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Maximal Subgroups of Infinite Symmetric Groups

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Abstract We prove that it is consistent that there exists a subgroup of the symmetric group $\mathrm{Sym}(\lambda)$ which is not included in a maximal proper subgroup of $\mathrm{Sym}(\lambda)$. We also consider the question of which subgroups of $\mathrm{Sym}(\lambda)$ stabilize a nontrivial ideal on λ.

1 Introduction The work in this paper was motivated by the following question, which was raised by Peter Neumann. If $\lambda \geq \omega$, does every proper subgroup of $\mathrm{Sym}(\lambda)$ lie in a maximal subgroup of $\mathrm{Sym}(\lambda)$? While a positive answer seems very unlikely, all of the results up to this point have concerned sufficient conditions for a subgroup $G < \mathrm{Sym}(\lambda)$ to lie in a maximal subgroup of $\mathrm{Sym}(\lambda)$. For example, the main theorem in MacPherson and Praeger [3] states that if $G < \mathrm{Sym}(\omega)$ is not highly transitive, then G is contained in a maximal subgroup. In Section 2, we shall prove the following result.

Theorem 1 (F_λ) *There exists a subgroup $G < \mathrm{Sym}(\lambda)$ such that the set $\mathbb{L} = \{H \mid G \leq H < \mathrm{Sym}(\lambda)\}$ is a well-ordering under inclusion of order-type 2^λ. In particular, G is not contained in a maximal subgroup of* $\mathrm{Sym}(\lambda)$.

It is not known whether this theorem can be proved in ZFC. Our extra hypothesis F_λ is the following statement. Let $\mathrm{Sym}_{<\lambda}(\lambda)$ be the group of all permutations π of λ such that $|\mathrm{Mov}(\pi)| < \lambda$, where $\mathrm{Mov}(\pi) = \{\alpha \mid \alpha^\pi \neq \alpha\}$. Let $S(\lambda) = \mathrm{Sym}(\lambda)/\mathrm{Sym}_{<\lambda}(\lambda)$.

(F_λ) If $T < S(\lambda)$ is a subgroup with $|T| < 2^\lambda$, then there exists an element of infinite order $\pi \in S(\lambda) \setminus T$ such that $\langle T, \pi \rangle = T * \langle \pi \rangle$.

Here $*$ denotes the free product. We shall also show that F_λ is consistent with but independent of ZFC.

Another result from [3] states that if I is a nontrivial ideal on λ which contains a set X with $|X| = |\lambda \setminus X| = \lambda$, and $G \leq S_{\{I\}} = \{\pi \in \mathrm{Sym}(\lambda) \mid I^\pi = I\}$,

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then G is contained in a maximal subgroup of $\mathrm{Sym}(\lambda)$. It is also shown in [3] that if $|G| \leq \lambda$, then there exists such an ideal I with $G \leq S_{\{I\}}$. In the third section of this paper, we shall obtain a stronger version of the latter result and also prove the independence of the strongest conceivable version. We shall see that the least size of a subgroup $G \leq \mathrm{Sym}(\lambda)$ which fails to stabilize such an ideal is bounded below by the size $B(\lambda)$ of the smallest family of uniform ultrafilters which cover $[\lambda]^{\lambda}$. In the final section, we shall prove that it is consistent that $B(\lambda)$ is much bigger than the size of any maximal almost disjoint family $\mathcal{F} \subseteq \mathcal{P}(\lambda)$.

Our notation follows that of Kunen [2]. Thus if $\mathbb{P}$ is a notion of forcing and $p, q \in \mathbb{P}$, then $q \leq p$ means that q is a strengthening of p. The notation $p \| q$ means that p and q are compatible conditions. A subset $X \subset \lambda$ is said to be a moiety if $|X| = |\lambda \backslash X| = \lambda$.

2 The main result Theorem 1 is an immediate consequence of the following result.

Theorem 2.1 *Let S be a group with $|S| = \kappa > \omega$. Suppose that whenever $T < S$ is a subgroup with $|T| < \kappa$, then there exists an element of infinite order $\pi \in S \backslash T$ such that $\langle T, \pi \rangle = T * \langle \pi \rangle$. Then there exists a subgroup $G < S$ such that the set $\mathbb{L} = \{H \mid G \leq H < S\}$ is a well-ordering under inclusion of order-type κ.*

Proof: Let $S = \{g_\alpha \mid \alpha < \kappa\}$. We shall define inductively a sequence of strictly increasing chains of subgroups $\langle H_\beta^\alpha \mid \beta \leq \alpha \rangle$ for $\alpha < \kappa$ such that the following condition is satisfied.

$$(*) \qquad \text{If } \beta \leq \gamma \leq \alpha, \text{ then } H_\beta^\alpha \cap H_\gamma^\gamma = H_\beta^\gamma.$$

