive (see Boffa [3]) if it is a structure of type $(M, \in)(\in_M$ is the "true" $\in$ and $=_M$ is the "true" $=)$ where M is a supertransitive class, that is, M is transitive and realizes: $x \subseteq y \in M \Longrightarrow x \in M$.

Comment 2.9 This is the analogue, with antifoundation in place of foundation, of Theorems 2 and 5 of [3].

Proof of Proposition 2.6: Working in Z + AFA + TC +$\exists\beth_\omega$, we can easily construct a set $b = \{b_i \mid i \in \omega\}$ such that $b_i = \{b_{i+1}\} \cup \beth_i$ (just decorate the adequate graph). Then take the class $M = M_{\beth_\omega}(\varnothing)$ as defined in [3]:

$$M = \{x \mid (\forall n \in \omega) \bigcup^n x \text{ is of power } < \beth_\omega\}.$$

Theorem 5 of [3] can be obviously adapted to work here and shows that this class gives a supertransitive interpretation of Z. The supertransitivity immediately guarantees AFA + WREP in M. But evidently TC is false in M because the transitive closure of b_0 (for example) is of cardinal $\beth_\omega$. □

Remark 2.10 To prove this proposition we have added to Z the axiom $\exists\beth_\omega$ which cannot be interpreted in Z. The next theorem shows that if we want the interpretation to be supertransitive, we cannot avoid this axiom. Theorem 2.12 gives nevertheless the relative consistency of Z + AFA + ¬TC with Z via a more complicated technique (permutation models).

Theorem 2.11 *Let* Z* *be an extension of* Z *such that there is a supertransitive interpretation of* Z + AFA + ¬TC *in* Z*+ TC. *Then there is an interpretation of* Z +$\exists\beth_\omega$ *in* Z*+ TC.

Proof: Let us work in Z*+ TC. Let $(M, \in)$ be a supertransitive interpretation of Z + AFA + ¬TC. By Proposition 2.3, $\#a$ exists in M for any $a \in M$. Since $(M, \in)$ is supertransitive, we have that the internal cardinal of any $a \in M$ agrees with its external cardinal; so we can speak about *the* cardinal of a without confusion. Let $a \in M$ be such that a has no transitive closure in M. For any $i \in \omega$, let $a_i = \cup^i a$ and let $\lambda_i = \#a_i$. Let us first prove the following preliminary fact:

$$(\forall i \in \omega)(\exists j \in \omega)(\#a_j \geqslant \beth_i). \tag{$*$}$$

If not, we should have

$$(\exists i \in \omega)(\forall j \in \omega)(\#a_j < \beth_i).$$

Then it would follow that the class $\mathrm{TC}(a)$ can be injected in $\beth_i$ and so $\mathrm{TC}(a)$ would be a set in $(M, \in)$ by Proposition 2.1, a contradiction.

Since the ground universe satisfies TC, $\mathrm{TC}(a)$ is a set in this ground universe. By $(*)$, the class $\beth_\omega$ can be injected in $\mathsf{TC}(a)$ and so, in $\mathsf{Z}^*+\mathsf{TC}$, there is a graph isomorphic to the structure $(\beth_\omega, \in)$. We interpret now AFA in $\mathsf{Z}^*+\mathsf{TC}$ as described in Proposition 2.5. It is now clear that this last interpretation satisfies $\exists\beth_\omega$. $\square$

The next theorem shows that in Z, if one does not have TC, one cannot prove that AFA $\Longrightarrow$ WREP. It gives also the relative consistency of $\mathsf{Z} + \mathsf{AFA} + \neg\mathsf{TC}$ relative to Z. We use the technique of permutations. Similar results for foundation instead of AFA can be found in Boffa [2].

Theorem 2.12 *There is an interpretation of* $\mathsf{Z} + \mathsf{AFA} + \neg\mathsf{WREP}$ *in* Z.

