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An Isomorphism Between Rings and Groups

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Abstract Bijective functors σ and ρ are constructed between a category $\mathcal{R}$ of commutative nonassociative rings and a category $\mathcal{G}$ of nilpotent groups, such that for all $R \in \mathcal{R}$ and $G \in \mathcal{G}$, $\rho\sigma R \cong R$ and $\sigma\rho G \cong G$. Furthermore, if $\mathcal{L}$ is a subclass of $\mathcal{R}$, then $\mathcal{L}$ has a decidable elementary theory iff $\sigma\mathcal{L}$ has a decidable elementary theory.

Among the methods used to prove that certain classes of models have undecidable elementary theories are the "interpretation of one class into another" (cf. [7]), "semantic embedding" (cf. [1]), and "syntactic inclusion" (cf. [3]). Mal'cev [3] used "syntactic inclusion" to show that the class of all nilpotent class n-groups (for $n \geq 2$) has an undecidable elementary theory. In fact, Mal'cev employed a notion which we will here call "syntactic isomorphism". If two classes of models are syntactically isomorphic, then their elementary theories are either simultaneously undecidable or simultaneously decidable. One of the results of [3] states that the ring of integers $\mathbf{Z}$ is syntactically isomorphic to the free nilpotent class 2-group on two free generators, and the class of all not necessarily associative rings with identity is syntactically isomorphic to a class of nilpotent class 2-groups with two constants. In the present paper, we show that certain classes of rings with identity are syntactically isomorphic to classes of nilpotent groups whose nilpotency class is at most 3; in particular, the ring of integers $\mathbf{Z}$ is syntactically isomorphic to the free nilpotent class 3-group with two free generators, and the class of all Boolean algebras is syntactically isomorphic to a class of nilpotent class 2-groups.

The phrase "nonassociative ring" will mean "a ring with identity (denoted by 1) that is not necessarily associative". The notations of decidability theory that will be used here are those of [7]. Unless otherwise stated, the notations of group theory that are used here are those of [4].

1 Let $\mathcal{K}$ be a class of models of type τ. The first-order language of type τ will be denoted by $L\tau$. The elementary theory of $\mathcal{K}$, i.e., all sentences of $L\tau$ that

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are valid in every member of $\mathcal{K}$, will be denoted by Th$\mathcal{K}$. The following notion was used informally in Mal'cev [3]:

Definition Let $\mathcal{K}_1$, $\mathcal{K}_2$ be classes of models of type τ_1, τ_2 respectively. The classes $\mathcal{K}_1$, $\mathcal{K}_2$ are called *syntactically isomorphic* if there is an algorithm that assigns to every $A \in \mathcal{K}_1$ a $\mathcal{K}_2$-model σA, to every $B \in \mathcal{K}_2$ a $\mathcal{K}_1$-model ρB, to every sentence $\Phi \in \mathrm{L}\tau_2$ a sentence $\sigma\Phi \in \mathrm{L}\tau_1$, and to every sentence $\Psi \in \mathrm{L}\tau_1$ a sentence $\rho\Psi \in \mathrm{L}\tau_2$ such that

(i) for every $A \in \mathcal{K}_1$ and $B \in \mathcal{K}_2$, $\rho\sigma A \cong A$ and $\sigma\rho B \cong B$
(ii) for all sentences $\Phi \in \mathrm{L}\tau_2$ and $\Psi \in \mathrm{L}\tau_1$, $\Phi \in \mathrm{Th}\sigma A$ iff $\sigma\Phi \in \mathrm{Th}A$ and $\Psi \in \mathrm{Th}\rho B$ iff $\rho\Psi \in \mathrm{Th}B$.

Thus, if $\mathcal{K}_1$ and $\mathcal{K}_2$ are syntactically isomorphic, then Th$\mathcal{K}_1$ is decidable iff Th$\mathcal{K}_2$ is decidable. If $\mathcal{L}$ is a subclass of $\mathcal{K}_1$ and $\sigma\mathcal{L} = \{\sigma A : A \in \mathcal{L}\}$, then $\mathcal{L}$ is syntactically isomorphic to $\sigma\mathcal{L}$. As the ring of integers **Z** has an undecidable elementary theory (cf. [7]), Mal'cev showed that the free nilpotent class 2-group with two free generators has an undecidable elementary theory by establishing a syntactic isomorphism between it and the ring of integers **Z**. We shall show here that the class of all commutative nonassociative rings R, satisfying the identity $2x(yz) = 2(xy)z$ and for every $x \in R$ there is a $t \in R$ such that $2t = x(x-1)$, is syntactically isomorphic to a class of nilpotent groups.

All the groups considered here will be groups with two fixed elements, a_1 and a_2. If G is such a group, we shall use the following notations:

(1) G_i is the centralizer of a_i in G, $i = 1,2$
(2) if A,B are subsets of G, then $AB = \{xy : x \in A,\ y \in B\}$ and $[A,B] = \{[x,y] = x^{-1}y^{-1}xy : x \in A,\ y \in B\}$
(3) $a_{21} = [a_2,a_1]$
(4) $\alpha G = [[a_2,G_1],a_1][[a_2,G_1],a_2]$
(5) $\beta G = [G_2,G_1]\alpha G$
(6) if $x \in G$, then
$\mathrm{U}_1 x = \{y \in G_1 : [y,a_2]x \in \alpha G\}$,
$\mathrm{U}_2 x = \{y \in G_2 : [a_1,y]x \in \alpha G\}$, and
$\mathrm{T}x = \{y \in \beta G : y^{-2}[s,t] \in \alpha G$ for some $s \in \mathrm{U}_2\, x$ and $t \in \mathrm{U}_1(xa_{21}^{-1})\}$.

The neutral element of G will be noted by e. The center of G will be denoted by ζG. The commutator subgroup of G will be denoted by $\gamma_2 G$, and in general $\gamma_n G$ will denote the nth member of the lower central series of G, $n = 2,3,\ldots$. We shall also use the abbreviations $[x,y,z]$ for $[[x,y],z]$ and $[A,B,C]$ for $[[A,B],C]$ where $x,y,z \in G$ and $A,B,C \subseteq G$.

2 Let $\mathcal{R}$ be the class of all commutative nonassociative rings R such that

(i) for all $x,y,z \in R$, $2x(yz) = 2(xy)z$
(ii) for every $x \in R$, there is a $t \in R$ such that $2t = x(x-1)$.

The class $\mathcal{R}$ contains the ring of integers **Z** and all Boolean rings; it also contains all associative and commutative algebras over fields of characteristic not 2. The class $\mathcal{R}$ is closed under homomorphic images and Cartesian products.

Let $\mathcal{G}$ be the class of all nilpotent groups G, of nilpotency class at most 3, with two fixed elements a_1, a_2 such that

(A1) for all $x \in G$ each of the sets $\mathrm{U}_1[a_2,x]$ and $\mathrm{U}_2[x,a_1]$ is not empty
(A2) $[a_2,G_1,a_1] \cap [a_2,G_1,a_2] = \{e\}$
(A3) if $x \in G$ and $[a_i,x] \in \alpha G$, $i = 1,2$, then $x \in \beta G$
(A4) if $x \in \beta G$, then the following conditions are equivalent:
  (i) $x^2 \in \alpha G$, (ii) $[a_1,x] = e$, (iii) $[a_2,x] = e$
(A5) there are mappings $f_i : \beta G \to G_i$, $i = 1,2$, such that
  (a) $f_i a_{21} = a_i$, $f_i x \in \mathrm{U}_i x$ for all $x \in \beta G$, $i = 1,2$
  (b) if $x,y,z \in \beta G$ and $z = xy$, then $f_i z = f_i x f_i y$, $i = 1,2$
  (c) if $x \in \alpha G$, then $f_i x = e$, $i = 1,2$
  (d) if $x,y \in \beta G$, then $[f_2x,f_1y][f_1x,f_2y] \in \alpha G$
  (e) if $x,y,z \in \beta G$, then $([f_2x,f_1[f_2y,f_1z]][f_1z,f_2[f_2x,f_1y]])^2 \in \alpha G$
  (f) if $x,y \in \beta G$, then there are $s \in \mathrm{T}[f_2x,f_1y]$, $t \in \mathrm{T}x$, $u \in \mathrm{T}y$ such that $[f_2x,f_1y] = [a_2,f_1[f_2x,f_1y]][a_1,s][f_2x,f_1u,a_1][f_2t, f_1y,a_2]$.

Now we can formulate the main theorem of this paper.

Theorem 1 *The class of nonassociative rings $\mathcal{R}$ is syntactically isomorphic to the class of groups $\mathcal{G}$.*

3 The proof of Theorem 1 depends on the following lemmas:

Lemma 1 *Let G be a nilpotent group whose nilpotency class is at most 3, and let $a,b,c \in G$. Then*

(i) *$\gamma_2 G$ is abelian*
(ii) *The mappings $x \to [x,b,c]$, $x \to [a,x,c]$, and $x \to [a,b,x]$ are homomorphisms of G into $\gamma_3 G$*
(iii) *The mapping $x \to [x,c]$ is a homomorphism of $\gamma_2 G$ into $\gamma_3 G$*
(iv) $[a,b,c][b,c,a][c,a,b] = e.$

Proof: Most of the statements of this lemma are probably well known. For the sake of completeness, however, we shall sketch a proof.

