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The Price of Universality

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Abstract We investigate the effect on the complexity of adding the universal modality and the reflexive transitive closure modality to modal logics. From the examples in the literature, one might conjecture that adding the reflexive transitive closure modality is at least as hard as adding the universal modality, and that adding either of these modalities to a multi-modal logic where the modalities do not interact can only increase the complexity to EXPTIME-complete. We show that the first conjecture holds under reasonable assumptions and that, except for a number of special cases which we fully characterize, the hardness part of the second conjecture is true. However, the upper bound part of the second conjecture fails miserably: we show that there exists a uni-modal, decidable, finitely axiomatizable, and canonical logic for which adding the universal modality causes undecidability and for which adding the reflexive transitive closure modality causes high undecidability.

1 Introduction The use of modal logics in fields like distributed systems, computational linguistics, and program verification has raised new questions about modal logics. For instance, although a logician might be satisfied by knowing that a logic is decidable, a typical "user" might want more precise information, for example *how* decidable that logic is, or, in other words, what the (computational) complexity of that logic is. These applied modal logics are usually multi-modal and contain modalities that are powerful enough to make global statements about models. The simplest form of such a modality is the universal modality $\boxed{u}$, with semantics $\boxed{u}\varphi$ is true if and only if φ is true in every world of the model (see, for example, Goranko and Passy [8]). Another powerful modality which occurs in various guises in the literature is the reflexive transitive closure modality, which we will denote by $\boxed{*}$. This modality occurs for instance in temporal logic, where the "always" operator is the reflexive transitive closure of the "nexttime" operator, and in logics of knowledge, where "common knowledge" is defined as the reflexive transitive closure of the **S5** logics that model the processors.

In this paper, we investigate what happens to the complexity of the satisfiability problem of a (multi-) modal logic when we add $\boxed{u}$ or $\boxed{*}$. If modalities interact, adding $\boxed{u}$ can increase the complexity of the satisfiability problem from decidable (even from as low as NP) to undecidable and adding $\boxed{*}$ can boost the complexity to highly undecidable, typically to Σ^1_1-complete. This occurs for example in two-dimensional logic (Harel [11]); various logics of knowledge and time with the property that processors do not forget, or do not remember (Halpern and Vardi [10], Ladner and Reif [19], Spaan [26]), and extended attribute value formalisms that allow identification of points (Blackburn and Spaan [4]). The situation is usually a lot better if the modalities do not interact. From the literature, we know that adding $\boxed{u}$ or $\boxed{*}$ to such multi-modal logics typically leads to EXPTIME-complete satisfiability problems. To state but a few examples:

- various logics for knowledge with an operator C for Common Knowledge (Halpern and Moses [9]),
- propositional dynamic logic (lower bound in Fischer and Ladner [7], upper bound in Pratt [22]),
- deterministic propositional dynamic logic (lower bound in Parikh [21], upper bound in Ben-Ari, Halpern, and Pnueli [2]),
- branching time logics (Emerson and Halpern [5]), and
- various attribute value description formalisms with the ability to express generalizations and recursive constraints (Blackburn and Spaan [4]).

From these examples, one might conjecture that adding $\boxed{u}$ or $\boxed{*}$ to a logic in which the modalities do not interact can only increase the complexity to EXPTIME-complete. However, in Section 3, we will refute this conjecture. We will show that there exists a uni-modal logic such that its satisfiability problem is in NP, but adding $\boxed{u}$ causes undecidability and adding $\boxed{*}$ causes high undecidability. We also show that there exists a uni-modal, finitely axiomatizable, decidable, and canonical logic for which adding $\boxed{u}$ causes undecidability (thereby refuting a conjecture from Goranko and Passy [8]), and for which adding $\boxed{*}$ causes high undecidability.

Section 4 will be devoted to the relationship between adding $\boxed{u}$ and adding $\boxed{*}$ to a logic. Intuitively, $\boxed{*}$ is at least as hard as $\boxed{u}$, and in this case, our intuition is correct. We will show that under reasonable assumptions, the complexity of a logic with $\boxed{*}$ is at least as high as the complexity of this logic with $\boxed{u}$. We also show that our "reasonable assumptions" are really necessary: if we drop any of our assumptions, adding $\boxed{u}$ can be arbitrarily harder than adding $\boxed{*}$.

Finally, in Section 5, we will show that there is a reason why EXPTIME shows up so often in this context. We show that, except for a number of special cases which we fully characterize, adding $\boxed{u}$ or $\boxed{*}$ to a multi-modal logic with independent modalities causes EXPTIME-hardness.

This paper is relatively self contained. In particular, all the necessary concepts from modal logic are presented in Section 2. However, we do assume that the reader understands what is meant by such complexity classes as NP, PSPACE, EXPTIME, and so on. Such definitions may be found in Balcázar, Díaz, and Gabarró [1], for example. For further information on modal logic, the reader is referred to Hughes and Cresswell [15].

2 Preliminaries

2.1 Syntax The language $\mathcal{L} = \mathcal{L}(I)$ is a language of propositional modal logic with an I indexed set of modal operators ($\boxed{a}$ for all $a \in I$). We assume a countable infinite set of propositional variables $\mathcal{P}$. The set of $\mathcal{L}$ formulas is inductively defined as follows: p is an $\mathcal{L}$ formula for every $p \in \mathcal{P}$, and if φ and ψ are $\mathcal{L}$ formulas, then so are $\neg\varphi$ and $\varphi \wedge \psi$, and $\boxed{a}\varphi$ for all $a \in I$. We define the other boolean connectives $\vee, \rightarrow, \leftrightarrow, \top$, and $\bot$ in the usual way. In addition, we define $\langle a \rangle\varphi := \neg\boxed{a}\neg\varphi$ for each $a \in I$. If $|I| = 1$, we usually use $\Box$ and $\Diamond$.

The *closure* of φ, denoted by $Cl(\varphi)$, is the least set of formulas containing φ, and closed under subformulas and single negations, that is, if $\psi \in Cl(\varphi)$ and ψ is not of the form $\neg\xi$, then $\neg\psi \in Cl(\varphi)$. Since the number of subformulas of φ is at most $|\varphi|$ (every connective and propositional variable in φ corresponds to a subformula of φ and vice versa), the size of $Cl(\varphi)$ is at most twice the length of φ.

2.2 Semantics An I *frame* is a tuple $F = \langle W, \{R_a\}_{a\in I}\rangle$ where W is a nonempty set of possible worlds, and for every $a \in I$, R_a is a binary relation on W. A frame F is *rooted at* w_0 if every world w is reachable from w_0. We call w_0 the *root* of F. An $\mathcal{L}$ *model* is of the form $M = \langle W, \{R_a\}_{a\in I}, \pi\rangle$ such that $\langle W, \{R_a\}_{a\in I}\rangle$ is an I frame (we say that M is *based on* this frame), and $\pi : \mathcal{P} \rightarrow Pow(W)$ is a valuation, that is, $w \in \pi(p)$ means that p is true at w. For φ an $\mathcal{L}$ formula, we will write $M, w \models \varphi$ for φ is *true* or *satisfied at* w *in* M. The truth relation $\models$ is defined with induction on φ in the following way:

- $M, w \models p$ if and only if $w \in \pi(p)$ for $p \in \mathcal{P}$,
- $M, w \models \neg\varphi$ if and only if not $M, w \models \varphi$,
- $M, w \models \varphi \wedge \psi$ if and only if $M, w \models \varphi$ and $M, w \models \psi$, and
- $M, w \models \boxed{a}\varphi$ if and only if $\forall w' \in W(wR_aw' \Rightarrow M, w' \models \varphi)$.

The notion of satisfiability can be extended to models and frames in the following way: φ is satisfied in M if $M, w \models \varphi$ for some world w in M, and φ is satisfiable in F (F-satisfiable) if φ is satisfied in M for some model M based on F.

In the sequel, we will talk about substructures of a frame $F = \langle W, \{R_a\}_{a\in I}\rangle$. We'll say that $\widehat{F} = \langle\widehat{W}, \{\widehat{R}_a\}_{a\in I}\rangle$ is a *subframe* of F if $\widehat{W} \subseteq W$ and $\widehat{R}_a = R_a \upharpoonright \widehat{W}$ for all $a \in I$. We'll say that $\widehat{F}$ is a *skeleton* subframe of F if $W \subseteq \widehat{W}$ and $\widehat{R}_a \subseteq R_a \upharpoonright \widehat{W}$ for all $a \in I$. Finally, we'll say that $\widehat{F}$ is a *generated* subframe of F if $\widehat{F}$ is a subframe of F and $\widehat{W}$ is closed under all accessibility relations, that is, for all $\widehat{w} \in \widehat{W}$ and $a \in I$, if $\widehat{w}R_aw$, then $w \in \widehat{W}$.

We usually look at satisfiability and validity with respect to a class of frames $\mathcal{F}$ instead of a single frame or model. All definitions on frames carry over to classes of frames in the obvious way: we say that φ is satisfiable with respect to $\mathcal{F}$ ($\mathcal{F}$-satisfiable) if φ is satisfiable in some frame $F \in \mathcal{F}$, and that $\widehat{F}$ is a (skeleton/rooted/generated) subframe of a class of frames $\mathcal{F}$ if $\widehat{F}$ is a (skeleton/rooted/generated) subframe of some frame $F \in \mathcal{F}$.

2.3 Adding $\boxed{u}$ ***and*** $\boxed{*}$ For $\mathcal{L}$ a modal language, let $\mathcal{L}_{\boxed{u}}$ be the language obtained from $\mathcal{L}$ by adding $\boxed{u}$, and let $\mathcal{L}_{\boxed{*}}$ be the language obtained from $\mathcal{L}$ by adding $\boxed{*}$. For $F = \langle W, \{R_a\}_{a\in I}\rangle$ an I frame, define $F_{\boxed{u}}$ as $\langle W, \{R_a\}_{a\in I}, R_u\rangle$ such that $R_u = W \times W$, and $F_{\boxed{*}}$ as $\langle W, \{R_a\}_{a\in I}, R_*\rangle$ such that $R_* = (\cup_{a\in I} R_a)^*$. ($R^*$ is the reflexive transitive closure of R; formally: R_0 = "=", $R^{n+1} = R; R^n$, where ";" is relation composition, and $R^* = \cup_{n\in\mathbb{N}} R^n$.) When no confusion arises, we will identify $F_{\boxed{u}}$ and $F_{\boxed{*}}$ with F.

For $\mathcal{F}$ a class of frames, we define $\mathcal{F}_{\boxed{u}}$ as the class of all frames $F_{\boxed{u}}$ such that $F \in \mathcal{F}$, and $\mathcal{F}_{\boxed{*}}$ as the class of all frames $F_{\boxed{*}}$ such that $F \in \mathcal{F}$.

3 Upper bounds In this section, we look at the following problems: given a class of frames $\mathcal{F}$ and an upper bound on the complexity of $\mathcal{F}$-satisfiability, what can we say about $\mathcal{F}_{\boxed{u}}$-satisfiability and $\mathcal{F}_{\boxed{*}}$-satisfiability? As mentioned in the introduction, the answer is: "not much."

As is shown in Harel [11], *tiling problems* provide a particularly elegant method of proving lower bounds for modal logics, so we will use such an approach here to prove our lower bounds. A tile T is a 1×1 square fixed in orientation with colored edges *right*(T), *left*(T), *up*(T), and *down*(T) taken from some denumerable set. A tiling problem takes the following form: given a finite set of $\mathcal{T}$ of tiles, can we cover a certain part of the integer grid $\mathbf{Z} \times \mathbf{Z}$, using only copies of tiles in $\mathcal{T}$, in such a way that adjacent tiles have the same color on the common edge, and such that the tiling obeys certain constraints? There exist complete tiling problems for many complexity classes (see for example Lewis [20] and van Emde Boas [27]). In the proofs that follow, we show undecidability for $\mathcal{F}_{\boxed{u}}$-satisfiability by constructing a reduction from a coRE-complete tiling problem, and high undecidability for $\mathcal{F}_{\boxed{*}}$-satisfiability by a reduction from a Σ^1_1-complete tiling problem.

3.1 Universal modality

Theorem 3.1 *There exists a uni-modal frame F such that F-satisfiability is NP-complete, while $F_{\boxed{u}}$-satisfiability is undecidable.*

Proof: Let $F = \langle \mathbb{N} \times \mathbb{N}, S \rangle$, where $\mathbb{N}$ denotes the natural numbers and S is the successor relation in the grid, i.e. $S = \{\langle\langle n, m\rangle, \langle n+1, m\rangle\rangle, \langle\langle n, m\rangle, \langle n, m+1\rangle\rangle \mid n, m \in \mathbb{N}\}$. We will show that F-satisfiability is NP-complete, but $F_{\boxed{u}}$-satisfiability is coRE-hard.

First note that F-satisfiability is certainly NP-hard, as it is a conservative extension of propositional satisfiability. To prove that F-satisfiability is in NP, suppose that φ is satisfied in $\langle \mathbb{N} \times \mathbb{N}, S\rangle$. We may assume that φ is satisfied at the origin. Now let k be the modal depth of φ. Then all relevant worlds $\langle n, m\rangle$ can be reached from the origin in at most k steps. Thus, satisfiability of φ can be verified by looking at the frame $\langle\{\langle n, m\rangle \mid n + m \leq k\}, S \restriction \{\langle n, m\rangle \mid n + m \leq k\}\rangle$, which is obviously of polynomial size in the length of φ.

It remains to show that $\mathcal{F}_{\boxed{u}}$-satisfiability is undecidable. We will construct a reduction from the following coRE-complete tiling problem $\mathbb{N} \times \mathbb{N}$ **tiling** (Berger [3], Robinson [23]) to $F_{\boxed{u}}$-satisfiability.

$\mathbb{N} \times \mathbb{N}$ **tiling**: Given a finite set $\mathcal{T}$ of tiles, can $\mathcal{T}$ tile $\mathbb{N} \times \mathbb{N}$?

That is, does there exist a function t from $\mathbb{N} \times \mathbb{N}$ to $\mathcal{T}$ such that $\mathit{right}(t(n, m)) = \mathit{left}(t(n + 1, m))$ and $\mathit{up}(t(n, m)) = \mathit{down}(t(n, m + 1))$?

Let $\mathcal{T} = \{T_1, \ldots, T_k\}$ be a set of tiles. We will construct a formula $\varphi_{\mathcal{T}}$ such that

$$\mathcal{T} \text{ tiles } \mathbb{N} \times \mathbb{N} \text{ if and only if } \varphi_{\mathcal{T}} \text{ is } F_{\boxed{u}}\text{-satisfiable.}$$

To encode the tiling, we use a propositional vector $\mathit{tile} \in \{1, \ldots, k\}$. That is, *tile* consists of sequence of $\ulcorner \log k \urcorner$ propositional variables and the values of these propositional variables will be interpreted as an integer between 1 and k. We need to ensure that adjacent tiles have the same color on their common edges. In order to enforce this, we have to be able to differentiate between upward and rightward successors. This would be easy if we knew the coordinates at each world, but as the relevant part of the frame can be infinite, this would take too much space. Let S_x and S_y stand for the rightward and upward successor relations respectively. Then we want the following to hold:

- $S = S_x \cup S_y$,
- S_x and S_y are deterministic, and
- $S_x S_y = S_y S_x$.

If S_x and S_y fulfill these conditions, then it is easy to see that one of the relations is the upward successor relation on $\mathbb{N} \times \mathbb{N}$, and the other the rightward successor relation on $\mathbb{N} \times \mathbb{N}$, which is what we were after. The requirement that $S_x S_y = S_y S_x$ seems the most difficult, for how can we force this?

This becomes clear if we look at the 2-step successors of a world w. Suppose that every world has an S_x and an S_y successor. Let $w S_x S_x w_{xx}$, $w S_x S_y w_{xy}$, $w S_y S_x w_{yx}$, and $w S_y S_y w_{yy}$. Since every world has exactly three 2-step successors, we know that two of these worlds must be equal. We will ensure that the only worlds that can be equal are w_{xy} and w_{yx}, which implies that $S_x S_y = S_y S_x$. We use propositional vector $w3 \in \{0, 1, 2\}$ and ensure that the values of $w3$ in w_{xy} and w_{yx} are the same, while the values of $w3$ in w_{xx}, w_{xy} and w_{yy} are all different. This is easy: intuitively, we let taking an S_x step correspond to adding 2 mod 3 to the value of $w3$, and taking an S_y step to addition of 1 mod 3. Then it is immediate that, for a the value of $w3$ at w, the value of $w3$ is $a + 1$ mod 3 at w_{xx}, $a + 2$ mod 3 at w_{yy}, and a at w_{xy} and w_{yx}. Formally, define

- $S_x := \bigcup_{0 \le a \le 2} \{\langle w, w' \rangle \in S \mid M, w \models (w3 = a)$ and $M, w' \models (w3 = (a + 2) \bmod 3)\}$, and
- $S_y := \bigcup_{0 \le a \le 2} \{\langle w, w' \rangle \in S \mid M, w \models (w3 = a)$ and $M, w' \models (w3 = (a + 1) \bmod 3)\}$.

