Notre Dame Journal of Formal Logic
Volume 33, Number 4, Fall 1992

What Evidence is There That 2^65536 is a Natural Number?

DAVID ISLES

Abstract The closure of the natural numbers under exponentiation $a\hat{}b$ is a fact which is central to results in metamathematics. The argument which purports to establish this closure involves a simple mathematical induction. An analysis of this proof shows that it may involve a new and subtle form of circularity.

1 Introduction[1] To most mathematicians, the title of this article will, I suppose, appear a bit strange: it is so obvious that 2^65536 *is* a natural number that there would seem to be no rational basis for questioning it. Yet there have been objections to the claim that all such exponential expressions name a natural number, two of the best known being due to Paul Bernays [1] and Edward Nelson [8]. Bernays, in "On Platonism in Mathematics", rhetorically questions whether 67^(257^729) can be represented by an "Arabic numeral" (he does not, however, press the discussion). By contrast, Nelson, in "Predicative Arithmetic", develops a large body of theory which he then advances to support his belief that 2^65536 is not a natural number or that, more generally, exponentiation is not a total function. His ideas will be discussed a bit more fully further on.

What I would like to try to do here is to shift the burden of proof onto those who would claim that 2^65536 does name or is equal to a natural number by examining the methods and/or arguments they might employ to convince an intelligent but untutored student of the fact. I have attempted to mention what I think are the main arguments and tried to give criticisms of those arguments. These criticisms have reinforced in me the belief that talking of *number* in the abstract, while useful at times, is a bit sloppy: to obtain more precision one must instead talk about *numerical notations*. This view is shared by others including Nelson, Rotman [9], and vanBendegem [10], and was exposited by vanDantzig in "Is 10^(10^10) a finite number?" [11].[2] Furthermore, the line of argument I have followed leads me to re-examine the circumstances under which a proof (at

Received August 19, 1991; Revised January 31, 1992

least in the area under consideration) can be regarded as "convincing": to be composed of logically correct steps which lead from accepted assumptions to the conclusion is not enough; in addition there are certain global conditions which the proof must satisfy in order to be seen as "making sense".

2 ***Preliminaries*** First, some terminology. By "natural number" or "numeral" I mean (unary) notations of the type $|, ||, |||, ||||, \ldots$, ordered as indicated; and by "numerical exponential notations" I mean the collection of terms built from these together with the function signs + (addition), $\cdot$ (multiplication), ^ (exponentiation) and left and right parentheses (although these last will frequently be omitted). $|+|$, $||\cdot(||+||)$, $||{}^\wedge((||+|||)\cdot|||)$, etc. are examples. If we add to these the usual recursion rules for evaluating these functions, viz.

$$A + | = A|$$
$$A \cdot | = A$$
$$A{}^\wedge| = A$$
$$A + (B|) = (A + B)|$$
$$A \cdot (B|) = (A \cdot B) + A$$
$$A{}^\wedge(B|) = (A{}^\wedge B) \cdot A$$

we obtain a reasonable model of these functions on the (informal) natural numbers. Notice that these rules provide a computing procedure which we could apply to any given numerical exponential notation. Thus $||+|| = ||\cdot|| = ||{}^\wedge|| = ||||$ and $||+(||+||) = ||||||$, $||\cdot(||\cdot||) = ||||||||$, $||{}^\wedge(||{}^\wedge||) = ||||||||||||||||$. But there is a sharp difference between the computations of, say, $||+(||+(||+(||+||)))$ or $||\cdot(||\cdot(||\cdot(||\cdot||)))$ and the computations of $||{}^\wedge(||{}^\wedge(||{}^\wedge(||{}^\wedge||)))$ (which we will abbreviate as 2^65536 using the usual notation). The former are quite feasible and can be performed in a short time whereas the latter represents a number which exceeds the total number of vibrations executed by all subatomic particles of size <10^-30 cm (smaller than a quark!) which would be needed to fill a universe of radius 10^12 light years (larger than the observational diameter of the universe!) were each to vibrate 10^50 times per second over a period of 10^12 years (longer than the surmised age of the universe!).[3] Neither now nor ever (as far as we can tell at present) is there likely to be a case where the computation that starts with 2^65536 and proceeds according to the recursion rules terminates. Thus even with respect to small numerical notations there are sharp differences between addition and multiplication on the one hand and exponentiation on the other.

It is true that the notation 2^65536 is of the same kind as the term $||{}^\wedge||$ or $||{}^\wedge(||{}^\wedge||)$ for which the analogous computations do terminate but what does that prove? In other areas of life we know that parallelism of form, and even parallelism with respect to forms of processes, does not imply similar behavior (unless the parallel is "exact"; but the appropriateness of the adjective in the present situation is what is being challenged here). After all, the activity of swimming is the same whether it is done in a pool or in the ocean. Yet from the fact that I can swim across a pool it hardly follows that I can swim across an ocean. Moreover the facts of experience belie the assumption that the processes of computing the numerical values of ||^|| and 2^65536 proceed in the same way. ||^|| = |||| can be done in a few steps with pencil and paper. But we know that to actually compute large numbers requires additional computational tools (logarithms, slide rules, calculators, computers, etc.), and any particular tool (e.g., computer system with programs and outside storage) is limited in the size of number it can compute. Eventually, due to the increasing amount of data involved, any completely specified computing scheme must break down and require revision; and it is by no means clear that the pattern of these revisions has any uniformity. Finally the putative fact that there is a Turing machine computation which shows that 2^65536 evaluates to a numeral is (at best) problematic. For Turing machine computations are defined as sequences of state-symbol-tape descriptions. Because these descriptions are isomorphic to the numerals, to claim that there exists a terminal one which encodes a numeral equal to 2^65536 is simply to claim that 2^65536 equals a numeral.