Since $[\gamma_2 G, \gamma_2 G] \subseteq \gamma_4 G = \{e\}$ (cf. [4], p. 122), $\gamma_2 G$ is abelian.
Let $x,y \in G$. Then

$$\begin{aligned}[xy,b] &= [x,b]^y[y,b] \text{ (cf. [4], p. 119)}\\ &= [x,b][x,b,y][y,b]\\ &= [x,b][y,b][x,b,y].\end{aligned}$$

So

$$\begin{aligned}[xy,b,c] &= [[x,b][y,b],c] \text{ (since } \gamma_3 G \text{ is central)}\\ &= [x,b,c]^{[y,b]}[y,b,c]\\ &= [x,b,c][y,b,c] \text{ (since } \gamma_3 G \text{ is central).}\end{aligned}$$

This establishes (ii) since the other mappings are similar.

Since every element of $\gamma_2 G$ is a product of commutators, (iii) will follow if we show that $[[x,y][z,t],c] = [x,y,c][z,t,c]$ for all $x,y,z,t \in G$. But $[[x,y][z,t],c] = [x,y,c]^{[z,t]}[z,t,c] = [x,y,c][z,t,c]$ (since $\gamma_3 G$ is central).

Actually (iv) is the Hall-Witt identity

$$[a,b^{-1},c]^b[b,c^{-1},a]^c[c,a^{-1},b]^a = e$$

(cf. [4], p. 119). By (ii) and the fact that $\gamma_3 G$ is central, $[a,b,c]^{-1}[b,c,a]^{-1}[c,a,b]^{-1} = e$, from which (iv) follows.

Lemma 2 *Let G be a nilpotent group whose nilpotency class is at most 3 and let $a_1, a_2 \in G$. Then*

(i) $[a_2, G_1, a_i]$ *is a subgroup of* G, $i = 1,2$
(ii) αG *is a central subgroup of* G.

Proof: Since G_1 is a subgroup of G and $[a_2, G_1, a_i]$ is a homomorphic image of G_1 by (ii) of Lemma 1, $[a_2, G_1, a_i]$ is a subgroup of $\gamma_3 G$. Since αG is the product of these subgroups of $\gamma_3 G$, condition (ii) follows.

Lemma 3 *Let G be a nilpotent group of nilpotency class at most 3, with elements a_1, a_2. Assume that for every $x \in [G_2, G_1]$ each of the sets $\mathrm{U}_1 x$, $\mathrm{U}_2 x$ is not empty. Then*

(i) $[G_2, G_1] \subseteq [a_2, G_1]\alpha G$
(ii) $[G_2, G_1] \subseteq [G_2, a_1]\alpha G$
(iii) $\beta G = [a_2, G_1]\alpha G = [G_2, a_1]\alpha G$
(iv) *if* $x \in G_1$, $y \in G_2$, *then*

$$[a_2,x,y] = [y,x,a_2] \text{ and } [y,a_1,x] = [y,x,a_1]$$

(v) $[G_2, G_1, G_i] = [a_2, G_1, a_i]$, $i = 1,2$
(vi) βG *is a subgroup of* G
(vii) *the mappings* $x \to [a_2,x]\alpha G$ *and* $y \to [y,a_1]\alpha G$ *are homomorphisms of the groups* G_1, G_2, *respectively, onto the quotient group* $\beta G/\alpha G$.

Proof: Let $x \in [G_2, G_1]$, $y \in \mathrm{U}_1 x$. Then $x = [a_2, y]u$ for some $u \in \alpha G$. This shows (i). Statement (ii) is similar. Statement (iii) follows from (i), (ii), and Lemma 2. Statement (v) follows from (i), (ii), and (iv). Thus, we need to show (iv), (vi), and (vii).

By (iv) of Lemma 1, $[a_2,x,y][x,y,a_2][y,a_2,x] = e$. If $y \in G_2$, then $[y,a_2] = e$ and $[a_2,x,y] = [x,y,a_2]^{-1} = [y,x,a_2]$ by (iii) of Lemma 1. The remaining statement of (iv) is similar.

Now, we need to show (vi) and (vii). Let $x,y \in G_1$. Then

$$\begin{aligned}[a_2,xy^{-1}] &= [a_2,y^{-1}][a_2,x][a_2,x,y^{-1}] \\ &= [a_2,x][a_2,y^{-1}][a_2,x,y^{-1}] \text{ (since } \gamma_2 G \text{ is abelian)} \\ &\in [a_2,x][a_2,y^{-1}]\alpha G \text{ (by (v)).}\end{aligned}$$

Thus $e \in [a_2,y][a_2,y^{-1}]\alpha G$, i.e., $[a_2,y]^{-1} \in [a_2,y^{-1}]\alpha G$. Hence $[a_2,xy^{-1}] \in [a_2,x][a_2,y]^{-1}\alpha G$. This proves (vi) and the first part of (vii). The second part of (vii) is similar.

Lemma 4 *Let G be a nilpotent group of nilpotency class at most 3, with elements a_1, a_2. Suppose that G satisfies the conditions:*

(i) *if* $x \in [G_2, G_1]$, *then each of the sets* $\mathrm{U}_1 x, \mathrm{U}_2 x$ *is not empty*
(ii) *if* $x \in G$ *and* $[a_i, x] \in \alpha G$, $i = 1,2$, *then* $x \in \beta G$.

Then βG is a subgroup of G, αG is a central subgroup of G, and $R = \rho G$ is a nonassociative ring whose additive group is $\beta G/\alpha G$; the ring multiplication of R is given by the following: if $x,y \in \beta G$, then $x\alpha G \times y\alpha G = [s,t]\alpha G$ where $s \in \mathrm{U}_2 x$, $t \in \mathrm{U}_1 y$; the ring identity element is $a_{21}\alpha G$. Moreover, if Ψ is a first-order sentence in the language of nonassociative rings, we can construct a sentence $\rho\Psi$ in the language of groups with constants a_1, a_2 such that $\Psi \in \mathrm{Th}R$ iff $\rho\Psi \in \mathrm{Th}G$.

Proof: From Lemma 3, βG is a subgroup of G. From Lemma 2, αG is a central subgroup of G. Since $\beta G \subseteq \gamma_2 G$ and $\gamma_2 G$ is abelian (by Lemma 1), $\beta G/\alpha G$ is an abelian group. We need to show first that multiplication on R is well-defined. Let $x \in \beta G$. Then $x = x'u$ where $x' \in [G_2, G_1]$, $u \in \alpha G$. Thus $\mathrm{U}_i x = \mathrm{U}_i x'$. Hence $\mathrm{U}_i x$ is not empty, $i = 1,2$. Let $x,y \in \beta G$, $s,s' \in \mathrm{U}_2 x$, $t,t' \in \mathrm{U}_1 y$. Then $[a_2, s^{-1}s'] = e$ and $[a_1, s^{-1}s'] \in [a_1,s]^{-1}[a_1,s']\alpha G = xx^{-1}\alpha G = \alpha G$, by (vii) of Lemma 3. Thus $[a_i, s^{-1}s'] \in \alpha G$, $i = 1,2$. Hence $v = s^{-1}s' \in \beta G$. Similarly $w = t^{-1}t' \in \beta G$. Thus $s' = sv$, $t' = tw$. Now $[s',t'] = [sv,tw] = [sv,w][sv,t]^w = [s,w][sv,t] = [s,w][s,t]^v[v,t] = [s,t][s,w][v,t]$ (since $\gamma_2 G$ is abelian). Since $[s,w] \in [s,\beta G] = [s,[G_2,G_1]] \subseteq [G_2,G_1,G_2]^{-1} \subseteq \alpha G$ by (v) of Lemma 3 and (ii) of Lemma 2, and similarly $[v,t] \in \alpha G$, $[s',t'] \in [s,t]\alpha G$. Thus multiplication on R is well-defined.

Moreover, $a_{21}\alpha G \times y\alpha G = [a_2,t]\alpha G = y\alpha G$ and $x\alpha G \times a_{21}\alpha G = [s,a_1]\alpha G = x\alpha G$ for all $x,y \in \beta G$. This shows that $a_{21}\alpha G$ is the identity element of R. We need to show that multiplication on R distributes over addition. Let $x,y,z \in \beta G$, $r \in \mathrm{U}_2 x$, $s \in U_1 y$, $t \in \mathrm{U}_1 z$. Then $st \in \mathrm{U}_1(yz)$. Indeed, $[a_2,st] \in [a_2,s][a_2,t]\alpha G$ by (vii) of Lemma 3. Thus $[a_2,st] \in yz\alpha G$ and $st \in \mathrm{U}_1(yz)$. Hence $x\alpha G \times (y\alpha G + z\alpha G) = x\alpha G \times (yz)\alpha G = [r,st]\alpha G = [r,s][r,t]\alpha G$ (by a method similar to the proof of (vii) of Lemma 3) $= x\alpha G \times y\alpha G + x\alpha G \times z\alpha G$. This shows left distributivity. Right distributivity is similar.

