Notre Dame Journal of Formal Logic
Volume 30, Number 1, Winter 1989

Some Modal Logics Based on a Three-Valued Logic

OSAMU MORIKAWA

1 Introduction A K-modal logic based on Łukasiewicz's three-valued logic has been formulated by Schotch [2]. In this paper we formulate K-, M-, $S4$-, and $S5$-modal logics based on a general three-valued logic by using the notion of a matrix in [3].

In Section 2, we define truth values, formulas, and matrices. In Section 3, we introduce three-valued Kripke models defined in [1]. In Section 4, we present the systems K, M, $S4$, and $S5$ of modal logic based on a general three-valued logic (3-K_3, 3-K_2, 3-M_3, 3-M_2, 3-$S4_3$, 3-$S4_2$, 3-$S5_3$, and 3-$S5_2$). 3-K_3, 3-M_3, 3-$S4_3$, and 3-$S5_3$ are modal logics based on a three-valued logic in which the modal operators take on all three of our truth-values. 3-K_2, 3-M_2, 3-$S4_2$, and 3-$S5_2$ are modal logics based on a three-valued logic in which the modal operators take only the two classical truth-values. In Section 5, we develop the syntax of 3-K_i, 3-M_i, 3-$S4_i$, and 3-$S5_i$ ($i = 2,3$) and it will be shown that the cut-elimination theorems no longer hold in 3-K_i, 3-M_i, 3-$S4_i$, and 3-$S5_i$. In Section 6, we prove the completeness theorems for 3-K_i, 3-M_i, 3-$S4_i$, and 3-$S5_i$.

2 Matrices

2.1 Truth values We take 1, 2, and 3 as truth-values. Intuitively '1' stands for 'true' and '3' for 'false', whereas '2' may be interpreted as 'undefined' or 'meaningless'.

We denote the set of all the truth values by **T**. $\mathbf{T} = \{1,2,3\}$.

2.2 Primitive symbols

(1) Propositional variables: p, q, r, etc.
(2) Propositional connectives:

$$F_i(*_1,\ldots,*_{\alpha_i}) = i = 1,2,\ldots,\beta,\alpha_i \geqq 1.$$

Received October 28, 1985; revised August 28, 1986 and April 9, 1987

With each F_i we associate a function f_i from $\mathbf{T}^{\alpha_i}$ into $\mathbf{T}$. We call f_i the truth function of F_i.

(3) Modal symbol: $\Box$.

(4) Auxiliary symbols: (,).

2.3 *Definition of a formula*

(1) A propositional variable is a formula.

(2) If $A_1, \ldots, A_{\alpha_i}$ are formulas, then $F_i(A_1, \ldots, A_{\alpha_i})$ is a formula ($i = 1, \ldots, \beta$).

(3) If A is a formula, then $\Box A$ is a formula.

2.4 Matrices Gentzen's sequent $A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n$ means intuitively that some formula of $A_1, \ldots, A_m$ is false or some formula of $B_1, \ldots, B_n$ is true.

The truth-value 1 corresponds to the succedent and the truth-value 3 corresponds to the antecedent. We extend the notion of a sequent to three-valued logic.

When $A_i^{(\mu)}$ ($\mu = 1,2,3$; $i = 1,2, \ldots, m_\mu$; $m_\mu \geq 0$) are formulas, we call the following ordered triple of finite sets of formulas a matrix:

$$\{A_1^{(1)}, \ldots, A_{m_1}^{(1)}\}_1 \cup \{A_1^{(2)}, \ldots, A_{m_2}^{(2)}\}_2 \cup \{A_1^{(3)}, \ldots, A_{m_3}^{(3)}\}_3.$$

We call $A_1^{(1)}, \ldots, A_{m_1}^{(1)}$ or $A_1^{(2)}, \ldots, A_{m_2}^{(2)}$ or $A_1^{(3)}, \ldots, A_{m_3}^{(3)}$ the 1-part or the 2-part or the 3-part respectively. The matrix $[A_1^{(1)}, \ldots, A_{m_1}^{(1)}\}_1 \cup \{A_1^{(2)}, \ldots, A_{m_2}^{(2)}\}_2 \cup \{A_1^{(3)}, \ldots, A_{m_3}^{(3)}\}_3$ means intuitively that some formula of $A_1^{(1)}, \ldots, A_{m_1}^{(1)}$ is false or some formula of $A_1^{(2)}, \ldots, A_{m_2}^{(2)}$ is undefined or some formula of $A_1^{(3)}, \ldots A_{m_3}^{(3)}$ is true.