3 Nelson's analysis The average primary school student to whom exponentiation is an unknown concept would, of course, find the claim that 2^65536 is a numeral incomprehensible. A junior high school student might find it comprehensible but problematic. But if the student continues her association with mathematics, by the time she has completed college she will have been convinced of its truth. Now for most students this occurs because they have been *told* that it is true by those whose credentials have given them the authority to authenticate such claims. However, for a smaller number (including some of those who eventually become professional mathematicians) this conviction is achieved (or at least secured) by means of a proof which says that the recursive procedures for evaluating exponential numerical notations always terminate in a numeral. Using standard logical notations, let us list such a proof for further examination.

Example 1

1. Assume that we have proved $(\forall n)(\forall m)(\exists p)\,[n\cdot m = p]$.
2. From the recursion equations for exponentiation we have $q\hat{}| = q$ (i.e., $(\exists s)\,[q\hat{}| = s]$ for any natural number q.
3. As the hypothesis of induction we assume $(\exists s)\,[q\hat{}r = s]$. Then $q\hat{}(r|) = (q\hat{}r)\cdot q$. Using the formula in Step 1 (with $q\hat{}r$ substituted for n and q for m), we conclude that $(\exists s)\,[q\hat{}(r|) = s]$.
4. By induction on r, then, it follows from Steps 2 and 3 that $(\forall q)(\forall r)(\exists s)$ $[q\hat{}r = s]$.

This is a very simple argument. Indeed, the only slightly problematic step seems to be the use of mathematical induction; and it is precisely at this point that Nelson [8] raises criticisms (following a tradition which goes back at least to Poincare). He states his reasons on page 1:

> The reason for mistrusting the induction principle is that it involves an impredicative concept of number. It is not correct to argue that induction only involves the numbers from 0 to n; the property of n being established may be a formula with bound variables that are thought of as ranging over all numbers. That is, the induction principle assumes that the natural number system is

> given. . . . But numbers are symbolic constructions; a construction does not exist until it is made; when something new is made, it is something new and not a selection from a preexisting collection.

(Nelson calls his viewpoint a modified formalism.) In order to avoid this putative "impredicativity" of induction, Nelson proceeds by building up a body of arithmetic statements which he calls "predicative arithmetic". He takes as his starting point the subsystem of Peano arithmetic called Robinson's system, which is given by the following six nonlogical axioms:

(Q1) $(\forall x)[x| \neq 0]$
(Q2) $(\forall x)(\forall y)[x| = y| \rightarrow x = y]$
(Q3) $(\forall x)[x + 0 = x]$
(Q4) $(\forall x)(\forall y)[x + y| = (x + y)|]$
(Q5) $(\forall x)[x \cdot 0 = 0]$
(Q6) $(\forall x)(\forall y)[x \cdot (y|) = (x \cdot y) + x]$.

He then proceeds to build up a sequence of theories by successively adding to Q a finite number of new axioms. For example, one could add to Q finitely many induction axioms whose quantified variables were bounded by terms in the language of Q. More generally, if $Q[T_1, \ldots, T_n]$ is the present theory then we can adjoin $T_{(n+1)}$ as long as $Q[T_1, \ldots, T_{(n+1)}]$ is interpretable in Q (via an interpretation of a certain sort). Nelson indicates some of the mathematical power of predicative arithmetic, studies some of its metamathematics, and then concludes with proofs of the fact that the formula $(\forall n)\mathrm{Exp}(n)$ cannot be proved in predicative arithmetic where $\mathrm{Exp}(n)$ says that, for any natural number $b, b\hat{\ }y$ is defined and satisfies the recursion equations for exponentiation for all numbers $y \leq n$. This result indicates that the use of induction in the above elementary proof of the closure of the natural numbers under exponentiation is impredicative in his sense—that is, that the existentially quantified variable s must range over "natural numbers" which go beyond any that can be shown to exist "predicatively". This evidence, added to the obvious practical impossibilities of computing the numeral value of even modest exponential expressions, leads him to conclude that exponentiation is not total. Whether these arguments would change the mind of someone who believed that 2^65536 evaluated to a natural number is not clear to me. Such a skeptic might counter that while Nelson's results illuminated interesting properties of subsystems of Peano arithmetic, there was no a priori reason to believe that exponentiation *should* be derivable in those systems. Moreover, this skeptic could turn some of Nelson's arguments against him as follows: You base predicative arithmetic on the formalized properties of addition and multiplication, both of which, presumably, you believe to be total. But what reasons do you have for believing *that*, especially as regards multiplication? For large enough n and m you will not be in a position to evaluate $n \cdot m$ no matter how much time, space, and so forth you assume that you have. In these cases also your conviction that the computations are total is secured by means of a proof and, in fact, by a proof of the very same form as that by which I am convinced that exponentiation is total. This is particularly clear if the proofs concern the addition, multiplication, and exponentiation relations on the numerals. Indicate these relations by $A(a, b, a)$, $M(a, b, c)$, and $P(a, b, c)$ respectively. Suppose we assume the following axioms about them:

(A1) $(\forall x)A(x,|,x|)$
(A2) $(\forall x)(\forall y)(\forall z)\,[A(x,y,z) \Rightarrow A(x,y|,z|)]$
(M1) $(\forall x)M(x,|,x)$
(M2) $(\forall x)(\forall y)(\forall z)(\forall w)\,[M(x,y,z)$ and $A(z,x,w) \Rightarrow M(x,y|,w)]$
(P1) $(\forall x)P(x,|,x)$
(P2) $(\forall x)(\forall y)(\forall z)(\forall w)\,[P(x,y,z)$ and $M(z,x,w) \Rightarrow P(x,y|,w)]$.

By induction on the second variable, we obtain the following proofs of $(\forall x)(\forall y)(\exists z)A(x,y,z)$, $(\forall x)(\forall y)(\exists z)M(x,y,z)$, and $(\forall x)(\forall y)(\exists z)P(x,y,z)$.

Example 2

Proof I:

1. Axiom (A1) gives $A(a,|,a|)$; hence $(\exists z)A(a,|,z)$.
2. Assume $(\exists z)A(a,b,z)$. From axiom (A2) it then follows that $A(a,b|,c|)$ and so $(\exists z)A(a,b|,z)$.
3. Using induction on the second variable, we conclude $(\forall x)(\forall y)(\exists z)A(x,y,z)$.

Proofs II (and III):

1. Axiom (M1) gives $M(a,|,a)$; hence $(\exists z)M(a,|,z)$.
(1′. Axiom (P1) gives $P(a,|,a)$; hence $(\exists z)P(a,|,z)$.)
2. Assume $(\exists z)M(a,b,z)$. From axiom (M2) plus the conclusion of Proof I we then conclude $M(a,b|,c)$ and so $(\exists z)M(a,b|,z)$.
(2′. Assume $(\exists z)P(a,b,z)$. From axiom (P2) plus the conclusion of Proof II we then conclude $P(a,b|,c)$ and so $(\exists z)P(a,b|,z)$.)
3. Using induction on the second variable, we conclude $(\forall x)(\forall y)(\exists z)M(x,y,z)$.
(3′. Using induction on the second variable we conclude $(\forall x)(\forall y)(\exists z)P(x,y,z)$.)

Clearly Proofs II and III are practically identical. How can Proof II be claimed to be convincing whereas Proof III is not?

4 Freeing the bound variable The challenge posed in the last paragraph is to find some intrinsic, *structural* difference between Proof III and Proof II (and I). In order to see where this structural difference might lie, let us look again at the proof of Example 1.[4] Steps 1 and 2 seem unassailable; but if we substitute 2^65535 and 2^65536 for the variables q and r in Step 3, we see something interesting. For although the implication $[(\exists s)(2\text{^}65535 = s) \rightarrow (\exists s)(2\text{^}65536 = s)]$ seems hard to deny (on the basis of the assumptions made), the truth of the conclusion can only follow from the truth of the antecedent $(\exists s)(2\text{^}65535 = s)$. But, since no argument is given to establish the truth of this formula, one is forced to conclude that *its truth is simply assumed.* That is, if this proof is to count as "correct reasoning", we must assume that the numerical exponential term 2^65535 equals a natural number value. In other words if we are to count this proof *as a proof*, we must change the meaning of "natural number" to include such expressions as 2^65535.

This conclusion is certainly compatible with Nelson's view that "numbers are symbolic constructions". (It does, however, include a very nontraditional feature—namely, that what you are referring to within an argument by the expression "natural number" may depend on the location within the argument at which you find yourself.) Unfortunately, this answer does not seem to meet the specific challenge posed earlier to find some intrinsic differences between Proof II and Proof III because neither of these two proofs seems to require anything other than references to the (unary) natural numbers (in particular there are no other numerical terms occurring). But if we recall the result presented in elementary logic texts as the "elimination of defined function constants"—i.e., the replacement of function terms $f(x,y,z,\dots)$ in locations $A(w,v,f(x,y,z,\dots))$ by formulas $(\exists s)\,[A(w,v,s)$ and $F(x,y,z,\dots,s)]$ (in the presence of certain axioms)—we see that although different terms may not occur explicitly within the body of a proof they may occur as referents of various quantified variables within the proof. Thus in proofs such as those in Example 2, one should consider the possibility of "freeing the bound variables"—that is, of allowing different bound variables in different locations to have different referents.