Now we need to prove the last part of Lemma 4. Let Ψ be a first-order sentence in the language of nonassociative rings. We can assume that $\Psi = (Q_1x_1)(Q_2x_2)\ldots(Q_nx_n)\Psi_0(x_1,x_2,\ldots,x_n)$, where $\{Q_1,Q_2,\ldots,Q_n\} \subseteq \{\exists,\forall\}$ and Ψ_0 is a quantifier-free formula built from atomic formulas in the language of nonassociative rings via logical connectives. To say that Ψ is valid in $R = \rho G$ is a demand on the group G. This demand, in turn, is equivalent to the validity in G of a first-order sentence in the language of groups with constants a_1, a_2. We shall denote such a sentence by $\rho\Psi$. The statements $x \in \alpha G$ and $x \in \beta G$ are expressible, respectively, by the following first-order formulas:

$$A(x) \equiv (\exists s)(\exists t)(a_1 s \approx s a_1 \wedge a_1 t \approx t a_1 \wedge x \approx [a_2,s,a_1][a_2,t,a_2]),$$
$$B(x) \equiv (\exists u)(\exists v)(a_1 u \approx u a_1 \wedge A(v) \wedge x \approx [a_2,u]v).$$

The sentence $\rho\Psi$ can be obtained from Ψ by replacing:

1. every atomic formula of the form $x \times y \approx z$ by
$$(B(x) \wedge B(y) \wedge B(z)) \to ((\exists s)(\exists t)(a_2 s \approx s a_2 \wedge a_1 t \approx t a_1 \wedge A(x[a_1,s]) \wedge A(y[t,a_2]) \wedge A(z[t,s]));$$
2. every atomic formula of the form $x + y \approx z$ by
$$(B(x) \wedge B(y) \wedge B(z)) \to A(z^{-1}xy); \text{ and}$$
3. the constant 1 by $[a_2, a_1]$.

4 Now we complete the proof of Theorem 1. Let $R \in \mathcal{R}$. The set of all elements $x \in R$ such that $2x = 0$ will be denoted by 0/2. It is clear that 0/2 is an ideal of R and the quotient ring $R/(0/2)$ is associative. Also, if $x \in R$, then the elements t such that $2t = x(x-1)$ are unique modulo 0/2. We denote by tx the coset $t + 0/2$ where $2t = x(x-1)$. The group σR will be constructed as follows: The carrier set is $R \times R \times R \times R/(0/2) \times R/(0/2)$. If $x \in \sigma R$, then x_i will denote the ith entry in x, $i = 1,\ldots,5$. The operation $\circ$ on σR is defined as follows. Let $x,y \in \sigma R$. Then

$$\begin{aligned}
&(x \circ y)_i = x_i + y_i,\ i = 1,2\\
&(x \circ y)_3 = x_3 + y_3 + x_2 y_1\\
&(x \circ y)_4 = x_4 + y_4 + x_3 y_1 + x_2 t y_1\\
&(x \circ y)_5 = x_5 + y_5 + x_3 y_2 + (x_2 y_1) y_2 + y_1 t x_2.
\end{aligned}$$

The groupoid just defined is a group belonging to the class $\mathcal{G}$. The neutral element of $G = \sigma R$ is $e = (0,0,0,0,0)$, and if $x \in G = \sigma R$, then the inverse of x is given by

$$\begin{aligned}
&(x^{-1})_i = -x_i,\ i = 1,2\\
&(x^{-1})_3 = -x_3 + x_1 x_2\\
&(x^{-1})_4 = -x_4 + x_1 x_3 - x_2 t (x_1 + 1)\\
&(x^{-1})_5 = -x_5 + x_2 x_3 - x_1 t (x_2 + 1).
\end{aligned}$$

If $x,y \in \sigma R$, then the commutator $[x,y]$ is given by

$$\begin{aligned}
&[x,y]_i = 0,\ i = 1,2\\
&[x,y]_3 = x_2 y_1 - x_1 y_2\\
&[x,y]_4 = x_3 y_1 - x_1 y_3 + x_2 t y_1 - y_2 t x_1\\
&[x,y]_5 = x_3 y_2 - x_2 y_3 + (x_2(y_1 - x_1)) y_2 + y_1 t x_2 - x_1 t y_2.
\end{aligned}$$

The following claim will be needed later:

Claim *Let $R \in \mathcal{R}$ and $G = \sigma R$. Let $a_1 = (1,0,0,0,0)$, $a_2 = (0,1,0,0,0)$. Then*

(i) $\beta G = \gamma_2 G$
(ii) $\alpha G = \gamma_3 G$
(iii) $\zeta G = G_1 \cap G_2 = \{x \in \beta G : x^2 \in \alpha G\}$
(iv) G_i *is an abelian group,* $i = 1,2$.

Proof: It is clear that $a_{21} = (0,0,1,0,0)$ and $G_1 = R \times 0 \times 0/2 \times R/(0/2) \times R/(0/2)$; $G_2 = 0 \times R \times 0/2 \times R/(0/2) \times R/(0/2)$; $[a_2,G_1] = \{(0,0,x,tx,0) : x \in R\}$; $[a_2,G_1,a_1] = 0 \times 0 \times 0 \times R/(0/2) \times 0$; $[a_2,G_1,a_2] = 0 \times 0 \times 0 \times 0 \times R/(0/2)$; $\alpha G = [a_2,G_1,a_1] \circ [a_2,G_1,a_2] = 0 \times 0 \times 0 \times R/(0/2) \times R/(0/2)$; $[G_2,G_1] = \{(0,0,xy,xty,ytx) : x,y \in R\}$; and $\beta G = [G_2,G_1] \circ \alpha G = 0 \times 0 \times R \times R/(0/2) \times R/(0/2)$.

Since $\beta G \subseteq \gamma_2 G$ and $\gamma_2 G$ is the subgroup of G generated by $[G,G] \subseteq 0 \times 0 \times R \times R/(0/2) \times R/(0/2)$, and βG is a subgroup of G from the definition of the operation $\circ$, $\beta G = \gamma_2 G$. Also, since $\alpha G \subseteq \gamma_3 G \subseteq 0 \times 0 \times 0 \times R/(0/2) \times R/(0/2) = \alpha G$, we conclude that $\alpha G = \gamma_3 G$. The center of G is the set of all $x \in G$ such that $[x,y] = e$ for all $y \in G$. Thus $x_2 y_1 - x_1 y_2 = 0$ for all $y_1, y_2 \in R$. Hence $x_1 = x_2 = 0$. Also $x_3 y_1 \in 0/2$ for all $y_1 \in R$. Hence $2x_3 = 0$. Conversely, if $x_1 = x_2 = 2x_3 = 0$, then $x \in \zeta G$. Thus $\zeta G = 0 \times 0 \times 0/2 \times R/(0/2) \times R/(0/2) = G_1 \cap G_2$. It is clear that $\zeta G \subseteq \beta G$. Let $x \in \beta G$. Then $x^2 \in \alpha G$ iff $2x_3 = 0$. Thus $\zeta G = \{x \in \beta G : x^2 \in \alpha G\}$. From the definition of the operation $\circ$, G_1, G_2 are abelian. This establishes the claim.

Since $\gamma_3 G \subseteq \zeta G$, where G is nilpotent of class at most 3, it is clear that $G = \sigma R$ is abelian iff the ring R is trivial ($0 = 1$). The group $G = \sigma R$ is nilpotent of class 2 iff $\gamma_2 G$ is nontrivial and $\gamma_3 G$ is trivial; i.e., iff R is nontrivial and $R = 0/2$. In other words, $G = \sigma R$ is nilpotent of class 2 iff R is nontrivial and R satisfies the identity $x = x^2$.

Condition (A2) is clearly satisfied by $G = \sigma R$. The group G satisfies condition (A3). Indeed, let $x \in G$, $[a_i, x] \in \alpha G$, $i = 1,2$. We have $[a_1, x]_3 = -x_2$ and $[a_2, x]_3 = x_1$. Thus $x_1 = x_2 = 0$ and $x \in \beta G$. The group G also satisfies condition (A4). Indeed, condition (A4)(i) is equivalent to $x \in \zeta G$ by (iii) of the Claim. Thus (A4)(i) implies (A4)(ii) and (iii). Conversely, if $x \in \beta G$ and $[a_1, x] = e$, then $x_1 = x_2 = 0$ and $-x_3 \in 0/2$. Thus $x \in \zeta G$, i.e., $x^2 \in \alpha G$ by (iii) of the Claim and so (A4)(ii) implies (A4)(i). Similarly, (A4)(iii) implies (A4)(i).