2.5 *Abbreviations*

(1) When L is a matrix, we denote the series of formulas occurring in the i-part of L by L_i.

(2) When $m_\mu = 0$ for all $\mu \in \mathbf{T}$, we denote this matrix by Φ and call it the empty matrix.

(3) Let $R \subseteq \mathbf{T}$. The matrix such that $m_\mu = 1$ and $A_1^{(\mu)} = A$ for all $\mu \in R$ and $m_\mu = 0$ for all $\mu \notin R$ is abbreviated by $\{A\}_R$. In particular, $\{A\}_{\mathbf{T}-\{\mu\}}$ is denoted by $\{A\}_{\bar{\mu}}$. $\{A\}_{\{\mu_1, \ldots, \mu_j\}}$ is denoted by $\{A\}_{\mu_1, \ldots, \mu_j}$.

(4) For matrices L, M we put $L \cup M = \{L_1, M_1\}_1 \cup \{L_2, M_2\}_2 \cup \{L_3, M_3\}_3$.

(5) We write $L \subset M$, if for all $\mu \in \mathbf{T}$ every formula which occurs in L_μ also occurs in M_μ.

3 *Kripke models*

3.1 Definition of a Kripke model A 3-K_3 model is a structure $\mathfrak{M} = (W, R, \phi)$ where

(1) W is a nonempty set

(2) R is a binary relation on W

(3) For all $s \in W$ and every propositional variable p, $\phi(p,s)$ assigns a truth-value in $\mathbf{T}$.

3.2 Given any 3-K_3 model $\mathfrak{M}$, the truth value $\phi(A,s)$ of a formula A at s is defined as follows:

(1) $\phi(F_i(A_1,\ldots,A_{\alpha_i}),s) = f_i(\phi(A_1,s),\ldots,\phi(A_{\alpha_i},s))$

$$(2)\quad \phi(\Box A,s) = \begin{cases} 1, \text{ if for all } t \text{ such that } sRt,\ \phi(A,t) = 1. \\ 2, \text{ if there exists a } t \text{ such that } sRt \text{ and } \phi(A,t) = 2. \\ 3, \text{ if for all } t \text{ such that } sRt,\ \phi(A,t) \neq 2 \text{ and there} \\ \quad \text{exists a } u \text{ such that } sRu \text{ and } \phi(A,u) = 3. \end{cases}$$

3.3 3-K_2 models Now, a 3-K_2 model is obtained from a 3-K_3 model by replacing (2) in 3.2 by the following (2′).

$$(2')\quad \phi(\Box A,s) = \begin{cases} 1, \text{ if for all } t \text{ such that } sRt,\ \phi(A,t) = 1 \\ 3, \text{ otherwise.} \end{cases}$$

3.4 A matrix $L = \{A_1^{(1)},\ldots,A_{m_1}^{(1)}\}_1 \cup \{A_1^{(2)},\ldots,A_{m_2}^{(2)}\}_2 \cup \{A_1^{(3)},\ldots,A_{m_3}^{(3)}\}_3$ is called 3-K_i valid if for all 3-K_i models $\mathfrak{M}_i$ and any $s \in W$, there exists an $A_j^{(\mu)}$ in L such that $\phi(A_j^{(\mu)},s) = \mu$. In the case where $m_2 = 0$, this definition is consistent with the classical definition of the validity of a sequent.

3.5 Let $\mathfrak{M}_i$ be a 3-K_i model. We say that $\mathfrak{M}_i$ is a 3-M_i model if R is reflexive, a 3-$S4_i$ model if R is reflexive and transitive, and a 3-$S5_i$ model if R is an equivalence relation.