And define the corresponding modalities

- $\boxed{x}\psi := \bigwedge_{a=0}^{2} ((w3 = a) \to \Box((w3 = (a + 2) \bmod 3) \to \psi)$, and
- $\boxed{y}\psi := \bigwedge_{a=0}^{2} ((w3 = a) \to \Box((w3 = (a + 1) \bmod 3) \to \psi)$.

Recall that we need to force that $S = S_x \cup S_y$, S_x and S_y are deterministic, and $S_x S_y = S_y S_x$. It suffices to force the first two requirements, since these imply that every world has an S_x and an S_y successor, which in turn implies, by the argument given above, that $S_x S_y = S_y S_x$. Thus we only have to force that $S = S_x \cup S_y$ and S_x and S_y are deterministic. Note that by definition, S_x and S_y are contained in S. Now look at the following formula, which states that every world has an S_x and an S_y successor:

$$\varphi_{succ} = \boxed{u}(\langle x \rangle \top \wedge \langle y \rangle \top).$$

Since S_x and S_y are by definition disjoint, and every world has exactly two S successors, this formula forces that $S = S_x \cup S_y$ and S_x and S_y are deterministic. We conclude that if φ_{succ} is satisfied on a model based on $F_{\boxed{u}}$, then one of S_x, S_y is the upward successor relation on $\mathbb{N} \times \mathbb{N}$, and the other the rightward successor relation on $\mathbb{N} \times \mathbb{N}$. Forcing a tiling is now trivial. Define φ_x and φ_y as follows.

$$\varphi_x = \boxed{u} \bigwedge_{i=1}^{k} ((tile = i) \to \bigvee_{right(T_i)=left(T_j)} \boxed{x}(tile = j))$$

$$\varphi_y = \boxed{u} \bigwedge_{i=1}^{k} ((tile = i) \to \bigvee_{up(T_i)=down(T_j)} \boxed{y}(tile = j))$$

Putting all this together, we define $\varphi_{\mathcal{T}}$ to be $\varphi_{succ} \wedge \varphi_x \wedge \varphi_y$. We will prove that $\mathcal{T}$ tiles $\mathbb{N} \times \mathbb{N}$ if and only if $\varphi_{\mathcal{T}}$ is $F_{\boxed{u}}$-satisfiable. The left to right direction follows from the arguments given above.

For the converse, suppose $t : \mathbb{N} \times \mathbb{N} \to \mathcal{T}$ is a tiling of $\mathbb{N} \times \mathbb{N}$. We construct the satisfying model for $\varphi_{\mathcal{T}}$ as follows: $M = \langle \mathbb{N} \times \mathbb{N}, S, \pi \rangle$ such that:

- $M, \langle n, m \rangle \models (tile = i)$ where $t(n, m) = T_i$, and
- $M, \langle n, m \rangle \models (w3 = (2n + m) \bmod 3)$.

Clearly, $\varphi_{\mathcal{T}}$ holds at any world $\langle n, m \rangle$ in M. This proves that $F_{\boxed{u}}$-satisfiability is coRE-hard, and therefore undecidable. $\square$

One could argue that frame F of Theorem 3.1 is an unfair example, because it contains so much structure. In particular, F is not even definable by a first order sentence. However, the next theorem shows that this is not the deciding factor. Even for universal first order definable classes of uni-modal frames, adding the universal modality to a decidable language can cause undecidability.

Theorem 3.2 *There exists a class of uni-modal frames $\mathcal{F}$ such that:*

- *$\mathcal{F}$-satisfiability is decidable,*
- *$\mathcal{F}_{\boxed{u}}$-satisfiability is undecidable,*
- *$\mathcal{F}$ is first order universal, and*
- *$\mathcal{F} = Fr(L)$ for L a uni-modal, finitely axiomatizable, and canonical logic.*

Proof: We need to construct a class $\mathcal{F}$ of uni-modal frames such that $\mathcal{F}$ is universal first order, $\mathcal{F} = Fr(L)$ for L a uni-modal, finitely axiomatizable, and canonical logic, and $\mathcal{F}$-satisfiability is decidable, but $\mathcal{F}_{\boxed{u}}$ is undecidable. The undecidability will be proved using the reduction constructed in the proof of Theorem 3.1, that is, we will construct $\mathcal{F}$ in such a way that $\mathcal{T}$ tiles $\mathbb{N} \times \mathbb{N}$ if and only if $\varphi_{\mathcal{T}}$ is $\mathcal{F}_{\boxed{u}}$-satisfiable. The most difficult restriction on $\mathcal{F}$ is the first order definability, for how can such a class of frames be forced to behave like $\mathbb{N} \times \mathbb{N}$? We do need some kind of diamond property, for instance $\forall xyy' \exists z(xRy \wedge xRy' \to yRz \wedge y'Rz)$. But diamond properties are certainly not universal first order.

However, $\mathcal{F}_{\boxed{u}}$ has to behave like $\mathbb{N} \times \mathbb{N}$ only if $\varphi_{\mathcal{T}}$ is $\mathcal{F}_{\boxed{u}}$-satisfiable. What does φ_{succ} force? That every world has an x and a y successor. Recall from the previous proof that we used the fact that every world in $\mathbb{N} \times \mathbb{N}$ has two successors, and three 2-step successors. Let $\mathcal{F}$ be the class of frames such that every world has at most two successors, and at most three 2-step successors. Then $\mathcal{F}$ is defined by the following universal first order sentence:

$$\varphi_\forall = \forall x\overline{y}(\bigwedge_{1\leq i\leq 3} xRy_i \rightarrow \bigvee_{1\leq i<j\leq 3} y_i = y_j) \wedge \forall x\overline{yz}(\bigwedge_{1\leq i\leq 4} xRy_iRz_i \rightarrow \bigvee_{1\leq i<j\leq 4} z_i = z_j).$$

We claim that $\mathcal{F}$ defined this way satisfies the requirements of the theorem. We start by proving that the reduction from the proof of Theorem 3.1 still works, that is, $\mathcal{T}$ tiles $\mathbb{N}\times\mathbb{N}$ if and only if $\varphi_\mathcal{T}$ is $\mathcal{F}_{\boxed{u}}$-satisfiable.

The left-to-right implication follows from the proof of Theorem 3.1. If $\mathcal{T}$ tiles $\mathbb{N}\times\mathbb{N}$ then $\varphi_\mathcal{T}$ is satisfiable on $\langle\mathbb{N}\times\mathbb{N}, S\rangle$, and it is obvious that $\varphi_\forall$ holds on this frame, and thus $\varphi_\mathcal{T}$ is $\mathcal{F}_{\boxed{u}}$-satisfiable.

To see that the converse also holds, suppose that $M = \langle W, R, \pi\rangle$ is a model such that $\langle W, R\rangle \models \varphi_\forall$ and M satisfies $\varphi_\mathcal{T}$, say at $w_0 \in W$. We reason in a similar way as in the proof of Theorem 3.1. Let R_x and R_y correspond to modalities $\boxed{x}$ and $\boxed{y}$:

- $R_x := \bigcup_{0\leq a\leq 2}\{\langle w, w'\rangle \in R \mid M, w \models (w3 = a)$ and $M, w' \models (w3 = (a+2) \bmod 3)\}$, and
- $R_y := \bigcup_{0\leq a\leq 2}\{\langle w, w'\rangle \in R \mid M, w \models (w3 = a)$ and $M, w' \models (w3 = (a+1) \bmod 3)\}$.

By definition, R_x and R_y are disjoint. By φ_{succ}, every world has an R_x and an R_y successor. Thus, by $\varphi_\forall$, it follows that every world has exactly one R_x and exactly one R_y successor. Since the second conjunct of $\varphi_\forall$ forces that every world has at most three 2-step successors, it follows in the same way as in proof of Theorem 3.1 that $R_xR_y = R_yR_x$. Now define the tiling as follows:

$$t(n, m) = T_i \text{ if and only if } M, w \models (tile = i) \text{ where } w_0R_x^nR_y^m w.$$

Since w exists and is unique, t is well-defined. To show that t is indeed a tiling, suppose $t(n, m) = T_i$ and $t(n+1, m) = T_j$. Let w and w' be the corresponding worlds, i.e. $w_0R_x^nR_y^m w$ and $w_0R_x^{n+1}R_y^m w'$. Then, by definition, $M, w \models (tile = i)$ and $M, w' \models (tile = j)$. That these tiles match follows from φ_x if we can show that wR_xw'. Since $R_xR_y = R_yR_x$, it follows that $R_x^{n+1}R_y^m = R_x^nR_y^mR_x$, and therefore, wR_xw' as required. That $t(n, m)$ and $t(n, m+1)$ match is immediate from the definition and φ_y. This proves that $\mathcal{F}_{\boxed{u}}$-satisfiability is coRE-hard, and thus undecidable.

Next we will show that $\mathcal{F}$-satisfiability is decidable. Let $M = \langle W, R, \pi\rangle$, $w_0 \in W$ be such that $M, w_0 \models \varphi$ and $\langle W, R\rangle \in \mathcal{F}$, that is, $\langle W, R\rangle \models \varphi_\forall$. For k the modal depth of φ, let $\widehat{W}$ be the set of worlds w in W such that $w_0R^{\leq k}w$. Then $M \restriction \widehat{W}, w_0 \models \varphi$, and $\langle W, R\rangle \restriction \widehat{W} \models \varphi_\forall$, since $\varphi_\forall$ is universal. At first sight, one might think that $\langle W, R\rangle$ must be grid-like so that the size of $\widehat{W}$ is at most $(k+1)^2$. But this is not true: consider for example the binary tree with the property that every left child has two children, and every right child has only a left child. Then every node has at most two successors, and at most three 2-step successors, but the size of $\widehat{W}$ is exponential in k. However, it is easy to see that it cannot be worse than that. Since each world has at most two successors, the size of $\widehat{W}$ is certainly less than 2^{k+1}. It follows that φ is $\mathcal{F}$-satisfiable if and only if φ is satisfiable on an $\mathcal{F}$ frame of size at most 2^{k+1}. Since $\mathcal{F}$ is first order definable, verifying that a frame is in $\mathcal{F}$ takes polynomial time (in the size of the frame). It is immediate that $\mathcal{F}$-satisfiability can be determined in nondeterministic exponential time.

To complete the proof of Theorem 3.2, we need to show that $\mathcal{F} = Fr(L)$ for L finitely axiomatizable and canonical. This is easy to prove, for L is defined by the following axioms:

- $\Diamond p_1 \wedge \Diamond p_2 \wedge \Diamond p_3 \to \Diamond(p_1 \wedge p_2) \vee \Diamond(p_1 \wedge p_3) \vee \Diamond(p_2 \wedge p_3)$, and
- $\bigwedge_{1 \le i \le 4} \Diamond\Diamond p_i \to \bigvee_{1 \le i < j \le 4} \Diamond\Diamond(p_i \wedge p_j)$.

The claim follows directly from Sahlqvist's theorem [24] but can easily be proven directly. To prove that $\mathcal{F} = Fr(L)$, we need to show that for all frames F, $F \models \varphi_\forall$ if and only if $F \models L$. We prove an equivalence between the second conjunct $\varphi_{\forall,2}$ of $\varphi_\forall$ $(\forall x\overline{yz}(\bigwedge_{1 \le i \le 4} xRy_iRz_i) \to (\bigvee_{1 \le i < j \le 4} z_i = z_j))$ and the second axiom of L. Proving an equivalence between the first conjunct of $\varphi_\forall$ and the first axiom of L can be done by similar arguments, from which $\mathcal{F} = Fr(L)$ follows.

First suppose that $M = \langle W, R, \pi\rangle$ and $\langle W, R\rangle \models \varphi_{\forall,2}$. Suppose $M, w \models \Diamond\Diamond p_1 \wedge \Diamond\Diamond p_2 \wedge \Diamond\Diamond p_3 \wedge \Diamond\Diamond p_4$. Let w_1, w_2, w_3, and w_4 be such that $M, w_i \models p_i$ and wR^2w_i. By $\varphi_{\forall,2}$, it holds that $w_i = w_j$ for some i, j with $1 \le i < j \le 4$. It follows that $M, w \models \Diamond\Diamond(p_i \wedge p_j)$ as required. For the converse, suppose that $\langle W, R\rangle$ is not an $\varphi_{\forall,2}$ frame. Let $w, w_1, \ldots, w_4$ be such that wR^2w_i and $w_i \neq w_j$ for $i \neq j$. Define valuation π in such a way that $\pi(p_i) = \{w_i\}$. Then $M, w \models \bigwedge_{1 \le i \le 4} \Diamond\Diamond p_i$ but $M, w \not\models \Diamond\Diamond(p_i \wedge p_j)$ for all $1 \le i < j \le 4$. It follows that $\langle W, R\rangle$ is not an L frame.

Finally, we show that the canonical model for L has an underlying $\mathcal{F}$ frame. For suppose it doesn't, and suppose we violate the second conjunct of $\varphi_\forall$. Then there exist maximal consistent sets $\Gamma, \Gamma_1, \ldots, \Gamma_4$ such that $\Box\Box\psi \in \Gamma \Rightarrow \psi \in \Gamma_i$, and all Γ_i are different. Since all Γ_i are different, there exist formulas ψ_i such that $\psi_i \in \Gamma_i$ and $\psi_i \notin \Gamma_j$ for all $j \neq i$. It follows that

$$\bigwedge_{1 \le i \le 4} \Diamond\Diamond(\psi_i \wedge \bigwedge_{j \neq i} \neg\psi_j) \in \Gamma.$$

By the second axiom of L, it follows that for some i, j with $1 \le i < j \le 4$

$$\Diamond\Diamond(\psi_i \wedge \bigwedge_{k \neq i} \neg\psi_k \wedge \psi_j \wedge \bigwedge_{k \neq j} \neg\psi_k) \in \Gamma.$$

But then $\Diamond\Diamond\bot \in \Gamma$, which contradicts the consistency of Γ. It follows that L is canonical. This completes the proof of Theorem 3.2. $\Box$

Goranko and Passy [8] also investigate enriching the modal language with a universal modality. They use an axiomatic approach. Given a uni-modal logic L, let $L_{\boxed{u}}$ consist of the following axioms:

- all L axioms,
- **S5** axioms for the universal box, and
- interaction axiom (containment): $\boxed{u}p \to \Box p$.

Among other things, they investigate what properties transfer from L to $L_{\boxed{u}}$. For instance, they show that if L is strongly complete, then so is $L_{\boxed{u}}$. They also conjecture that decidability transfers. However, the logic L defined above provides a counterexample.

Theorem 3.3 *There exists a uni-modal logic L, such that L is decidable, and $L_{\boxed{u}}$ is undecidable.*

Proof: Let L be the logic from Theorem 3.2. Since L is canonical, it follows that L is strongly complete. By the above mentioned transfer result, $L_{\boxed{u}}$ is strongly complete as well. Since $Fr(L) = \mathcal{F}$, it follows that L is decidable, being the the complement of $\mathcal{F}$-satisfiability (up to negating the formula), and $L_{\boxed{u}}$ is undecidable, being the complement of $\mathcal{F}_{\boxed{u}}$-satisfiability. □

3.2 Transitive closure We will now investigate what happens to upper bounds on satisfiability if we add $\boxed{*}$ to the language. Intuitively, $\boxed{*}$ is at least as hard as $\boxed{u}$ (this issue will be addressed in greater detail in the next section), and thus we would expect the situation to be as least as bad as in the previous subsection. This is indeed the case: Theorems 3.1 and 3.2 also hold if we replace $\boxed{u}$ by $\boxed{*}$. Indeed, we even show that the enriched logics are highly undecidable.

Theorem 3.4 *There exists a uni-modal frame F such that F-satisfiability is NP-complete, while $F_{\boxed{*}}$-satisfiability is Σ^1_1-complete.*

Proof: Let F be as defined in the proof of Theorem 3.1. Then F-satisfiability is NP-complete. It remains to prove that $F_{\boxed{*}}$-satisfiability is Σ^1_1-complete. The Σ^1_1 upper bound is immediate, since $F_{\boxed{*}}$ is countable. For the corresponding lower bound, we construct a reduction from the following Σ^1_1-complete tiling problem from Harel [12].