We set $H_0^0 = 1$. If λ is a limit ordinal, then we define

$$H_\beta^\lambda = \bigcup_{\beta \leq \alpha < \lambda} H_\beta^\alpha \quad \text{if } \beta < \lambda$$

$$H_\lambda^\lambda = \bigcup_{\alpha < \lambda} H_\alpha^\alpha.$$

Assume that H_β^γ has been defined for all $\beta \leq \gamma \leq \alpha$. Our intention is that, at the end of the construction, we will have that

$$\{H \mid H_0^\kappa \leq H < S\} = \{H_\beta^\kappa \mid \beta < \kappa\}$$

where $H_\beta^\kappa = \bigcup_{\beta \leq \alpha < \kappa} H_\beta^\alpha$. To accomplish this, we take steps to ensure that for all $\beta < \kappa$, if $g \in H_{\beta+1}^\kappa \backslash H_\beta^\kappa$, then $\langle H_0^\kappa, g \rangle = H_{\beta+1}^\kappa$. So suppose that there exist $\beta + 1 \leq \alpha$, $g \in H_{\beta+1}^\alpha \backslash H_\beta^\alpha$ and $h \in H_{\beta+1}^\alpha$ such that $h \notin \langle H_0^\alpha, g \rangle$. By hypothesis, there exist elements of infinite order $\pi_1, \pi_2 \in S \backslash H_\alpha^\alpha$ such that $\langle H_\alpha^\alpha, \pi_1, \pi_2 \rangle = H_\alpha^\alpha * \langle \pi_1 \rangle * \langle \pi_2 \rangle$. Let $\varphi = h\pi_1^{-1} g^{-1} \pi_2^{-1} g$; so that $h = \varphi g^{-1} \pi_2 g \pi_1$. For $0 \leq \gamma \leq \alpha$, define $H_\gamma^{\alpha+1} = \langle H_\gamma^\alpha, \pi_1, \pi_2, \varphi \rangle$. We must check that if $0 \leq \gamma \leq \alpha$, then

$$(**) \qquad H_\gamma^{\alpha+1} \cap H_\alpha^\alpha = H_\gamma^\alpha.$$

There are three possibilities to consider.

Case 1. Suppose that $g \in H_\gamma^\alpha$, and hence also $h \in H_\gamma^\alpha$. Then $H_\gamma^{\alpha+1} = H_\gamma^\alpha * \langle \pi_1 \rangle * \langle \pi_2 \rangle$, and $(**)$ is obvious.

Case 2. Suppose that $h \in H^\alpha_\gamma$, but $g \notin H^\alpha_\gamma$. It is easily checked that

$$H^{\alpha+1}_\gamma = H^\alpha_\gamma * \langle \pi_1 \rangle * \langle \pi_2 \rangle * \langle g^{-1}\pi_2 g \rangle.$$

Furthermore, if $z \in H^{\alpha+1}_\gamma$, $z = a_1 \cdots a_n$ is the unique reduced sequence expression with respect to the above free product decomposition, and m is the length of the unique reduced sequence expression of z with respect to the decomposition $H^\alpha_\alpha * \langle \pi_1 \rangle * \langle \pi_2 \rangle$, then $m \geq n$. Hence $(**)$ holds.

Case 3. Suppose that $g, h \notin H^\alpha_\gamma$. Then the proof that $(**)$ holds is similar to that in Case 2, using the free product decomposition

$$H^{\alpha+1}_\gamma = H^\alpha_\gamma * \langle \pi_1 \rangle * \langle \pi_2 \rangle * \langle \varphi \rangle.$$

Finally, let $\delta = \min\{\xi \,|\, g_\xi \notin H^{\alpha+1}_\alpha\}$, and define $H^{\alpha+1}_{\alpha+1} = \langle H^{\alpha+1}_\alpha, g_\delta \rangle$.

It is now clear that we can perform the construction successfully. This completes the proof of Theorem 2.1.

The following result, which is an easy exercise, establishes the consistency of F_λ.

Theorem 2.2 (GCH) *For all* $\lambda \geq \omega$, F_λ *holds.*

We now prove the independence of F_λ for $cf(\lambda) > \omega$ and for $\lambda = \omega$. We first deal with the case when $\lambda = \omega$.

Theorem 2.3 *Let* $M \models \kappa^\omega = \kappa$. *Then there exists a generic extension* $M[G]$ *in which the following are true.*

(i) $2^\omega = \kappa$.

(ii) *There exists a subgroup* $T < S(\omega)$ *of cardinality* ω_1 *such that for all* $\pi \in S(\omega) \setminus T$, *there exist* $g, h \in T \setminus 1$ *with* $[g^\pi, h] = 1$.

Proof: By first adding κ Cohen reals if necessary, we can suppose that $M \models 2^\omega = \kappa$. We now perform an iterated finite support construction M_α, $\alpha \leq \omega_1$. We pass from M_α to $M_{\alpha+1}$ via a 2-step c.c.c. iteration, say

$$M_\alpha \subset M^0_{\alpha+1} \subset M_{\alpha+1}.$$

First let

$$\mathbb{P} = \{p \,|\, p : \omega \to \omega \text{ is a finite injective function}\}.$$

Then $M^0_{\alpha+1} = M_\alpha[G]$, where G is a generic subset of $\mathbb{P}$. Let $\pi = \bigcup G$ and $\Gamma_\alpha = \mathrm{Sym}(\omega)^{M_\alpha}$. Clearly $\pi \in \mathrm{Sym}(\omega)$.

Claim 2.4 *If* $g_1, \ldots, g_n \in \Gamma_\alpha$, *then* $\bigcap_{1 \leq i \leq n} fix(\pi^{g_i})$ *is an infinite subset of* ω.