By (i) of the Claim, if we show that G satisfies (A5)(a), then G satisfies (A1). Thus $G \in \mathcal{G}$, if we show that G satisfies (A5). Define $f_1(0,0,x_3,x_4,x_5) = (x_3,0,0,0,0)$, $f_2(0,0,x_3,x_4,x_5) = (0,x_3,0,0,0)$. It is clear that f_i is a homomorphism of βG into G_i, $f_i a_{21} = a_i$, $i = 1,2$. If $x \in \beta G$, then

$$\begin{aligned}[][f_1, x, a_2] \circ x &= [(x_3,0,0,0,0),(0,1,0,0,0)] \circ (0,0,x_3,x_4,x_5) \\ &= (0,0,-x_3,*,*) \circ (0,0,x_3,*,*) \\ &= (0,0,0,*,*) \in \alpha G.\end{aligned}$$

Similarly, $[a_1, f_2 x] \circ x \in \alpha G$. Thus G satisfies (A5)(a), (b), and (c). We shall show that G satisfies (A5)(d). Let $x, y \in \beta G$. Then

$$\begin{aligned}&[f_2 x, f_1 y] \circ [f_1 x, f_2 y] \\ &\quad = [(0,x_3,0,0,0),(y_3,0,0,0,0)] \circ [(x_3,0,0,0,0),(0,y_3,0,0,0)] \\ &\quad = (0,0,x_3 y_3,*,*) \circ (0,0,-x_3 y_3,*,*) \\ &\quad = (0,0,0,*,*) \in \alpha G.\end{aligned}$$

To show that G satisfies (A5)(e), let $x, y, z \in \beta G$. Then

$$\begin{aligned}[][f_2 x, f_1[f_2 y, f_1 z]] &= [(0,x_3,0,0,0), f_1(0,0,y_3 z_3,*,*)] \\ &= [(0,x_3,0,0,0),(y_3 z_3,0,0,0,0)] \\ &= (0,0,\ x_3(y_3 z_3),*,*); \\ [f_1 z, f_2[f_2 x, f_1 y]] &= [(z_3,0,0,0,0),(0,x_3 y_3,0,0,0)] \\ &= (0,0,-z_3(x_3 y_3),*,*).\end{aligned}$$

Thus the left hand side of (A5)(e) is

$$\begin{aligned}&((0,0,x_3(y_3 z_3),*,*) \circ (0,0,-z_3(x_3 y_3),*,*))^2 = \\ &\qquad (0,0,2(x_3(y_3 z_3) - z_3(x_3 y_3)),*,*) = (0,0,0,*,*) \in \alpha G\end{aligned}$$

since R satisfies the identity $2u(vw) = 2(uv)w$ and R is commutative.

The proof that $G = \sigma R \in \mathcal{G}$ will be complete if we show that G satisfies (A5)(f). We show first that if $x,t \in \beta G$ and $t_3 \in tx_3$, then $t \in \mathrm{T}x$. Indeed, $t^{-2} = (0,0,t_3,*,*)^{-2} = (0,0,-2t_3,*,*) = (0,0,-x_3(x_3-1),*,*)$ and $[f_2x,f_1(x \circ a_{21}^{-1})] = [(0,x_3,0,0,0),(x_3-1,0,0,0,0)] = (0,0,x_3(x_3-1),*,*)$. Thus $t^{-2} \circ [f_2x,f_1(x\circ a_{21}^{-1})] \in \alpha G$, i.e., $t \in \mathrm{T}x$. Let $x,y,s,t,u \in \beta G$, $t_3 \in tx_3$, $u_3 \in ty_3$, and $s_3 \in t[f_2x,f_1y]_3 = t(x_3y_3)$. Then $t \in \mathrm{T}x, u \in \mathrm{T}y$ and $s \in \mathrm{T}[f_2x,f_1y]$. The right hand side of (A5)(f) is

$$\begin{aligned}
[f_2x,f_1y] &= [(0,x_3,0,0,0),(y_3,0,0,0,0)] \\
&= (0,0,x_3y_3,x_3ty_3,y_3tx_3). \\
[a_2,f_1[f_2x,f_1y]] &= [(0,1,0,0,0),(x_3y_3,0,0,0,0)] \\
&= (0,0,x_3y_3,t(x_3y_3),0), \\
[a_1,s] &= [(1,0,0,0,0),(0,0,s_3,*,*)] \\
&= (0,0,0,-s_3+0/2,0) = (0,0,0,-t(x_3y_3),0), \\
[f_2x,f_1u,a_1] &= [(0,x_3,0,0,0),(u_3,0,0,0,0),(1,0,0,0,0)] \\
&= [(0,0,x_3u_3,*,*),(1,0,0,0,0)] \\
&= (0,0,0,x_3u_3+0/2,0) = (0,0,0,x_3ty_3,0), \\
[f_2t,f_1y,a_2] &= [(0,t_3,0,0,0),(y_3,0,0,0,0),(0,1,0,0,0)] \\
&= [(0,0,t_3y_3,*,*),(0,1,0,0,0)] \\
&= (0,0,0,0,t_3y_3+0/2) = (0,0,0,0,y_3tx_3).
\end{aligned}$$

Thus the right hand side of (A5)(f) is $(0,0,x_3y_3,t(x_3y_3),0) \circ (0,0,0,-t(x_3y_3),0) \circ (0,0,0,x_3ty_3,0) \circ (0,0,0,0,y_3tx_3) = (0,0,x_3y_3,0,0) \circ (0,0,0,x_3ty_3,y_3tx_3) = (0,0,x_3y_3,x_3ty_3,y_3tx_3)$. Thus the two sides of (A5)(f) are equal and $G = \sigma R \in \mathcal{G}$.

5 If $G \in \mathcal{G}$, then G satisfies the conditions of Lemma 4 and $R = \rho G$ is a ring. Condition (A5)(d) implies that R is commutative and condition (A5)(e) implies that R satisfies the identity $2x \times (y \times z) = 2(x \times y) \times z$. From condition (A5)(f), for every $x \in \beta G$ there is a $t \in \mathrm{T}x$; i.e., $t^{-2}[y,z] \in \alpha G$ and $t \in \beta G$ where $y \in \mathrm{U}_2x$ and $z \in \mathrm{U}_1(xa_{21}^{-1})$. Thus the nonassociative ring $\rho G = \beta G/\alpha G$ satisfies the condition: For every $x \in \rho G$, there is a $t \in \rho G$ such that $2t = x \times (x-1)$, i.e., $\rho G \in \mathcal{R}$.

Let Φ be a first-order sentence in the language of groups with constants a_1,a_2 and let $R \in \mathcal{R}$. The validity of Φ in σR is equivalent to some demand on the nonassociative ring R. This demand, in turn, is equivalent to the validity of some first-order sentence in the language of nonassociative rings. We shall denote such a sentence by $\sigma\Phi$. We can assume that $\Phi = (Q_1x_1)(Q_2x_2)\ldots(Q_nx_n)\Phi_0(x_1,x_2,\ldots,x_n)$, where $\{Q_1,Q_2,\ldots,Q_n\} \subseteq \{\forall,\exists\}$and Φ_0 is a quantifier-free formula built from atomic formulas in the language of nonassociative rings via logical connectives. The sentence $\sigma\Phi$ can be obtained from Φ by replacing

1. every quantifier (Qx) by $(Qx_1)(Qx_2)(Qx_3)(Qx_4)(Qx_5)$
2. every atomic formula $x\cdot y \approx z$ by the conjunction of $x_1+y_1 \approx z_1$, $x_2+y_2 \approx z_2$, $x_3+y_3+x_2y_1 \approx z_3$, $2(x_4+y_4+x_3y_1)+x_2(y_1(y_1-1)) \approx 2z_4$, $2(x_5+y_5+x_3y_2+(x_2y_1)y_2)+(x_2(x_2-1))y_1 \approx 2z_5$

3. every constant a_1 by $(a_1)_1 = 1$, $(a_1)_j = 0$ if $j = 2,3,4,5$
4. every constant a_2 by $(a_2)_2 = 1$, $(a_2)_j = 0$ if $j = 1,3,4,5$.

6 Now we shall show that if $R \in \mathcal{R}$ then $\rho\sigma R \cong R$. Indeed, $\beta\sigma R = 0 \times 0 \times R \times R/(0/2) \times R/(0/2)$ and $\alpha\sigma R = 0 \times 0 \times 0 \times R/(0/2) \times R/(0/2)$. The mapping $\delta : \rho\sigma R \to R$ defined by $\delta((0,0,x,*,*) \circ \alpha\sigma R) = x$ is a homomorphism of the nonassociative ring $\rho\sigma R$ onto R and its kernel is trivial.

7 The proof of Theorem 1 will be concluded if we show that for every $G \in \mathcal{G}$, $\sigma\rho G \cong G$. To establish this, we define for any given $G \in \mathcal{G}$ homomorphisms f_3, f_4, f_5 from the group βG into βG, $[a_2, G_1, a_1]$, $[a_2, G_1, a_2]$, respectively, and show that the mapping $\theta : \sigma\rho G \to G$ defined by $\theta x = f_1x_1f_2x_2f_3x_3f_4x_4f_5x_5$ is the required isomorphism.