Proof: We begin to interpret the theory $\mathsf{Z} + \mathsf{AFA} + \mathsf{TC}$ in Z (see Proposition 2.5). Let $M = R_{\omega+\omega}$. Recall that the R_α's are defined by induction on $\alpha \in \mathsf{On}$ in the following way: $R_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}R_\beta$. Using Proposition 2.1, we easily see that $R_{\omega+\omega}$ exists as a class. Let $a_i = \beth_i \times \{\beth_i\}$ (for $i \in \omega$). Let us consider the permutation φ on M defined in the following way for $i \in \omega$:

$$\begin{cases} \varphi(a_i) & = & \{a_{i+1}\} \\ \varphi(\{a_{i+1}\}) & = & a_i \\ \varphi(x) & = & x \quad \text{if } (\forall i \in \omega)(x \neq \{a_{i+1}\} \ \& \ x \neq a_i) \end{cases}.$$

Define $\in_\varphi$ on M by: $x \in_\varphi y \Longleftrightarrow \varphi(x) \in y$. Various set-theoretic operations will be used with an index φ when they are to be considered in the sense of $(M, \in_\varphi)$.

The permutation φ satisfies conditions 1 and 2 of [2], so $(M, \in_\varphi) \models \mathsf{Z}$. Let us prove that $(M, \in_\varphi) \models \neg\mathsf{TC}$. For example, $\{a_0\}$ has no transitive closure in $(M, \in_\varphi)$. Otherwise, put $t = \mathrm{TC}_\varphi(\{a_0\})$. We have

$$\cdots \in_\varphi \{a_2\} \in_\varphi \{a_1\} \in_\varphi \{a_0\},$$

thus $\forall i\ \{a_i\} \in_\varphi t$, which gives $(\forall i \geqslant 1)(a_{i-1} \in t)$ which is impossible since $\#t$ would be equal to $\beth_\omega$.

Let us prove now that, in $(M, \in_\varphi)$, each graph has a decoration. It amounts to proving that for each graph $(G, \in_G)$, there is a $\in_\varphi$-decoration d^φ; that is, d^φ is a function of domain G such that $\big(\forall x, y \in G\big)\big(\{y \mid y \in_\varphi d^\varphi(x)\} = \{d^\varphi(z) \mid z \in_G x\}\big)$.

Let d be a decoration (in the usual sense) of G; for technical reasons, we suppose also that $(\forall g \in G)(g \notin \mathrm{TC}(\mathrm{dom}(\varphi))$—by $\mathrm{dom}(\varphi)$, we mean $\{x \in M \mid \varphi(x) \neq x\}$. To each $g \in G$ with $d(g) = a_i$, we associate an apg $K(g) = (K(g), \in_{K(g)}, n(K(g)))$ as follows:

$$\begin{cases} K(G) & = & \mathrm{TC}(\{\{a_{i+1}\}\}) \\ n(K(g)) & = & \{a_{i+1}\} \\ \in_{K(g)} & = & \in \cap\, (G \times G) \end{cases}.$$

Notice that $\{K(g) \mid g \in G \cap \mathrm{dom}(K)\}$ is a set ($\mathrm{dom}(K)$ is the domain of the function $K : g \to K(g)$, that is, $\mathrm{dom}(K) = \{g \in G \mid d(g) = a_i\}$). We define a graph G' (it clearly suffices to define $\in_{G'}$). For $x, y \in M$, we define $x \in_{G'} y$ to be the disjunction of the following conditions:

(i) $x \in G \;\&\; y \in G \;\&\; \forall i\; x \neq a_i \;\&\; x \in_G y$;
(ii) $(\exists g \in G)\big(x \in K(g) \;\&\; y \in K(g) \;\&\; x \in y\big)$;
(iii) $(\exists g \in G)\big(x = n(K(g)) \;\&\; y \in G \;\&\; g \in_G y\big)$.

Let d' be a decoration of G'. Notice that if $e \in K(g)$ for a $g \in G$, then $d'(e) = e$. We define d^φ on G in the following way:

(i) if $d(g) = a_i,\; d^\varphi(g) = a_i$;
(ii) if $d(g) \notin \{a_i \mid i \in \omega\},\; d^\varphi(g) = d'(g)$.

In order to prove that d^φ is an $\in_\varphi$-decoration of G, let us prove the following preliminary fact.

Fact 2.13 *If $d(g) \notin \{a_i \mid i \in \omega\}$ then $d^\varphi(g) \notin \operatorname{dom}(\varphi)$.*

Proof: Suppose that $\big(\forall h \in \mathrm{TC}_G(g_G)\big)\big(d(h) \notin \operatorname{dom}(\varphi)\big)$. In this case, we see that $d^\varphi\restriction_{\mathrm{TC}_G(g_G)}$ is a decoration of $\mathrm{TC}_G(g_G)$ and thus we have $d^\varphi(g) = d(g) \notin \operatorname{dom}(\varphi)$.