$\mathbb{N} \times \mathbb{N}$ **recurrent tiling**: Given a finite set $\mathcal{T}$ of tiles, and a tile $T_1 \in \mathcal{T}$, can $\mathcal{T}$ tile $\mathbb{N} \times \mathbb{N}$ such that T_1 occurs in the tiling infinitely often on the first row.

That is, does there exist a function t from $\mathbb{N} \times \mathbb{N}$ to $\mathcal{T}$ such that: $\mathit{right}(t(n,m)) = \mathit{left}(t(n+1,m))$, $\mathit{up}(t(n,m)) = \mathit{down}(t(n,m+1))$, and the set $\{i \mid t(i,0) = T_1\}$ is infinite?

Let $\mathcal{T} = \{T_1, \ldots, T_k\}$ be a set of tiles. We construct a formula φ_{rt} such that:

$$\langle \mathcal{T}, T_1 \rangle \in \mathbb{N} \times \mathbb{N} \text{ recurrent tiling} \quad \text{if and only if} \quad \varphi_{rt} \text{ is } F_{\boxed{*}}\text{-satisfiable.}$$

To ensure that φ_{rt} forces a tiling of $\mathbb{N} \times \mathbb{N}$, we use the formula $\varphi_{\mathcal{T}}$ constructed in the proof of Theorem 3.2. Let $\varphi'_{\mathcal{T}}$ be the result of replacing every occurrence of $\boxed{u}$ by $\boxed{*}$ in $\varphi_{\mathcal{T}}$. Then, as in the proof of Theorem 3.2, the following hold:

- if $\varphi'_{\mathcal{T}}$ is not satisfiable, then $\mathcal{T}$ does not tile $\mathbb{N} \times \mathbb{N}$, and
- if $M, w_0 \models \varphi'_{\mathcal{T}}$, then there exists a tiling t defined as follows:

$$t(n,m) = T_i \text{ if and only if } M, w \models (\mathit{tile} = i) \text{ where } w_0 R_x^n R_y^m w.$$

Now we force the recurrence. We will use a new propositional variable row_0, which can only be true at worlds of the form $\langle n, 0 \rangle$, and we will ensure that there exist an infinite number of worlds where row_0 holds and tile T_1 is placed. Define

$$\varphi_{rec} = \mathit{row}_0 \wedge \boxed{*}\boxed{y}\boxed{*}\neg \mathit{row}_0 \wedge \boxed{*}(\mathit{row}_0 \rightarrow \langle * \rangle \boxed{*}(\mathit{row}_0 \wedge (\mathit{tile} = 1))).$$

Let φ_{rt} be the conjunction of $\varphi'_{\mathcal{T}}$ and φ_{rec}. It is easy to prove that $\langle \mathcal{T}, T_1 \rangle \in \mathbb{N} \times \mathbb{N}$ recurrent tiling if and only if φ_{rt} is $F_{\boxed{*}}$-satisfiable. This proves Theorem 3.4. □

Theorem 3.5 *There exists a class of uni-modal frames $\mathcal{F}$ such that:*

- *$\mathcal{F}$-satisfiability is decidable,*
- *$\mathcal{F}_{\boxed{*}}$-satisfiability is Σ^1_1-complete,*
- *$\mathcal{F}$ is first order universal, and*
- *$\mathcal{F} = Fr(L)$ for L a uni-modal, finitely axiomatizable, and canonical logic.*

Proof: Let $\mathcal{F}$ and L be as defined in Theorem 3.2. It remains to prove that $\mathcal{F}_{\boxed{*}}$-satisfiability is Σ^1_1-complete. The Σ^1_1 upper bound is immediate, since any $\mathcal{F}_{\boxed{*}}$-satisfiable formula is satisfiable in a countable $\mathcal{F}_{\boxed{*}}$ frame. The reduction from the proof of Theorem 3.4 witnesses the Σ^1_1-hardness. □

4 Universal modality versus transitive closure Intuitively, $\boxed{*}$ is a more difficult modality than $\boxed{u}$. After all, $\boxed{u}$ behaves like **S5**, while $\boxed{*}$ behaves like **S4**, and **S5**-satisfiability is NP-complete, whereas **S4**-satisfiability is PSPACE-complete (Ladner [17]). And indeed, in all the examples that we have seen, $\mathcal{F}_{\boxed{*}}$-satisfiability is at least as hard as $\mathcal{F}_{\boxed{u}}$-satisfiability. In this section, we will show that this is a general phenomenon: for well-behaved classes of frames $\mathcal{F}$ and many complexity classes C, if $\mathcal{F}_{\boxed{*}}$-satisfiability is in C then so is $\mathcal{F}_{\boxed{u}}$-satisfiability.

We first prove that for well-behaved classes of frames $\mathcal{F}$, $\mathcal{F}_{\boxed{u}}$-satisfiability nondeterministic polynomial time conjunctive truth-table ($\leq^{NP}_{ctt}$) reduces to $\mathcal{F}_{\boxed{*}}$-satisfiability, where $\leq^{NP}_{ctt}$ is defined as follows. $A \leq^{NP}_{ctt} B$ if and only if there exists an NP machine M with an output tape such that $x \in A$ if and only if for some computation on input x, M outputs $y_1 \# y_2 \# \cdots \# y_k$, and $\{y_1, \ldots, y_k\} \subseteq B$ (Ladner, Lynch, and Selman [18]).

Theorem 4.1 *If $\mathcal{F}$ is closed under isomorphism, disjoint union, and generated subframes, then $\mathcal{F}_{\boxed{u}}$-satisfiability is $\leq^{NP}_{ctt}$ reducible to $\mathcal{F}_{\boxed{*}}$-satisfiability.*

Corollary 4.2 *Let $\mathcal{F}$ be closed under isomorphism, disjoint union, and generated subframes, and let C be a complexity class closed under $\leq^{NP}_{ctt}$ reductions. If $\mathcal{F}_{\boxed{*}}$-satisfiability is in C then so is $\mathcal{F}_{\boxed{u}}$-satisfiability.*

Corollary 4.2 is often applicable, since many complexity classes that we commonly encounter when proving complexity for modal satisfiability problems, such as NP, PSPACE, EXPTIME, NEXPTIME, etc., are closed under $\leq^{NP}_{ctt}$ reductions.

Before proving Theorem 4.1, note that demanding closure of the class of frames under isomorphism, disjoint union, and generated subframes is not restrictive in the $\boxed{*}$ case.

Lemma 4.3 *If $\widehat{\mathcal{F}}$ is the closure under isomorphism, disjoint union and generated subframes of $\mathcal{F}$, then $\mathcal{F}_{\boxed{*}}$-satisfiability = $\widehat{\mathcal{F}}_{\boxed{*}}$-satisfiability.*

The situation is different for $\mathcal{L}_{\boxed{u}}$ formulas. After the proof of Theorem 4.1, we will show that it is necessary to require that the class of frames be closed under isomorphism, disjoint union, and generated subframes. We will show that there exist counterexamples of arbitrarily high complexity if we fail to meet any of the three requirements.

Proof of Theorem 4.1: We have to use $\boxed{\ast}$ to simulate $\boxed{u}$, but we cannot just replace $\boxed{u}$ by $\boxed{\ast}$. For a very simple counterexample, consider the class of all frames that consist of the disjoint union of singletons and the formula $p \wedge \langle u\rangle\neg p$. This formula is satisfiable on this class, but $p \wedge \langle\ast\rangle\neg p$ is not.

One of the problems is that every $\mathcal{F}$-satisfiable $\mathcal{L}_{\boxed{\ast}}$ formula is satisfiable in a rooted generated $\mathcal{F}$ subframe, but that this is not the case for $\mathcal{L}_{\boxed{u}}$-formulas. However, as the next lemma shows, $\mathcal{L}_{\boxed{u}}$ formulas *are* satisfiable on generated subframes with a small number of roots.

Lemma 4.4 *Let $\mathcal{F}$ be closed under isomorphism, disjoint union, and generated subframes and let φ be an $\mathcal{L}_{\boxed{u}}$ formula. If φ is $\mathcal{F}_{\boxed{u}}$-satisfiable, then there exist a model M, an integer $k \leq$ the number of $\boxed{u}$s in φ, and worlds $w_0, w_1, \ldots, w_k$ in M such that*

- *$M, w_0 \models \varphi$,*
- *M is based on an $\mathcal{F}$ frame,*
- *all worlds in M are reachable from $\{w_0, w_1, \ldots, w_k\}$, and*
- *for all $\boxed{u}\psi \in Cl(\varphi)$, if $M, w_0 \not\models \boxed{u}\psi$, then $M, w_i \not\models \psi$ for some $0 \leq i \leq k$.*

Proof: The construction is reminiscent of the proof that **S5**-satisfiability is in NP from Ladner [17]. This is not surprising, since the $\boxed{u}$ operator behaves like the **S5** operator. Suppose that φ is $\mathcal{F}_{\boxed{u}}$-satisfiable. Let $M_0 = \langle W_0, \{R_i\}_{i\in I}, \pi\rangle$ and $w_0 \in W_0$ be such that $M_0, w_0 \models \varphi$ and $\langle W_0, \{R_i\}_{i\in I}\rangle \in \mathcal{F}$. Let $\boxed{u}\psi_1, \boxed{u}\psi_2, \ldots, \boxed{u}\psi_k$ be an enumeration of all $\boxed{u}\psi \in Cl(\varphi)$ that do *not* hold in w_0. Note that $k \leq$ the number of $\boxed{u}$s in φ. Let $w_1, w_2, \ldots, w_k$ be worlds such that $M_0, w_i \not\models \psi_i$, let W be the set of worlds reachable from $w_0, w_1, \ldots, w_k$, and let M be the restriction of M_0 to W. We claim that M fulfills the requirements of Lemma 4.4.

Since $\mathcal{F}$ is closed under generated subframes, M is based on an $\mathcal{F}$ frame. We will now show that every world in W satisfies the same set of $Cl(\varphi)$ formulas in M as in M_0. We will use induction on the structure of the formula. The only nontrivial case is for $\boxed{u}\psi$.

So suppose that $w \in W$, and that $M_0, w \models \boxed{u}\psi$. Then $\forall w' \in W_0$, $M_0, w' \models \psi$. Using the induction hypothesis and the fact that $W \subseteq W_0$, it follows that $M, w \models \boxed{u}\psi$. For the converse, suppose that $M_0, w \not\models \boxed{u}\psi$. By definition, for some $1 \leq i \leq k$, $M_0, w_i \not\models \psi$. Since $w_i \in W$, again, by induction, $M, w_i \not\models \psi$, and therefore $M, w \not\models \boxed{u}\psi$.

From this, it follows immediately that $M, w_0 \models \varphi$, and that for all $\boxed{u}\psi \in Cl(\varphi)$, if $M, w_0 \not\models \boxed{u}\psi$, then for some i, $M, w_i \not\models \psi$. □

But there are more problems to replacing $\boxed{u}$ by $\boxed{\ast}$, even if we look at rooted frames. For example, the formula $\langle\ast\rangle\boxed{\ast}p \wedge \langle\ast\rangle\neg p$ is certainly satisfiable, but $\langle u\rangle\boxed{u}p \wedge \langle u\rangle\neg p$ is not satisfiable on any frame. This problem is caused by the simple fact that nested $\boxed{\ast}$ operators behave very differently from nested $\boxed{u}$ operators. This is why we will first bring an $\mathcal{L}_{\boxed{u}}$ formula φ in a form that restricts the depth of $\boxed{u}$ nesting. This is pretty simple: first we introduce propositional variables $p_{\boxed{u}\psi}$ for all $\boxed{u}\psi \in Cl(\varphi)$. Now define φ' inductively as follows:

$$p' = p;\ (\neg\psi)' = \neg\psi';\ (\psi \wedge \xi)' = \psi' \wedge \xi';\ (\boxed{a}\psi)' = \boxed{a}\psi';\ (\boxed{u}\psi)' = p_{\boxed{u}\psi}.$$

Note that φ' does not contain $\boxed{u}$. The following lemma shows how to convert φ into a formula φ_{flat} of small $\boxed{u}$ nesting depth.

Lemma 4.5 *φ is $\mathcal{F}_{\boxed{u}}$-satisfiable if and only if the following formula φ_{flat} is $\mathcal{F}_{\boxed{u}}$-satisfiable.*

$$\varphi_{flat} = \varphi' \wedge \boxed{u} \bigwedge_{\boxed{u}\psi \in Cl(\varphi)} (p_{\boxed{u}\psi} \leftrightarrow \boxed{u}\psi').$$

Proof: Let $M = \langle W, \{R_i\}_{i \in I}, \pi \rangle$ and $w_0 \in W$ be such that $M, w_0 \models \varphi$. Extend π such that for each $\boxed{u}\psi \in Cl(\varphi)$ and $w \in W$, $M, w \models p_{\boxed{u}\psi}$ if and only if $M, w \models \boxed{u}\psi$. By induction on the structure of ψ, it is easy to prove that for all $\psi \in Cl(\varphi)$ and all $w \in W$, $M, w \models \psi$ if and only if $M, w \models \psi'$. From this it follows that $M, w_0 \models \varphi'$. It also follows that for all $w \in W$ and for all $\boxed{u}\psi \in Cl(\varphi)$, $M, w \models p_{\boxed{u}\psi}$ if and only if $M, w \models \boxed{u}\psi$ if and only if $\forall w' \in W$, $M, w' \models \psi$ if and only if $\forall w' \in W$, $M, w' \models \psi'$ if and only if $M, w \models \boxed{u}\psi'$. And thus, $M, w_0 \models \varphi_{flat}$.

For the converse, let $M = \langle W, \{R_i\}_{i \in I}, \pi \rangle$ and $w_0 \in W$ be such that $M, w_0 \models \varphi_{flat}$. We will show by induction that for all $\psi \in Cl(\varphi)$ and $w \in W$, $M, w \models \psi$ if and only if $M, w \models \psi'$. The only nontrivial step is for formulas of the form $\boxed{u}\psi$. It holds that $M, w \models \boxed{u}\psi$ if and only if $\forall w' \in W$, $M, w' \models \psi$ if and only if $\forall w' \in W$, $M, w' \models \psi'$ if and only if $\forall w' \in W$, $M, w' \models \boxed{u}\psi'$ if and only if $M, w \models \boxed{u}\psi'$ if and only if $M, w \models p_{\boxed{u}\psi}$ if and only if $M, w \models (\boxed{u}\psi)'$. Thus, $M, w_0 \models \varphi$. □

Lemma 4.5 brings an $\mathcal{L}_{\boxed{u}}$ formula in such a form that $\boxed{u}$ can be simulated by $\boxed{*}$ on rooted frames. Lemma 4.4 limits the number of rooted frames needed to satisfy an $\mathcal{L}_{\boxed{u}}$ formula in such a way that the behavior of the $\boxed{u}\psi$ subformulas depends solely on these roots. These two facts lead to the following lemma.