Let $x \in \beta G$ and $t \in \mathrm{T}x$. Define $f_3x = [a_2, f_1x][a_1, t]$. We shall show that f_3 is an endomorphism of βG whose kernel is αG and which satisfies $x^{-1}f_3x \in \alpha G$. Indeed, f_3 is well defined. Let $x, t, t' \in \beta G$, $u, u' \in \mathrm{U}_2x$, $v, v' \in \mathrm{U}_1(xa_{21}^{-1})$, and $t^{-2}[u,v]$, $t'^{-2}[u',v'] \in \alpha G$. From the proof of Lemma 4, $[u,v]\alpha G = [u',v']\alpha G$. Since βG is abelian (by Lemma 1), we get $(t^{-1}t')^2 \in \alpha G$. But then, by (A4), $[a_1, t^{-1}t'] = e$. Thus $[a_1, t] = [a_1, t(t^{-1}t')] = [a_1, t']$. Now let $f_3x = e$. Then $[a_2, f_1x] = [t, a_1] \in \alpha G$. Hence $x \in \alpha G$. Conversely, if $x \in \alpha G$, then $f_1x = e$ by (A5)(c) and $t \in \mathrm{T}x = \alpha G$. Thus $f_3x = e$. Moreover, $x^{-1}f_3x = x^{-1}[a_2, f_1x][a_1, t] \in \alpha G$. It remains to show that if $x, y \in \beta G$, then $f_3(xy) = f_3xf_3y$. Indeed,

$$\begin{aligned} [a_2, f_1(xy)] &= [a_2, f_1xf_1y] = [a_2, f_1x][a_2, f_1y][a_2, f_1x, f_1y] \\ &= [a_2, f_1x][a_2, f_1y][x, f_1y] \\ &= [a_2, f_1x][a_2, f_1y][f_2x, a_1, f_1y] \\ &= [a_2, f_1x][a_2, f_1y][f_2x, f_1y, a_1] \end{aligned}$$

(by (A5)(a) and (iv) of Lemma 3); and

$$\begin{aligned} [f_2(xy), f_1(xya_{21}^{-1})] &= [f_2xf_2y, f_1(xya_{21}^{-1})] \\ &\in [f_2x, f_1(xya_{21}^{-1})][f_2y, f_1(xya_{21}^{-1})]\alpha G \\ &= [f_2x, f_1(xa_{21}^{-1})][f_2x, f_1y][f_2y, f_1(ya_{21}^{-1})][f_2y, f_1x]\alpha G \\ &= [f_2x, f_1y]^2[f_2x, f_1(xa_{21}^{-1})][f_2y, f_1(ya_{21}^{-1})]\alpha G \end{aligned}$$

(by (A5)(d) and (vii) of Lemma 3).

Thus if $s \in \mathrm{T}x$, $t \in \mathrm{T}y$, $u \in \mathrm{T}(xy)$, then $u^2 \in [f_2x, f_1y]^2s^2t^2\alpha G$; i.e., $(u^{-1}st[f_2x, f_1y])^2 \in \alpha G$. Hence, by (A4), $[a_1, u^{-1}st[f_2x, f_1y]] = e$ and $[a_1, u] = [a_1, s][a_1, t][a_1, [f_2x, f_1y]]$ (by (iii) of Lemma 1). So

$$\begin{aligned} f_3(xy) &= [a_2, f_1(xy)][a_1, u] \\ &= [a_2, f_1x][a_2, f_1y][f_2x, f_1y, a_1][a_1, s][a_1, t][a_1, [f_2x, f_1y]] \\ &= [a_2, f_1x][a_1, s][a_2, f_1y][a_1, t] = f_3xf_3y. \end{aligned}$$

Define $f_{3+i}x = [x, a_i]$ for $x \in \beta G$, $i = 1,2$. By Lemmas 1 and 3 f_{3+i} is a homomorphism of βG onto $[a_2, G_1, a_i]$, $i = 1,2$. The kernel of f_{3+i} is $\{x \in$

$\beta G : x^2 \in \alpha G\}$ (by A4). Thus if $x \in \sigma\rho G$ then $f_j x_j$ is a well-defined element of G for all $j = 1,2,3,4,5$. Thus θ is a well-defined mapping of $\sigma\rho G$ into G. The homomorphisms f_j satisfy the following relations for all $x,y,z \in \beta G/\alpha G$:

$$(*) \qquad [f_2 x, f_1 y] = f_3(x \times y) f_4(x \times ty) f_5(y \times tx)$$
$$(**) \qquad [f_3 x, f_1 y, f_2 z] = f_5((x \times y) \times z)$$
$$(***) \qquad [f_3 x, f_i y] = f_{3+i}(x \times y),\ i = 1,2.$$

The equality $(*)$ follows from (A5)(f). The left hand side of equation $(**)$ is equal to $[a_2, f_1[f_2 x, f_1 y], f_2 z]$ due to the fact that $u \in [a_2, f_1 u]\alpha G$ and αG is a central subgroup of G. Thus

$$\begin{aligned}[f_2 x, f_1 y, f_2 z] &= [a_2, f_1[f_2 x, f_1 y], f_2 z] = [f_2 z, f_1[f_2 x, f_1 y], a_2] \\ &= f_5(z \times (x \times y)) = f_5((x \times y) \times z) \text{ (by (iv) of Lemma 3).}\end{aligned}$$

Since $f_3 x \in [a_2, f_1 x]\alpha G = [f_2 x, a_1]\alpha G$,

$$\begin{aligned}[f_3 x, f_1 y] = [f_2 x, a_1, f_1 y] &= [f_2 x, f_1 y, a_1] \text{ (by (iv) of Lemma 3)} \\ &= f_4(x \times y); \\ [f_3 x, f_2 y] = [a_2, f_1 x, f_2 y] &= [f_2 y, f_1 x, a_2] \text{ (by (iv) of Lemma 3)} \\ &= f_5(y \times x) = f_5(x \times y).\end{aligned}$$

Now we show that for any $x,y \in \sigma\rho G$, $\theta x \theta y = \theta(x \circ y)$. Since $f_4 x_4 f_5 x_5$ is central, it is sufficient to consider the case $x_4 = x_5 = y_4 = y_5 = \sqrt{\alpha}G$, where $\sqrt{\alpha}G = \{x \in \beta G : x^2 \in \alpha G\}$. First,

$$\begin{aligned}f_3 x_3 f_1 y_1 f_2 y_2 &= f_1 y_1 f_2 y_2 f_3 x_3 [f_3 x_3, f_1 y_1 f_2 y_2] \\ &= f_1 y_1 f_2 y_2 f_3 x_3 [f_3 x_3, f_1 y_1][f_3 x_3, f_2 y_2] \\ &= f_1 y_1 f_2 y_2 f_3 x_3 f_4(x_3 \times y_1) f_5(x_3 \times y_2) \text{ (by } (***)\text{)},\end{aligned}$$

$$\begin{aligned}f_2 x_2 f_1 y_1 &= f_1 y_1 f_2 x_2 [f_2 x_2, f_1 y_1] \\ &= f_1 y_1 f_2 x_2 f_3(x_2 \times y_1) f_4(x_2 \times ty_1) f_5(y_1 \times tx_2) \text{ (by } (*)\text{)}.\end{aligned}$$

Thus

$$\begin{aligned}\theta x \theta y &= f_1 x_1 f_2 x_2 f_3 x_3 f_1 y_1 f_2 y_2 f_3 y_3 \\ &= f_1 x_1 f_2 x_2 f_1 y_1 f_2 y_2 f_3 x_3 f_4(x_3 \times y_1) f_5(x_3 \times y_2) f_3 y_3 \\ &= f_1 x_1 f_2 x_2 f_1 y_1 f_2 y_2 f_3(x_3 + y_3) f_4(x_3 \times y_1) f_5(x_3 \times y_2) \\ &\quad \text{(since } \alpha G \text{ is central and } \beta G \text{ is abelian)} \\ &= f_1 x_1 f_1 y_1 f_2 x_2 f_3(x_2 \times y_1) f_4(x_2 \times ty_1) f_5(y_1 \times tx_2) \\ &\quad f_2 y_2 f_3(x_3 + y_3) f_4(x_3 \times y_1) f_5(x_3 \times y_2) \\ &= f_1(x_1 + y_1) f_2 x_2 f_3(x_2 \times y_1) f_2 y_2 f_3(x_3 + y_3) f_4(x_3 \times y_1 + x_2 \times ty_1) \\ &\quad f_5(x_3 \times y_2 + y_1 \times tx_2) \\ &= f_1(x_1 + y_1) f_2 x_2 f_2 y_2 f_3(x_2 \times y_1)[f_3(x_2 \times y_1), f_2 y_2] f_3(x_3 + y_3) \\ &\quad f_4(x_3 \times y_1 + x_2 \times ty_1) f_5(x_3 \times y_2 + y_1 \times tx_2) \\ &= f_1(x_1 + y_1) f_2(x_2 + y_2) f_3(x_2 \times y_1) f_5((x_2 \times y_1) \times y_2) f_3(x_3 + y_3) \\ &\quad f_4(x_3 \times y_1 + x_2 \times ty_1) f_5(x_3 \times y_2 + y_1 \times tx_2) \text{ (by } (***)\text{)} \\ &= f_1(x_1 + y_1) f_2(x_2 + y_2) f_3(x_3 + y_3 + x_2 \times y_1) f_4(x_3 \times y_1 + x_2 \times ty_1) \\ &\quad f_5(x_3 \times y_2 + (x_2 \times y_1) \times y_2 + y_1 \times tx_2) \text{ (since } \alpha G \text{ is central)} \\ &= \theta(x \circ y).\end{aligned}$$