3.6 We define 3-M_i validity, 3-$S4_i$ validity, and 3-$S5_i$ validity in the same manner as we defined 3-K_i validity.

4 Formal systems Now we introduce the formal systems 3-K_3, 3-M_3, 3-$S4_3$, and 3-$S5_3$ by using Takahashi's matrix. Henceforth K, L, M, etc. stand for matrices.

4.1 3-K_3

(1) A matrix of the form $\{A\}_1 \cup \{A\}_2 \cup \{A\}_3$ is called a beginning matrix.

(2) Inference rules:

(1) Weakening

$$\frac{L}{K}\ (\text{if } L \subset K)$$

(2) Cut

$$\frac{L \cup \{A\}_\mu,\ K \cup \{A\}_\nu}{L \cup K}\ (\mu \neq \nu)$$

(3) Inference for propositional connectives: Let $f_i(\mu_1,\ldots,\mu_{\alpha_i}) = \mu$

$$\frac{L \cup \{A_1\}_{\mu_1},\ldots,L \cup \{A_{\alpha_i}\}_{\mu_{\alpha_i}}}{L \cup \{F_i(A_1,\ldots,A_{\alpha_i})\}_\mu}$$

(4) Inferences for modal connectives:

$$\frac{\{A\}_1 \cup \{A,\Gamma,\Sigma\}_2 \cup \{\Sigma\}_3}{\{\Box A\}_1 \cup \{\Box A,\Box\Gamma,\Box\Sigma\}_2 \cup \{\Box\Sigma\}_3} \ (\Box^K_{1,2})$$

$$\frac{\{A\}_1 \cup \{\Gamma,\Sigma\}_2 \cup \{A,\Gamma\}_3}{\{\Box A\}_1 \cup \{\Box\Gamma,\Box\Sigma\}_2 \cup \{\Box A,\Box\Gamma\}_3} \ (\Box^K_{1,3})$$

where Γ, Σ, etc. mean (void or nonvoid) series of formulas and $\Box\Gamma$ denotes the set $\{\Box B\colon B \in \Gamma\}$.

(3) Provable matrices: A matrix is called 3-K_3-provable if it is obtained from beginning matrices by a finite number of applications of the above inference rules. We write $\vdash L$ (in 3-K_3) if L is provable in 3-K_3.

4.2 3-M_3 3-M_3 is obtained from 3-K_3 by adding the following rules $\Box^M_{2,3}$ and $\Box^M_2$.

$$\frac{\{\Gamma\}_1 \cup \{\Delta,\Sigma\}_2 \cup \{\Delta,\Pi\}_3}{\{\Gamma\}_1 \cup \{\Box\Delta,\Sigma\}_2 \cup \{\Box\Delta,\Pi\}_3} \ (\Box^M_{2,3})$$

$$\frac{\{\Gamma\}_1 \cup \{A,\Sigma\}_2 \cup \{\Delta\}_3}{\{\Gamma\}_1 \cup \{\Box A,\Sigma\}_2 \cup \{\Delta\}_3} \ (\Box^M_2).$$

4.3 3-$S4_3$ 3-$S4_3$ is obtained from 3-M_3 by replacing the rules $\Box^K_{1,2}$ and $\Box^K_{1,3}$ by the following rules:

$$\frac{\{A\}_1 \cup \{\Box\Gamma,\Box\Sigma\}_2 \cup \{\Box\Gamma\}_3}{\{\Box A\}_1 \cup \{\Box\Gamma,\Box\Sigma\}_2 \cup \{\Box\Gamma\}_3} \ (\Box^{S4}_1)$$

$$\frac{\{A\}_1 \cup \{\Box\Gamma,\Box\Sigma\}_2 \cup \{A,\Box\Gamma\}_3}{\{\Box A\}_1 \cup \{\Box\Gamma,\Box\Sigma\}_2 \cup \{\Box A,\Box\Gamma\}_3} \ (\Box^{S4}_{1,3}).$$