It seems to me that if you abandon the notion of a single unique series of natural numbers in favor of a multiplicity of numerical notations, you are almost forced into this nontraditional stance. Imagine a (precocious) 12 year old whose powers of logical argumentation were well developed but for whom the world of natural numbers consisted of a disconnected collection of numerical notations for which she had learned various algorithms.[5] Such a person might be expected to ask at each step of the proofs in Example 2 questions like "To which numbers or notations are you referring at this step?" If she were skeptical, she might reject assurances that the references were always to "the" natural numbers and ask "Since unary notation isn't the only way we write numbers, isn't it possible that your arguments remain valid if we consider other numerical notations as well?" In other words, the child would be saying that the steps in the proof might remain valid even if we allow the references to "numbers" which occur in the proof to vary over collections of possible numerical notations (as long as we observe certain structural restrictions on these references which the proof rules impose). For example, if the inductive step in the proof of $(\forall x)(\forall y)(\exists z)A(x,y,z)$ from Axioms (A1) and (A2) is to be valid, the range of the bound variable y in the second argument place must be the numerals; but no such constraints apply to the first argument place. Likewise in the proof of $(\forall x)(\forall y)(\exists z)M(x,y,z)$ from Axioms (A1), (A2), (M1), and (M2), the range of the second argument place y must again be the numerals. Furthermore, because in the formula

(M2) $\qquad (\forall x)(\forall y)(\forall z)(\forall w)\,[M(x,y,z) \text{ and } A(z,x,w) \Rightarrow M(x,y|,w)]$

x in the first argument place of $M(x,y,z)$ occurs in the second argument place of $A(z,x,w)$ (over which induction was carried out in the preceding proof that $(\forall x)(\forall y)(\exists z)A(x,y,z)$), it as well as y must have the numerals as range. Finally, in the proof of $(\forall x)(\forall y)(\exists z)\mathrm{P}(x,y,z)$ from Axioms (A1) to (P2), not only must the range of y be the numerals, but so too must that of the third argument place z; for in Axiom (P2) $(\forall x)(\forall y)(\forall z)(\forall w)\,[P(x,y,z)$ and $M(z,x,w) \Rightarrow P(x,y|,w)]$, z occupies the first argument place of M, which we have already shown must have the natural numbers as range.

To examine this argument as well as other more complex ones (see Example 5) more closely, we must analyze the proofs in some detail. In the case of the inductive arguments which concern us, they can be modeled as derivations quite naturally and convincingly within a formal system of first-order logic. Consider the following "natural-deduction" derivation (D1) of the implication

$$(\forall x)A(x,|,x|),(x^*)(y^*)(z^*)\,[A(x^*,y^*,z^*) \\ \rightarrow A(x^*,y^*|,z^*|)] \vdash (\forall w)(\forall v)(\exists u)A(w,v,u)$$

which models the statement that the closure of the addition relation follows from Axioms (A1) and (A2). Here x, y, z, x^*, w, u, etc. are bound variables with $(\forall x)$ being the universal quantifier and $(\exists u)$ the existential quantifier; a, b, and α are parameters (free variables). The logical steps are indicated as follows: $\exists$I = existential introduction, $\exists$E = existential elimination, $\forall$I and $\forall$E = universal introduction and elimination, respectively, and $\rightarrow$E = implication elimination. An arrow such as $x^* \rightarrow a$ indicates that *in order for the logical step which accompanies it to be valid* (*preserve truth*) the range of possible values of a must be contained within the range of possible values of x^*. Finally to write $y = u$ means that at the step in question the ranges of the two bound variables are identified (again in order to preserve validity of the logical step (Isles [2])).

Example 3

$$(\forall x^*)(\forall y^*)(\forall z^*)\,[A(x^*,y^*,z^*) \rightarrow A(x^*,y^*|,z^*|)]$$

$$x^* \rightarrow a \quad y^* \rightarrow \alpha \quad z^* \rightarrow b \qquad \forall\mathrm{E}$$

$$\text{①}\ A(a,\alpha,b) \qquad A(a,\alpha,b) \rightarrow A(a,\alpha|,b|)$$

$$\rightarrow\mathrm{E}$$

$$A(a,\alpha|,b|)$$

$$u \rightarrow b| \qquad \exists\mathrm{I}$$

$$(\exists u)A(a,\alpha|,u)$$

$$(\forall x)A(x,|,x|)$$

$$\forall\mathrm{E} \qquad x \rightarrow a$$

$$A(a,|,a|) \qquad \text{②}\ (\exists y)A(a,\alpha,y) \qquad (\exists u)A(a,\alpha|,u)$$

$$\exists\mathrm{I} \qquad u' \rightarrow a| \qquad\qquad y = u \quad b \rightarrow y$$

$$\exists\mathrm{E}\ \text{①}$$

$$(\exists u')A(a,|,u') \qquad (\exists u)A(a,\alpha|,u)$$

$$\text{②}\ \text{Induction} \qquad u' = u$$

$$(\exists u)A(a,\alpha,u)$$

$$\forall\mathrm{I} \qquad \alpha \rightarrow v \quad a \rightarrow w$$

$$(\forall w)(\forall v)(\exists u)A(w,v,u)$$

The arrows here include $x \to a \to w$, $u \to a|$, $a \to w$, $\alpha \to v$, $u \to b|$, and $b \to u$. These indicate the inclusion relations which must hold among sets of terms if they are to be ranges for the various free and bound variables in the derivation. In order that induction be valid, the induction parameter α must range over a(n initial) segment of the numerals (notice that this forces the range of v to contain only numeral-valued notations). Because of the arrows $u \to b|$ and $b \to u$, the range of u contains all the terms $r, r|, r||, r|||, \ldots$ if it contains any term r at all. Otherwise there are very few restrictions on the possible ranges for the other variables. In the standard Tarski interpretation of this derivation, the range of all the variables is the complete set of numerals. *The central assertion of this paper is that this confounds a semantical assertion (there is a "unique" set of natural numbers) with the purely logical requirement that the derivation rules preserve truth.* But if we consider a purely logical derivation and permit the variables and parameters of (D1) to have *different reference ranges*, we can describe alternative (nontraditional) models for the formulas *in this derivation*. For example, consider the structure *NS* of exponential numerical notations. Its domain will consist of the set of such notations and it will contain a one place function $|^*$, a binary relation $=^*$ and three ternary relations A^*, M^*, and P^* which will provide interpretations for a language containing the corresponding function and relation symbols $|$, $=$, A, M, and P. Further, this interpretation will be a (Tarski) model for the nonlogical axioms expressing