Thus θ is a group homomorphism. It is clear that θ is also a homomorphism of groups with constants a_1, a_2. We need to show that θ is bijective. First we prove that θ is injective. Let $x \in \sigma\rho G$ and $\theta x = e$. Then $e = f_1x_1f_2x_2f_3x_3f_4x_4f_5x_5$. Hence $f_1(-x_1) = (f_1x_1)^{-1} = f_2x_2f_3x_3f_4x_4f_5x_5$, and $[a_2,f_1(-x_1)] = [a_2, f_2x_2f_3x_3f_4x_4f_5x_5] = [a_2,f_3x_3] \in \alpha G$. Thus $x_1 = \alpha G$ and $f_1x_1 = e$. Hence $f_2x_2 \in \beta G$ and $[f_2x_2,a_1] \in \alpha G$, and $x_2 = \alpha G$ and $f_2x_2 = e$. Thus $f_3x_3 \in \alpha G$. But $x_3 = (f_3x_3)\alpha G = \alpha G$. Thus $f_3x_3 = e$. Hence $f_4x_4f_5x_5 = e$, but by (A2) $f_4x_4 = f_5x_5 = e$. Hence $x_4 = x_5 = \sqrt{\alpha}G$. This shows that x is the neutral element of $\sigma\rho G$, i.e., θ is injective. It remains to show that θ is surjective. Let $G \in \mathcal{G}$ and $s \in G$. We need to find an $x \in \sigma\rho G$ such that $s = \theta x$. Let $s_1 = [a_2,s]$, $s_2 = [s,a_1]$. By (A1) there are $t_i \in G_i$, $i = 1,2$, such that $s_1 \in [a_2,t_1]\alpha G$ and $s_2 \in [t_2,a_1]\alpha G$. Thus $s_1,s_2 \in \beta G$ and f_1s_1, f_2s_2 are defined. Let $s_3 = (f_1s_1f_2s_2)^{-1}s$. Then $s_3 \in \beta G$. Indeed, $[a_1,s_3] = [a_1,f_2s_2^{-1}f_1s_1^{-1}s] \in [a_1,f_2s_2^{-1}][a_1,s]\alpha G = [a_1,f_2s_2]^{-1}[a_1,s]\alpha G = s_2s_2^{-1}\alpha G = \alpha G$. Similarly, $[a_2,s_3] \in \alpha G$. By (A3) $s_3 \in \beta G$. Let $z = (f_3s_3)^{-1}s_3$. Then $z \in \alpha G$. Hence there are $u,v \in G_1$ such that $z = [a_2,u,a_1][a_2,v,a_2] = f_4[a_2,u]f_5[a_2,v] = f_4s_4f_5s_5$. Thus $s = \theta x$ where $x_i = s_i\alpha G$ if $i = 1,2,3$ and $x_j = s_j\sqrt{\alpha}G$ if $j = 4,5$. Thus θ is surjective, which concludes the proof of Theorem 1.

8

Corollary 1 *If $G \in \mathcal{G}$, then G_1, G_2 are abelian subgroups of G, $G_1 \cap G_2 = \zeta G = \{x \in \beta G : x^2 \in \alpha G\}$, $\beta G = \gamma_2 G$, and $\alpha G = \gamma_3 G$.*

This follows from the Claim of the proof of Theorem 1 and the fact that $G \cong \sigma\rho G$.

Corollary 2 *The ring of integers* **Z** *is syntactically isomorphic to the free nilpotent class 3-group on two free generators.*

Proof: Let F be the free nilpotent class 3-group with the two free generators a_1, a_2. Every element of F can be written uniquely as $x = a_1^s a_2^t a_3^u a_4^v a_5^w$ where $s,t,u,v,w \in \mathbf{Z}$, $a_3 = [a_2,a_1]$, $a_4 = [a_3,a_1]$, and $a_5 = [a_3,a_2]$. The mapping that sends the above element x to the element (s,t,u,v,w) of $\sigma\mathbf{Z}$ is an isomorphism of F onto $\sigma\mathbf{Z}$.

Since the elementary theory of the ring of integers **Z** is undecidable (cf. [7]), the elementary theory of F is also undecidable. Another proof of this is given in [3]. Also, since **Z** is syntactically isomorphic to the free nilpotent class 2-group with two free generators (cf. [3]), we conclude that the free nilpotent class 2-group on two free generators and the free nilpotent class 3-group on two free generators are syntactically isomorphic.

Corollary 3 *The class of all nontrivial nonassociative rings satisfying the identity $x = x^2$ is syntactically isomorphic to the class of all nilpotent class 2-groups G with elements a_1, a_2 and satisfying*

(B1) $G_1 \cap G_2 = \zeta G$
(B2) *there are homomorphisms $f_i : \zeta G \to G_i$ such that $f_i a_{21} = a_i$, $i = 1,2$, and for every $x \in \zeta G$, $[f_2x,a_1] = x = [a_2,f_1x]$*
(B3) *for every $x \in \zeta G$, $[f_2x,f_1x\} = x$.*

Proof: Let G be a group satisfying all the conditions of Corollary 3. Then $G \in \mathcal{G}$. Indeed, (A1) follows from (B2) since $\gamma_2 G \subseteq \zeta G$ for all nilpotent class 2-groups. Condition (A2) is trivial since $\alpha G \subseteq \gamma_3 G = \{e\}$. To show (A3), let $x \in G$ and $[a_i,x] = e$, $i = 1,2$. Then $x \in \zeta G$ by (B1) and $x = [a_2,f_1 x] \in \beta G$ by (B2). Thus $\beta G = \zeta G = [a_2, G_1] = [G_2, a_1] = [G_2, G_1]$. Conditions (A5)(a),(b),(c) follow from (B2). We need to show that conditions (A4), (A5)(d),(e),(f) hold in G. Since G satisfies the conditions of Lemma 4, $\rho G = \beta G/\alpha G$ is a ring satisfying the identity $x = x \times x$ by (B3). Thus every element of ζG is of order 2. Hence (A4) and (A5)(e) hold, and for every $x \in \zeta G$, $e \in \mathrm{T}x$. Hence the right hand side of (A5)(f) is $[a_2,f_1[f_2x,f_1y]] = [f_2x,f_1y]$ by (B2). This proves (A5)(f). Since $x \times x = x$ in ρG, ρG is a commutative nonassociative ring and so (A5)(d) follows. Thus $G \in \mathcal{G}$. Conversely, if R is a nonassociative ring satisfying $x = x^2$, then σR satisfies (B1), (B2), and (B3) since such an R belongs to $\mathcal{R}$ and $0/2 = R$. Thus σR is nilpotent of class 2.

There are infinitely many varieties of nonassociative Boolean algebras, i.e., nonassociative rings satisfying the identity $x = x^2$ (see [2]).

Corollary 4 *The class of all nontrivial Boolean algebras is syntactically isomorphic to the class of all nilpotent class* 2*-groups G with elements a_1, a_2 and satisfying* (B1), (B2), (B3), *and*

(B4) *for all* $x,y,z \in \zeta G$, $[f_2x,f_1[f_2y,f_1z]] = [f_2[f_2x,f_1y],f_1z]$.

Proof: This follows from Corollary 3 since a Boolean algebra is polynomially equivalent to a Boolean ring. A Boolean ring is an associative ring satisfying the identity $x = x^2$. Condition (B4) is equivalent to the associativity of multiplication in ρG.

From [6], the class of all Boolean algebras has a decidable elementary theory. Hence the class of groups in Corollary 4 has a decidable elementary theory.

From [8], a variety of associative rings has a decidable elementary theory iff it satisfies $x = x^n$ for some integer $n > 1$. All such rings belong to the class $\mathcal{R}$. Thus the corresponding classes of groups have decidable elementary theories.

It may be noted that the correspondences $R \to \sigma R$ and $G \to \rho G$ provide a bijective equivalence between the category of rings $\mathcal{R}$ with ring homomorphisms preserving the identity elements and the category of groups $\mathcal{G}$ with elements a_1, a_2 and homomorphisms preserving a_1, a_2. Thus σ, ρ preserve homomorphic images and Cartesian products. The algorithm given in Theorem 1 is uniform between the categories $\mathcal{G}$ and $\mathcal{R}$.