Lemma 4.6 *Let $\mathcal{F}$ be closed under isomorphism, disjoint union, and generated subframes, and let φ be an $\mathcal{L}_{\boxed{u}}$ formula. Then φ is $\mathcal{F}_{\boxed{u}}$-satisfiable if and only if there exist an integer $k \leq |\varphi|$ and sets $\Gamma_0, \Gamma_1, \ldots, \Gamma_k \subseteq \{\psi' \mid \psi \in Cl(\varphi)\} \cup \{\boxed{*}\psi' \mid \boxed{u}\psi \in Cl(\varphi)\}$ such that the following hold:*

1. *$\varphi' \in \Gamma_0$,*
2. *for $0 \leq i \leq k$, the following formula is $\mathcal{F}_{\boxed{*}}$-satisfiable:*

$$\bigwedge \Gamma_i \wedge \bigwedge_{\psi \in Cl(\varphi) \setminus \Gamma_i} \neg\psi \wedge \bigwedge_{\boxed{u}\psi \in Cl(\varphi)} ((p_{\boxed{u}\psi} \rightarrow \boxed{*}p_{\boxed{u}\psi}) \wedge (\neg p_{\boxed{u}\psi} \rightarrow \boxed{*}\neg p_{\boxed{u}\psi})),$$

3. *for all $\boxed{u}\psi \in Cl(\varphi)$, $p_{\boxed{u}\psi} \in \Gamma_i$ if and only if $\boxed{*}\psi' \in \Gamma_j$ for all j, and*
4. *for all $\boxed{u}\psi \in Cl(\varphi)$, if $\neg p_{\boxed{u}\psi} \in \Gamma_i$ then $\neg\psi' \in \Gamma_j$ for some j.*

Now we can finish the proof of Theorem 4.1, i.e., we can show that $\mathcal{F}_{\boxed{u}}$-satisfiability is $\leq_{ctt}^{NP}$ reducible to $\mathcal{F}_{\boxed{*}}$-satisfiability. Let M be a nondeterministic Turing machine with an output tape that on input φ guesses an integer $k \leq |\varphi|$ and sets $\Gamma_0, \Gamma_1, \ldots, \Gamma_k \subseteq \{\psi' \mid \psi \in Cl(\varphi)\} \cup \{\boxed{*}\psi' \mid \boxed{u}\psi \in Cl(\varphi)\}$, verifies that conditions 1, 3, and 4 of Lemma 4.6 hold, and if so, writes the $k+1$ formulas of condition 2 on its output tape, separated by #'s. Since the size of $Cl(\varphi)$ is linear in the length of φ, and 1, 3, and 4 can be checked in deterministic polynomial time in the length of φ, M witnesses the $\leq_{ctt}^{NP}$ reduction from $\mathcal{F}_{\boxed{u}}$-satisfiability to $\mathcal{F}_{\boxed{*}}$-satisfiability. □

Proof of Lemma 4.6: First suppose that φ is $\mathcal{F}_{\boxed{u}}$-satisfiable. By Lemma 4.5, so is $\varphi_{flat} = \varphi' \wedge \boxed{u} \bigwedge_{\boxed{u}\psi \in Cl(\varphi)} (p_{\boxed{u}\psi} \leftrightarrow \boxed{u}\psi')$. By Lemma 4.4, there exist a model M, an integer $k \leq$ number of $\boxed{u}$'s in $\varphi_{flat} \leq |\varphi|$, and worlds $w_0, w_1, \ldots, w_k$ in M such that

- $M, w_0 \models \varphi_{flat}$,
- M is based on an $\mathcal{F}$ frame,
- all worlds in M are reachable from $\{w_0, w_1, \ldots, w_k\}$, and
- for all $\boxed{u}\psi \in Cl(\varphi)$, if $M, w_0 \not\models \boxed{u}\psi'$, then $M, w_i \not\models \psi'$ for some $0 \leq i \leq k$.

Let Γ_i be the set of relevant $\mathcal{L}_{\boxed{*}}$ formulas that are satisfied in M at w_i. That is, $\Gamma_i = \{\psi' \mid \psi \in Cl(\varphi) \text{ and } M, w_i \models \psi'\} \cup \{\boxed{*}\psi' \mid \boxed{u}\psi \in Cl(\varphi) \text{ and } M, w_i \models \boxed{*}\psi'\}$. We claim that these Γ_i's fulfill the requirements of the lemma.

1. $\varphi' \in \Gamma_0$, since $\varphi \in Cl(\varphi)$ and $M, w_0 \models \varphi'$.
2. First of all, $M, w_i \models \bigwedge \Gamma_i \wedge \bigwedge_{\psi \in Cl(\varphi) \setminus \Gamma_i} \neg\psi$ by definition. In addition, for all $\boxed{u}\psi \in Cl(\varphi)$ and $w \in W$, $M, w \models p_{\boxed{u}\psi} \leftrightarrow \boxed{u}\psi'$. This implies that either $M, w \models p_{\boxed{u}\psi}$ for all w, or that $M, w \models \neg p_{\boxed{u}\psi}$ for all w. It follows immediately that for all $w \in W$, $M, w \models \bigwedge_{\boxed{u}\psi \in Cl(\varphi)}((p_{\boxed{u}\psi} \rightarrow \boxed{*}p_{\boxed{u}\psi}) \wedge (\neg p_{\boxed{u}\psi} \rightarrow \boxed{*}\neg p_{\boxed{u}\psi}))$.
3. Let $\boxed{u}\psi \in Cl(\varphi)$. Note that $p_{\boxed{u}\psi} \in \Gamma_i$ if and only if $M, w_i \models p_{\boxed{u}\psi}$ if and only if $M, w_i \models \boxed{u}\psi'$ if and only if $\forall w \in W, M, w \models \psi'$ if and only if for all j, $M, w_j \models \boxed{*}\psi'$.
4. Finally, suppose that $\boxed{u}\psi \in Cl(\varphi)$ and that $p_{\boxed{u}\psi} \notin \Gamma_i$. Then $M, w_i \not\models p_{\boxed{u}\psi}$, and thus $M, w_i \not\models \boxed{u}\psi'$. It follows that for some j, $0 \leq j \leq k$, $M, w_j \not\models \psi'$, and therefore $\psi' \notin \Gamma_j$.

To show the converse, suppose that $\Gamma_0, \Gamma_1, \ldots, \Gamma_k$ fulfill the conditions of Lemma 4.6. Let $M_0, M_1, \ldots, M_k$ be models based on frames in $\mathcal{F}$ and $w_0, w_1, \ldots, w_k$ be worlds such that w_i is a world in M_i and $M_i, w_i \models \bigwedge \Gamma_i \wedge \bigwedge_{\psi \in Cl(\varphi)\setminus\Gamma_i} \neg\psi \wedge \bigwedge_{\boxed{u}\psi \in Cl(\varphi)}((p_{\boxed{u}\psi} \rightarrow \boxed{*}p_{\boxed{u}\psi}) \wedge (\neg p_{\boxed{u}\psi} \rightarrow \boxed{*}\neg p_{\boxed{u}\psi}))$. Suppose that M_i is generated by w_i and that the models are disjoint. Now, let M be the union of these models. This model is based on an $\mathcal{F}$ frame as well, since $\mathcal{F}$ is closed under disjoint union. We will show that $M, w_0 \models \varphi_{flat}$. This implies that φ is $\mathcal{F}_{\boxed{u}}$-satisfiable by Lemma 4.5 and completes the proof of Lemma 4.6.

First of all, note that $M, w_0 \models \varphi'$, since φ' does not contain $\boxed{u}$ or $\boxed{*}$. It remains to show that for all $w \in W$ and $\boxed{u}\psi \in Cl(\varphi)$, $M, w \models p_{\boxed{u}\psi} \leftrightarrow \boxed{u}\psi'$. Suppose that w is reachable from w_i.

First suppose that $M, w \models p_{\boxed{u}\psi}$. By 2, $M, w_i \models \neg p_{\boxed{u}\psi} \rightarrow \boxed{*}\neg p_{\boxed{u}\psi}$. It follows that $M, w_i \models p_{\boxed{u}\psi}$, and by definition of M, $p_{\boxed{u}\psi} \in \Gamma_i$. It follows from 3 that $\boxed{*}\psi' \in \Gamma_j$ for all j and therefore also $M, w_j \models \boxed{*}\psi'$ for all j. This implies that $\forall w' \in W$, $M, w' \models \psi'$, and thus $M, w \models \boxed{u}\psi'$.

Finally, suppose that $M, w \not\models p_{\boxed{u}\psi}$. Since $M, w_i \models p_{\boxed{u}\psi} \rightarrow \boxed{*}p_{\boxed{u}\psi}$, it follows that $M, w_i \models \neg p_{\boxed{u}\psi}$. Since $Cl(\varphi)$ is closed under single negations, $\neg\boxed{u}\psi \in Cl(\varphi)$. It follows that $\neg p_{\boxed{u}\psi} \in \Gamma_i$, and therefore, by 4, $\neg\psi' \in \Gamma_j$ for some j. It follows that $M, w_j \models \neg\psi'$, which implies that $M, w \not\models \boxed{u}\psi'$. □

As mentioned in the beginning of this section, the requirements in Theorem 4.1 that $\mathcal{F}$ be closed under isomorphism, disjoint union, and generated subframes are all necessary. In Theorems 4.7, 4.10, and 4.11, we will construct arbitrarily hard counterexamples for classes of frames that have exactly two of the three closure properties.

Theorem 4.7 *For every set $A \subseteq \mathbb{N}$, there exists a class of frames $\mathcal{F}$ closed under isomorphism and disjoint union such that $\mathcal{F}_{\boxed{*}}$-satisfiability is in* PSPACE *and A*

in unary is polynomial-time many-one reducible to $\mathcal{F}_{\boxed{u}}$*-satisfiability.*

Proof: For all $i \in \mathbb{N}$, let F_i be the linear irreflexive frame on $\{0, \ldots, i\}$. That is, $F_i = \langle\{0, \ldots, i\}, \{\langle j, j{+}1\rangle\} \mid j < i\}\rangle$. Let $\mathcal{F}_i$ be the closure under isomorphism of F_i, and let $\mathcal{F}$ be the closure under disjoint union of the class of frames $\bigcup_{i \in A} \mathcal{F}_i$.

We first show that A in unary is polynomial-time many-one reducible to $\mathcal{F}_{\boxed{u}}$-satisfiability. Let φ_i be the formula $p \wedge \boxed{u}\Box\neg p \wedge \Diamond^i\top \wedge \Box^{i+1}\bot$. This formula is exactly satisfiable in worlds that have no predecessors, that have a sequence of i successors, and have no sequence of $i+1$ successors. Since we look only at frames that consist of the disjoint union of frames in $\mathcal{F}_j$, φ_i is exactly satisfiable in $\mathcal{F}$ frames that contain a frame in $\mathcal{F}_i$ as a disjoint.

It follows that φ_i is $\mathcal{F}$-satisfiable if and only if a frame in $\mathcal{F}_i$ occurs as a disjoint in some frame in $\mathcal{F}$ if and only if $i \in A$. Since φ_i is clearly computable in polynomial time in i, this shows that A in unary is polynomial-time many-one reducible to $\mathcal{F}_{\boxast}$-satisfiability.

It remains to show that $\mathcal{F}_{\boxast}$-satisfiability is in PSPACE. First suppose A is finite. Then, by Lemma 4.3, $\mathcal{F}_{\boxast}$-satisfiability amounts to determining satisfiability with respect to a finite set of finite frames, which is in NP, and therefore certainly in PSPACE. Now suppose that A is infinite. Then, by Lemma 4.3, $\mathcal{F}_{\boxast}$-satisfiability = $[\{F_i \mid i \in \mathbb{N}\}]_{\boxast}$-satisfiability. According to the following lemma, this is in PSPACE. $\Box$

Lemma 4.8 *For all* $i \in \mathbb{N}$*, let* $F_i = \langle\{0, \ldots, i\}, \{\langle j, j{+}1\rangle\} \mid j < i\}\rangle$*.* $[\{F_i \mid i \in \mathbb{N}\}]_{\boxast}$*-satisfiability is in* PSPACE*.*

$[\{F_i \mid i \in \mathbb{N}\}]_{\boxast}$ is very close to linear temporal logic with operators "nexttime," and "always in the future," the satisfiability problem of which is PSPACE-complete (Sistla and Clarke [25]). Reformulating their result in our notation yields the following theorem.

Theorem 4.9 ([25]) *Let* $\mathbb{N}$ *denote the natural numbers, and let* S *be the successor relation on the natural numbers, i.e.,* $S = \{\langle i, i+1\rangle \mid i \in \mathbb{N}\}$*.* $[\langle\mathbb{N}, S\rangle]_{\boxast}$*-satisfiability is* PSPACE*-complete.*

Proof of Lemma 4.8: First suppose φ is satisfiable on F_k for some $k \in \mathbb{N}$. Let $M = \langle F_k, \pi\rangle$, and suppose that $M, 0 \models \varphi$. To encode M into a model $M' = \langle\mathbb{N}, S, \pi'\rangle$, we will use a new propositional variable w. w will be true in worlds that correspond to worlds in M. Formally, we encode M by model $M' = \langle\mathbb{N}, S, \pi'\rangle$ as follows: π and π' coincide on all propositional variables in φ on all worlds in W, and $M', i \models w$ if and only if i is a world in M if and only if $i \le k$.

Define φ' by replacing all subformulas of the form $\Box\psi$ by $\Box(w \to \psi')$, and all subformulas of the form $\boxast\psi$ by $\boxast(w \to \psi')$. Then, for all $i \le k$, $M, i \models \varphi$ if and only if $M', i \models \varphi'$.

Not all valuations on a $\langle\mathbb{N}, S\rangle$ frame correspond to a finite prefix of $\mathbb{N}$, so we still need to ensure that the encoding model behaves properly. We need to enforce that if $M', 0 \models f(\varphi)$, then $\{i \mid M', i \models w\}$ is a nonempty, finite prefix of $\mathbb{N}$. Define $f(\varphi)$ as $\varphi' \wedge w \wedge \diamondast\neg w \wedge \boxast(\neg w \to \boxast\neg w)$. It is easy to verify that φ is satisfiable on F_i for some $i \in \mathbb{N}$ if and only if $f(\varphi)$ is satisfiable on $\langle\mathbb{N}, S\rangle$. Since f is clearly polynomial-time computable, and $[\langle\mathbb{N}, S\rangle]_{\boxast}$-satisfiability is in PSPACE, this proves that $[\{F_i \mid i \in \mathbb{N}\}]_{\boxast}$-satisfiability is in PSPACE. $\square$

Theorem 4.10 *For every set $A \subseteq \mathbb{N}$, there exists a class of frames $\mathcal{F}$ closed under isomorphism and generated subframes such that $\mathcal{F}_{\boxast}$-satisfiability is in PSPACE and A in unary is polynomial-time many-one reducible to $\mathcal{F}_{\boxed{u}}$-satisfiability.*

Proof: For all $i \in \mathbb{N}$, let $\widehat{F_i}$ be the frame F_i from Lemma 4.8 with extra edge $\langle 0, 0\rangle$, that is, $\widehat{F_i} = \langle\{0, \ldots, i\}, \{\langle 0, 0\rangle\} \cup \{\langle j, j{+}1\rangle\} \mid j < i\}\rangle$. Let $\widehat{\mathcal{F}_i}$ be the closure under isomorphism of $\widehat{F_i}$. Let $\mathcal{F}$ be the closure under generated subframes of $\bigcup_{i\in\mathbb{N}} \widehat{\mathcal{F}_i}$ and the disjoint union of $\widehat{\mathcal{F}_i}$ and $\widehat{\mathcal{F}_i}$ for all $i \in A$. Note that if A is infinite, $\mathcal{F}$ consists exactly of these frames and frames in the disjoint union of $\widehat{\mathcal{F}_i}$ and $\mathcal{F}_j$ for $i \in A$, $j \in \mathbb{N}$, the disjoint union of $\mathcal{F}_i$ and $\mathcal{F}_j$ for $i, j \in \mathbb{N}$, and $\mathcal{F}_i$ for $i \in \mathbb{N}$.

We first show that A in unary is polynomial-time many-one reducible to $\mathcal{F}_{\boxed{u}}$-satisfiability. Let φ_i be the following formula.

$$p \wedge \Diamond p \wedge \Diamond(\neg p \wedge \Diamond^{i-1}\top \wedge \Box^i\bot) \wedge \langle u\rangle(\neg p \wedge \Diamond\neg p \wedge \Diamond(p \wedge \Diamond^{i-1}\top \wedge \Box^i\bot)).$$

The formula $p \wedge \Diamond p \wedge \Diamond(\neg p \wedge \Diamond^{i-1}\top \wedge \Box^i\bot)$ is satisfiable on a frame $F \in \mathcal{F}$ in world w if and only if F contains a frame in $\widehat{\mathcal{F}_i}$ as a disjoint, and w is the root of this disjoint. The same is true for formula $\neg p \wedge \Diamond\neg p \wedge \Diamond(p \wedge \Diamond^{i-1}\top \wedge \Box^i\bot)$. Since both formulas cannot be satisfied in the same world, it follows that φ_i is satisfiable on frame $F \in \mathcal{F}$ if and only if F contains two disjoints from $\widehat{\mathcal{F}_i}$ if and only if $i \in A$. This proves that A in unary polynomial-time many-one reduces to $\mathcal{F}_{\boxed{u}}$-satisfiability.

It remains to show that $\mathcal{F}_{\boxast}$-satisfiability is in PSPACE. By Lemma 4.3, $\mathcal{F}_{\boxast}$-satisfiability = $[\{\widehat{F_i} \mid i \in \mathbb{N}\}]_{\boxast}$-satisfiability. It follows that φ is $\mathcal{F}_{\boxast}$-satisfiable if and only if

- φ is satisfiable with respect to $\{F_i \mid i \in \mathbb{N}\}$, or
- φ is satisfiable with respect to $\widehat{F_0}$, or
- φ is satisfiable in the root of $\widehat{F_i}$ for some $i \geq 1$.