4.4 3-$S5_3$ 3-$S5_3$ is obtained from 3-$S4_3$ by replacing the rules $\Box^{S4}_1$ and $\Box^{S4}_{1,3}$ by the following rules:

$$\frac{\{A,\Box\Gamma\}_1 \cup \{\Box\Delta\}_2 \cup \{\Box\Sigma\}_3}{\{\Box A,\Box\Gamma\}_1 \cup \{\Box\Delta\}_2 \cup \{\Box\Sigma\}_3} \ (\Box^{S5}_1)$$

$$\frac{\{A,\Box\Gamma\}_1 \cup \{\Box\Delta\}_2 \cup \{A,\Box\Sigma\}_3}{\{\Box A,\Box\Gamma\}_1 \cup \{\Box\Delta\}_2 \cup \{\Box A,\Box\Sigma\}_3} \ (\Box^{S5}_{1,3}).$$

4.5 We define 3-M_3 provability, 3-$S4_3$ provability and 3-$S5_3$ provability in the same manner as we defined 3-K_3 provability.

4.6 3-K_2 is obtained from 3-K_3 by adding the following beginning matrix:

$$\{\Box A\}_1 \cup \{\Box A\}_3.$$

4.7 We define 3-M_2, 3-$S4_2$, and 3-$S5_2$ in the same manner as we defined 3-K_2.

5 Syntax of the systems We can easily prove the following lemmas.

5.1 Lemma
(1) *The rules* $\Box^K_{1,2}$ *and* $\Box^K_{1,3}$ *are admissible in* 3-$S4_i$.
(2) *The rules* $\Box^{S4}_1$ *and* $\Box^{S4}_{1,3}$ *are admissible in* 3-$S5_i$.
(3) *The following rule* $\Box^{S5}_{1,2}$ *is admissible in* 3-$S5_i$.

$$\frac{\{A,\Box\Gamma\}_1 \cup \{A,\Box\Sigma\}_2 \cup \{\Box\Delta\}_3}{\{\Box A,\Box\Gamma\}_1 \cup \{\Box A,\Box\Sigma\}_2 \cup \{\Box\Delta\}_3}\ (\Box^{S5}_{1,2})$$

5.2 Lemma
(1) $\vdash\{A\}_1 \cup \{\Box A\}_2 \cup \{A\}_3$ *in* 3-M_i, 3-$S4_i$, *and* 3-$S5_i$.
(2) $\vdash\{A\}_1 \cup \{\Box A\}_2 \cup \{\Box A\}_3$ *in* 3-M_i, 3-$S4_i$, *and* 3-$S5_i$.
(3) $\vdash\{\Box\Box A\}_1 \cup \{\Box A\}_2 \cup \{\Box A\}_3$ *in* 3-$S4_i$ *and* 3-$S5_i$.
(4) $\vdash\{\Box\Box A\}_1 \cup \{\Box A\}_2 \cup \{\Box\Box A\}_3$ *in* 3-$S4_i$ *and* 3-$S5_i$.

5.3 Theorem 1 *The cut inference rule cannot be eliminated in* 3-K_i, 3-M_i, 3-$S4_i$, *and* 3-$S5_i$.

Proof: We give an example of a provable matrix which is not provable without using the cut inference rules.

(1) In the case of 3-K_i and 3-M_i: Let $F(*)$ be a propositional connective with the associated function f from $\mathbf{T}$ to $\mathbf{T}$ which is defined by $f(1) = f(2) = f(3) = 1$.

$$\cfrac{\cfrac{\cfrac{\{A\}_1 \cup \{A\}_2 \cup \{A\}_3}{\{A,F(A)\}_1 \cup \{A\}_2}}{\{F(A),F(A)\}_1 \cup \{A\}_2}}{\{F(A)\}_1} \qquad \therefore \vdash\{F(A)\}_1.$$

Hence by weakening $\vdash\{F(A)\}_1 \cup \{F(A)\}_2$ and $\vdash\{F(A)\}_1 \cup \{F(A)\}_3$.

$$\cfrac{\cfrac{\cfrac{\{F(A)\}_1 \cup \{F(A)\}_2}{\{\Box F(A)\}_1 \cup \{\Box F(A)\}_2}\ (\Box^K_{1,2}) \qquad \cfrac{\{F(A)\}_1 \cup \{F(A)\}_3}{\{\Box F(A)\}_1 \cup \{\Box F(A)\}_3}\ (\Box^K_{1,3})}{\{\Box F(A)\}_1 \cup \{\Box F(A)\}_1}\ (2 \neq 3)}{\{\Box F(A)\}_1}\ \text{(Weakening)}$$

Therefore $\vdash\{\Box F(A)\}_1$ in 3-K_i and 3-M_i. But it is evident that $\{\Box F(A)\}_1$ is not provable without the cut inference rules.