Proof: Fix $t \in \omega$. Let $\mathfrak{D}$ consist of those $q \in \mathbb{P}$ for which there exists $m > t$ such that for all $1 \leq i \leq n$, $g_i^{-1} q g_i(m) = m$. It is enough to show that $\mathfrak{D}$ is a dense subset of $\mathbb{P}$. Let $p \in \mathbb{P}$. For each $1 \leq i \leq n$, there are finitely many r such that $g_i(r) \in \mathrm{dom}\, p \cup \mathrm{ran}\, p$. So there exists $m > t$ with

$$\{g_i(m) \,|\, 1 \leq i \leq n\} \cap [\mathrm{dom}\, p \cup \mathrm{ran}\, p] = \varnothing.$$

Let $q < p$ satisfy $q(g_i(m)) = g_i(m)$ for $1 \leq i \leq n$. Clearly $q \in \mathfrak{D}$. This proves Claim 2.4.

Now we explain how to pass from $M^0_{\alpha+1}$ to $M_{\alpha+1}$. Let $\mathcal{F} = \{\mathrm{fix}(\pi^g) \mid g \in \Gamma_\alpha\}$. By Kunen's A10 [2] (p. 289), there exists a c.c.c. notion of forcing such that the generic extension $M_{\alpha+1}$ has the following property: there exists an infinite subset $S \subset \omega$ such that $|S \backslash F| < \omega$ for all $F \in \mathcal{F}$. Choose an infinite cycle φ such that $\mathrm{Mov}(\varphi) = S$. Then for each $g \in \Gamma_\alpha$, $|\mathrm{Mov}(\pi^g) \cap \mathrm{Mov}(\varphi)| < \omega$. Hence, when regarded as elements of $S(\omega)$, we have that $[\pi^g, \varphi] = 1$. Now write $\pi_\alpha = \pi$ and $\varphi_\alpha = \varphi$, and let $T = \langle \pi_\alpha, \varphi_\alpha \mid \alpha < \omega_1 \rangle$. Then clearly T satisfies the requirements of the theorem. This completes the proof of Theorem 2.3.

Theorem 2.5 *Suppose that $M \models$ GCH and $cf(\lambda) > \omega$. Then there exists a generic extension $M[G]$ such that $M[G] \models \neg F_\lambda$.*

Proof: Let $\lambda = \omega_\alpha$. For each $i \leq \omega$, let $\mu_i = \omega_{\alpha+i}$. Let $\mathbb{P} = Fn(\mu_\omega, 2)$ be the set of finite functions p from μ_ω to 2, and let $\mathbb{P}_n = Fn(\mu_n, 2)$ for $n < \omega$. Let G be a generic subset of $\mathbb{P}$ and let $G_n = G \cap \mathbb{P}_n$. Note that for $1 \leq n < \omega$, $M[G_n] \models 2^\lambda = \mu_n$: while $M[G] \models 2^\lambda = (\mu_\omega)^+$.

Let $\pi \in \mathrm{Sym}(\lambda)^{M[G]}$, and let $\tilde{\pi}$ be a $\mathbb{P}$-name of π. For each $n < \omega$, let $\pi_n = \{\langle \alpha, \beta \rangle \mid (\exists p \in G_n) p \Vdash \tilde{\pi}(\alpha) = \beta\}$. Then $\pi_n \in M[G_n]$ and $\pi_n \subseteq \pi$. Also $\pi = \bigcup_{n \in \omega} \pi_n$. Since $cf(\lambda) > \omega$, there exists $n < \omega$ such that $|\mathrm{dom}(\pi_n)| = \lambda$. By taking a suitable subset of π_n if necessary, we can suppose that $|\lambda \backslash \mathrm{dom}(\pi_n)| = |\lambda \backslash \mathrm{ran}(\pi_n)| = \lambda$. Hence there exist $\psi, \theta \in \mathrm{Sym}(\lambda)^{M[G_n]}$ such that $\psi \supset \pi_n$ and $\mathrm{Mov}(\theta) = \mathrm{dom}(\pi_n)$. Then $\mathrm{fix}(\psi^{-1}\pi) \supseteq \mathrm{Mov}(\theta)$, so that $[\psi^{-1}\pi, \theta] = 1$.

Let $G = \bigcup_{n \in \omega} \mathrm{Sym}(\lambda)^{M[G_n]}$, and let T be the corresponding subgroups of $S(\lambda)^{M[G]}$. Then $|T| = \mu_\omega < 2^\lambda$, and T witnesses the failure of F_λ in $M[G]$.

3 *Small subgroups of* Sym(λ)

In [3], the following observation is made.

Lemma 3.1 *Let $G \leq \mathrm{Sym}(\lambda)$. Then the following are equivalent.*
(i) *For some proper ideal I on λ which contains a moiety of λ, $G \leq S_{\{I\}}$.*
(ii) *There exists a moiety A of λ such that for all $g_1, \ldots, g_n \in G$,*

$$\lambda \neq \bigcup_{1 \leq i \leq n} A^{g_i}.$$

If condition (ii) holds, we say that λ is not G-covered.

Definition 3.2

$$C(\lambda) = \min\{|G| \mid G \leq \mathrm{Sym}(\lambda),\ \lambda \text{ is } G\text{-covered}\}.$$

In [3], it is proved that $C(\lambda) > \lambda$. To explain what is going on here, it is useful to introduce three more cardinal functions.