1. the transitive, reflexive, and symmetric properties of equality
2. the replacement property of equality with respect to successor terms and atomic formulas
3. the functionality of the successor function and the relations A, M, and P
4. the recursion equations for A, M, and P
5. the formula $(\forall x)\neg[x| = |]$,

if we read these (universally quantified) axioms as giving inductive definitions of the interpreting relations $=^*$, A^*, M^*, and E^*. If we also add the following defining clauses for all terms r, s

$$A^*(r, s, r + s)$$

$$M^*(r, s, r \cdot s)$$

$$P^*(r, s, r \hat{\ } s)$$

we could use *NS* to provide an interpretation for the formulas in (D1) *which would not need to assume the existence of numeral values for the addition relation* A^*. All that is needed is to assign ranges to the variables and parameters in (D1) in a way that is consonant with the arrow conditions. As none of these inclusion relations require that the range of u consist of numeral-valued notations (even if the ranges of w and v do) one can simply let range$(u) = \{r + s \mid r$ in range(w), s in range$(v)\} \cup \{|, ||, |||, \ldots\}$. In this way we avoid using an inductive proof (for the existence of numeral values for A^*) of the sort which we are investigating.

NS can also serve as the universe for the variables and parameters of the inductive derivation (D2) of (A1),(A2),(M1),(M2) $\Rightarrow (w_1)(v_1)(\exists u_1)M(w_1, v_1, u_1)$. In this case, the "arrow conditions" include both those of (D1) (which is a subproof of (D2)) plus some additional ones: $w \to u_1 \to w_1$, $u_1 \to u$, $\beta \to v_1$, $v_1 \to w_1$, $\alpha \to v$, and $u \to u|$. As before, these conditions permit the range of u_1 to include non-numeral numerical exponential terms. Thus both derivations (D1) and (D2) can be shown to be satisfiable (i.e., to possess models of the sort described above) without assuming the putative fact established by the proof which we are investigating.[6]

Parallel in form to (D2) is the following inductive proof (D3) of

$$(\mathrm{A1}),\ldots,(\mathrm{P2}) \Rightarrow (\forall w_2)(\forall v_2)(\exists u_2)P(w_2,v_2,u_2).$$

See Example 4 on next page. Among the reference arrows of (D3) are $\gamma \to v_2$, $u_2 \to u_1$, $u_2 \to w_2$, $v_1 \to w_2$, and $w_1 \to u_2$. These, together with the reference arrows of (D2) (which are part of (D3)), result in the following inclusion conditions:

$$a \to v \to w_1 \to u_2 \to u_1 \to u \to u|$$

$$u_2 \to w_2$$

$$u \to w|$$

$$\beta \to v_1$$

$$\gamma \to v_2.$$

The ranges of the induction parameters a, β, and γ must consist of numerals. Therefore, if *NS* is to be a model of the formulas in (D3), the numerals of *NS* must be closed under the relations A^*, M^*, and P^*, which means that the *numerals of NS must be closed under exponentiation.* Because closure of the numerals must be either assumed or proved (using a derivation essentially like (D3)), it is hard for me to see how this derivation can be viewed as supporting the claim that the numerals are closed under exponentiation. That claim is, in the sense indicated, circular. Thus the response I propose to the challenge posed earlier to show a structural difference between derivations (D1) and (D2) on the one hand and (D3) on the other is this: in the case of the first two, one can construct a satisfying interpretation which shows them to be consistent in a non-question-begging way. This does not seem to be possible for (D3) (or any analogous derivation with which I am acquainted).[7] Inasmuch as (D3) (or its corresponding versions in sequent or other first-order calculi) is a reasonably natural and direct formulation of the informal proof which would usually be presented as evidence that the natural numbers *are* closed under exponentiation, the onus of presenting a non-question-begging proof of this fact is thrown back upon those who would assert it.

The previous conclusions can, of course, be avoided by refusing to accept the central notion of the variability of reference. But enough has been said, I believe, to at least raise the possibility that this refusal has its consequences. It will not suffice to point to a result such as

$$(\forall x)(\forall y)(\forall z)\,[N(x) \wedge N(y) \wedge P(x,y,z) \to N(z)]$$

and claim that this shows that the numerals are closed under exponentiation. For the (first-order) proofs of this formula not only require the assumption of the functionality of the addition, multiplication, and exponentiation relations, but

Example 4

① $M(c_2, a_2, b_2)$ ② $P(a_2, \gamma, c_2)$

$\wedge$I

$M(c_2, a_2, b_2) \wedge P(a_2, \gamma, c_2)$

$(\forall x_2^*)(\forall y_2^*)(\forall z_2^*)(\forall w_2^*)\,[M(z_2^*, x_2^*, w_2^*) \wedge P(x_2^*, y_2^*, z_2^*) \rightarrow P(x_2^*, y_2^* |, w_2^*)]$

