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Constructive Ultraproducts and Isomorphisms of Recursively Saturated Ultrapowers

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Abstract Various models of a first order theory T are obtained from given models of that theory by generalizations of the ultraproduct construction. It is demonstrated that for a model complete theory this construction can be carried out using as functions for the ultraproduct exactly those functions defined by terms in an extension of the original language. In this way one obtains countable nonstandard models of T which can be endowed with other desirable properties such as being recursively saturated. These constructions use only the most basic ideas of model theory and recursion theory. Two countable elementarily equivalent models are shown to have recursive ultrapowers which are isomorphic and recursively saturated.

0 Introduction and recursive ultraproducts The ultraproduct construction is one of the basic constructions of model theory and its applications are numerous (see Chang & Keisler [1]). It is particularly useful in that it enables one to prove that the notion of two structures in a first order language being elementary equivalent is an algebraic property via the Keisler–Shelah Theorem, which equates this property with that of having isomorphic ultrapowers. Another use of this construction is to show that a given mathematical property of structures is not a first order property by showing that there is an ultraproduct of structures with the given property which does not inherit that property.

Another application of ultraproducts and ultrapowers is the following: in order to attempt to understand what all models of a given first order theory T look like, we might start with some well understood subset of models of T and consider what models can be constructed from them using ultraproducts and ultrapowers. If we had started with an initial class which contained a model of each complete extension of T and had infinite wisdom, then in view of the Keisler–Shelah Theorem we would see all other models of T as an elementary substructure of these models. If we did not have a representation of each complete extension of T initially, then under suitable assumptions we still might be

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able, using ultraproducts of a relatively small class of models of T, to obtain representatives of each complete extension of T from which we could proceed as above and obtain all models of T. Even though this method has obvious shortcomings, such as the fact that nontrivial ultraproducts usually give us models which have very large cardinality and in general we do not have a systematic way to extract all elementary substructures of a given model even when it is denumerable, it still does work well in finding representative models of each complete extension of T in a substantial number of cases.

It is clear that the ultraproduct and ultrapower construction are important enough in model theory to want to have a construction which yields the same first order theories as the usual construction but which remain countable as well as being manageable in other ways. In particular it would be desirable to have the elements of the domains of these ultraproducts encoded by natural numbers or, even better, given by certain terms in some first order language. This is what we intend to do in this paper.

We assume that T is a first order theory in a countable language $\mathcal{L}$ with equality (we assume that the symbols of $\mathcal{L}$ are all natural numbers). Moreover we assume that T is model complete in $\mathcal{L}$ and observe that this always could be accomplished by replacing it by its Morleyization (see Sacks [10]). The fact that T is model complete will make everything that we do much easier. It is this simplification over the other approaches such as those in [1] or Keisler [5], which require enough built-in Skolem functions, or those in Cleave [2] or Cutland [3], which require one to deal with a hyperarithmetic hierarchy of sets in the ultrafilter under consideration, that makes our approach easier. Of course we realize that in carrying out the Morleyization for a theory such as the usual system of natural numbers one obtains a diagram which is hyperarithmetic. Moreover it can be shown that all universalizations of model complete theories do essentially have built-in Skolem functions, which is the feature that makes our method work. But on the other hand there are an abundance of known model complete theories as well as ones which require fairly small language extensions to make them into model complete theories to which our method would automatically apply. Our approach when restricted to ultrapowers is close in structure to that in Hirschfeld [4], but not much is done in [4] on how one can obtain such classes of functions satisfying Hirschfeld's conditions except by taking an elementary substructure of a given model as a starting point. In general most of the approaches in the literature deal just with a more general version for ultrapowers, whereas here we handle ultraproducts which contain as a special case ultrapowers. Before going on we introduce a few definitions and preliminary results.