9 We shall now consider the special cases of rings of prime characteristic and algebras over fields in general.

Lemma 6 *Let m be a positive integer. Then the following conditions on a group $G \in \mathcal{G}$ are equivalent:*

1. *The element a_1 is of order m*
2. *The element a_2 is of order m*
3. *The element $[a_2, a_1]$ is of order m*
4. *The ring ρG is of characteristic m.*

Under these conditions, the following also hold: The exponent of G is $(6,m)m$; the groups $\gamma_2 G$, G_1, G_2 are each of exponent m; the exponent of $\gamma_3 G$ is $m/(2,m)$; the exponent of ζG is $m/2$ if m is divisible by 4 and is m otherwise.

Proof: Let $R \in \mathcal{R}$ and $x \in \sigma R$. By induction on the positive integer n, we can show that

$$(x^n)_i = nx_i,\ i = 1,2$$
$$(x^n)_3 = nx_3 + (n(n-1)/2)x_1x_2$$
$$(x^n)_4 = nx_4 + (n(n-1)/2)x_1x_3 + (n(n-1)(n-2)/6)x_1^2x_2 + (n(n-1)/2)x_2tx_1$$
$$(x^n)_5 = nx_5 + (n(n-1)/2)x_2x_3 + (n(n-1)(2n-1)/6)x_1x_2^2 + (n(n-1)/2)x_1tx_2.$$

It is clear that $a_1^k = (k,0,0,0,0)$ and a_1 is of order m iff R is of characteristic m. Also $a_{21}^k = (0,0,k,0,0)$ and a_{21} is of order m iff R is of characteristic m. Thus, for $G = \sigma R$, conditions 1,3,4 are equivalent. The equivalence of conditions 2,4 is similar. Since, by Theorem 1, for every $G \in \mathcal{G}$ $G \cong \sigma\rho G$, the first part of Lemma 6 is proved.

If R is of characteristic m, then $x^{(6,m)m} = e$ for all $x \in \sigma R$. The element a_1a_2 is of order $(6,m)m$. Indeed, $((a_1a_2)^{km})_3 = km(km-1)/2 = 0$ iff $(2,m)|k$; $((a_1a_2)^{km})_4 = km(km-1)(km-2)/6 + 0/2 = 0/2$ iff $(3,m)|k$; and $((a_1a_2)^{km})_5 = km(km-1)(2km-1)/2 + 0/2 = 0/2$ iff $(3,m)|k$. This shows that $G = \sigma R$ is of exponent $(6,m)m$. Since $G_1 = R \times 0 \times 0/2 \times R/(0/2) \times R/(0/2)$, for $x \in G_1$ we have $(x^m)_3 = mx_3 = 0$, $(x^m)_4 = mx_4 + (m(m-1)/2)x_1x_3 + 0/2 = 0/2$, and $(x^m)_5 = mx_5 + 0/2 = 0/2$. Thus $x^m = e$ for all $x \in G_1$. Since a_1 is of order m, G_1 is of exponent m. Similarly, G_2 is of exponent m. As $\gamma_2 G = 0 \times 0 \times R \times R/(0/2) \times R/(0/2)$, if $x \in \gamma_2 G$, then $x^m = e$. Hence $\gamma_2 G$ is of exponent m as a_{21} is of order m. That the exponent of $\gamma_3 G$ is $m/(2,m)$ can be proved similarly. Now $\zeta G = 0 \times 0 \times 0/2 \times R/(0/2) \times R/(0/2)$. If $x \in \zeta G$, then $x_1 = x_2 = 2x_3 = 0$. Thus $(x^k)_3 = kx_3$ and $(x^k)_j = kx_j + 0/2$ if $j = 4,5$. Thus $x^k = e$ iff $2|k$ and $m|2k$; i.e., $2|k$ and $m/(2,m)|k$. Thus the exponent of ζG is $m/2$ if $4|m$ and m otherwise.

In the case of rings of prime characteristic, the conditions on the groups $G \in \mathcal{G}$ can be rewritten without recourse to the mappings f_1, f_2. First, we consider the case of rings of characteristic 2.

Theorem 2 *The class of all nontrivial nonassociative rings satisfying the identity $x = x^2$ is syntactically isomorphic to the class of all nilpotent class 2-groups G with elements a_1, a_2 such that*

(C1) *G_1, G_2 are of exponent* 2
(C2) $G_1 \cap G_2 = \zeta G$
(C3) *for all $x \in \zeta G$, there are $x_i \in G_i$, $i = 1,2$, such that $[a_2,x_1] = [x_2,a_1] = [x_2,x_1] = x$.*

Theorem 3 *The class of all nontrivial Boolean algebras is syntactically isomorphic to the class of all nilpotent class 2-groups G with elements a_1, a_2 satisfying* (C1), (C2), (C3), *and*

(C4) *for all* $x,y,z \in \zeta G$, *there are* $y_1, t_1, z_1, v_1 \in G_1$, $x_2, y_2, v_2 \in G_2$ *such that* $[a_2,y_1] = [y_2,a_1] = y$, $[a_2,z_1] = z$, $[x_2,a_1] = x$, $[a_2,v_1] = [y_2,z_1]$, $[v_2,a_1] = [x_2,y_1]$, *and* $[x_2,v_1] = [v_2,z_1]$.

Now we consider the case of rings of odd prime characteristic.

Theorem 4 *Let p be an odd prime. Then the class of all nontrivial associative and commutative rings of characteristic p is syntactically isomorphic to the class of all nilpotent class* 3*-groups G with elements* a_1, a_2 *satisfying* (A1), (A2), (A3), *and* (D4)-(D9).

(D4) *For* $x \in \beta G$, *the following conditions are equivalent:*
(i) $x \in \alpha G$ (ii) $[a_1,x] = e$ (iii) $[a_2,x] = e$.
(D5) G_1, G_2 *are abelian of exponent p*
(D6) *if* $x \in \beta G$, *then* $x^p \in \alpha G$
(D7) *for all* $x,y \in \beta G$, *there are* $x_i \in \mathrm{U}_i x$, $y_i \in \mathrm{U}_i y$, $i = 1,2$, *such that* $[x_2,y_1][x_1,y_2] \in \alpha G$
(D8) *for all* $x,y,z \in \beta G$, *there are* $x_2 \in \mathrm{U}_2 x$, $y_i \in \mathrm{U}_i y$, $i = 1,2$, $z_1 \in \mathrm{U}_1 z$, $u \in \mathrm{U}_1[y_2,z_1]$, *and* $v \in \mathrm{U}_2[x_2,y_1]$ *such that* $[x_2,u][z_1,v] \in \alpha G$
(D9) *for all* $x,y \in \beta G$, *there are* $x_2 \in \mathrm{U}_2 x$, $y_1 \in \mathrm{U}_1 y$, $r \in \mathrm{U}_1[x_2,y_1]$, $s \in \mathrm{T}[x_2,y_1]$, $t \in \mathrm{T}x$, $u \in \mathrm{T}y$, $u_1 \in \mathrm{U}_1 u$, *and* $t_2 \in \mathrm{U}_2 t$ *such that* $[x_2,y_1] = [a_2,r][a_1,s][x_2,u_1,a_1][t_2,y_1,a_2]$.

(The class of groups defined in Theorem 4 is an elementary class. This is the reason for introducing αG and βG.)

Proof: If R is an associative and commutative ring of odd characteristic, then $R \in \mathfrak{R}$. We need to show that the groups satisfying the conditions of Theorem 4 belong to $\mathcal{G}$. Theorem 4 will then follow from Lemma 6 and Theorem 1 since $\beta G/\alpha G$ is of exponent p.

Condition (A4) is equivalent to (D4) since $\beta G/\alpha G$ is of exponent p (an odd prime) by (D6). Also, from (D5) and (D6), we can consider G_1, G_2, and $\beta G/\alpha G$ as vector spaces over the field of integers modulo p, since $\beta G \subseteq \gamma_2 G$ is abelian (by Lemma 1). By Lemma 3 (vii) the mappings $g_i : G_i \to \beta G/\alpha G$, $i = 1,2$, defined by $g_1 x = [a_2,x]\alpha G$, $g_2 x = [x,a_1]\alpha G$, are homomorphisms of abelian groups. These mappings can be considered as surjective linear transformations of the given vector spaces. Choose a basis of $\beta G/\alpha G$ containing $a_{21}\alpha G$. For every element b of this basis choose $x_i \in G_i$ such that $g_i x_i = b$ and $g_i a_i = a_{21}\alpha G$, $i = 1,2$. The mapping $b \to x_i$ can be extended to a linear transformation h_i of $\beta G/\alpha G$ into G_i; moreover, $g_i h_i$ is the identity mapping on $\beta G/\alpha G$, $i = 1,2$. The composition f_i of the natural homomorphism of βG onto $\beta G/\alpha G$ and h_i is a homomorphism of βG into G_i such that $f_i x \in \mathrm{U}_i x$, $f_i a_{21} = a_i$, $i = 1,2$, $x \in \beta G$. Thus G satisfies (A5)(a),(b),(c).