$[\{F_i \mid i \in \mathbb{N}\}]_{\boxast}$-satisfiability is in PSPACE by Lemma 4.8, and $[\widehat{F_0}]_{\boxast}$-satisfiability is in NP. It remains to show that determining if an $\mathcal{L}_{\boxast}$ formula is satisfiable in the root of $\widehat{F_i}$ for some $i \geq 1$ is in PSPACE. We claim that this is the case if and only if there exist subsets Γ and Δ of $Cl(\varphi)$ such that:

- $\varphi \in \Gamma$,
- $\forall\neg\psi \in Cl(\varphi)$, $\neg\psi \in \Gamma$ if and only if $\psi \notin \Gamma$,
- $\forall\psi \wedge \xi \in Cl(\varphi)$, $\varphi \wedge \xi \in \Gamma$ if and only if $\psi \in \Gamma$ and $\xi \in \Gamma$,
- $\forall\Box\psi \in Cl(\varphi)$, $\Box\psi \in \Gamma$ if and only if $\psi \in \Gamma$ and $\psi \in \Delta$,
- $\forall\boxast\psi \in Cl(\varphi)$, $\boxast\psi \in \Gamma$ if and only if $\psi \in \Gamma$ and $\boxast\psi \in \Delta$, and
- $\bigwedge\Delta \wedge \bigwedge_{\psi\in Cl(\varphi)\setminus\Delta} \neg\psi$ is $[\{F_i \mid i \in \mathbb{N}\}]_{\boxast}$-satisfiable.

Since subsets of $Cl(\varphi)$ can be represented in space linear in the length of φ, and $[\{F_i \mid i \in \mathbb{N}\}]_{\boxast}$-satisfiability is in PSPACE by Lemma 4.8, it follows that $\mathcal{F}_{\boxast}$-satisfiability is in PSPACE. It remains to prove the claim.

First suppose φ is satisfiable in the root of $\widehat{F_i}$ for some $i \geq 1$. Let M be the model based on $\widehat{F_i}$ such that $M, 0 \models \varphi$. Let Γ be the set of $Cl(\varphi)$ formulas satisfied in M at

0, and let Δ be the set of $Cl(\varphi)$ formulas satisfied at world 1. It is immediate that Γ and Δ fulfill the requirements.

For the converse, suppose there exist sets Γ and Δ that fulfill the requirements. Let $k \geq 0$ and $M = \langle F_k, \pi \rangle$ be such that $M, 0 \models \bigwedge \Delta \wedge \bigwedge_{\psi \in Cl(\varphi) \setminus \Delta} \neg \psi$. Let $\widehat{M} = \langle \widehat{F}_{k+1}, \widehat{\pi} \rangle$. Define $\widehat{\pi}$ on all propositional variables p in φ such that $\widehat{M}, 0 \models p$ if and only if $p \in \Gamma$ and for all $i \leq k$, $M, i \models p$ if and only if $\widehat{M}, i+1 \models p$. With induction, it is easy to show that for all $\psi \in Cl(\varphi)$, $\widehat{M}, 0 \models \psi$ if and only if $\psi \in \Gamma$. Since $\varphi \in \Gamma$, it follows that $\widehat{M}, 0 \models \varphi$ as required. $\square$

Theorem 4.11 *For every set $A \subseteq \mathbb{N}$, there exists a class of frames $\mathcal{F}$ closed under disjoint union and generated subframes such that $\mathcal{F}_{\boxed{*}}$-satisfiability is in* PSPACE *and A in unary is polynomial time many-one reducible to $\mathcal{F}_{\boxed{u}}$-satisfiability.*

Proof: For all $i \in \mathbb{N}$, let $\widehat{F}_i$ be the frame from the proof of Theorem 4.10, that is, $\widehat{F}_i = \langle \{0, \ldots, i\}, \{\langle 0, 0\rangle\} \cup \{\langle j, j+1\rangle\} \mid j < i\}\rangle$ and let $\widehat{G}_i = \langle \{0', \ldots, i'\}, \{\langle 0', 0'\rangle\} \cup \{\langle j', (j+1)'\rangle\} \mid j < i\}\rangle$. Define $\mathcal{F}$ as the closure under generated subframes and disjoint union of $\bigcup_{i \in \mathbb{N}} \widehat{F}_i \cup \{\widehat{F}_i \cup \widehat{G}_i \mid i \in A\}$. The same reduction as in the proof of Theorem 4.10 reduces A in unary to $\mathcal{F}_{\boxed{u}}$-satisfiability. In addition, $\mathcal{F}_{\boxed{*}}$-satisfiability = $[\{\widehat{F}_i \mid i \in \mathbb{N}\}]_{\boxed{*}}$-satisfiability, which is in PSPACE by the proof of Theorem 4.10. $\square$

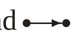
5 Lower bounds As we have shown in Section 3, adding $\boxed{u}$ or $\boxed{*}$ to a language can increase the complexity of the satisfiability problem dramatically. In this section, we will study the following related question of whether the complexity always increases, and if so, whether we can give a lower bound on the complexity of the resulting logic. From the examples in the introduction, it seems that EXPTIME is a prime candidate for multi-modal logics. Note that we certainly cannot do better, since the satisfiability problems for the examples in the introduction are EXPTIME-complete. As we shall see in this section, it is indeed the case that adding $\boxed{u}$ or $\boxed{*}$ to almost all multi-modal logics forces EXPTIME-hardness. We will give a criterion which exactly characterizes when the resulting logic will be EXPTIME-hard.

We first look at the simplest multi-modal case: bi-modal logics with two independent modalities. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two classes of uni-modal frames. The *join* of $\mathcal{F}_1$ and $\mathcal{F}_2$, denoted by $\mathcal{F}_1 \oplus \mathcal{F}_2$ is the class $\{\langle W, R_1, R_2\rangle \mid \langle W, R_1\rangle \in \mathcal{F}_1$ and $\langle W, R_2\rangle \in \mathcal{F}_2\}$. To avoid anomalies, we will require that the frame classes are closed under isomorphism and disjoint union. This is essential, since for example $\{\bullet\!\!\rightarrow\!\!\bullet\} \oplus \{\bullet\} = \emptyset$. For the relationship between the join and its uni-modal fragments, see Fine and Schurz [6], Kracht and Wolter [16], and Hemaspaandra [14]. We are interested in the complexity of $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{*}}$-satisfiability and $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiability.

Here is how we will proceed in this section. In Theorem 5.1, we will show that $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{*}}$-satisfiability and $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiability are EXPTIME-hard if one class of frames contains a rooted subframe of size three, and the other class of frames contains a rooted subframe of size two. Next, we will show in Theorem 5.2 that a class of singleton frames does not contribute to the complexity. The remaining cases are when both classes of frames contain a rooted subframe of size two, but not larger. We will show that in all these cases $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{*}}$-satisfiability is PSPACE-complete (Theorem 5.3), and that $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiability is PSPACE-complete if $\mathcal{F}_1$ and $\mathcal{F}_2$ are closed under generated subframes (Theorem 5.10). In the previous section, we showed that there are cases when $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiability is harder than $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{\circledast}}$-satisfiability. Surprisingly, we can find an example of this phenomenon even in this very restricted case (Theorem 5.11). We end this section by showing that all these results generalize quite well to the join of an arbitrary number of uni-modal logics (Theorem 5.13).

The EXPTIME lower bound proofs of the examples stated in the introduction are all variations of the reduction in the lower bound proof for propositional dynamic logic from Fischer and Ladner [7]. Loosely speaking, this technique can be applied if (sub)frames can look like binary trees. We won't go into the details of the proof, but we will show in what way our frames can look like binary trees.

Theorem 5.1 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be closed under isomorphism and disjoint union. If $\mathcal{F}_1$ contains a rooted subframe of size three, and $\mathcal{F}_2$ contains a rooted subframe of size two, then $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiability and $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{\circledast}}$-satisfiability are* EXPTIME*-hard.*

Proof: For case 1, note that  or is a skeleton subframe of $\mathcal{F}_1$, and  is a skeleton subframe of $\mathcal{F}_2$. It follows that one of the two frames in Figure 1 is a skeleton subframe of $\mathcal{F}_1 \oplus \mathcal{F}_2$.

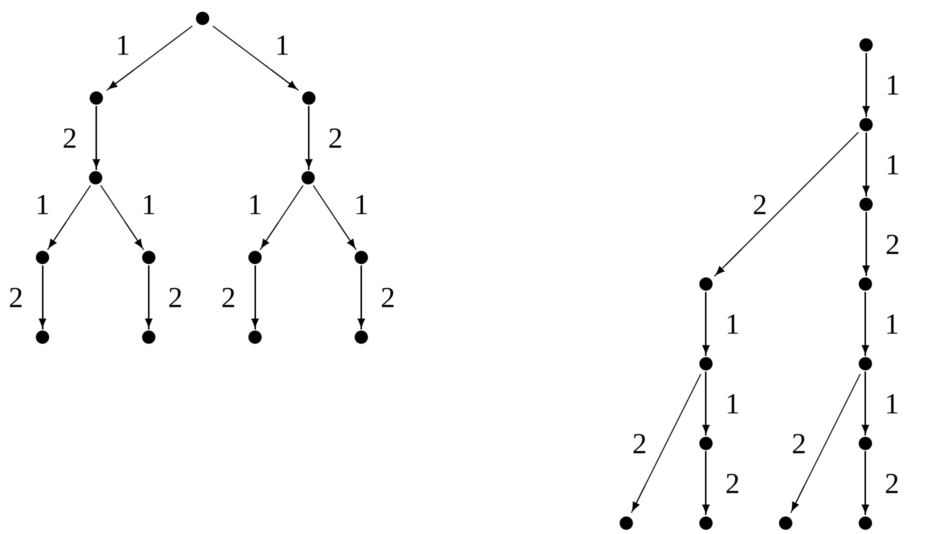


Figure 1:

These structures look like binary trees. Note that it might be necessary to add some edges to these structures to obtain an $\mathcal{F}_1 \oplus \mathcal{F}_2$ subframe, but all new 1 (2) edges will be between nodes that are already connected by a 1 (2) path. It can be shown that adding these edges will keep the structures tree-like enough to immediately apply the EXPTIME-hardness proof of propositional dynamic logic from Fischer and Ladner [7]. □

What happens if we cannot apply Theorem 5.1? First note that if one of the classes of frames, say $\mathcal{F}_2$, does not contain a rooted subframe of size two, then every frame in $\mathcal{F}_2$ consists of the disjoint union of singletons. The following theorem states that such a class of frames does not increase the complexity.

Theorem 5.2 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be closed under isomorphism and disjoint union. If every frame in $\mathcal{F}_2$ consists of the disjoint union of singletons, then*

1. *$[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiability polynomial-time reduces to $[\mathcal{F}_1]_{\boxed{u}}$-satisfiability, and*
2. *$[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{*}}$-satisfiability polynomial-time reduces to $[\mathcal{F}_1]_{\boxed{*}}$-satisfiability.*

Proof: The idea for both reductions is the following. Let φ be a formula, and suppose $M = \langle W, R_1, R_2, \pi\rangle$ is a model based on an $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame. Since every frame in $\mathcal{F}_2$ consists of the disjoint union of singletons, $R_2 \subseteq \{\langle w, w\rangle \mid w \in W\}$. We will encode R_2 by a propositional variable r not in φ, that will be true in worlds that are R_2 reflexive. Formally, we encode M by model $M' = \langle W, R_1, \pi'\rangle$ where π' and π coincide on all propositional variables in φ, and $M', w \models r$ if and only if wR_2w.

Define φ' by replacing all subformulas of the form $\boxed{2}\psi$ in φ by $(r \to \psi')$. Then, $M, w \models \varphi$ if and only if $M', w \models \varphi'$.

It may seem that this is the desired reduction. Certainly, if φ is $\mathcal{F}_1 \oplus \mathcal{F}_2$-satisfiable, then φ' is $\mathcal{F}_1$-satisfiable. However, the converse does not necessarily hold. For example, suppose that all frames in $\mathcal{F}_2$ are reflexive. Then $\boxed{2}\bot$ is not $\mathcal{F}_1 \oplus \mathcal{F}_2$-satisfiable, but $(r \to \bot)$ *is* $\mathcal{F}_1$-satisfiable.

Obviously, our reductions need to restrict the valuation of r in an appropriate manner. The situation is different for $\boxed{u}$ and $\boxed{*}$, and we will start with $\boxed{u}$. We claim that f is a polynomial-time reduction from $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiability to $[\mathcal{F}_1]_{\boxed{u}}$-satisfiability, where f is defined as follows:

- $f(\varphi) = \varphi'$ if $\mathcal{F}_2$ contains a reflexive frame and an irreflexive frame,
- $f(\varphi) = \varphi' \wedge \boxed{u}r$ if all worlds in $\mathcal{F}_2$ are reflexive,
- $f(\varphi) = \varphi' \wedge \boxed{u}\neg r$ if all worlds in $\mathcal{F}_2$ are irreflexive,
- $f(\varphi) = \varphi' \wedge \langle u\rangle r$ if $\mathcal{F}_2$ contains a reflexive frame, but no irreflexive frames,
- $f(\varphi) = \varphi' \wedge \langle u\rangle\neg r$ if $\mathcal{F}_2$ contains an irreflexive frame, but no reflexive frames, and
- $f(\varphi) = \varphi' \wedge \langle u\rangle r \wedge \langle u\rangle\neg r$ if $\mathcal{F}_2$ contains neither reflexive frames nor irreflexive frames.

f is obviously computable in polynomial time, and it is clear that $M, w \models \varphi$ implies that $M', w \models f(\varphi)$, with M' defined as before. It remains to show that if $f(\varphi)$ is $[\mathcal{F}_1]_{\boxed{u}}$-satisfiable, then φ is $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiable.

Let $M' = \langle W, R_1, \pi\rangle$ and $w \in W$ be such that $\langle W, R_1\rangle \in \mathcal{F}_1$ and $M', w \models f(\varphi)$. Let $M = \langle W, R_1, R_2, \pi\rangle$ where $R_2 = \{\langle w, w\rangle \mid M', w \models r\}$. Then $M, w \models \varphi$. $\langle W, R_1, R_2\rangle$ is not necessarily an $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame, but if we take enough disjoint copies of M, the resulting model will be based on an $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame and will of course still satisfy φ.

In a similar way, the polynomial-time reduction from $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{*}}$-satisfiability to $[\mathcal{F}_1]_{\boxed{*}}$-satisfiability is defined as follows:

- $g(\varphi) = \varphi'$ if $\mathcal{F}_2$ contains reflexive and irreflexive worlds,
- $g(\varphi) = \varphi' \wedge \boxed{*}r$ if all worlds in $\mathcal{F}_2$ are reflexive, and
- $g(\varphi) = \varphi' \wedge \boxed{*}\neg r$ if all worlds $\mathcal{F}_2$ are irreflexive. $\square$

There is one case left: both $\mathcal{F}_1$ and $\mathcal{F}_2$ contain a rooted subframe of size two, but not of size three. We will first look at $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxast}$-satisfiability.

Theorem 5.3 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be closed under isomorphism and disjoint union. If $\mathcal{F}_1$ and $\mathcal{F}_2$ contain a rooted subframe of size two, but do not contain a rooted subframe of size three, then $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxast}$-satisfiability is* PSPACE*-complete.*

The main idea of the proof is the following observation. In a way to be made more precise below, the situation is very close to linear temporal logic with operators "next-time," and "always in the future," the satisfiability problem of which is PSPACE-complete (Sistla and Clarke [25]).

Theorem 5.4 ([25]) *Let $\mathbb{N}$ denote the natural numbers, and let S be the successor relation on the natural numbers, i.e., $S = \{\langle i, i+1\rangle \mid i \in \mathbb{N}\}$. $[\langle \mathbb{N}, S\rangle]_{\boxast}$-satisfiability is* PSPACE*-complete, even if we look only at formulas of the form $\varphi_1 \wedge \boxast\varphi_2$, with φ_1, φ_2 $\boxast$-less.*

Proof: By careful inspection from the proof of Sistla and Clarke [25] and the realization that their conjunct $\diamondast$ (accepting state) can be replaced by the equivalent $\boxast$(halting state $\to$ accepting state). Alternatively, note that the EXPTIME-hardness proof for propositional dynamic logic from Fischer and Ladner [7] degenerates to a PSPACE-hardness proof for $[\langle \mathbb{N}, S\rangle]_{\boxast}$-satisfiability and that their proof has the right formula property. □

Lemma 5.5 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be closed under isomorphism and disjoint union. If $\mathcal{F}_1$ and $\mathcal{F}_2$ contain a rooted subframe of size two but do not contain a rooted subframe of size three, then $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$ and $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxast}$-satisfiability are* PSPACE*-hard.*

Proof: We will construct polynomial time computable functions f and g such that for all formulas φ of the form $\varphi_1 \wedge \boxast\varphi_2$, with φ_1, φ_2 $\boxast$-less, φ is $[\langle \mathbb{N}, S\rangle]_{\boxast}$-satisfiable if and only if $f(\varphi)$ is $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiable if and only if $g(\varphi)$ is $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxast}$-satisfiable. Our construction is close to the proof that **S5** $\oplus$ **S5**-satisfiability is PSPACE-hard from Halpern and Moses [9].