For $S \subseteq \omega$ we say that the language $\mathcal{L}$ is *recursive in S* if its set S_1 of relation symbols, its set S_2 of function symbols, and its set S_3 of logical symbols are disjoint sets of natural numbers each of which is recursive in S. Moreover it is required that the functions $\sigma_i(s)$ giving the arity of s in S_i for $i = 1,2$ are recursive in S and that there is a recursive in S procedure which decides whether or not s in S_3 is a variable symbol. We are treating constant symbols of $\mathcal{L}$ as 0-ary function symbols. It is easy to carry out the usual Gödel numbering and obtain the expected result that the syntax of $\mathcal{L}$ is recursive in S.

For $S \subseteq \omega$ we say that the theory T is *S-decidable* if its language $\mathcal{L}$ is recursive in S and the set of consequences of T is recursive in S. Usually $\mathfrak{A}$ with or without subscripts will denote a model of T and its domain $|\mathfrak{A}| = A$ is assumed to be ω. We note that this assumption is made only for notational convenience and that there are versions of each of the results here which hold for finite models $\mathfrak{A}$ just as well as for denumerable models $\mathfrak{A}$. The results of Section 1 hold for all models $\mathfrak{A}$ regardless of cardinality. By the *oracle* for $\mathfrak{A}$ we mean the following set $O = \bigcup_{s \in S_1} \tau(\{s\} \times \tau^{\sigma_1(s)}(s^{\mathfrak{A}})) \cup \bigcup_{s \in S_2} \tau(\{s\} \times \tau^{\sigma_2(s)+1}(s^{\mathfrak{A}}))$ where τ is the usual pairing function from recursion theory.

Definition 0.1 We say $\mathfrak{A}$ is *recursive in S* if each of its relations and functions is recursive in S. We say that $\mathfrak{A}$ is *uniformly recursive in S* if the oracle of $\mathfrak{A}$ is recursive in S and $\mathfrak{L}$ is recursive in S.

The next result is easy to verify since any formula of $\mathfrak{L}$ is provably equivalent in a model complete theory T to an existential formula.

Lemma 0.2 *If $\mathfrak{A}$ is a recursive in S model of T, φ is any formula of $\mathfrak{L}$ in at most n free variables, and $f_1, \ldots, f_n$ are recursive in S functions, then $\{i \in \omega : \mathfrak{A} \models \varphi[f_1(i), \ldots, f_n(i)]\}$ is recursive in S.*

The next result is due to Ershov and appears in Peretyat'kin [7] (Theorem 1).

Lemma 0.3 *If $\mathfrak{A}$ is a uniformly recursive in S model of T and T is S-decidable, then the satisfaction relation on $\mathfrak{A}$ is recursive in S uniformly, i.e., there is a recursive procedure which when presented an S-index for the oracle of $\mathfrak{A}$ as its input gives as its output an S-index for a recursive in S procedure for the satisfaction relation on $\mathfrak{A}$ which when presented a Gödel number of a formula φ in n variables and an n-tuple m from ω determines whether or not $\mathfrak{A} \models \varphi[m]$.*

Let $\mathfrak{A}_i$ for i in ω be models of T. Let O_i denote the oracle of $\mathfrak{A}_i$ for each i in ω and let S be a subset of ω such that $\bigcup_{i \in \omega} \tau(\{i\} \times O_i)$ and T are both recursive in S. We set $R(\{\mathfrak{A}_i : i \in \omega\}) = \{f : f$ maps ω into ω and f is recursive in $S\}$. We let D denote an ultrafilter of the Boolean algebra of recursive in S subsets of ω. We define *the recursive in S ultraproduct of the* $\{\mathfrak{A}_i; i \in \omega\}$ *modulo the ultrafilter* D denoted by $R(\mathfrak{A}_i)/D$ to be that structure one obtains by using in the definition of ultraproduct in [1] the above set of functions and ultrafilter D. In this way one obtains a well defined structure on $\mathfrak{L}$ and one needs to see that it behaves like an ultraproduct. This we show next by verifying that Los's property is satisfied.