Definition 3.3
(i) $A(\lambda)$ is the least cardinal κ such that if $\mathcal{A} \subset \mathcal{P}(\lambda)$ is an almost disjoint family, then $|\mathcal{A}| \leq \kappa$.
(ii) $B(\lambda)$ is the least size $|I|$ of a family of ultrafilters $\mathcal{U}_i \subseteq \mathcal{P}(\lambda)$, $i \in I$, such that
  (a) for all $i \in I$ and $X \in \mathcal{U}_i$, $|X| = \lambda$;
  (b) $\{X \subseteq \lambda \mid |X| = \lambda\} \subseteq \bigcup_{i \in I} \mathcal{U}_i$.
(iii) $D(\lambda)$ is the least size $|J|$ of a family of sets $\{Y_j \mid j \in J\} \subseteq \mathcal{P}(\lambda)$ such that

(a) for all $j \in J$, $|Y_j| = \lambda$;
(b) if $X \subseteq \lambda$ with $|X| = \lambda$, then there exists $j \in J$ with $Y_j \subseteq X$.

Theorem 3.4

$$\lambda < A(\lambda) \leq B(\lambda) \leq C(\lambda) \leq D(\lambda) \leq 2^{\lambda}.$$

Corollary 3.5 *If $G < \mathrm{Sym}(\omega)$ and $|G| < 2^{\omega}$, then ω is not G-covered.*

It is clear that $\lambda < A(\lambda) \leq B(\lambda)$. We prove the other inequalities in the next two lemmas.

Lemma 3.6

$$B(\lambda) \leq C(\lambda)$$

Proof: Suppose $G \leq \mathrm{Sym}(\lambda)$ is such that λ is G-covered. Let $\mathcal{U}$ be a uniform ultrafilter on λ; i.e., $|X| = \lambda$ for all $X \in \mathcal{U}$. Suppose that there exists a moiety $X \in \mathcal{U}$ such that $g[X] \cap X \in \mathcal{U}$ for all $g \in G$. Then for all $g_1, \dots, g_n \in G$, $\bigcap_{1 \leq i \leq n} g_i[X] \in \mathcal{U}$. Let I be the ideal which is dual to the filter

$$\mathcal{F} = \left\{ Z \in \mathcal{P}(\lambda) \,\middle|\, \text{There exist } g_1, \dots, g_n \in G \text{ with } \bigcap_{1 \leq i \leq n} g_i[X] \subseteq Z \right\}.$$

Then $G \leq S_{\{I\}}$ and I is a proper ideal which contains a moiety of λ, a contradiction. Hence for each moiety $X \in \mathcal{U}$, there exists $g \in G$ such that $X \setminus g[X] \in \mathcal{U}$.

Fix an element $g \in G$ and let

$$S(g) = \{X \in \mathcal{U} \mid X \setminus g[X] \in \mathcal{U}\}.$$

If $X_1, \dots, X_n \in S(g)$, then

$$\bigcap_{1 \leq i \leq n} [X_i \setminus g[X_i]] = \left(\bigcap_{1 \leq i \leq n} X_i \right) \Big\backslash \left(\bigcup_{1 \leq i \leq n} g[X_i] \right) \in \mathcal{U}.$$

In particular, $\bigcup_{1 \leq i \leq n} g[X_i] = g[\bigcup_{1 \leq i \leq n} X_i]$ must be a moiety of λ, so that $\bigcup_{1 \leq i \leq n} X_i$ is a moiety. Hence $\lambda \setminus \bigcup_{1 \leq i \leq n} X_i = \bigcap_{1 \leq i \leq n} (\lambda \setminus X_i)$ is a moiety. Consequently, there exists a uniform ultrafilter $\mathcal{U}(g) \supseteq \{\lambda \setminus X \mid X \in S(g)\}$. So every moiety of λ lies in one of the uniform ultrafilters $\{\mathcal{U}\} \cup \{\mathcal{U}(g) \mid g \in G\}$. Hence $B(\lambda) \leq |G|$, and so $B(\lambda) \leq C(\lambda)$.

Lemma 3.7

$$C(\lambda) \leq D(\lambda)$$

Proof: Let $\mathcal{F} \subseteq \mathcal{P}(\lambda)$ satisfy the following:

(a) $|X| = \lambda$ for $X \in \mathcal{F}$;
(b) if $Y \subseteq \lambda$ with $|Y| = \lambda$, then there exists $X \in \mathcal{F}$ with $X \subseteq Y$;
(c) $|\mathcal{F}| = D(\lambda)$.

Clearly we can also suppose that

(d) each $X \in \mathcal{F}$ is a moiety.

For each $X \in \mathcal{F}$, let $\pi_X \in \mathrm{Sym}(\lambda)$ be an involution such that $\pi_X[X] = \lambda \setminus X$, and let $G = \langle \pi_X \mid X \in \mathcal{F} \rangle$.

Now let $A \subseteq \lambda$ be a moiety. Then there exists $X \in \mathcal{T}$ with $X \subseteq A$. Thus $\pi_X[A] \supseteq \lambda \backslash X \supseteq \lambda \backslash A$, so that $\lambda = A \cup \pi_X[A]$. Hence λ is G-covered, and so $C(\lambda) \leq D(\lambda)$.

The final result in this section shows that it is consistent that there exists $G < \mathrm{Sym}(\lambda)$ with $|G| < 2^\lambda$ such that λ is G-covered. It also demonstrates the consistency of $B(\lambda) < C(\lambda)$.