$\forall$E | $x_2^* \rightarrow a_2,\ y_2^* \rightarrow \gamma,\ z_2^* \rightarrow c_2,\ w_2^* \rightarrow b_2$

$[M(c_2, a_2, b_2) \wedge P(a_2, \gamma, c_2) \rightarrow P(a_2, \gamma |, b_2)]$

$\rightarrow$E

$P(a_2, \gamma |, b_2)$

$\exists$I | $u_2 \rightarrow b_2$

$(\exists u_2)P(a_2, \gamma |, u_2)$

D2

$(\forall w_1)(\forall v_1)(\exists u_1)M(w_1, v_1, u_1)$

$w_1 \rightarrow c_2,\ v_1 \rightarrow a_2$ | $\forall$E

$(\exists u_1)M(c_2, a_2, u_1)$

$\exists$E ① $b_2 \rightarrow u_1$

$(\exists u_2)P(a_2, \gamma |, u_2)$ ③ $(\exists u_2^*)P(a_2, \gamma, u_2^*)$

$\exists$E ② $c_2 \rightarrow u_2^*$

$(\exists u_2)P(a_2, \gamma |, u_2)$

$(\forall x_2')P(x_2', |, x_2')$

$\forall$E | $x_2' \rightarrow a_2$

$P(a_2, |, a_2)$

$\exists$I | $u_2' \rightarrow a_2$

$(\exists u_2')P(a_2, |, u_2')$

Induction ③ $u_2 = u_2^* = u_2'$

$(\exists u_2)P(a_2, \gamma, u_2)$

$\forall$I | $\gamma \rightarrow v_2,\ a_2 \rightarrow w_2$

$(\forall w_2)(\forall v_2)(\exists u_2)P(w_2, v_2, u_2)$

also the proofs of the thirteen relativized versions of the theorems we have been examining—i.e., of theorems of the form

$$(\forall x)(\forall y)\,[N(x) \wedge N(y) \rightarrow (\exists z)\{N(z) \wedge P(x,y,z)\}].$$

Aside from the inclusion of the axiom $(\forall x)\,[N(|) \wedge \{N(x) \rightarrow N(x|)\}]$, the proofs of these are practically identical to the proofs we have examined, and they involve the same reference arrows. Consequently, the arguments we made before will carry over to these derivations with the one minor change that the interpretation N^* of N in the model must include the numerals.

Example 5 This last example is included to show that one cannot (apparently) avoid the circularity implicit in the proof of the closure of the exponentiation relation by defining that relation *à la* Gödel in elementary number theory. If one examines the proof that exponentiation is numeralwise representable in elementary number theory as given in Kleene [4], one finds that an essential step is the proof of the existence of a common multiple of the numbers in the sequence $1,2,\ldots,n$ (Kleene's formal theorem 157 in [4], p. 192). This derivation is given below in two parts with some trivial steps left out. Assumptions I, II, and III are derived theorems. The formula to be proved using induction on the parameter α is $(\exists x)F(\alpha,x)$ which is

$$(\exists x)\,[0 < x \wedge (\forall z)\,[\{0 < z \wedge z \leq \alpha\} \rightarrow (\exists w)\,[z \cdot w = x]]]$$

(where $z \leq \alpha$ abbreviates $(z < \alpha) \vee (z = \alpha)$). The main step is an or-elimination where the two side formulas express the existence of such a multiple for the two cases $b < a|$ and $b = a|$. The derivation for the first of these two cases is shown in Diagram 1. The completion of the derivation is shown in Diagram 2 (with the derivation of $(\exists x_2)\ F(0,x_2)$ not given). The reference arrows of this derivation include, among others, $x \rightarrow |$, $x \rightarrow x \cdot \alpha|$ and $q \rightarrow x$. Because the induction parameter α has as its range (a segment of) the numerals, this means that the range of x includes terms with an arbitrarily high number of consecutive multiplications. Examination of the derivations of Theorems I, II, and III shows that the ranges of the variables j, k, m, n, p, and q must all be numerals (because they are included within the ranges of induction parameters). Hence a model for this derivation would require that terms of arbitrarily large multiplicative degree be equal to numerals (i.e., that the numerals be closed under exponentiation).

Is 2^65536 a natural number? From the viewpoint urged here, to answer Yes is to recognize that the natural numbers are no longer coextensive with the numerals but rather, say, with numerical exponential (or "similar") notations. This does not mean that you cannot use ordinary mathematical induction in reasoning about these notations. For example, the inductive proof of the formula $(\forall x)(\exists y)H(x,y)$ (where $H(x,y)$ expresses the "hyperexponential relation" $h(0) = 1$, $h(n+1) = 2\hat{\ }(h(n))$ which proceeds from the axioms $H(0,1)$ and $(\forall u)(\forall v)(\forall w)\,[H(u,v) \wedge w = 2\hat{\ }v \rightarrow H(u|,w)]$) generates as reference arrows only $\alpha \rightarrow x$, $y \rightarrow |$, and $y \rightarrow 2\hat{\ }y$ (α is an induction parameter). Consequently an interpretation of this derivation does not require that exponential terms equal numerals. By contrast, the derivation of $(\forall x)(\forall y)(\exists u)Ac(x,y,u)$ (where $Ac(x,y,u)$ stands for the "Ackermann" relation $ac(0,y) = 2\hat{\ }y$, $ac(x,0) = 1$,