(2) In the case of 3-$S4_i$ and 3-$S5_i$: Let $G(*)$ be a propositional connective with the associated function g from $\mathbf{T}$ to $\mathbf{T}$ which is defined by $g(1) = g(2) = g(3) = 3$. Similarly we can prove $\{G(A)\}_1 \cup \{G(A)\}_3$ and $\{G(A)\}_2 \cup \{G(A)\}_3$.

$$\cfrac{\cfrac{\cfrac{\{G(A)\}_1 \cup \{G(A)\}_3}{\{\Box G(A)\}_1 \cup \{\Box G(A)\}_3}\ (\Box^K_{1,3}) \qquad \cfrac{\{G(A)\}_2 \cup \{G(A)\}_3}{\{\Box G(A)\}_2 \cup \{\Box G(A)\}_3}\ (\Box^M_{2,3})}{\{\Box G(A)\}_3 \cup \{\Box G(A)\}_3}\ (1 \neq 2)}{\{\Box G(A)\}_3}\ \text{(Weakening)}$$

By Lemma 5.1 $\vdash\{\Box G(A)\}_3$ in 3-$S4_i$ and 3-$S5_i$. But it is evident that $\{\Box G(A)\}_3$ is not provable without using the cut inference rules.

6 Semantics of 3-K_i, 3-M_i, 3-$S4_i$, and 3-$S5_i$.

6.1 Theorem 2 (Soundness Theorem) *If a matrix is provable in* 3-K_i, 3-M_i, 3-$S4_i$, *or* 3-$S5_i$, *then it is valid in* 3-K_i, 3-M_i, 3-$S4_i$, *or* 3-$S5_i$, *respectively.*

Proof: This can easily be proved by induction on the construction of a proof of the given matrix.

6.2 Lemma *If L is G-unprovable, then for any formula A, $L \cup \{A\}_1 \cup \{A\}_2$ or $L \cup \{A\}_1 \cup \{A\}_3$ or $L \cup \{A\}_2 \cup \{A\}_3$ is G-unprovable.*

Proof: Suppose that $L \cup \{A\}_1 \cup \{A\}_2$, $L \cup \{A\}_1 \cup \{A\}_3$, and $L \cup \{A\}_2 \cup \{A\}_3$ are G-provable. By using the cut inference rules we can prove that L is G-provable.

6.3 Let the matrix K be fixed. We denote the set of subformulas of formulas occurring in K by $FL(K)$. If the matrix L is G-unprovable and for any $A \in FL(K)$, $A \in L_1 \cap L_2$ or $A \in L_1 \cap L_3$ or $A \in L_2 \cap L_3$, we call L G-complete. We denote the set of G-complete matrices by $C_G(K)$.

6.4 Lemma (Lindenbaum's Lemma) *If L is G-unprovable, there exists an N such that*
(1) $N \in C_G(K)$
(2) $N_\mu \supset L_\mu$ for any $\mu \in \mathbf{T}$.

Proof: We fix an enumeration of $FL(K), B_1, B_2, \ldots, B_m$. We define $N_n (n = 0, 1, \ldots, m)$ as follows:

$$N_0 = L$$

$$N_{n+1} = \begin{cases} N_n \cup \{B_{n+1}\}_1 \cup \{B_{n+1}\}_2, \text{ if } N_n \cup \{B_{n+1}\}_1 \cup \{B_{n+1}\}_2 \text{ is consistent} \\ N_n \cup \{B_{n+1}\}_1 \cup \{B_{n+1}\}_3, \text{ if } N_n \cup \{B_{n+1}\}_1 \cup \{B_{n+1}\} \text{ is consistent} \\ N_n \cup \{B_{n+1}\}_2 \cup \{B_{n+1}\}_3, \text{ otherwise.} \end{cases}$$

We put $N = \bigcup_{n=0}^{m} N_n$. It is evident that N satisfies (1) and (2).