Theorem 0.4 *For any formula φ of $\mathfrak{L}$ with n free variables and any $f_1, \ldots, f_n$ recursive in S functions, $R(\mathfrak{A}_i)/D \models \varphi[[f_1], \ldots, [f_n]]$ iff $\{i \in \omega : \mathfrak{A}_i \models \varphi[f_1(i), \ldots, f_n(i)]\} \in D$.*

Proof: First we observe that for any such φ and fs recursive in S the set on the right of the iff is recursive in S by means of Lemma 0.3. The argument proceeds by induction on how φ is constructed, and there is only one nontrivial step. So suppose that φ is $\exists y \psi$ and that $V = \{i \in \omega : \mathfrak{A}_i \models \exists y \psi[f_1(i), \ldots, f_n(i)]\} \in D$. We define for $i \in V$ $g(i) = \mu k[\mathfrak{A}_i \models \psi[f_1(i), \ldots, f_n(i), k]]$ and for $i \notin V$ define $g(i) = 0$. It follows by Lemma 0.3 that g is recursive in S and now apply the inductive hypothesis.

Let $\mathfrak{A}$ be as above and suppose now that it is merely recursive in S. We define *the recursive in S ultrapower of* $\mathfrak{A}$ in a manner analogous to the above. Let $R(\mathfrak{A}) = \{f: f$ is a function from ω into ω which is recursive in $S\}$. Let D be any ultrafilter on the Boolean algebra of recursive in S subsets of ω. Just as before one obtains a structure for $\mathfrak{L}$ denoted by $R(\mathfrak{A})/D$. We state next the fact that Los' property holds in this setting which can be proved in the same way as the above theorem but now using Lemma 0.2. Apparently this construction can be attributed to Skolem as remarked in [5] or [4].

Theorem 0.5 *For any formula φ of $\mathfrak{L}$ and recursive in S functions $f_1, \ldots, f_n$, $R(\mathfrak{A})/D \models \varphi[[f_1], \ldots, [f_n]]$ iff $\{i \in \omega : \mathfrak{A} \models \varphi[f_1(i), \ldots, f_n(i)]\} \in D$.*

As usual as in [1] one obtains.

Corollary 0.6 $\mathfrak{A} < R(\mathfrak{A})/D$.

1 Ultraproducts using the terms of a language Let T be a model complete theory in the first order language $\mathfrak{L}$ with equality. Let $S(T)$ be a universalization of T in some language $\mathfrak{L}^*$, where $\mathfrak{L}^*$ is obtained from $\mathfrak{L}$ by adding only new constant symbols or new function symbols to $\mathfrak{L}$ as described in Winkler [11]. In brief $S(T)$ has a set of universal axioms which are obtained from a set of universal existential axioms for T. The new universal axioms in $S(T)$ are introduced using the new function symbols to eliminate the existential quantifiers in each axiom of T, e.g., an axiom of the form $\forall x \exists y\, \varphi(x, y)$ is essentially replaced by $\forall x\, \varphi(x, f(x))$ where f is a new function symbol introduced in $\mathfrak{L}^*$ for this purpose. Clearly any model $\mathfrak{A}$ of T can be expanded, using the axiom of choice, to a model of $S(T)$, and any model of $S(T)$ has a retraction to $\mathfrak{L}$ which is a model of T. Let $\mathfrak{A}$ be a fixed model of $S(T)$ ($\mathfrak{L}^*$ is intentionally left vague since we may want to consider that $\mathfrak{L}^*$ has more function symbols in it than is necessary to carry out the universalization of T, e.g., constants for each element of A may be in $\mathfrak{L}^*$). Let $S(T)^*$ be the set of universal sentences of $\mathfrak{L}^*$ which are true in $\mathfrak{A}$. We state a result which is proven in Robinson [8] (Theorem 9.1.5).