Diagram 1

$(\forall m)(\forall n)[n < m| \to n \leq m]$ III

$\forall$E $n \to b,\ m \to \alpha$

③ $(0 < b) \wedge (b \leq \alpha|)$ ② $b < \alpha|$ $(b < \alpha|) \to (b \leq \alpha|)$

$\wedge$E $\to$E

$0 < b$ $b \leq \alpha$ $F(\alpha, d)$ ④

$\wedge$I $\wedge$E, $\forall$E $z \to b$

$(0 < b) \wedge (b \leq \alpha)$ $\{(0 < b) \wedge (b \leq \alpha)\} \to (\exists w)[b \cdot w = d]$

$\to$E

$(\exists w)[b \cdot w = d]$

I $(\forall j)(\forall k)(\forall z_1)[(j \cdot k) \cdot z_1 = j \cdot (k \cdot z_1)]$

$j \to b,\ k \to c$

$\forall$E

$z_1 \to \alpha|$

$(b \cdot c) \cdot \alpha| = b \cdot (c \cdot \alpha|)$

① $b \cdot c = d$

$\vdots$

$(b \cdot c) \cdot \alpha| = d \cdot \alpha|$

$b \cdot (c \cdot \alpha|) = d \cdot \alpha|$

$w_1 \to c \cdot \alpha|$ $\exists$I

$(\exists w_1)[b \cdot w_1 = d \cdot \alpha|]$

① $\exists$E $c \to w$

$(\exists w_1)[b \cdot w_1 = d \cdot \alpha|]$

Diagram 2

II $\quad (\forall p)(\forall q)[p\cdot q = q\cdot p]$ $\qquad\qquad$ $b = \alpha|$ ②

$p \to \alpha|,\ q \to c_1$ $\mid$ $\forall$E

$\alpha|\cdot c_1 = c_1\cdot\alpha|$ $\qquad$ $b\cdot c_1 = \alpha|\cdot c_1$

$b\cdot c_1 = c_1\cdot\alpha|$

$\exists$I $\mid$ $w_2 \to c_1$

③ $(0 < b) \wedge (b \le \alpha|)$

$\wedge$E

$(b \le \alpha|)$

Diagram 1

$(b < \alpha|) \vee (b = \alpha|)$ $\qquad$ $(\exists w_1)[b\cdot w_1 = d\cdot\alpha|]$ $\qquad$ $(\exists w_2)[b\cdot w_2 = c_1\cdot\alpha|]$

② $\vee$E $\mid$ $d = c_1,\ w_1 = w_2$

④ $F(\alpha, d)$ $\qquad$ $(\exists w_1)[b\cdot w_1 = d\cdot\alpha|]$

$\to$I ③, $\forall$I $b \to z_1$

$0 < d\cdot\alpha|$ $\qquad$ $(\forall z_1)[\{0 < z_1 \wedge z_1 \le \alpha|\} \to (\exists w_1)[z_1\cdot w_1 = d\cdot\alpha|]]$

$\wedge$I, $\exists$I $x_1 \to d\cdot\alpha|$

$(\exists x_1)[0 < x_1 \wedge (\forall z_1)[\{0 < z_1 \wedge z_1 \le \alpha|\} \to (\exists w_1)[z_1\cdot w_1 = x_1|]]]$ $\qquad$ $(\exists x)F(\alpha, x)$ ⑤

$\exists$E ④ $\mid$ $d \to x$

$x_2 \to 1$ $\quad$ $(\exists x_2)F(0, x_2)$ $\qquad$ $(\exists x_1)[0 < x_1 \wedge (\forall z_1)[\{0 < z_1 \wedge z_1 \le \alpha|\} \to (\exists w_1)[z_1\cdot w_1 = x_1|]]]$

IND ⑤ $\mid$ $x = x_1 = x_2,\ z = z_1$

$a \to y$

$(\forall y)(\exists x)F(y, x)$

$ac(x + 1, y + 1) = ac(x, ac(x + 1, y)))$ includes among its arrows $u \rightarrow |$ and $\beta \rightarrow u \rightarrow 2^\wedge u$ (β an induction parameter) and so forces on any interpretation the existential requirement that its numerals be closed under exponentiation.

This view of the matter is both more and less radical than that of Nelson. For while it does not limit the use of induction it does imply that the effect of induction is context-dependent. It also implies that when the objects of discussion are linguistic entities (and in this paper the position advocated is that "natural numbers" or better "natural number notations" are linguistic entities) then that collection of entities may vary as a result of discussion about them. A consequence of this is that the "natural numbers" of today are not the same as the "natural numbers" of yesterday. Although the possibility of such denotational shifts remains inconceivable to most mathematicians, it seems to be more compatible with the history both of the cultural growth (and of growth in individuals) of the number concept than is the traditional, Platonic picture of an unchanging, timeless, and notation-independent sequence of numbers.[8]

NOTES

1. Investigator: "In the Far North, where there is snow, all bears are white. Novaya Zemlya is in the Far North and there is always snow there. What color are the bears there?" Kashgar peasant: "You've seen them, you know. I haven't seen them, so how could I say?" (from Luria [7], p. 112)