Suppose φ is $[\langle \mathbb{N}, S\rangle]_{\boxast}$-satisfiable. Without loss of generality, we assume that φ is satisfiable in world 0. The frame $0R_1 1R_2 2R_1 3R_2 \ldots$ is a skeleton subframe of $\mathcal{F}_1 \oplus \mathcal{F}_2$. This frame is very close to $\langle \mathbb{N}, S\rangle$. Our reductions will simulate the satisfying $\langle \mathbb{N}, S\rangle$ model by a frame that contains this skeleton subframe in the following way. We will use $R_1 R_2$ to simulate S, and we will let world i in the satisfying model correspond to world $2i$ in the $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame. Let $F = \langle W, R_1, R_2\rangle$ be an $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame that contains $0R_1 1R_2 2R_1 3R_2 \ldots$ as a skeleton subframe. Note that every world in F has at most one nonreflexive R_1 successor and at most one nonreflexive R_2 successor.

We will use new propositional variables p_{even} to denote that a world is even and p_{odd} to denote that a world is odd. Define ψ' as follows on formulas with $\Box$ as only modal operator.

$$p' = p;\ (\neg\psi)' = \neg\psi';\ (\psi \wedge \xi)' = \psi' \wedge \xi';$$
$$(\Box\psi)' = p_{even} \to \boxed{1}(p_{odd} \to \boxed{2}(p_{even} \to \psi')).$$

Define reductions f and g as follows:

$$f(\varphi_1 \wedge \boxast\varphi_2) = \varphi_1' \wedge \boxed{u}(p_{even} \to \varphi_2') \wedge p_{even} \wedge$$

$$\wedge \boxed{u}(\neg(p_{odd} \wedge p_{even}) \wedge (p_{even} \to \langle 1 \rangle p_{odd}) \wedge (p_{odd} \to \langle 2 \rangle p_{even}))$$

$$\begin{aligned} g(\varphi_1 \wedge \boxed{\circledast}\varphi_2) = \varphi_1' \wedge \boxed{\circledast}(p_{even} \to \varphi_2') \wedge p_{even} \wedge \\ \boxed{\circledast}(\neg(p_{odd} \wedge p_{even}) \wedge (p_{even} \to \langle 1 \rangle p_{odd}) \wedge (p_{odd} \to \langle 2 \rangle p_{even})) \end{aligned}$$

Let $M = \langle \mathbb{N}, S, \pi \rangle$ be a model such that $M, 0 \models \varphi$, and let $F = \langle W, R_1, R_2 \rangle$ be the $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame defined above. Define $\widehat{M} = \langle F, \widehat{\pi} \rangle$ such that for all $i \in \mathbb{N}$ and all propositional variables p, $M, i \models p$ if and only if $\widehat{M}, 2i \models p$, and such that for all $w \in W$, $\widehat{M}, w \models p_{even}$ if and only $w \in \mathbb{N}$ and w is even, and $\widehat{M}, w \models p_{odd}$ if and only $w \in \mathbb{N}$ and w is odd. Then, for all $i \in \mathbb{N}$ and all formulas ψ with $\Box$ as only modal operator, $M, i \models \psi$ if and only if $\widehat{M}, 2i \models \psi'$. In particular, $\widehat{M}, 0 \models \varphi_1'$ and $\widehat{M}, 2i \models \varphi_2'$ for all $i \in \mathbb{N}$. It follows immediately that $\widehat{M}, 0 \models f(\varphi)$ and $\widehat{M}, 0 \models g(\varphi)$.

It remains to show that if $f(\varphi)$ is $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiable or $g(\varphi)$ is $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{\circledast}}$-satisfiable, then φ is $[\langle \mathbb{N}, S \rangle]_{\boxed{\circledast}}$-satisfiable. Let $\widehat{M} = \langle W, R_1, R_2, \widehat{\pi} \rangle$ and $w_0 \in W$ be such that $\widehat{M}, w_0 \models f(\varphi)$ or $\widehat{M}, w_0 \models g(\varphi)$ and $\langle W, R_1, R_2 \rangle \in \mathcal{F}_1 \oplus \mathcal{F}_2$. By definition of f and g, there exists a sequence $w_0, w_1, w_2, w_3, \ldots$ of (not necessarily distinct) worlds in W such that $\widehat{M}, w_i \models p_{odd}$ if and only if i is odd, and $\widehat{M}, w_i \models p_{even}$ if and only if i is even. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ do not contain generated subframes of size larger than two, this sequence is unique. Define $M = \langle \mathbb{N}, S, \pi \rangle$ so that $M, i \models p$ if and only if $\widehat{M}, w_{2i} \models p$. A simple induction will show that $M, 0 \models \varphi$ as required. $\Box$

To finish the proof of Theorem 5.3, it remains to show that $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{\circledast}}$-satisfiability is in PSPACE.

Lemma 5.6 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be closed under isomorphism and disjoint union. If $\mathcal{F}_1$ and $\mathcal{F}_2$ have rooted subframes of size two, but not of size three, then $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{\circledast}}$-satisfiability is in* PSPACE.

The proof is not that hard, but involves a lot of messy encoding details. This is often the case in PSPACE upper bound proofs, but especially so in this case, since we have to prove the lemma for a whole bunch of logics at the same time.

From Lemma 4.3, we may assume that $\mathcal{F}_1 \oplus \mathcal{F}_2$ is closed under generated subframes. Suppose φ is satisfiable in world w on the $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame $\langle W, R_1, R_2 \rangle$. Every world has at most one R_1 successor other than itself, and at most one R_2 successor other than itself. Also, since φ contains only $\boxed{1}$, $\boxed{2}$, and $\boxed{\circledast}$ as modal operators, φ will still be satisfied if we restrict the frame to the set of worlds reachable from w.

From these observations, it follows that φ is $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{\circledast}}$-satisfiable if and only if φ is satisfiable in the root of a generated $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame with an underlying skeleton of the form depicted in Figure 2, where both branches can be finite or infinite.

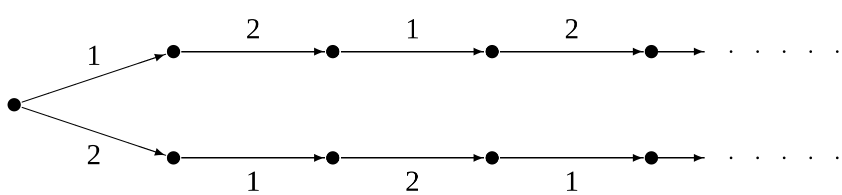


Figure 2:

We will call the class of frames of this form $\mathcal{F}_{2la}$ (for two linear alternating). Note that the extra edges needed to turn an $\mathcal{F}_{2la}$ frame into a generated $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame are all "local," i.e., an extra 1 (2) edge can only be a reflexive edge or a symmetric backward edge, that is, a 1 (2) edge connecting two worlds that are already connected by a 1 (2) edge in the skeleton.

As a first step, we will show that satisfiability with respect to linear alternating frames is in PSPACE. That is, satisfiability with respect to finite and infinite frames of the form $0R_1 1R_2 2R_1 3R_2 4R_1 5R_2 \ldots$. We will use this result to show that $\mathcal{F}_{2la}$-satisfiability is in PSPACE, and then, finally, prove Lemma 5.6.

Lemma 5.7 *Let $\mathcal{F}_{la}$ be the class of frames $\langle W, R_1, R_2\rangle$, where W a prefix of $\mathbb{N}$, $R_1 = \{\langle 2i, 2i+1\rangle \mid i \in \mathbb{N}, 2i+1 \in W\}$, and $R_2 = \{\langle 2i+1, 2i+2\rangle \mid i \in \mathbb{N}, 2i+2 \in W\}$. $[\mathcal{F}_{la}]_{\circledast}$-satisfiability is in* PSPACE.

We will use the fact that satisfiability with respect to finite and infinite frames of the form $0R1R2R3R4R5R\ldots$ is in PSPACE.

Lemma 5.8 *$[\{\langle W, S\rangle \mid W$ prefix of $\mathbb{N}$ and $S = \{\langle i, i+1\rangle \mid i+1 \in W\}]_{\circledast}$-satisfiability is in* PSPACE.

Proof: Immediate from Lemma 4.8 and from Sistla and Clarke [25] as stated in Theorem 5.4. $\square$

Proof of Lemma 5.7: First suppose φ is $\mathcal{F}_{la}$-satisfiable. Let $M = \langle W, R_1, R_2, \pi\rangle$, where W is a prefix of $\mathbb{N}$, $R_1 = \{\langle 2i, 2i+1\rangle \mid i \in \mathbb{N}, 2i+1 \in W\}$, and $R_2 = \{\langle 2i+1, 2i+2\rangle \mid i \in \mathbb{N}, 2i+2 \in W\}$, and suppose that $M, k \models \varphi$ for some $k \in W$.

To encode M into a model $M' = \langle W, S, \pi'\rangle$, we will use a new propositional variable f_1. f_1 will be true in worlds that have an R_1 successor in M. Formally, we encode M by model $M' = \langle W, S, \pi'\rangle$ as follows: π and π' coincide on all propositional variables in φ, and $M', i \models f_1$ if and only if $iR_1 i{+}1$. It is immediate that the modality $\Box$ plays the role of $\boxed{1}$ in worlds where f_1 holds, and the role of $\boxed{2}$ in worlds where f_1 does not hold. Furthermore, the transitive closures of corresponding frames coincide. These observations lead to the following. Define φ' by replacing all subformulas of the form $\boxed{1}\psi$ by $f_1 \to \Box\psi'$, and all subformulas of the form $\boxed{2}\psi$ by $\neg f_1 \to \Box\psi'$. Then for all $i \in W$, $M, i \models \varphi$ if and only if $M', i \models \varphi'$.

Not all valuations on a frame $\langle W, S\rangle$ with W a prefix of $\mathbb{N}$ correspond to an $\mathcal{F}_{la}$ frame, so we still need to ensure that the linear encoding model behaves like an $\mathcal{F}_{la}$ frame. We need to construct a formula φ_{lin} such that for all $M' = \langle W, S, \pi'\rangle$ with W a prefix of $\mathbb{N}$, if $M', k \models \varphi_{lin}$, then M' starting at k corresponds to a $\mathcal{F}_{la}$ frame, in the sense as described above. It is easy to see that this is equivalent to the following two conditions for all $i \in W$, $i \geq k$:

- if $M', i \models f_1$, then $i{+}1 \in W$ and $M, i{+}1 \models \neg f_1$, and
- if $M', i \models \neg f_1$, then $i{+}2 \notin W$ or $M, i{+}1 \models f_1$.

Define the reduction f as follows:

$$f(\varphi) = \varphi' \wedge \circledast((f_1 \to \Diamond\top \wedge \Box\neg f_1) \wedge (\neg f_1 \to \Box(f_1 \vee \Box\bot)).$$

It is easy to verify that an $\mathcal{L}_{\circledast}$ formula φ is satisfiable on an $\mathcal{F}_{la}$ frame if and only if $f(\varphi)$ is satisfiable on a frame $\langle W, S\rangle$ with W a prefix of $\mathbb{N}$. Since f is obviously polynomial time computable, it follows from Lemma 5.8 that $\mathcal{F}_{la}$-satisfiability is in PSPACE. $\square$

Next, we will show that $[\mathcal{F}_{2la}]_{\boxed{*}}$-satisfiability is in PSPACE as well, where $\mathcal{F}_{2la}$ is defined after the statement of Lemma 5.6. This satisfiability problem has PSPACE written all over it, since $\mathcal{F}_{2la}$ frames consist of two $\mathcal{F}_{la}$ branches.

Lemma 5.9 *$[\mathcal{F}_{2la}]_{\boxed{*}}$-satisfiability is in* PSPACE.

Proof: First note that φ is $\mathcal{F}_{2la}$-satisfiable if and only if φ is $\mathcal{F}_{la}$-satisfiable or φ is satisfiable in the root of an $\mathcal{F}_{2la}$ frame, and this root has an R_1 and an R_2 successor. Since $\mathcal{F}_{la}$-satisfiability is in PSPACE by Lemma 5.7, it remains to show that determining whether φ is satisfiable in the root of an $\mathcal{F}_{2la}$ frame that has an R_1 and an R_2 successor is in PSPACE as well. We claim that this is the case if and only if there exist subsets Γ, Γ_1, and Γ_2 of $Cl(\varphi)$ such that

- $\varphi \in \Gamma$,
- $\forall \neg\psi \in Cl(\varphi)$, $\neg\psi \in \Gamma$ if and only if $\psi \notin \Gamma$,
- $\forall \psi \wedge \xi \in Cl(\varphi)$, $\varphi \wedge \xi \in \Gamma$ if and only if $\psi \in \Gamma$ and $\xi \in \Gamma$,
- $\forall \boxed{a}\psi \in Cl(\varphi)$, $\boxed{a}\psi \in \Gamma$ if and only if $\psi \in \Gamma_a$,
- $\forall \boxed{*}\psi \in Cl(\varphi)$, $\boxed{*}\psi \in \Gamma$ if and only if $\psi \in \Gamma$, $\boxed{*}\psi \in \Gamma_1$, and $\boxed{*}\psi \in \Gamma_2$, and
- $\bigwedge \Gamma_1 \wedge \bigwedge_{\psi \in Cl(\varphi)\setminus\Gamma_1} \neg\psi \wedge \boxed{1}\bot$ and $\bigwedge \Gamma_2 \wedge \bigwedge_{\psi \in Cl(\varphi)\setminus\Gamma_2} \neg\psi \wedge \boxed{2}\bot$ are $[\mathcal{F}_{la}]_{\boxed{*}}$-satisfiable.

Since subsets of $Cl(\varphi)$ can be represented in space polynomial in the length of φ, and $[\mathcal{F}_{la}]_{\boxed{*}}$-satisfiability is in PSPACE, it follows that $[\mathcal{F}_{2la}]_{\boxed{*}}$-satisfiability is in PSPACE as well. It remains to prove the claim.

First suppose φ is satisfiable in the root of an $\mathcal{F}_{2la}$ frame, and this root has an R_1 successor and an R_2 successor. Let M be the model and w the world that witness this and let w_1 be the R_1 successor and w_2 be the R_2 successor of w. Let Γ be the set of $Cl(\varphi)$ formulas satisfied in M at w, let Γ_1 be the set of $Cl(\varphi)$ formulas satisfied at w_1, and let Γ_2 be the set of $Cl(\varphi)$ formulas satisfied at w_2. It is immediate that Γ, Γ_1, and Γ_2 fulfill the requirements.

For the converse, suppose there exist sets Γ, Γ_1, and Γ_2 that fulfill the requirements. Let $M = \langle W, R_1, R_2, \pi\rangle$ and $M' = \langle W', R_1', R_2', \pi'\rangle$ be two models based on $\mathcal{F}_{la}$ frames such that $W \cap W' = \varnothing$, w is the root of M, w' is the root of M', $M, w \models \bigwedge \Gamma_1 \wedge \bigwedge_{\psi \in Cl(\varphi)\setminus\Gamma_1} \neg\psi \wedge \boxed{1}\bot$, and $M', w' \models \bigwedge \Gamma_2 \wedge \bigwedge_{\psi \in Cl(\varphi)\setminus\Gamma_2} \neg\psi \wedge \boxed{2}\bot$. Let $\widehat{w}$ be a new world and define $\widehat{M}$ as follows: $\widehat{M} = \langle W \cup W' \cup \{\widehat{w}\}, R_1 \cup R_1' \cup \{\langle \widehat{w}, w\rangle\}, R_2 \cup R_2' \cup \{\langle \widehat{w}, w'\rangle\}, \widehat{\pi}\rangle$. $\widehat{M}$ is based on a $\mathcal{F}_{2la}$ frame, since w doesn't have an R_1 successor and w' doesn't have an R_2' successor. Define $\widehat{\pi}$ on all propositional variables p in φ such that $\widehat{\pi}$ agrees with π on worlds in W and with π' in worlds in W' and so that $\widehat{M}, \widehat{w} \models p$ if and only if $p \in \Gamma$. With induction, we can show that for all $\psi \in Cl(\varphi)$, $\widehat{M}, w \models \psi$ if and only if $\psi \in \Gamma$. Since $\varphi \in \Gamma$, it follows that $\widehat{M}, w \models \varphi$ as required. $\square$

Proof of Lemma 5.6: Suppose φ is $\mathcal{F}_1 \oplus \mathcal{F}_2$-satisfiable. Then φ is satisfiable in the root of a generated $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame $F = \langle W, R_1, R_2\rangle$ with an underlying $\mathcal{F}_{2la}$ skeleton $\widehat{F} = \langle W, \widehat{R}_1, \widehat{R}_2\rangle$ in such a way that the extra edges needed to turn this structure into a generated $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame are all "local," i.e., an extra 1 (2) edge can only be a reflexive edge or a symmetric backward edge, that is, an 1 (2) edge connecting two worlds that are already connected by a 1 (2) edge in the skeleton.