Lemma 1.1 *If T^* is a universal theory in a language $\mathfrak{L}^*$, $\varphi(x_1, \ldots, x_n, y)$ is an existential formula in $\mathfrak{L}^*$, and $T^* \vdash \forall x \exists y\, \varphi$, then there are terms $\tau_1, \ldots, \tau_m$ in the variables x in $\mathfrak{L}^*$ such that $T^* \vdash \forall x(\varphi(x, \tau_1) \vee \ldots \vee \varphi(x, \tau_m))$.*

We built our new model of T in $\mathfrak{L}$ from the set $\mathfrak{T}(x)$ of terms $\tau(x)$ of $\mathfrak{L}^*$, which have at most the variable x occurring in them, and the set Boolean algebra of subsets of A, which has as elements exactly those subsets S of A for which there is some formula $\varphi(x_1, \ldots, x_n)$ of $\mathfrak{L}$ and there are terms $\tau_1, \ldots, \tau_n$ in $\mathfrak{T}(x)$ such that $S = \{a \in A : \mathfrak{A} \models \varphi(\tau_1, \ldots, \tau_n)[a]\}$. With each element τ of $\mathfrak{T}(x)$ there is an associated function from A into A, namely $a \mapsto \tau^{\mathfrak{A}}(a)$, and in this way $\mathfrak{T}(x)$ plays the role of the set of functions in this general version of the ultrapower construction. We assume that U is a fixed ultrafilter in the above Boolean algebra. Now we define *the ultrapower of term functions of* $\mathfrak{A}$ *modulo* U which will be denoted by $\mathfrak{A}^*$ or $\mathfrak{T}(x)/U$ as follows: introduce an equivalence relation of the τ's according to their taking the same value on a set S in U and specify that the set of these equivalence classes be the domain of $\mathfrak{A}^*$. On these equivalence classes define for each n-ary function symbol F of $\mathfrak{L}$, $F^{\mathfrak{A}^*}([\tau_1], \ldots,$

$[\tau_n]) = [F(\tau_1, \ldots, \tau_n)]$, and for each n-ary relation symbol R of $\mathfrak{L}$ define $R^{\mathfrak{A}^*} = \{([\tau_1], \ldots, [\tau_n]) : \{a \in A : (\tau_1(a), \ldots, \tau_n(a)) \in r^{\mathfrak{A}}\} \in U\}$. This gives a structure on $\mathfrak{L}$, and it remains to be shown that it satisfies Los's property which is our next result.

Theorem 1.2 *For any formula* $\varphi(x_1, \ldots, x_n)$ *of* $\mathfrak{L}$ *and any terms* τ_i *in* $\mathfrak{T}(x)$, $\mathfrak{A}^* \models \varphi[[\tau_1], \ldots, [\tau_n]]$ *iff* $\{a \in A : \mathfrak{A} \models \varphi[\tau_1(a), \ldots, \tau_n(a)]\} \in U$.

Proof: It is routine until one reaches the step where one considers a formula φ of the form $\exists z\psi$. It is sufficient to show that if $S = \{a \in A : \mathfrak{A} \models \exists z\psi[\tau_1(a), \ldots, \tau_n(a)]\} \in U$, then there is an element $\tau(x)$ in $\mathfrak{T}(x)$ such that $\{a \in A : \mathfrak{A} \models \psi[\tau_1(a), \ldots, \tau_n(a), \tau(a)]\} \in U$. For notational ease we assume $n = 1$. Since T is model complete, every formula of $\mathfrak{L}$ is provably equivalent to an existential formula of $\mathfrak{L}$ as well as to a universal formula of $\mathfrak{L}$. Let $\theta(x)$ be a universal formula of $\mathfrak{L}$ which is provably equivalent in T to $\exists z\psi(x, z)$ and let $\psi'(x, z)$ be a formula of $\mathfrak{L}$ which is existential and is provably equivalent in T to $\psi(x, z)$. It follows from the above since $\mathfrak{A}$ is a model of T that $\mathfrak{A} \models \forall x(\exists z\psi'(\tau_1, z) \rightarrow \theta(\tau_1))$. However since this sentence of $\mathfrak{L}^*$ is universal it is an element of $S(T)^*$. It follows since $S(T)^*$ "extends" T that $S(T)^* \vdash \forall x(\theta(\tau_1) \rightarrow \exists z\psi'(\tau_1, z))$ and hence $S(T)^* \vdash \forall x \exists z(\theta(\tau_1) \rightarrow \psi'(\tau_1, z))$. By Lemma 1.1 there are terms $\tau_1', \ldots, \tau_m'$ in $\mathfrak{L}^*$ such that $S(T)^* \vdash \forall x((\theta(\tau_1) \rightarrow \psi'(\tau_1, \tau_1')) \vee \ldots \vee (\theta(\tau_1) \rightarrow \psi'(\tau_1, \tau_m')))$ and since $\mathfrak{A}$ is a model of $S(T)^*$ we have that the union of the sets $S_i = \{a \in A : \mathfrak{A} \models (\theta(x) \rightarrow \psi'(x, z))[\tau_1(a), \tau_i'(a)]$ for $i = 1, \ldots, m\}$ equals A, and since U is an ultrafilter $S_k \in U$ for some $k = 1, \ldots, m$. It follows that $\{a \in A : \mathfrak{A} \models \psi(x, z)[\tau_1(a), \tau_k'(a)]\} \in U$.