We can thus suppose that $\big(\exists h \in \mathrm{TC}_G(g_G)\big)\big(d(h) \in \operatorname{dom}(\varphi)\big)$. We can find a path $g_0 \in_G g_1 \in_G \cdots \in_G g_n = g$ with $d(g_0) = a_i$ for a $i \in \omega$ and $g_i \neq a_i$ (for $1 \leqslant i \leqslant n$). By construction, we have $n(K(g_0)) \in_{G'} g_1 \in_{G'} \cdots \in_{G'} g_n = g$ and also $d'(n(K(g_0))) \in d'(g_1) \in \cdots \in d'(g_n) = d'(g)$. As $d'(n(K(g_0))) = \{a_{i+1}\}$ and $d'(g) = d^\varphi(g)$, we have $\{a_{i+1}\} \in \mathrm{TC}(d^\varphi(g))$. Looking at the definition of φ, we see that no elements of $\operatorname{dom}(\varphi)$ satisfy this condition. This achieves the proof of Fact 2.13. $\square$

Let us prove that d^φ is an $\in_\varphi$-decoration of G. Suppose $x \in_G y$.

(i) If $d(x) = a_i$, we have that $d^\varphi(x) = a_i$ and $d^\varphi(y) = d'(y)$. By the construction of G', we have $\{a_{i+1}\} \in d'(y)$ and thus $d^\varphi(x) = a_i \in_\varphi d^\varphi(y)$.
(ii) If $d(x) \notin \{a_i \mid i \in \omega\}$, we have $d^\varphi(x) = d'(x),\; d^\varphi(y) = d'(y)$, and $d'(x) \in d'(y)$. By Fact 2.13 $d'(x) \notin \operatorname{dom}(\varphi)$ and $d^\varphi(x) \in_\varphi d^\varphi(y)$.

We have thus proved that

$$\big(\forall x, y \in G\big)\big(\{y \mid y \in_\varphi d^\varphi(x)\} \supseteq \{d^\varphi(z) \mid z \in_G x\}\big).$$

It is easy to see that we have the equality.

Let us now prove the uniqueness of the decoration. Suppose we have a graph G and two $\in_\varphi$-decorations: d^φ and $\widetilde{d}^\varphi$. Consider the graph $(G', \in_G)$ as previously and consider a decoration d of G.

Fact 2.14 $d^\varphi(g) \in \operatorname{dom}(\varphi) \Longrightarrow \big(d^\varphi(g) \in \{a_i \mid i \in \omega\} \;\&\; d^\varphi(g) = d(g)\big)$ *for all* $g \in G$.

Proof: We have that $d^\varphi(g) \notin \{\{a_i\} \mid i \in \omega\}$ since $\{a_i\}$ has no transitive closure in $(M, \in_\varphi)$. Suppose that $d^\varphi(g) = a_i$ for a $i \in \omega$. In this case $d^\varphi\restriction_{\mathrm{TC}_G(g_G)}$ is a decoration of $\mathrm{TC}_G(g_G)$ and thus $d^\varphi(g) = d(g)$. $\square$

Let us now define two functions h and h' of domain G'. Let $e \in G'$. We define the following:

(i) if $e \in K(g)$ for a $g \in G$: $h(e) = h'(e) = e$;
(ii) if $e \notin K(g)$ for all $g \in G$: $h(e) = d^\varphi(e),\; h'(e) = \widetilde{d}^\varphi(e)$.

Using Fact 2.14 and by similar arguments as those used before, we see that h and h' are two decorations of G' and that $h \neq h'$.

Let us prove now that $(M, \in_\varphi) \not\models$ WREP. Consider the class $T = \mathrm{TC}_\varphi(\{a_0\})$. We easily see that $T = \{\{a_i\} \mid i \geqslant 1\}$ which is clearly in bijection with ω. But T is not a set. This achieves the proof of Theorem 2.12. $\square$

3 *Other forms of antifoundation* In [1], other forms of antifoundation are considered, namely, FAFA and SAFA, respectively, in relation to Finsler (strong) extensionality and Scott (strong) extensionality, and we express (in the local version here) as: a graph has an injective decoration if and only if this graph is strongly extensional (respectively, in the sense of Finsler/in the sense of Scott). Our proofs can easily be adapted to show that our results are still true for FAFA or SAFA in place of AFA.