6.5 Lemma *For any $A \in FL(K)$, $L \in C_G(K)$, and $\lambda, \mu, \nu \in \mathbf{T}$ where λ, μ,* ν are distinct,

$$A \in L_\mu \textit{ iff } \vdash L_\mu \cup \{A\}_\lambda \cup \{A\}_\nu.$$

Proof: Left-to-right is trivial. For right-to-left, suppose that $A \notin L_\mu$ and $\vdash L_\mu \cup \{A\}_\lambda \cup \{A\}_\nu$. Since $L \in C_G(K)$, $A \in L_\lambda \cap L_\nu$. So $\vdash L$. This is a contradiction.

We can easily prove the following lemmas.

6.6 Lemma *For any $\Box A \in FL(K)$ and $L \in C_{3\text{-}M_i}(K)$*
(1) *If $\Box A \in L_2$, then $A \in L_2$.*
(2) *If $\Box A \in L_2 \cap L_3$, then $A \in L_2 \cap L_3$.*

6.7 Lemma *For any $\Box A \in FL(K)$ and $L \in C_{3\text{-}S4_i}(K)$*
(1) *If $\Box A \in L_2$, then $\Box\Box A \in L_2$.*
(2) *If $\Box A \in L_2 \cap L_3$, then $\Box\Box A \in L_2 \cap L_3$.*

6.8 We prove the completeness theorem by a powerful method of a canonical model for $G(G = 3\text{-}K_i,\ 3\text{-}M_i,\ 3\text{-}S4_i,\ 3\text{-}S5_i)$. We define the canonical G-model $\mathfrak{C}_G = (C_G(K), R_G, \phi_G)$ $(G = 3\text{-}K_i,\ 3\text{-}M_i,\ 3\text{-}S4_i)$ as follows:

(1) LR_GN iff $\Box A \in L_2$ implies $A \in N_2$ and $\Box A \in L_2 \cap L_3$ implies $A \in N_2 \cap N_3$.
(2) $\phi_G(p,L) = \mu$ iff $p \in L_{\hat{\mu}} (\mu = 1,2,3)$.

Similarly we define the canonical $3\text{-}S5_i$-model $\mathfrak{C}_{3\text{-}S5_i} = (C_{3\text{-}S5_i}(K), R_{3\text{-}S5_i}, \phi_{3\text{-}S5_i})$ as follows:

(1) $LR_{3\text{-}S5_i}N$ iff $\Box A \in L_\mu$ implies $\Box A \in N_\mu$ $(\mu = 1,2,3)$.
(2) $\phi_{3\text{-}S5_i}(p,L) = \mu$ iff $p \in L_{\hat{\mu}}$ $(\mu = 1,2,3)$.

6.9 Lemma *$\mathfrak{C}_G$ is a G model.*

Proof: (1) In the case $G = 3\text{-}K_i$: immediate from the definition.
(2) In the case $G = 3\text{-}M_i$: by Lemma 6.6 $\mathfrak{C}_G$ is a G-model.
(3) In the case $G = 3\text{-}S4_i$: by Lemmas 6.6 and 6.7 $\mathfrak{C}_G$ is a G-model.
(4) In the case $G = 3\text{-}S5_i$: it is sufficient to show that LR_GN implies NR_GL. Suppose it is not the case that $\Box A \in L_\mu$. Because $L \in C_G(K)$, $\Box A \in L_\lambda \cap L_\nu$, by the assumption $\Box A \in N_\lambda \cap N_\nu$. Therefore it is not the case that $\Box A \in N_\mu$.