We observe that the theory $S(T)^*$ in the above proof is used only to prove the existence of a suitable term in $\mathfrak{T}(x)$ and of course plays no role in the definition of $\mathfrak{A}^*$. Now as in [1] we obtain as an immediate consequence the next result.

Corollary 1.3 $\mathfrak{A} \equiv \mathfrak{A}^*$.

Now let $\mathfrak{L}^*$ contain a constant symbols $\bar{a}$ for each element a in A the domain of $\mathfrak{A}$ where $\mathfrak{A}$ is a model of $S(T)$. Forming $\mathfrak{A}^*$ just as above from the terms $\mathfrak{T}(x)$ of $\mathfrak{L}^*$ and an ultrafilter U as above, we now obtain the following as in [1].

Corollary 1.4 $\mathfrak{A} < \mathfrak{A}^*$.

By choosing U to be a nonprincipal ultrafilter, one is always able to obtain a proper elementary extension of $\mathfrak{A}$ in this manner when A is infinite. Next we show how to use Theorem 1.2 to deduce Theorem 0.5. Let $\mathfrak{A}$ be a model of T where $A = \omega$ and let $\mathfrak{A}$ be recursive in S. Let $\mathfrak{L}^*$ contain a function symbol $\bar{f}$ for each $f : \omega \rightarrow \omega$ such that f is recursive in S. We also add to $\mathfrak{L}^*$ enough functions perhaps of several variables as names for functions on A which accomplish a universalization of a set of axioms for T, i.e., for each $\forall_2$ axiom of T $\forall x \exists y \varphi$ with φ quantifier-free we add a function symbol $F_{y(i)}$ to $\mathfrak{L}^*$ for each existential variable $y(i)$ in front of φ where $i = 1, \ldots, j$ and define its interpretation in $\mathfrak{A}$ successively to be $F_{y(i)}(n) = \mu k[\mathfrak{A} \models \exists y' \varphi[n, F_{y(1)}(n), \ldots, F_{y(i-1)}(n), k]]$, which by Lemma 0.2 gives us a function that is recursive in S. So now we have an extension of $\mathfrak{A}$ to $\mathfrak{A}^*$, a structure for $\mathcal{L}^*$ which is a model of the universalization $S(T)$ of T, and with $\mathfrak{T}(x)$ as before and D an ultrafilter of the recursive in S subsets of ω, it is obvious that $\mathfrak{T}(x)/D = R(\mathfrak{A})/D$. In this way we see that the recursive ultrapower construction of Section 0 is a special case of our ultrapower of term functions of $\mathfrak{A}$ construction.