Our results also hold with the local universality axiom U_ℓ (see Boffa [4]) except Proposition 2.1 and Theorem 2.11. Let us recall that U_ℓ is defined by 'each extensional graph has an injective decoration'. It should be noticed that some proofs need more adaptations here than for FAFA or SAFA. Proposition 2.5 furnishes a good example: the construction (as explained there) does not work simply by replacing "strongly extensional" by "extensional." One has to modify the proof like this: start with the class M^* of all extensional pointed graphs (i.e., structures of type $(G, \mathcal{E}, z)$, where G is a set, $\mathcal{E} \subseteq G \times G$ and $z \in G$) and define $\in^*$ (on M^*) by: $(G, \mathcal{E}, z) \in^* (G', \mathcal{E}', z')$ if and only if $G = G'$ & $\mathcal{E} = \mathcal{E}'$ & $z\mathcal{E}z'$. The structure $(M^*, \in^*)$ is *universal* in the sense that any extensional graph is isomorphic to some M^*-transitive subset of M^*, but $(M^*, \in^*)$ is not itself extensional, so (a fortiori) not an interpretation for Z. It suffices, however, to *complete* M^* by adding copies of $\mathcal{P}M^*, \mathcal{P}^2M^*, \mathcal{P}^3M^*, \ldots$ such that all these $\mathcal{P}^kM^*$ are disjoint (for $k \in \omega$). Notice that, for a class X, we define $\mathcal{P}X$ as the class of all *subsets* of X and that the union M of the copies of the $\mathcal{P}^kM^*$ can indeed be defined in Z. Naturally one *completes* also $\in^*$ in the obvious way (e.g., if $a \in \mathcal{P}M^*$, we want that $\overline{x} \in_M \overline{a}$ if and only if $x \in a$, where $\overline{z}$ is the *copy* of z and $\in_M$ is the *completion* of $\in^*$). One can easily check that $(M, \in_M)$ is an interpretation for Z, except extensionality, and that it is still a universal structure. It suffices now to define $=_M$ as the minimum contraction on $(M, \in_M)$, that is, the least (for $\subseteq$) bisimulation $\sim$ on $(M, \in_M)$ such that the quotient $(M, \in_M)/\!\sim$ is extensional, to get an interpretation of Z $+\mathsf{U}_\ell+$ATC (details about these considerations can be found in [8], [5], and [1]; technically the situation here is relatively simple because we do not need to modelize the replacement scheme); this proof is close to the one of von Rimscha [9] adapted to Z.

4 *A few words about* ZF *without infinity* In Hauschild [6] it is shown that ZF_0 (i.e., the Zermelo-Fraenkel set theory without infinity, but with foundation for sets) cannot prove TC (because it cannot prove the foundation scheme, i.e., foundation for classes).

The main results proved in Section 2 can be adapted to work with ZF_0^- (i.e., ZF without infinity nor foundation) instead of Z. We give hereafter the statements of the adapted results. First notice that trivially $\mathsf{ZF}_0^- \models$ WREP.

Proposition 4.1 (see Proposition 2.5) *There is an interpretation of* $\mathsf{ZF}_0^- +$ AFA $+$ TC *in* ZF_0^-.

In ZF_0^-, the axiom $\exists\beth_\omega$ introduced in Proposition 2.6 is clearly equivalent to the axiom of infinity. Denote by ZF^- the theory ZF without foundation. Proposition 2.6 becomes the following.

Proposition 4.2 *There is a supertransitive interpretation of* $\mathsf{ZF}_0^- + \mathsf{AFA} + \neg\mathsf{TC}$ *in* $\mathsf{ZF}^- + \mathsf{AFA}$.

Proof: Replace the class M in Proposition 2.6 by the following:

$$M = \{x \mid (\forall n \in \omega) \bigcup^n x \text{ is finite}\}.$$

□

Theorem 2.11 becomes the following proposition.

Proposition 4.3 *Let* ZF_0^{-*} *be an extension of* ZF_0^- *such that there is a supertransitive interpretation of* $\mathsf{ZF}_0^- + \mathsf{AFA} + \neg\mathsf{TC}$ *in* $\mathsf{ZF}_0^{-*} + \mathsf{TC}$, *then* $\mathsf{ZF}_0^{-*} \models \mathsf{ZF}^-$.

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