If $x \in \beta G$, $s,s' \in \mathrm{U}_1 x$, then $s^{-1}s' \in \alpha G$. Indeed, $[a_2,s^{-1}s'] \in [a_2,s]^{-1}[a_2,s']\alpha G = x^{-1}x\alpha G = \alpha G$. Furthermore, $[a_1,s^{-1}s'] = e \in \alpha G$. Thus $s^{-1}s' \in \beta G$ by (A3). Hence by (D4) $s^{-1}s' \in \alpha G$. Similarly, if $t,t' \in \mathrm{U}_2 x$, then $t^{-1}t' \in \alpha G$. Let $x,y \in \beta G$. By (D7), there are $x_i \in \mathrm{U}_i x$, $y_i \in \mathrm{U}_i y$, $i = 1,2$, such that $[x_2,y_1][x_1,y_2] \in \alpha G$. But $[f_2 x, f_1 y] = [x_2 c, y_1 d]$ where $c,d \in \alpha G$. Thus $[f_2 x, f_1 y] = [x_2,y_1]$. Similarly $[f_1 x, f_2 y] = [x_1,y_2]$, and so (A5)(d) holds. By the same argument we can show that (D8) implies (A5)(e). Thus the operation $\times$ on

ρG is associative and commutative. From condition (D9) we have $[x_1,y_1] = [f_2x,f_1y]$, $[a_1,r] = [a_2,f_1[f_2x,f_1y]]$, $[x_2,u_1,a_1] = [f_2x,f_1u,a_1]$, $[t_2,y_1, a_2] = [f_2t,f_1y,a_2]$. Also $\mathrm{T}[x_2,y_1] = \mathrm{T}[f_2x,f_1y]$. Thus G satisfies (A5)(f) and $G \in \mathcal{G}$.

The proof of Theorem 2 follows from a similar argument. Theorem 3 follows from Theorem 2 since a Boolean algebra is polynomially equivalent to a Boolean ring, and condition (C4) states that the ring ρG is associative.

The class of all nontrivial associative and commutative rings of characteristic 3 corresponds, by Theorem 4, to a class of nilpotent class 3-groups of exponent 9. The class of all nontrivial associative and commutative rings of characteristic p, where p is a prime greater than 3, corresponds via Theorem 4 to a class of nilpotent class 3-groups of exponent p. This follows from Lemma 6. Since the class of associative and commutative rings of characteristic p contains the ring of polynomials over the field of integers modulo p, such classes have undecidable elementary theories (cf. [5]). Thus, for every odd prime p, the class of groups described in Theorem 4 has an undecidable elementary theory.

10 Now we consider algebras over fields of characteristic not 2 or 3. Let F be a field. A group G is called an F-group if the elements of F act as operators on G such that $x^a x^b = x^{a+b}$, $(x^a)^b = x^{ab}$ for all $x \in G$, $a,b \in F$. For algebras over fields we have the following:

Theorem 5 *Let F be a field of characteristic not belonging to* $\{2,3\}$. *Then the class of all nontrivial associative and commutative algebras over F is syntactically isomorphic to the class of all nilpotent class 3-F-groups G with elements a_1,a_2 satisfying* (A1), (A2), (A3), (D4), (D7), (D8), (D9) *and*

(E1) *for all $x,y \in G$, $a,b \in F$,*
$[x^a,y^b] = [x,y]^{ab}[x,y,x]^{a(a-1)b/2}[x,y,y]^{ab(b-1)/2}$

(E2) *for all $(x,y) \in (G_1 \times G_1) \cup (G_2 \times G_2) \cup (\beta G \times \beta G)$ and $a \in F$, $(xy)^a = x^a y^a$*

(E3) *if F is of characteristic* 0, *then αG is closed under F.*

Proof: Let G satisfy the conditions of Theorem 5. First, G_1 and G_2 are F-groups. Indeed, let $x \in G_i$, $a \in F$. Then by (E1), $[a_i,x^a] = [a_i,x]^a[a_i,x,a_i]^0 [a_i,x,x]^{a(a-1)/2} = e$, i.e., $x^a \in G_i$.

If F is of prime characteristic p, then G satisfies $x^p = e$. Also, from (E2), $(xy)^2 = x^2y^2$ in G_1 and G_2. Thus G_1, G_2 are abelian and G satisfies conditions (D5) and (D6). Hence, by Theorem 4, ρG is an associative and commutative ring of characteristic p and the groups $\sigma\rho G$ and G are isomorphic. Hence $\alpha G = \zeta G = G_1 \cap G_2$. Thus αG is closed under F. From this and (E3) αG is an F-group regardless of the characteristic of F. We shall show that βG is also an F-subgroup of G. Let $x \in G_1$, $a \in F$. Then $[a_2,x^a] = [a_2,x]^a[a_2,x,x]^{a(a-1)/2}$. Hence $[a_2,x]^a \in [a_2,x^a]\alpha G$ since αG is F-closed. But G_1 is also F-closed, and thus $[a_2,x]^a \in \beta G$. Hence, by (E2), βG is closed under F. Also due to (E2), G_1, G_2, βG, and αG can be considered as vector spaces over F. Since $[a_2,x^a] \in [a_2,x]^a\alpha G$ for every $x \in G_1$, the mapping $x \to [a_2,x]\alpha G$ is a linear transformation of the vector space G_1 onto the vector space $\beta G/\alpha G$. The proof after this point follows the argument of the proof of Theorem 4. The ring ρG inherits the F-algebra structure: If $x, y \in \beta G$, $x_2 \in U_2 x$, $y_1 \in U_1 y$, and $a \in F$, then $a\cdot((x\alpha G) \times (y\alpha G)) = [x_2, y_1]^a \alpha G = [x_2, y_1^a]\alpha G = (x\alpha G) \times (a\cdot(y\alpha G))$ since $[a_2, y_1^a] \in [a_2, y_1]^a \alpha G = y^a \alpha G$; i.e., $y_1^a \in U_1 y^a$.

The group σR is an F-group whenever R is an associative and commutative F-algebra. The action of F on σR is defined as follows: if $x \in \sigma R$ and $a \in F$, then

$$\begin{aligned}
(x^a)_i &= a\cdot x_i,\ i = 1,2\\
(x^a)_3 &= a\cdot x_3 + (a(a-1)/2)\cdot x_1 x_2\\
(x^a)_4 &= a\cdot x_4 + (a(a-1)/2)\cdot x_1 x_3 + (a(a-1)(a-2)/6)\cdot x_1^2 x_2\\
&\quad + (a(a-1)/4)\cdot x_2 x_1 (x_1 - 1)\\
(x^a)_5 &= a\cdot x_5 + (a(a-1)/2)\cdot x_2 x_3 + (a(a-1)(2a-1)/6)\cdot x_1 x_2^2\\
&\quad + (a(a-1)/4)\cdot x_1 x_2 (x_2 - 1).
\end{aligned}$$

The F-group σR satisfies all the conditions of Theorem 5. Furthermore, the isomorphism $G \cong \sigma\rho G$ is an isomorphism of F-groups with constants a_1, a_2 and the isomorphism $R \cong \rho\sigma R$ is an isomorphism of F-algebras.

The construction of $\sigma\Phi$ and $\rho\Psi$, where Φ is a first-order sentence in the language of F-groups and Ψ is a first-order sentence in the language of algebras over F, is the same as that in the proof of Theorem 1 with the following additions:

- every atomic formula in Ψ of the form $a\cdot x \approx z$ is replaced by $(B(x) \wedge B(z)) \rightarrow A(z^{-1}x^a)$;
- every atomic formula in Φ of the form $z \approx x^a$ is replaced by the conjunction of $z_1 \approx a\cdot x_1$, $z_2 \approx a\cdot x_2$, $z_3 \approx a\cdot x_3 + (a(a-1)/2)\cdot x_1 x_2$, $z_4 \approx a\cdot x_4 + (a(a-1)/2)\cdot x_1 x_3 + (a(a-1)(a-2)/6)\cdot x_1^2 x_2 + (a(a-1)/4)\cdot x_2 x_1 (x_1 - 1)$, and $z_5 \approx a\cdot x_5 + (a(a-1)/2)\cdot x_2 x_3 + (a(a-1)(2a-1)/6)\cdot x_1 x_2^2 + (a(a-1)/4)\cdot x_1 x_2 (x_2 - 1)$.

11 In conclusion, let $F(k,n)$ be the free nilpotent class k-group on n free generators. As we mentioned above, the groups $F(2,2)$ and $F(3,2)$ are syntactically isomorphic. It is our opinion that by using methods similar to those of [3] and to those of the present paper, one can show that the groups $F(j,m)$ and $F(k,n)$ are syntactically isomorphic for any $j,k,m,n \geq 2$.

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