With a little more elaboration we can carry out a similar generalization of the ultraproduct construction. Let T in $\mathcal{L}$ be model complete and $S(T)$ be its universalization in $\mathcal{L}^*$ as above. We assume that $\mathcal{L}^*$ has a constant symbol c in it, otherwise we just add a new constant symbol c to $\mathcal{L}^*$ and call the result $\mathcal{L}^*$. Let $\mathfrak{A}_i$ for i in I be models of $S(T)$ in $\mathcal{L}^*$. Let $\mathfrak{T}$ be the set of terms of $\mathcal{L}^*$ with no variables, i.e., $\mathfrak{T}$ is the set of constant terms of $\mathcal{L}^*$. Let $\mathfrak{B}$ be the Boolean algebra of subsets of I whose elements consist of all sets S such that for some formula $\varphi(x_1,\ldots,x_n)$ of $\mathcal{L}$ and some terms $\tau_1,\ldots,\tau_n$ in $\mathfrak{T}$ $S = \{i : \mathfrak{A}_i \models \varphi[\tau_1^{\mathfrak{A}_i},\ldots,\tau_n^{\mathfrak{A}_i}]\}$. Let U be an ultrafilter of $\mathfrak{B}$. We let $S(T)^*$ be the set of universal sentences α of $\mathcal{L}^*$ which are true in every $\mathfrak{A}_i$. Now we wish to define *the term ultraproduct of* $\{\mathfrak{A}_i : i \in I\}$ *with respect* U which we denote by $\mathfrak{T}_c(\mathfrak{A}_i)/U$ or $\mathfrak{P}$ as follows: for τ, τ' in $\mathfrak{T}$ we define $\tau \equiv \tau'$ to mean that $\{i : \tau^{\mathfrak{A}}i = \tau'^{\mathfrak{A}}i\} \in U$, and the set of these equivalence classes of constant terms of $\mathcal{L}^*$ is the domain of our ultraproduct $\mathfrak{P}$. For an n-ary function symbol F of $\mathcal{L}$ we define $F^{\mathfrak{P}}([\tau_1],\ldots,[\tau_n]) = [F(\tau_1,\ldots,\tau_n)]$. For an n-ary relation symbol R of $\mathcal{L}$ we define $R^{\mathfrak{P}}([\tau_1],\ldots,[\tau_n])$ to hold iff $\{i : \mathfrak{A}_i \models R[\tau_1,\ldots,\tau_n]\} \in U$. In this way we obtain a well-defined structure $\mathfrak{P}$ for $\mathcal{L}$. In the same way as Theorem 1.2 was proven, we obtain the next result.

Theorem 1.5 *For any formula* $\varphi(x_1,\ldots,x_n)$ *of* $\mathcal{L}$ *and terms* $\tau_1,\ldots,\tau_n$ *in* $\mathfrak{T}$, $\mathfrak{P} \models \varphi[[\tau_1],\ldots,[\tau_n]]$ *iff* $\{i : \mathfrak{A}_i \models \varphi[\tau_1,\ldots,\tau_n]\} \in U$.

Just as we did above for the ultrapower we can obtain the recursive in S ultraproduct construction of Section 0 from the term ultraproduct construction. One needs to add function symbols to $\mathcal{L}^*$ for each recursive in S function on ω; one needs also to interpret c in $\mathfrak{A}_i$ for i in ω as i. Of course the term ultrapower construction is a special case of the term ultraproduct construction where one interprets the value of c in the a^{th} copy of $\mathfrak{A}$ as a.

2 Isomorphism of recursive ultrapowers Let $\mathfrak{A}$ and $\mathfrak{B}$ be two denumerable models of the countable language $\mathcal{L}$ and assume that $A = B = \omega$ and $\mathfrak{A} \equiv \mathfrak{B}$. We suppose that $T = \mathrm{Th}(\mathfrak{A})$ is model complete. Our next result shows that there is an oracle set S such that by using it we obtain recursive ultrapowers of $\mathfrak{A}$ and $\mathfrak{B}$ which are isomorphic. This gives a countable version of the Keisler–Shelah Theorem that any two elementary equivalent structures have isomorphic ultrapowers.