Theorem 3.8 *Suppose that $M \models$ GCH and $\lambda > \omega$ is regular. Let $\lambda = \omega_\alpha$ and $\kappa = \omega_{\alpha+\omega}$. Let $\mathbb{P} = Fn(\kappa,2)$ be the partial order of finite functions from κ to 2, and let G be a generic subset of $\mathbb{P}$. Then the following statements are true in $M[G]$.*
(a) $2^\lambda = \kappa^+$
(b) $A(\lambda) = B(\lambda) = \lambda^+$
(c) $C(\lambda) = D(\lambda) = \kappa$.

Proof: The facts that $2^\lambda = \kappa^+$ and $A(\lambda) = \lambda^+$ are included in Theorem 5.6 of Baumgartner [1]. Arguing as in the proof of Theorem 2.5, we easily obtain that $D(\lambda) \leq \kappa$. Thus to prove (c), it is enough to show that $C(\lambda) \geq \kappa$.

So suppose that there exists $\Gamma < \mathrm{Sym}(\lambda)^{M[G]}$ with $\lambda < |\Gamma| = \theta < \kappa$ such that λ is Γ-covered. Then there exists $I \subset \kappa$ of cardinality θ such that $\Gamma \in M[G \cap Fn(I,2)] = N$. Let $\mathbb{Q} = Fn(\lambda,2)$ and let $H \subset \mathbb{Q}$ be generic over N. We shall show that λ is not Γ-covered in $N[H]$, which yields the desired contradiction.

Let $f = \bigcup\{p \mid p \in H\}$ and let $S = \{\alpha \in \lambda \mid f(\alpha) = 1\}$. Clearly S is a moiety of λ. Let $\pi_1, \ldots, \pi_n \in \Gamma$ and let $\mathfrak{D}$ consist of the $q \in \mathbb{Q}$ satisfying:

(+) There exists $\beta \in \lambda$ and $\gamma_1, \ldots, \gamma_n \in \lambda$ such that

(i) $\pi_i(\gamma_i) = \beta$ for $1 \leq i \leq n$;
(ii) $q(\gamma_i) = 0$ for $1 \leq i \leq n$.

Clearly $\mathfrak{D}$ is dense in $\mathbb{Q}$, and if $q \in \mathfrak{D}$ then $q \Vdash \bigcup_{1 \leq i \leq n} \pi_i[S] \neq \lambda$. Thus λ is not Γ-covered in $N[H]$.

It only remains to compute $B(\lambda)$. We shall do this via the following series of lemmas.

Definition 3.9 A $\mathbb{P}$-name σ is simple if it has the form

$$\sigma = \{\langle \check{\alpha}, q_\alpha \rangle \mid \alpha \in X\}$$

where

(a) $X \subseteq \lambda$ has cardinality λ.
(b) If $\alpha \neq \beta$, then $\mathrm{dom}\, q_\alpha \cap \mathrm{dom}\, q_\beta = \varnothing$.
(c) There exists $n < \omega$ and $f_\sigma : n \to 2$ such that for all $\alpha \in X$.
  (i) $\mathrm{dom}\, q_\alpha = \{\alpha_0, \ldots, \alpha_{n-1}\}$
  (ii) $q_\alpha(\alpha_i) = f_\sigma(i)$ for $i < n$.

Lemma 3.10 *If σ is a simple $\mathbb{P}$-name, then $\Vdash \sigma \in [\lambda]^\lambda$.*

A straightforward Δ-system argument yields the next result.

Lemma 3.11 *Suppose that $G \subseteq \mathbb{P}$ is generic and that $M[G] \models \tau_G \in [\lambda]^\lambda$. Then there exists a simple $\mathbb{P}$-name σ such that $M[G] \models \sigma_G \subseteq \tau_G$.*

Thus it suffices to find λ^+ uniform ultrafilters in $M[G]$ such that σ_G is contained in one of them for each simple $\mathbb{P}$-name σ. We shall also make use of the following well-known result.

Lemma 3.12 *For any cardinal $\theta \geq \omega$, $Fn(2^\theta,2)$ is the union of θ centered subsets.*

Clearly it is enough to show that $B(\lambda) \leq \lambda^+$. Initially we will work inside M. Let $\langle \mathcal{U}_\alpha \mid \alpha < \lambda^+ \rangle \in M$ be a sequence of uniform ultrafilters on λ such that for each $X \in [\lambda]^\lambda \cap M$, there exists $\alpha \leq \lambda^+$ with $X \in \mathcal{U}_\alpha$. Let $\sigma = \{\langle \check{\alpha}, q_\alpha \rangle \mid \alpha \in X\}$ be a simple $\mathbb{P}$-name, and let $\operatorname{dom} q_\alpha = \{\alpha_0, \ldots, \alpha_{n-1}\}$ for each $\alpha \in X$. Then $X \in \mathcal{U}_\gamma$ for some $\gamma < \lambda^+$. Define an equivalence relation $\equiv_\gamma$ on ${}^\lambda\kappa$ by:

$$\psi \equiv_\gamma \theta \text{ iff } \{\alpha \mid \psi(\alpha) = \theta(\alpha)\} \in \mathcal{U}_\gamma.$$