To encode the extra edges, we will use new propositional variables r_1, r_2, b_1, and b_2. For $a = 1, 2$, r_a will be true in worlds that are R_a reflexive, and b_a will be true in worlds that have an R_a backedge. To ensure that an $\mathcal{F}_{2la}$ frame indeed encodes a generated $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame, we first of all ensure that no world can have both a forward and backward R_a edge at the same time. This is forced by the following formula:

$$\boxed{*}(\langle a \rangle\top \to \neg b_a).$$

To ensure that the $\mathcal{F}_{2la}$ frame encodes a generated $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame, it suffices to force that every world with no R_a backedge generates a frame in $\mathcal{F}_a$. For every generated frame $F \in \mathcal{F}_a$, we can construct a formula φ_F that will be true exactly in those worlds that generate F. For $F = \langle \{w\}, R'_a\rangle \in \mathcal{F}_a$, let φ_F be the formula encoding the situation at w:

$$\varphi_F = \boxed{a}\bot \wedge r_a \text{ if } w R'_a w\ ;\ \varphi_F = \boxed{a}\bot \wedge \neg r_a \text{ if } \neg w R'_a w.$$

And for $F = \langle \{w, w'\}, R'_a\rangle$ a frame in $\mathcal{F}_a$ such that $w R'_a w'$, let φ_F be the formula encoding the situation at w:

$$\langle a \rangle\top \wedge \bigwedge_{w' R'_a w} \langle a \rangle b_a \wedge \bigwedge_{\neg w' R'_a w} \neg\langle a \rangle b_a \wedge \bigwedge_{w R'_a w} r_a \wedge$$
$$\wedge \bigwedge_{\neg w R'_a w} \neg r_a \wedge \bigwedge_{w' R'_a w'} \langle a \rangle r_a \wedge \bigwedge_{\neg w' R'_a w'} \neg\langle a \rangle r_a.$$

Now add the following formula for $a = 1, 2$.

$$\boxed{*}(\neg b_a \to (\bigvee_{F \in \mathcal{F}_a \text{generated}} \varphi_F)).$$

To construct a polynomial time reduction from $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{*}}$-satisfiability to $[\mathcal{F}_{2la}]_{\boxed{*}}$-satisfiability, let ψ' be the propositional version of ψ:

$$p' = p;\ (\neg\psi)' = \neg\psi';\ (\psi \wedge \xi)' = \psi' \wedge \xi';\ (\boxed{a}\psi)' = p_{\boxed{a}\psi};\ (\boxed{*}\psi)' = p_{\boxed{*}\psi}.$$

Now define $f(\varphi)$ as the conjunction of φ', the frame formulas given above, and the following formulas which force proper behavior of the new propositional variables. We first treat the case for $p_{\boxed{a}\psi}$ for $a \in \{1, 2\}$ and $\boxed{a}\psi \in Cl(\varphi)$. This is relatively straightforward, as all R_a successors are given by $\widehat{R}_a$ and the variables b_a and r_a. We treat all occurring combinations. First suppose that $w \widehat{R}_a w'$. If w is R_a reflexive, then r_a is true at w, and w and w' are the R_a successors of w. If w is R_a irreflexive, then r_a is false at w, and w' is the only R_a successor of w. This is enforced by the following formula:

$$\boxed{*}((\langle a \rangle\top \wedge r_a \to (p_{\boxed{a}\psi} \leftrightarrow \psi' \wedge \Box\psi')) \wedge (\langle a \rangle\top \wedge \neg r_a \to (p_{\boxed{a}\psi} \leftrightarrow \Box\psi'))).$$

We argue in a similar way in the case that $w' \widehat{R}_a w$ and $w R_a w'$, that is, b_a true at w.

$$\boxed{*}((\Diamond b_a \wedge \Diamond r_a \to (\Diamond p_{\boxed{a}\psi} \leftrightarrow \psi' \wedge \Diamond\psi')) \wedge (\Diamond b_a \wedge \neg\Diamond r_a \to (\Diamond p_{\boxed{a}\psi} \leftrightarrow \psi'))).$$

And if w does not have any nonreflexive R_a successors:

$$\boxed{*}(\neg\langle a \rangle\top \wedge \neg b_a \to (p_{\boxed{a}\psi} \leftrightarrow (r_a \to \psi'))).$$

Finally, we ensure the proper behavior of $p_{\boxed{*}\psi}$ for $\boxed{*}\psi \in Cl(\varphi)$. For worlds without backedges, the transitive closure on generated $\mathcal{F}_1 \oplus \mathcal{F}_2$ frames coincides with the transitive closure on the underlying $\mathcal{F}_{2la}$ frame:

$$\boxed{*}((\neg b_1 \wedge \neg b_2) \rightarrow (p_{\boxed{*}\psi} \leftrightarrow \boxed{*}\psi')).$$

On the other hand, if $w\widehat{R}_a w'$ and $w' R_a w$, then $\boxed{*}\psi$ holds at w if and only if $\boxed{*}\psi$ holds at w':

$$\boxed{*}(\Diamond(b_1 \vee b_2) \rightarrow (p_{\boxed{*}\psi} \leftrightarrow \Diamond p_{\boxed{*}\psi})).$$

It is easy to verify that φ is satisfiable in the root of a generated $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame if and only if $f(\varphi) \wedge \neg b_1 \wedge \neg b_2$ is satisfiable on the underlying $\mathcal{F}_{2la}$ frame. Since f is obviously polynomial time computable, this proves that $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{*}}$-satisfiability is in PSPACE. $\square$

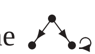
Now that we have completely classified $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{*}}$-satisfiability, we turn our attention to $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiability.

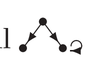
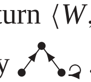
Theorem 5.10 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be closed under isomorphism, disjoint union, and generated subframes. If $\mathcal{F}_1$ and $\mathcal{F}_2$ contain a rooted subframe of size two, but do not contain a rooted subframe of size three, then $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiability is* PSPACE-*complete.*

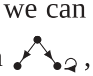
Proof: From Theorem 5.3, we know that $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{*}}$-satisfiability is in PSPACE. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are closed under isomorphism, disjoint union, and generated subframes, so is $\mathcal{F}_1 \oplus \mathcal{F}_2$. Since PSPACE is closed under $\leq^{NP}_{ctt}$ reductions, the theorem follows from Corollary 4.2. $\square$

In the previous section, we showed that there are cases when $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiability is harder than $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{*}}$-satisfiability. Surprisingly, we can find an example of this phenomenon even in the restricted case where $\mathcal{F}_1$ and $\mathcal{F}_2$ do not contain a rooted subframe of size three, under the assumption that EXPTIME $\neq$ PSPACE.

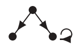
Theorem 5.11 *Let $\mathcal{F}_1$ consist of the closure under disjoint union of*  *and let $\mathcal{F}_2$ consist of the closure under disjoint union of $\bullet\!\!\longrightarrow\!\!\bullet$. Then $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiability is* EXPTIME-*hard.*

Proof: Let $\widehat{\mathcal{F}_1}$ consist of the closure under disjoint union of the frame . We will construct a reduction from $[\widehat{\mathcal{F}_1} \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiability to $[\mathcal{F}_1 \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiability. This proves the theorem, since $[\widehat{\mathcal{F}_1} \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiability is EXPTIME-hard by Theorem 5.1.

First suppose that φ is $[\widehat{\mathcal{F}_1} \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiable. Let $\widehat{M} = \langle W, \widehat{R}_1, R_2, \widehat{\pi}\rangle$ and $w_0 \in W$ be such that $\widehat{M}, w_0 \models \varphi$, $\langle W, \widehat{R}_1\rangle$ consists of the disjoint union of , and $\langle W, R_2\rangle$ consists of the disjoint union of $\bullet\!\!\longrightarrow\!\!\bullet$. The easiest way to turn $\langle W, \widehat{R}_1, R_2\rangle$ into an $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame, is by replacing all  frames in $\langle W, \widehat{R}_1\rangle$ by . That is, we will look at model $M = \langle W, \widehat{R}_1^{-1}, R_2, \pi\rangle$, where π and $\widehat{\pi}$ coincide on propositional variables in φ.

What do we do with formula φ? The only thing changed in the model are the 1 edges, so it stands to reason that we can expect difficulties with subformulas of the form $\boxed{1}\psi$. In a model of the form , the following hold for all $\boxed{1}\psi \in Cl(\varphi)$:

- the root satisfies $\boxed{1}\psi$ if and only if both children satisfy ψ,
- the irreflexive child satisfies $\boxed{1}\psi$ no matter what, and
- the reflexive child satisfies $\boxed{1}\psi$ if and only if it satisfies ψ.

Since we are simulating  by , we will introduce two new propositional variables $irref_\psi$ and ref_ψ for all $\boxed{1}\psi \in Cl(\varphi)$. In the root, $irref_\psi$ will denote that the irreflexive child satisfies ψ, and ref_ψ that the reflexive child satisfies ψ. Note that worlds that are roots are exactly those worlds satisfying $\boxed{1}\bot$ in M, that irreflexive children are exactly those worlds satisfying $\langle 1\rangle\top \wedge \boxed{1}\boxed{1}\bot$ in M, and that reflexive children are exactly those worlds satisfying $\langle 1\rangle\langle 1\rangle\top$ in M. First define ψ' inductively as follows:

$$p' = p;\ (\neg\psi)' = \neg\psi';\ (\psi \wedge \xi)' = \psi' \wedge \xi';\ (\boxed{2}\psi)' = \boxed{2}\psi';\ (\boxed{u}\psi)' = \boxed{u}\psi';$$
$$(\boxed{1}\psi)' = (\boxed{1}\bot \to irref_\psi \wedge ref_\psi) \wedge (\langle 1\rangle\langle 1\rangle\top \to \psi').$$

Extend π such that for all $\boxed{1}\psi \in Cl(\varphi)$, and for all roots $w \in W$, $M, w \models irref_\psi$ if and only if ψ' holds in w's irreflexive child, and $M, w \models ref_\psi$ if and only if ψ' holds in w's reflexive child. A simple induction will show that $\forall w \in W, \psi \in Cl(\varphi)$, $M, w \models \psi'$ if and only if $\widehat{M}, w \models \psi$.

It remains to force the proper behavior for $irref_\psi$ and ref_ψ. That is, for all $\boxed{1}\psi \in Cl(\varphi)$ and for all $w \in W$, we need to ensure that if w is a root, then $irref_\psi$ holds if and only if the irreflexive child satisfies ψ', and that ref_ψ holds if and only if the reflexive child satisfies ψ'. Reformulating this, we need to ensure that for all $\boxed{1}\psi \in Cl(\varphi)$ and for all $w \in W$, if w is an irreflexive child, then ψ' holds if and only if w's parent satisfies $irref_\psi$, and if w is a reflexive child, then ψ' holds if and only if w's parent satisfies ref_ψ. This requirement can be enforced by the following formula:

$$\bigwedge_{\boxed{1}\psi \in Cl(\varphi)} \boxed{u}(\langle 1\rangle\top \wedge \boxed{1}\boxed{1}\bot \to (\psi' \leftrightarrow \langle 1\rangle irref_\psi) \wedge$$
$$\wedge (\langle 1\rangle\langle 1\rangle\top \to (\psi' \leftrightarrow \boxed{1}(\boxed{1}\bot \to ref_\psi))).$$

Define $f(\varphi)$ as the conjunction of φ' and this formula. It is obvious that $M, w_0 \models f(\varphi)$.

For the converse, suppose $M = \langle W, R_1, R_2, \pi\rangle$ is a model based on an $\mathcal{F}_1 \oplus \mathcal{F}_2$ frame, and suppose that $M, w_0 \models f(\varphi)$. We have to show that φ is $[\widehat{\mathcal{F}_1} \oplus \mathcal{F}_2]_{\boxed{u}}$-satisfiable. We will turn M into an $\widehat{\mathcal{F}_1} \oplus \mathcal{F}_2$ frame in the same way as in the first part of the proof. Define $\widehat{M} = \langle W, R_1^{-1}, R_2, \pi\rangle$. Then $\langle W, R_1^{-1}, R_2\rangle \in \widehat{\mathcal{F}_1} \oplus \mathcal{F}_2$. It remains to show that $\widehat{M}, w_0 \models \varphi$. We will show by induction on ψ that for all $\psi \in Cl(\varphi)$ and $w \in W$, $M, w \models \psi'$ if and only if $\widehat{M}, w \models \psi$. The crucial case is of course for $\boxed{1}\psi$. We have to show that $M, w \models (\boxed{1}\psi)'(= (\boxed{1}\bot \to irref_\psi \wedge ref_\psi) \wedge (\langle 1\rangle\langle 1\rangle\top \to \psi'))$ if and only if $\widehat{M}, w \models \boxed{1}\psi$. There are three situations to consider, depending on whether w is a an irreflexive child, a reflexive child, or a root.

1. If w is an irreflexive child, then $M, w \models \neg\boxed{1}\bot \wedge \neg\langle 1\rangle\langle 1\rangle\top$ and therefore $M, w \models (\boxed{1}\psi)'$. Since w has no R_1^{-1} successors, it also holds that $\widehat{M}, w \models \boxed{1}\psi$.
2. If w is a reflexive child, then $M, w \models \neg\boxed{1}\bot \wedge \langle 1\rangle\langle 1\rangle\top$. It follows that $M, w \models (\boxed{1}\psi)'$ if and only if $M, w \models \psi'$ if and only if (by induction) $\widehat{M}, w \models \psi$. Since w is the only R_1^{-1} successor of w, this is equivalent to $\widehat{M}, w \models \boxed{1}\psi$.

3. Finally, suppose that w is a root. Then $M, w \models \boxed{1}\bot \wedge \neg\langle 1\rangle\langle 1\rangle\top$. We have to prove that $M, w \models \mathit{irref}_\psi \wedge \mathit{ref}_\psi$ if and only if $\widehat{M}, w \models \boxed{1}\psi$. Let w_1 be w's irreflexive child, and w_2 be w's reflexive child. Then $M, w_1 \models \langle 1\rangle\top \wedge \boxed{1}\boxed{1}\bot$, $M, w_2 \models \langle 1\rangle\langle 1\rangle\top$, $w_1 R_1 w$, and $w_2 R_1 w$.
   First suppose that $M, w \models \mathit{irref}_\psi \wedge \mathit{ref}_\psi$. Then $M, w_1 \models \langle 1\rangle\mathit{irref}_\psi$, and therefore $M, w_1 \models \psi'$, and $M, w_2 \models \boxed{1}(\boxed{1}\bot \to \mathit{ref}_\psi)$, and therefore $M, w_2 \models \psi'$. Now look at $\widehat{M}$. By induction, $\widehat{M}, w_1 \models \psi$ and $\widehat{M}, w_2 \models \psi$. Since w_1 and w_2 are the only worlds reachable from w by R_1^{-1}, it follows that $\widehat{M}, w \models \boxed{1}\psi$.
   For the converse, suppose that $\widehat{M}, w \models \boxed{1}\psi$. Then $\widehat{M}, w_1 \models \psi$ and $\widehat{M}, w_2 \models \psi$, and, with induction, $M, w_1 \models \psi'$ and $M, w_2 \models \psi'$. It follows that $M, w_1 \models \langle 1\rangle\mathit{irref}_\psi$ and that $M, w_2 \models \boxed{1}(\boxed{1}\bot \to \mathit{ref}_\psi)$. It is immediate that $M, w \models \mathit{irref}_\psi \wedge \mathit{ref}_\psi$. □

5.1 General join So far, we have investigated what happens with the complexity if we add $\boxed{u}$ or $\boxed{*}$ to the join of two uni-modal logics. The use of the join in the literature however, is not restricted to this simple case. We will now investigate to what extent our results for the join of two uni-modal logics go through for the join of an arbitrary number of uni-modal logics.