2. John Locke, in chapter 16 of *An Essay Concerning Human Understanding*, emphasizes the centrality of notation in dealing with numbers:

   > For, the several simple modes of numbers being in our minds but so many combinations of units, which have no variety, nor are capable of any difference but more or less, names or marks for each distinct combination seem more necessary than in any other sort of ideas. For, without such names or marks, we can hardly well make use of numbers in reckoning, especially where the combination is made up of any great multitude of units. ([6], p. 272)

3. Archimedes in his article "The Sand Reckoner" must have been the first to make a similar estimate.

4. Frequently, induction in a case like this is justified by regarding it as a repeated modus ponens—i.e., if F is the formula being proved, what we have is a tree:

   $$\dfrac{F(|) \qquad F(|) \rightarrow F(||)}{F(||)} \qquad F(||) \rightarrow F(|||)$$

   $$\dfrac{F(2^\wedge 65535) \qquad F(2^\wedge 65535) \rightarrow F(2^\wedge 65536)}{F(2^\wedge 65536)}$$

   Clearly this "explanation" begs the question.

5. Compare Locke again:

   > Thus children, either through want of names to mark the several progressions of numbers, or not yet having the faculty to collect scattered ideas into complex ones, . . . , do not be-

gin to number very early, nor proceed in it very far or steadily, till a good while after they are furnished with good store of other ideas: and one may often observe them discourse and reason pretty well, . . . , before they can tell twenty. ([6], p. 273)

6. Notice that one speaks of the *satisfiability of a derivation* rather than the satisfiability of a set of formulas. This is necessary with the semantics being proposed because a single logical step may introduce new identifications or reference arrows and thus cause a derivation to become unsatisfiable. As an example, notice that although $(\forall x)A(x)$ and $\neg(\forall y)A(y)$ are satisfiable when x and y are given different ranges, an unsatisfiable derivation leading to a contradiction can be written with them as axioms.

7. An elaboration and exploration of the technical ideas in this paper is contained in [2]. Although the discussion here is framed in terms of the construction of interpretations for derivations, it might be better done in terms of interpreting one derivation within another theory. Thus for (D2), this latter theory would be in a language which would include predicates $N(x)$ ("x is a numeral"), $Add(x)$ ("x is a term built with $|$ and $+$"), $Mult(x)$ ("x is a term built with $|$, $+$, and $\cdot$"), and $Exp(x)$ ("x is a term built with $|$, $+$, $\cdot$, and ^"). It would also include axioms for induction over terms of these classes (e.g., $\{F(|) \wedge (\forall z)[F(z) \to F(z|)]\} \to (\forall z)[N(z) \to F(z)]$, as well as axioms like $(\forall x)[U_1(x) \to W(x)]$ and $(\forall y)[U(y) \to U_1(y)]$ that reflect the arrow conditions $w \to u_1$ and $u_1 \to u$. However, it would not include the axiom $(\forall x)[Exp(x) \to N(x)]$ which *would* be necessary to interpret (D3).

8. It is, however, commonplace in the practice of law. One of the activities of a lawyer is to see how legal terms can be interpreted in novel ways so as to adapt a statute or case to her client's needs. If such an interpretation eventually becomes sufficiently widespread, it finally becomes part of the accepted meaning of the term in question. A study of parallels and divergences between standards of proof, evidence, etc. in the two fields would be interesting.

   Moreover, the shift in reference which occurs during the course of a derivation, with the consequent alteration of the possible interpretations of a predicate are (at least superficially) similar to the process of "concept-formation" which Lakatos argues occurs historically as a result of the interplay between proofs of a theorem and attempts to refute or confirm that proof [5].

REFERENCES

[1] Bernays, P., "On platonism in mathematics," pp. 258–271 in *Philosophy of Mathematics (2nd edition)*, edited by P. Benacerraf and H. Putnam, Cambridge University Press, Cambridge, 1983.

[2] Isles, D., "A finite analog to the Loewenheim-Skolem Theorem," under review.

[3] Isles, D., "On the notion of standard non-isomorphic natural number series," pp. 111–134 in *Constructive Mathematics*, edited by F. Richman, Springer-Verlag, Berlin, 1981.

[4] Kleene, S., *Introduction to Metamathematics*, North-Holland, Amsterdam, 1971.

[5] Lakatos, I., *Proofs and Refutations*, Cambridge University Press, 1976.

[6] Locke, J., *An Essay Concerning Human Understanding (vol 1)*, collated and annotated by A. C. Fraser, Dover, New York, 1959.

[7] Luria, A., *Cognitive Development: Its Cultural and Social Foundations*, Harvard University Press, Cambridge, 1982.

[8] Nelson, E., *Predicative Arithmetic*, Princeton University Press, Princeton, 1986.

[9] Rotman, B., "Towards a semiotics of mathematics," *Semiotica*, vol. 72 (1988), pp. 1–35.

[10] VanBendegem, J., "Finite, empirical mathematics: outline of a model," Preprint #17 of the Seminarie voor Logica en Wijsbegeerte van de Wetenschappen, Rijksuniversiteit-Gent, Belgium, 1985.

[11] VanDantzig, D., "Is 10^(10^10) a finite number?" *Dialectica*, vol. 19 (1956), pp. 273–277.

Department of Mathematics
Tufts University
Medford, Massachusetts 02155