Let $[\psi]_\gamma$ be the $\equiv_\gamma$-class containing $\psi \in {}^\lambda\kappa$, and let ${}^\lambda\kappa/\mathcal{U}_\gamma = \{[\psi]_\gamma \mid \psi \in {}^\lambda\kappa\}$. Then σ determines $p_\sigma \in Fn({}^\lambda\kappa/\mathcal{U}_\gamma,2)$ as follows. For $i < n$, define $h_i \in {}^\lambda\kappa$ by

$$\begin{aligned} h_i(\alpha) &= \alpha_i \text{ if } \alpha \in X \\ &= 0 \text{ if } \alpha \in \lambda \setminus X. \end{aligned}$$

Let $\operatorname{dom} p_\sigma = \{[h_0]_\gamma, \ldots, [h_{n-1}]_\gamma\}$ and set $p_\sigma([h_i]_\gamma) = f_\sigma(i)$.

Lemma 3.13 *Suppose that $\sigma_j = \{\langle \check{\alpha}, q_\alpha^j \rangle \mid \alpha \in X_j\}$ is a simple $\mathbb{P}$-name for $j < k$. Suppose further that:*
(1) $X_j \in \mathcal{U}_\gamma$ for $j < k$;
(2) $p_{\sigma_0}, \ldots, p_{\sigma_{k-1}}$ have a common strengthening $p \in Fn({}^\lambda\kappa/\mathcal{U}_\gamma,2)$.

Then $\Vdash \sigma_0 \cap \cdots \cap \sigma_{k-1} \in [\lambda]^\lambda$.

Proof: For each $j < k$ and $\alpha \in X_j$, let $\operatorname{dom} q_\alpha^j = \{\alpha_0^j, \ldots, \alpha_{n_j-1}^j\}$. Let $Z \in \mathcal{U}_\gamma$ consist of those $\alpha < \lambda$ satisfying

(a) $\alpha \in X_0 \cap \cdots \cap X_{k-1}$.
(b) If $s < t < k$, $l < n_s - 1$ and $m < n_t - 1$, then

$$\alpha_l^s = \alpha_m^t \text{ iff } [h_l^s]_\gamma = [h_m^t]_\gamma.$$

Let $r \in \mathbb{P}$ be arbitrary and $\beta < \lambda$. Then there exists $\alpha \in Z$ such that

(c) $\beta < \alpha < \lambda$.
(d) $\operatorname{dom} r \cap \operatorname{dom} q_\alpha^j = \emptyset$ for all $j < k$.

We define $q = r \cup \bigcup_{j<k} q_\alpha^j$. Assuming that $q \in \mathbb{P}$, we have that $q \leq r$ and that $q \Vdash \alpha \in \sigma_0 \cap \cdots \cap \sigma_{k-1}$. Thus it is enough to show that q is a well-defined function. Suppose that $\alpha_l^s = \alpha_m^t$ for some $s < t < k$. Then, since $[h_l^s]_\gamma = [h_m^t]_\gamma$ and $p_{\sigma_s}, p_{\sigma_t}$ are compatible, we must have $p_{\sigma_s}([h_l^s]_\gamma) = p_{\sigma_t}([h_m^t]_\gamma)$ and hence $q_\alpha^s(\alpha_l^s) = q_\alpha^t(\alpha_m^t)$.

For each $\gamma < \lambda^+$, let $\mathbb{Q}_\gamma = Fn({}^\lambda\kappa/\mathcal{U}_\gamma,2) \in M$. In the remainder of the proof, we will work inside $M[G]$. Notice that the cardinality of $({}^\lambda\kappa/\mathcal{U}_\gamma)^M$ is at most 2^λ in $M[G]$. So by Lemma 3.12, we can express $\mathbb{Q}_\gamma = \bigcup_{\xi<\lambda} A_{\gamma\xi}$ as a union of λ centered sets. Let $S = \{\sigma_G \mid \sigma \text{ is a simple } \mathbb{P}\text{-name}\}$. For each $\gamma < \lambda^+$ and $\xi < \lambda$, define $\mathcal{U}_{\gamma\xi} = \{\sigma_G \mid p_\sigma \in A_{\gamma\xi}\}$. Then

$$S = \bigcup_{\substack{\gamma<\lambda^+ \\ \xi<\lambda}} \mathcal{U}_{\gamma\xi}.\text{ If } (\sigma_0)_G, \ldots, (\sigma_{k-1})_G \in \mathcal{U}_{\gamma\xi}$$

then $p_{\sigma_0}, \ldots, p_{\sigma_{k-1}}$ have a common strengthening in $\mathbb{Q}_\gamma$ and so $\Vdash \sigma_0 \cap \cdots \cap \sigma_{k-1} \in [\lambda]^\lambda$. Thus $\mathcal{U}_{\gamma\xi}$ can be completed to a uniform ultrafilter on λ. This completes the proof that $B(\lambda) = \lambda^+$.