6.10 Lemma *For any $L \in C_G(K)$ and $A \in FL(K)$*

$$\phi_G(A,L) = \mu \text{ if } A \in L_{\hat{\mu}}.$$

Proof: We prove it by induction on the length of A. In the case of $A = F_i(B_1, \ldots, B_{\alpha_i})$, we can prove it as in [3]. Therefore we only consider the case of $A = \Box B$.

I. In the case of $G = 3\text{-}K_3$ or $3\text{-}M_3$:

I(1). $\mu = 1$: Suppose $\Box B \in L_{\hat{1}} = L_2 \cap L_3$. For any N such that LR_GN, $B \in N_{\hat{1}} = N_2 \cap N_3$. By the induction hypothesis, $\phi_G(B,N) = 1$. Hence $\phi_G(\Box B,L) = 1$.

I(2). $\mu = 2$: Suppose $\Box B \in L_{\hat{2}} = L_1 \cap L_3$. Since $\{\Box B\}_1 \cup \{\Box C \in L_2, \Box D \in L_2 \cap L_3\}_2 \cup \{\Box B, \Box D \in L_2 \cap L_3\}_3$ is G-unprovable as a restriction of L, $\{B\}_1 \cup \{C; \Box C \in L_2\}_2 \cup \{D; \Box D \in L_2 \cap L_3\}_2 \cup \{B\}_3 \cup \{D; \Box D \in L_2 \cap L_3\}_3$ is also G-unprovable. By Lemma 6.4 there exists an N such that LR_GN, $B \in N_1 \cap N_3 = N_{\hat{2}}$. By the induction hypothesis $\phi_G(B,N) = 2$. Hence $\phi_G(\Box B,L) = 2$.

I(3). $\mu = 3$: Similar to I(1), I(2).

II. In the case of $G = 3\text{-}S4_3$: By Lemma 5.1, we can prove it as in I.

III. In the case of $G = 3\text{-}S5_3$:

III(1). $\mu = 1$: Suppose $\Box B \in L_{\hat{1}} = L_2 \cap L_3$. Let N be such that LR_GN. By the definition of R_G and Lemma 6.6 $B \in N_2 \cap N_3$. Therefore, by the induction hypothesis, $\phi_G(B,N) = 1$. Hence $\phi_G(\Box B,L) = 1$.

III(2). $\mu = 2$: Suppose $\Box B \in L_{\hat{2}} = L_1 \cap L_3$. Since $\{\Box B, \Box C \in L_1\}_1 \cup \{\Box D \in L_2\}_2 \cup \{\Box B, \Box E \in L_3\}_3$ is G-unprovable as a restriction of L, by $\Box^{S5}_{1,3}\{B, \Box C \in L_1\}_1 \cup \{\Box D \in L_2\}_2 \cup \{B, \Box E \in L_3\}_3$ is also G-unprovable. By Lemma 6.4, there exists an N such that LR_GN and $B \in N_1 \cap N_3$. By the induction hypothesis $\phi_G(B,N) = 2$. Hence $\phi_G(\Box B,L) = 2$.

III(3). $\mu = 3$: We can prove it as in III(1), III(2).

IV. In the case of $G = 3\text{-}K_2$ or $G = 3\text{-}M_2$ or $G = 3\text{-}S4_2$ or $G = 3\text{-}S5_2$: We now show that $\Box B \in L_{\hat{2}}$ cannot hold, so that in view of cases I, II, and III above, $\phi_G(\Box B,L) = 2$ cannot obtain. If $\Box B \in L_{\hat{2}}$, then by the beginning matrix $\{\Box B\}_1 \cup \{\Box B\}_3$ we can prove that L is G-provable. This is a contradiction.

6.11 From Lemmas 6.9 and 6.10 we have the following completeness theorem:

Theorem III (Completeness Theorem) *If a matrix is valid in* $3\text{-}K_i$, $3\text{-}M_i$, $3\text{-}S4_i$, *or* $3\text{-}S5_i$, *then it is provable in* $3\text{-}K_i$, $3\text{-}M_i$, $3\text{-}S4_i$, *or* $3\text{-}S5_i$ *respectively.*

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Fukushima Technical College
Nagao 30, Kamiarakawa Taira
Iwaki, Fukushimaken, Japan