Let Ω be a prefix of $\mathbb{N}^+$ of size at least two. As before, we will look at the satisfiability problem with respect to a class of frames. For $\{\mathcal{F}_i\}_{i\in\Omega}$ classes of frames, the join of $\{\mathcal{F}_i\}_{i\in\Omega}$, denoted by $\bigoplus_{i\in\Omega} \mathcal{F}_i$, consists of the frames $\langle W, \{R_i\}_{i\in\Omega}\rangle$ such that for all $i \in \Omega$, $\langle W, R_i\rangle \in \mathcal{F}_i$. We will look at the complexity of $[\bigoplus_{i\in\Omega} \mathcal{F}_i]_{\boxed{u}}$ and $[\bigoplus_{i\in\Omega} \mathcal{F}_i]_{\boxed{*}}$-satisfiability.

As pointed out in Fine and Schurz [6] and Hemaspaandra [14], a problem is that the permutation of the $\mathcal{F}_i$'s can have an impact on the complexity. In fact, as pointed out in [6], it can be the case that the join of decidable logics is undecidable. Consider for instance the following example. Let A be an arbitrary subset of $\mathbb{N}^+$, and let $\Omega = \mathbb{N}^+$. For all $i \in \mathbb{N}^+$, let $\mathcal{F}_i$ consist of the closure under disjoint union of the reflexive singleton if $i \in A$, and of the closure under disjoint union of the irreflexive singleton if $i \notin A$. Obviously, for all $i \in \mathbb{N}^+$, $\mathcal{F}_i$-satisfiability is in NP. Furthermore, every frame in $\bigoplus_{i\in\mathbb{N}^+} \mathcal{F}_i$ consists of the disjoint union of singletons. In this sense, the join is trivial, but A is reducible to $\bigoplus_{i\in\mathbb{N}^+} \mathcal{F}_i$-satisfiability, by $\lambda i.\langle i\rangle\top$.

To avoid this problem, we will restrict the choice of the classes of frames $\{\mathcal{F}_i\}_{i\in\Omega}$ in such a way that the permutation of the $\mathcal{F}_i$'s does not contribute to the complexity. We want these restrictions to be reasonable, in the sense that the logics encountered in the literature should satisfy these restrictions. The problem sketched above can informally be stated as follows: given i, determining $\mathcal{F}_i$ should not contribute to the complexity. Note that this problem only occurs when Ω is infinite. We will ensure that there exist a finite number of classes of frames such that for every $i \in \Omega$, $\mathcal{F}_i$-satisfiability is isomorphic to the satisfiability problem with respect to one of these classes, and that these isomorphisms can be computed in polynomial time. Formalizing the above, we obtain the following.

Definition 5.12 Let Ω be a prefix of $\mathbb{N}^+$, and for every $i \in \Omega$, let $\mathcal{F}_i$ be a class of frames. We call $\{\mathcal{F}_i\}_{i\in\Omega}$ *well-behaved* if

1. for all i, $\mathcal{F}_i$ is nonempty and closed under isomorphism and disjoint union, and

2. there exist $i_1, \dots, i_k \in \Omega$ and a polynomial time computable function f from Ω to $\{i_1, \dots, i_k\}$ such that for all $i \in \Omega$, $\mathcal{F}_i$-satisfiability is isomorphic to $\mathcal{F}_{i_j}$-satisfiability by f.

Under these restrictions, we obtain the following general analog of the results from earlier in this section.

Theorem 5.13 *Let Ω be a prefix of $\mathbb{N}^+$ of size at least two, and for all $i \in \Omega$, let $\mathcal{F}_i$ be a class of frames such that $\{\mathcal{F}_i\}_{i\in\Omega}$ is well-behaved in the sense of Definition 5.12. Then we are in one of the following four cases.*

1. *There exist two distinct indices $i, j \in \Omega$ such that $\mathcal{F}_i$ contains a rooted subframe of size three, and $\mathcal{F}_j$ contains a rooted subframe of size two. In this case, $[\bigoplus_{i\in\Omega} \mathcal{F}_i]_{\boxed{u}}$-satisfiability and $[\bigoplus_{i\in\Omega} \mathcal{F}_i]_{\boxed{*}}$-satisfiability are* EXPTIME*-hard.*
2. *There exist three distinct indices $i, j, k \in \Omega$ such that $\mathcal{F}_i$, $\mathcal{F}_j$, and $\mathcal{F}_k$ have rooted subframes of size two. In this case, $[\bigoplus_{i\in\Omega} \mathcal{F}_i]_{\boxed{u}}$-satisfiability and $[\bigoplus_{i\in\Omega} \mathcal{F}_i]_{\boxed{*}}$-satisfiability are* EXPTIME*-hard.*
3. *There exists an index $i \in \Omega$ such that for all $j \in \Omega$ with $j \neq i$, every frame in $\mathcal{F}_j$ consists of the disjoint union of singletons. In this case, $[\bigoplus_{i\in\Omega} \mathcal{F}_i]_{\boxed{u}}$-satisfiability is polynomial time reducible to $[\mathcal{F}_i]_{\boxed{u}}$-satisfiability, and $[\bigoplus_{i\in\Omega} \mathcal{F}_i]_{\boxed{*}}$-satisfiability is polynomial time reducible to $[\mathcal{F}_i]_{\boxed{*}}$-satisfiability.*
4. *There exist two distinct indices $i, j \in \Omega$ such that $\mathcal{F}_i$ and $\mathcal{F}_j$ are closed under disjoint union, contain a rooted subframe of size two, but not of size three, and for all $k \in \Omega$ with $k \neq i, j$, every frame in $\mathcal{F}_k$ consists of the disjoint union of singletons. In this case, $[\bigoplus_{i\in\Omega} \mathcal{F}_i]_{\boxed{*}}$-satisfiability is* PSPACE*-complete. If $\mathcal{F}_i$ and $\mathcal{F}_j$ are also closed under generated subframes, then $[\bigoplus_{i\in\Omega} \mathcal{F}_i]_{\boxed{u}}$-satisfiability is* PSPACE*-complete.*

Proof Sketch

1. This case follows with the same argument as Theorem 5.1.
2. In this case, $\mathcal{F}_i \oplus \mathcal{F}_j$ has a rooted subframe of size three, and $\mathcal{F}_k$ has a rooted subframe of size two. The claim follows with the same argument as Theorem 5.1.
3. For this case, we can follow the construction of Theorem 5.2. For every $j \in \Omega$, $j \neq i$, we will encode R_j by a new propositional variable r_j. Define φ' by replacing all subformulas of the form $\boxed{j}\psi$ in φ by $(r_j \to \psi')$ for all $j \in \Omega$, $j \neq i$. φ' can be computed in polynomial time.

   Again, our reductions need to restrict the valuation of the r_j's in an appropriate manner. The situation is different for $\boxed{u}$ and $\boxed{*}$, and we will start with $\boxed{u}$. We claim that f is a polynomial time reduction from $[\bigoplus_{i\in\Omega} \mathcal{F}_i]_{\boxed{u}}$-satisfiability to $[\mathcal{F}_i]_{\boxed{u}}$-satisfiability, where f is defined as follows:

   $$\begin{aligned} f(\varphi) = {} & \varphi' \wedge \bigwedge\{\boxed{u} r_j \mid j \neq i, \boxed{j} \text{ occurs in } \varphi \text{ and all worlds in } \mathcal{F}_j \text{ are reflexive}\} \wedge \\ & \bigwedge\{\boxed{u}\neg r_j \mid j \neq i, \boxed{j} \text{ occurs in } \varphi \text{ and all worlds in } \mathcal{F}_j \text{ are irreflexive}\} \wedge \\ & \bigwedge\{\Diamond_u r_j \mid j \neq i, \boxed{j} \text{ occurs in } \varphi, \mathcal{F}_j \text{ contains no irreflexive frames}\} \wedge \end{aligned}$$

$\bigwedge\{\langle u\rangle\neg r_j \mid j \neq i, \boxed{j}$ occurs in φ, $\mathcal{F}_j$ contains no reflexive frames$\}$.

That f is reduction follows with the same argument as in Theorem 5.2. That f is polynomial time computable follows from clause 2 of well-behavedness.

Similarly, polynomial time reduction g from $[\bigoplus_{i\in\Omega}\mathcal{F}_i]_{\boxed{*}}$-satisfiability to $[\mathcal{F}_i]_{\boxed{*}}$-satisfiability is defined as follows:

$$g(\varphi) = \varphi' \wedge \bigwedge\{\boxed{*}r_j \mid j \neq i, \boxed{j}\text{ occurs in } \varphi \text{ and all worlds in } \mathcal{F}_j \text{ are reflexive}\} \wedge \bigwedge\{\boxed{*}\neg r_j \mid j \neq i, \boxed{j}\text{ occurs in } \varphi \text{ and all worlds in } \mathcal{F}_j \text{ are irreflexive}\}.$$

4. Finally, note that if we are not in case 1, 2, or 3, then there exist exactly two indices i and j in Ω such that $\mathcal{F}_i$ and $\mathcal{F}_j$ contain a rooted subframe of size two and not of size three, and for all $k \neq i, j$, $\mathcal{F}_k$ consists of the disjoint union of singletons. By the construction from case 3, $[\bigoplus_{i\in\Omega}\mathcal{F}_i]_{\boxed{u}}$-satisfiability is polynomial time reducible to $[\mathcal{F}_i \oplus \mathcal{F}_j]_{\boxed{u}}$-satisfiability, and $[\bigoplus_{i\in\Omega}\mathcal{F}_i]_{\boxed{*}}$-satisfiability is polynomial time reducible to $[\mathcal{F}_i \oplus \mathcal{F}_j]_{\boxed{*}}$-satisfiability. The claim now follows from Theorems 5.3 and 5.10.

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REFERENCES

[1] Balcázar, J., J. Díaz, and J. Gabarró, *Structural Complexity I*, EATCS Monographs on Theoretical Computer Science vol. 11, Springer-Verlag, Berlin, 1995. Zbl 0638.68040 MR 91f:68058 1

[2] Ben-Ari, M., J. Halpern, and A. Pnueli, "Deterministic propositional dynamic logic: finite models, complexity and completeness," *Journal of Computer and System Sciences*, vol. 25 (1982), pp. 402–417. Zbl 0512.03013 MR 84c:03033 1

[3] Berger, R., "The undecidability of the dominoe problem," *Memoirs of the American Mathematical Society*, vol. 66 (1966). MR 36:49 3.1

[4] Blackburn, P., and E. Spaan, "A modal perspective on the computational complexity of attribute value grammar," *Journal of Logic, Language and Information*, vol. 2 (1993), pp. 129–169. Zbl 0793.03018 MR 95f:03040 1, 1

[5] Emerson, E., and J. Halpern, "Decision procedures and expressiveness in the temporal logic of branching time," *Journal of Computer and System Sciences*, vol. 30 (1985), pp. 194–211. Zbl 0559.68051 MR 86j:68080 1

[6] Fine, K., and R. Schurz, "Transfer theorems for multimodal logics," *Proceedings Arthur Prior Memorial Conference*, Christchurch, New Zealand, 1989. Zbl 0919.03015 MR 98f:03016 5, 5.1, 5.1

[7] Fischer, M., and R. Ladner, "Propositional dynamic logic of regular programs," *Journal of Computer and System Sciences*, vol. 18 (1979), pp. 194–211. Zbl 0408.03014 MR 80f:68013 1, 5, 5, 5

[8] Goranko, V., and S. Passy, "Using the universal modality: gains and questions," *Journal of Logic and Computation*, vol. 2 (1992), pp. 5–30. Zbl 0774.03003 MR 93h:03016 1, 1, 3.1

[9] Halpern, J., and M. Moses, "A guide to completeness and complexity for modal logics of knowledge and belief," *Artificial Intelligence*, vol. 54 (1992), pp. 319–379. Zbl 0762.68029 MR 93c:03019 1, 5

[10] Halpern, J., and M. Vardi, "The complexity of reasoning about knowledge and time, I: Lower Bounds," *Journal of Computer and System Sciences*, vol. 38 (1989), pp. 195–237. Zbl 0672.03015 MR 91f:68146 1

[11] Harel, D., "Recurring dominoes: making the highly undecidable highly understandable," pp. 177–194 in *Proceedings of the Conference on Foundations of Computing Theory*, Springer Lecture Notes in Computer Science 158, Springer-Verlag, Berlin, 1983. Zbl 0531.68002 MR 734719 1, 3

[12] Harel, D., "Effective transformations on infinite trees, with applications to high undecidability, dominoes, and fairness," *Journal of the ACM*, vol. 33 (1986), pp. 224–248. MR 88a:68075 3.2

[13] Harel, D., A. Pnueli, and J. Stavi, "Propositional dynamic logic of nonregular programs," *Journal of Computer and System Sciences*, vol. 26 (1983), pp. 222–243. Zbl 0536.68041 MR 85b:03041

[14] Hemaspaandra, E., "Complexity transfer for modal logic," pp. 164–173 in *Proceedings of the Ninth IEEE Symposium on Logic In Computer Science (LICS'94)*, IEEE Computer Society Press, Los Alamitos, 1994. 5, 5.1

[15] Hughes, G., and M. Cresswell, *A Companion to Modal Logic*, Methuen, London, 1984. Zbl 0625.03005 MR 86j:03014 1

[16] Kracht, K., and F. Wolter, "Properties of independently axiomatizable bimodal logics," *The Journal of Symbolic Logic*, vol. 56 (1991), pp. 1469–1485. Zbl 0743.03013 MR 93c:03021 5

[17] Ladner, R., "The computational complexity of provability in systems of modal logic," *SIAM Journal on Computing*, vol. 6 (1977), pp. 467–480. Zbl 0373.02025 MR 56:8326 4, 4

[18] Ladner, R., N. Lynch, and A. Selman, "A comparison of polynomial time reductions," *Theoretical Computer Science*, vol. 1 (1975), pp. 103–123. Zbl 0321.68039 MR 52:16116 4

[19] Ladner, R., and J. Reif, "The logics of distributed protocols," pp. 207–221 in *Theoretical Aspects of Reasoning About Knowledge: Proceedings of the 1986 Conference*, Morgan Kaufmann, Los Altos, 1986. MR 88k:03008 1

[20] Lewis, H., "Complexity of solvable cases of the decision problem for the predicate calculus," pp. 35–47 in *Proceedings of the 19th IEEE Symposium on Foundations of Computer Science*, IEEE Computer Society Press, Los Alamitos, 1978. hrefhttp://www.ams.org/mathscinet-getitem?mr=80e:03041MR 80e:03041 3

[21] Parikh, R., "Propositional logics of programs: systems, models, and complexity," pp. 186–192 in *Seventh Annual ACM Symposium on Principles of Programming Languages*, ACM Press, New York, 1980. 1

[22] Pratt, V., "Models of program logics," pp. 115–122 in *Proceedings of the 20th IEEE Symposium on Foundations of Computer Science*, IEEE Computer Society Press, Los Alamitos, 1979. 1

[23] Robinson, R., "Undecidability and nonperiodicity for tilings of the plane," *Inventiones Mathematicæ*, vol. 12 (1971), pp. 177–209. Zbl 0197.4680 MR 45:6626 3.1

[24] Sahlqvist, H., "Completeness and correspondence in the first and second order semantics for modal logics," pp. 110–143 in *Proceedings of the third Scandinavian Logic Symposium*, North-Holland, Amsterdam, 1975. Zbl 0319.02018 MR 52:7855 3.1

[25] Sistla, A., and E. Clarke, "The complexity of propositional linear temporal logics," *Journal of the ACM*, vol. 32 (1985), pp. 733–749. Zbl 0632.68034 MR 87j:68077 4, 4.9, 5, 5.4, 5, 5

[26] Spaan, E., "Nexttime is not necessary," pp. 241–256 in *Proceedings of the Third Conference on Theoretical Aspects of Reasoning About Knowledge*, Morgan Kaufmann, Los Altos, 1990. 1

[27] van Emde Boas, P., "Dominoes are forever," pp. 75–95 in *1st GTI Workshop*, Paderborn, 1983. 3

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