4 Covering families of ultrafilters

Theorem 4.1 *Let $M \models$ GCH. Let λ and $\kappa > \lambda^{+++}$ be regular cardinals. Then there exists a notion of forcing $\mathbb{P}$, which preserves cofinalities and cardinalities, such that if $G \subseteq \mathbb{P}$ is generic then*

$$M[G] \models \lambda^+ = A(\lambda) < B(\lambda) = \kappa = 2^\lambda.$$

Definition 4.2 $\mathbb{P}$ consists of all conditions $p = \langle a, h, f, g\rangle$ satisfying
(i) $a \in [\kappa]^{\le\lambda^{++}}$.
(ii) $h : [a]^2 \to \lambda$.
(iii) There exist finite $u \subseteq a$, $v \subseteq \lambda$ such that $f : u \times v \to 2$ and $g : [u]^2 \to 2$.
(iv) If $g(\alpha,\beta) = f(\alpha,\gamma) = f(\beta,\gamma) = 1$, then $\gamma < h(\alpha,\beta)$.

The order relation is the natural one.

The intuitive meaning is that we are adjoining the sets $A_\alpha = \{\gamma < \lambda \mid f(\alpha,\gamma) = 1\}$ for $\alpha < \kappa$. The function h gives a vague promise that $A_\alpha \cap A_\beta \subseteq h(\alpha,\beta)$. But h is unreliable, and should only be taken seriously when $g(\alpha,\beta) = 1$.

Definition 4.3
(a) $q = \langle a_1, h_1, f_1, g_1\rangle \underset{pr}{\le} p = \langle a_0, h_0, f_0, g_0\rangle$ iff $q \le p$, $f_0 = f_1$ and $g_0 = g_1$.
(b) $q = \langle a_1, h_1, f_1, g_1\rangle \underset{ap}{\le} p = \langle a_0, h_0, f_0, g_0\rangle$ iff $q \le p$, $a_0 = a_1$ and $h_0 = h_1$.

Lemma 4.4 *If $q \le p$, then there exists $r \in \mathbb{P}$ such that $q \underset{ap}{\le} r \underset{pr}{\le} p$.*

An easy Δ-system argument yields the next result.

Lemma 4.5 *If $p \in \mathbb{P}$, then $\{q \in \mathbb{P} \mid q \underset{ap}{\le} p\}$ satisfies the c.c.c.*

Lemma 4.6 *If $p \in \mathbb{P}$ and $\tilde\tau$ is a $\mathbb{P}$-name of an ordinal, then there exists $q \in \mathbb{P}$ such that*
(i) $q \underset{pr}{\le} p$;
(ii) *if $r \le q$ and $r \Vdash \tilde\tau = \gamma$, then there exists $r' \parallel r$ such that $r' \underset{ap}{\le} q$ and $r' \Vdash \tilde\tau = \gamma$.*

Proof: We define inductively p_i, and also r_j, γ_j for successor j such that:

(i) $p_0 = p$;
(ii) $p_i \underset{pr}{\le} p$ and the chain $\{p_k \mid k \le i\}$ is strictly decreasing and continuous;
(iii) $r_j \underset{ap}{\le} p_j$ and $r_j \Vdash \tilde\tau = \gamma_j$;
(iv) if $j_1 < j_2$ then $r_{j_1} \not\parallel r_{j_2}$.

Suppose that the construction can be continued for all $i < \omega_1$. Then there exists $p^* \in \mathbb{P}$ with $p^* \underset{pr}{\leq} p_i$ for all $i < \omega_1$. Notice that for each successor $j < \omega_1$, there exists $r_j^* \in \mathbb{P}$ such that $r_j^* \leq r_j$ and $r_j^* \underset{ap}{\leq} p^*$. But then $\{r_j^* \mid j < \omega_1$ is a successor$\}$ is an uncountable antichain, which contradicts Lemma 4.5.

So where does the inductive construction break down? Since $\{q \in \mathbb{P} \mid q \underset{pr}{\leq} p\}$ is λ^{+++}-closed, the construction cannot fail at a limit stage. Thus we can suppose that p_i has been constructed, but that it is impossible to construct p_{i+1}, r_{i+1}, γ_{i+1}. We claim that $q = p_i$ satisfies our requirements. Suppose not. Then there exists γ and $r \leq p_i$ with $r \Vdash \tilde{\tau} = \gamma$ such that there is no $r' \underset{ap}{\leq} p_i$ satisfying $r' \parallel r$ and $r' \Vdash \tilde{\tau} = \gamma$. Let $r_{i+1} = r \underset{ap}{\leq} p_{i+1} \underset{pr}{\leq} p_i$, and let $\gamma_{i+1} = \gamma$. Then (iv) must fail, and so there exists $j \leq i$ with $r_j \parallel r_{i+1}$. In particular, $\gamma_j = \gamma_{i+1} = \gamma$ and $r_j \Vdash \tilde{\tau} = \gamma$. But now there exists $r_j^* \underset{ap}{\leq} p_i$ with $r_j^* \leq r_j$ and $r_j^* \parallel r$, which is a contradiction.

Using the fact that $\{q \in \mathbb{P} \mid q \underset{pr}{\leq} p\}$ is λ^{+++}-closed for each $p \in \mathbb{P}$, we easily obtain the following result.