Theorem 2.1 *There exists a set* $S \subseteq \omega$ *such that* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are both uniformly recursive in S and T is S-decidable. Moreover, for any such S and any nonprincipal ultrafilter D of the recursive in S subsets of* ω, $R(\mathfrak{A})/D \cong R(\mathfrak{B})/D$.

Proof: The existence of S is easy, since all one need do is take the joins of the sets of numbers and functions one needs to form $\mathcal{L}$, the oracle for $\mathfrak{A}$, the oracle for $\mathfrak{B}$, and the set of Gödel numbers of consequences of T, see Rogers [9].

Since $R(\mathfrak{A})/D$ and $R(\mathfrak{B})/D$ are both denumerable, let $[f_1],\dots,[f_n],\dots$ and $[g_1],\dots,[g_n],\dots$ be listings of their distinct elements, respectively. We construct the isomorphism using the usual back and forth argument as described in [10] or [1] together with Los's property. It is sufficient to show how to extend a given partial isomorphism with the following properties:

Inductive Hypothesis. Suppose we are given a map σ such that $\sigma([f_i']) = [g_i']$ for $i = 1,\dots,n$ where $[f_i']$ belong to $R(\mathfrak{A})$ and the $[g_i']$ belong to $R(\mathfrak{B})$. Moreover, for any formula φ of $\mathcal{L}$ with at most n free variables, $\{i:\mathfrak{A}\models\varphi[f_1'(i),\dots,f_n'(i)]\}\in D$ iff $\{i:\mathfrak{B}\models\varphi[g_1'(i),\dots,g_n'(i)]\}\in D$.

Now let $[f]$ be the first element in the above list such that $[f]\neq[f_i']$ for $i=1,\dots,n$. Let $\varphi_1,\dots,\varphi_k,\dots$ be a recursive in S listing of all formulas of $\mathcal{L}$ with at most the first $n+1$ variables free in them. Next we define g in $R(\mathfrak{B})$ as follows:

$$g(i) = \mu b \text{ such that for the largest } k\le i+1\,\forall j(1\le j\le k\to(\mathfrak{A}\models\varphi_j[f_1'(i),\dots,f_n'(i),f(i)] \text{ iff } \mathfrak{B}\models\varphi_j[g_1'(i),\dots,g_n'(i),b])); \text{ and}$$

$$g(i) = 0 \text{ otherwise.}$$

It follows by Lemma 0.3 that g is recursive in S. Now define $f_{n+1}' = f$ and $g_{n+1}' = g$. Extend σ by setting $\sigma([f_{n+1}']) = [g_{n+1}']$. We claim now that the inductive hypothesis is now satisfied with $n+1$ replacing n.

Let φ_m be any formula with at most $x_1,\dots,x_{n+1}$ as its free variables. It is sufficient to show that if $\{i:\mathfrak{A}\models\varphi_m[f_1'(i),\dots,f_{n+1}'(i)]\}\in D$ then $\{i:\mathfrak{B}\models\varphi_m[g_1'(i),\dots,g_{n+1}'(i)]\}\in D$. Let θ_i be the formula defined for $1\le i\le m$ by

$$\theta_i = \varphi_i \text{ if } R(\mathfrak{A})/D\models\varphi_i[[f_1'],\dots,[f_{n+1}']] \text{ and}$$

$$\theta_i = \neg\varphi_i \text{ if not } R(\mathfrak{A})/D\models\varphi_i[[f_i'],\dots,[f_{n+1}']].$$