Lemma 4.7 *If $\tilde{\tau}_i, i < \lambda^{++}$, are $\mathbb{P}$-names for ordinals and $p \in \mathbb{P}$, then there exists $q \underset{pr}{\leq} p$ such that if $i < \lambda^{++}$ and $r \leq q$ with $r \Vdash \tilde{\tau}_i = \gamma$, then there exists $r' \parallel r$ such that $r' \underset{ap}{\leq} q$ and $r' \Vdash \tilde{\tau}_i = \gamma$.*

Lemma 4.8 *$\mathbb{P}$ preserves all cardinals and cofinalities less than or equal to λ^{+++}.*

Proof: For example, suppose that $p \Vdash \tilde{f}: \lambda^{++} \to \lambda^{+++}$. Let $q \underset{pr}{\leq} p$ satisfy the conclusion of Lemma 4.7 with respect to the $\mathbb{P}$-names $\tilde{f}(\check{\alpha})$, $\alpha < \lambda^{++}$. Since $\{r \in \mathbb{P} \mid r \underset{ap}{\leq} q\}$ satisfies the c.c.c., we see that $q \Vdash \tilde{f}$ is not a cofinal map in λ^{+++}.

An easy Δ-system argument (which makes use of the assumption that $M \models$ GCH) yields the next result.

Lemma 4.9 *$\mathbb{P}$ is λ^{++++}-c.c.; and hence $\mathbb{P}$ preserves all cardinals and cofinalities.*

Lemma 4.10

$$\Vdash A(\lambda) = \lambda^{+}.$$

Proof: Suppose that $p \Vdash$ "$\langle \tilde{T}_i \mid i < \lambda^{++} \rangle$ is an almost disjoint family in $\mathcal{P}(\lambda)$." For each $i < j < \lambda^{++}$, let $\tilde{\tau}_{ij} = \sup(\tilde{T}_i \cap \tilde{T}_j)$. Then $p \Vdash \tilde{\tau}_{ij} < \lambda$. Choose $q \underset{pr}{\leq} p$ satisfying the conclusion of Lemma 4.7 with respect to the $\mathbb{P}$-names $\tilde{\tau}_{ij}, i < j < \lambda^{++}$. Using Lemma 4.5, we see that there exists $\beta_{ij} < \lambda$ such that $q \Vdash \tilde{T}_i \cap \tilde{T}_j \subseteq \beta_{ij}$.

Since $M \models$ GCH, $\lambda^{++} \to (\lambda^{+})^2_{\lambda}$. Hence there exists $H \subset \lambda^{++}$ with $|H| = \lambda^{+}$ and $\beta < \lambda$ such that for all distinct $i,j \in H$, $q \Vdash \tilde{T}_i \cap \tilde{T}_j \subseteq \beta$. Let $G' \ni q$ be generic and $T_i = (\tilde{T}_i)_{G'}$. Then in $M[G']$, $\{T_i \setminus \beta \mid i \in H\}$ is a collection of λ^+ nonempty pairwise disjoint subsets of λ, which is a contradiction.

Definition 4.11 For each $\alpha < \kappa$, $\tilde{A}_\alpha = \{\langle \check{\gamma}, \langle a,h,f,g \rangle\rangle \mid f(\alpha,\gamma) = 1\}$.

Lemma 4.12
(i) $\Vdash |\tilde{A}_\alpha| = \lambda$.
(ii) *If* $p = \langle a,h,f,g \rangle$ *and* $g(\alpha,\beta) = 1$, *then* $p \Vdash \tilde{A}_\alpha \cap \tilde{A}_\beta \subseteq h(\alpha,\beta) < \lambda$.

Lemma 4.13

$$\Vdash B(\lambda) = \kappa = 2^\lambda.$$

Proof: Suppose not, and let $\theta = \lambda^{++++}$. Then there exists a $\mathbb{P}$-name $\tilde{\mathfrak{D}}$ for a uniform ultrafilter on λ, distinct ordinals $\alpha_i < \kappa$ for $i < \theta$, and conditions $p_i \in \mathbb{P}$ such that $p_i \Vdash \tilde{A}_{\alpha_i} \in \tilde{\mathfrak{D}}$. Let $p_i = \langle a_i, h_i, f_i, g_i \rangle$. We can suppose that $\alpha_i \in a_i$ for each $i < \theta$.

Since $M \models \mathrm{GCH}$, we can also suppose that the following hold.

(i) $\{a_i \mid i < \theta\}$ forms a Δ-system with root A; and the h_i are pairwise compatible functions.
(ii) $\{u_i \mid i < \theta\}$ forms a Δ-system with root U, $\{v_i \mid i < \theta\}$ forms a Δ-system with root V; and the f_i, g_i are pairwise compatible functions. Since $|A| \leq \lambda^{++}$, we can also suppose that
(iii) $\alpha_i \notin A$ for all $i < \theta$.

Fix $i < j < \theta$. Since $\alpha_i, \alpha_j \notin A$, we can form a condition $q = \langle a,h,f,g \rangle \leq p_i, p_j$ such that $g(\alpha_i,\alpha_j) = 1$ and $h(\alpha_i,\alpha_j)$ is given a sufficiently large value. But then

$$q \Vdash \tilde{A}_{\alpha_i} \cap \tilde{A}_{\alpha_j} \subseteq h(\alpha_i,\alpha_j) < \lambda,$$

which is a contradiction.

This completes the proof of Theorem 4.1. The following problems remain open.

Question 4.14 Suppose that $G < \mathrm{Sym}(\lambda)$ and $|G| < 2^\lambda$. Is G contained in a maximal subgroup of $\mathrm{Sym}(\lambda)$?

Question 4.15 Does $C(\lambda) = D(\lambda)$?

Question 4.16 Is it consistent that $C(\omega_1) = \omega_2 < 2^{\omega_1}$?

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