By Theorem 0.4 for $1\le i\le m$, $A_i = \{i:\mathfrak{A}\models\theta_i[f_1'(i),\dots,f_{n+1}'(i)]\}\in D$. Let $\theta = \exists x_{n+1}\bigwedge_{i=1}^m\theta_i$. Clearly $U=\bigcap_{j=1}^m A_j\subseteq U^* = \{i:\mathfrak{A}\models\theta[f_1'(i),\dots,f_n'(i)]\}\in D$. Thus by our original inductive hypothesis we have that $U^{**} = \{i:\mathfrak{B}\models\theta[g_1'(i),\dots,g_n'(i)]\}\in D$. Now if $i\in U\cap U^{**}$ and $i\ge m-1$, then it follows from the definition of g above that $\mathfrak{B}\models\varphi_m[g_1'(i),\dots,g_n'(i),g(i)]$. Now since D is a nonprincipal ultrafilter of the sets recursive in S we have $\{i:\mathfrak{B}\models\varphi_m[g_1'(i),\dots,g_n'(i),g_{n+1}'(i)]\}\in D$. This verifies the claim.

Recall from [1] (new edition 1990) or Keisler [6] (p. 69) that a structure $\mathfrak{M}$ is *recursively saturated* if for any finite subset $Y\subseteq M$ every recursive set $\Psi(x)$ of formulas of $\mathcal{L}(Y)$ which is finitely satisfiable in $(\mathfrak{M}, Y)$ is satisfiable in $(\mathfrak{M}, Y)$. This leads to our next result.

Theorem 2.2 *If $\mathfrak{A}$ is uniformly recursive in S and T is S-decidable, then for any nonprincipal ultrafilter D of the recursive in S subsets of ω, $R(\mathfrak{A})/D$ is recursively saturated.*

Proof: Let $Y=\{[f_1],\dots,[f_n]\}$ be a finite set of elements of $R(\mathfrak{A})/D$ and let $\Psi(x)$ be a recursive set of formulas in $\mathcal{L}(Y)$ which are finitely satisfiable in $(R(\mathfrak{A})/D, Y)$. We imitate a portion of the proof of Theorem 2.1. Let $\varphi_1(y_1,\dots,y_n,x),\dots,\varphi_m(y_1,\dots,y_n,x),\dots$ be a recursive listing of all the formulas in $\Psi(x)$. We define g in $R(\mathfrak{A})$ as follows:

$g(i) = \mu b$ such that for the largest $k \leq i + 1$ $\forall j(1 \leq j \leq k \rightarrow \mathfrak{A} \models \varphi_j[f_1(i), \ldots, f_n(i), b])$; and

$g(i) = 0$, otherwise.

It follows by Lemma 0.3 that g is recursive in S. Moreover we know by hypothesis and Los's property that for $\theta_j = \bigwedge_{i=1}^{j} \varphi_i$ that $U_j = \{i : \mathfrak{A}(\models \exists x\theta_j[f_1(i), \ldots, f_n(i)]\} \in D$. We wish to show that for any $m \geq 1$ that $U_m = \{i : \mathfrak{A} \models \varphi_m[f_1(i), \ldots, f_n(i), g(i)]\} \in D$. Let $i \geq m - 1$ and suppose $i \in U_m$ then in the definition of g $k \geq m$, and hence $\mathfrak{A} \models \varphi_m[f_1(i), \ldots, f_n(i), g(i)]$; but then since D is nonprincipal, one has that $U_m \in D$. It then follows that $[g]$ satisfies $\Psi(x)$ in $(R(\mathfrak{A})/D, Y)$.

We point out that the above proof shows that any recursive in S set of formulas of $\mathcal{L}(Y)$ which is finitely satisfiable in $R(\mathfrak{A})/D$ is also satisfiable there. In combination with Theorem 2.1 we obtain the result in our title that any two countable elementary equivalent structures have recursive ultrapowers which are recursively saturated and isomorphic. In closing we point out that it is possible to extend our constructions to allow uncountable oracle sets $\mathfrak{S} = \{S_i : i \in I\}$ by defining a function or set to be *recursive in* $\mathfrak{S}$ if it is recursive in some finite subset of $\mathfrak{S}$. By taking $\mathfrak{S} = \mathcal{P}(\omega)$ our construction becomes the same as the usual ultraproduct and ultrapower construction which was our starting point.

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