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Paraconsistency and the C-Systems of da Costa

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Abstract It is argued that, both in view of da Costa's conditions for paraconsistent systems and for independent reasons, the failure of the C-systems to enjoy the property of intersubstitutivity of provable equivalents constitutes a deficiency which it is reasonable to attempt to remedy. Two extensions of the C-systems are considered, and both are shown to collapse all but the base system C_ω into classical logic. The general result is established that there is no extension of the stronger C-systems which both enjoys the intersubstitutivity property and is weaker than classical logic. Methods of constructing alternative but related hierarchies for which this property might more successfully be secured are suggested.

1 The C-systems and the paraconsistency conditions The systems $C_n (1 \leq n \leq \omega)$ are among the best-known contributions of da Costa and his collaborators to the program of constructing paraconsistent logics, i.e., logics capable of supporting inconsistent theories without collapse into triviality.

In [5], da Costa and Alves state that, in general, systems of paraconsistent logic must satisfy the following conditions:

(I) From two contradictory formulas A and $\neg A$, it must not be possible in general to deduce an arbitrary formula B
(II) Such systems should contain most of the schemata and deduction rules of classical logic that do not interfere with (I).

In considering the C-systems specifically, da Costa states in [4] that "it seems natural that they satisfy" not only (I) and (II), but also the following:

(III) In these systems, the schema $\neg(A \;\&\; \neg A)$ must not be derivable
(IV) It must be simple to extend the systems to first-order predicate calculi (with or without equality).

Of these four conditions, only two are beyond controversy: (I) is unanimously accepted as a necessary condition for paraconsistent systems, and (IV) is also uncontroversial, if only because paraconsistency researchers have stan-

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dardly sought to guarantee the stability of their systems under the burden of inconsistency at the propositional level, rather than by tampering with the usual (classical) rules governing the quantifiers.

Conditions (II) and (III), on the other hand, have variously been either rejected outright (see, for example, [3] and [9]), or modified (see [13]). Certainly, the construction of paraconsistent logics incorporating $\neg(A \mathbin{\&} \neg A)$ is not unusual, even in Brazilian circles, as the J-systems of Arruda and da Costa testify (see [2]). And the extensive research into relevant logics represents a significant departure from the view that classical logic is an ideal to be approximated, insofar as paraconsistency considerations allow (see [1] or the more recent [11]).

Even if all four conditions are accepted, however, it is not absolutely clear that they are all satisfied by the C-systems. Conditions (I) and (III) present no problem, for $\neg(A \mathbin{\&} \neg A)$ is indeed not derivable in $C_n (1 \leq n \leq \omega)$, nor is an arbitrary proposition B derivable from contradictory formulas A and $\neg A$. And the first-order extension of the systems, as described in [4], is evidently simple enough to satisfy (IV). Rather, it is with respect to (II) that some room for doubt emerges.

One of the problems in assessing whether the C-systems satisfy (II) is that, as da Costa himself notes in [4], this condition is somewhat vague. Certainly, it is not exactly clear what measure of containment of classical schemata and rules constitutes satisfaction of (II). Nor is the condition determinative in the construction of paraconsistent systems, for it suggests no means of deciding between candidate schemata or rules which could singly but not jointly be added to a base system without interfering with (I).

Notwithstanding these and other reservations about (II) expressed in [13], the tenor of this condition is clear enough. But there is one important point of divergence between the C-systems and classical logic which provides a ground for concern, both in light of (II) and for independent reasons. For unlike classical logic, and for that matter, virtually every other reasonably familiar system of logic, the C-systems do not enjoy the property of intersubstitutivity of provable equivalents (see [6], Corollary to Theorem 1). This property at the very least appears to be required for the systematic behavior of the connectives in a logic; its absence from the C-systems is held in [8] and [12] to be responsible for the difficulty in obtaining natural and elegant algebraic and semantic perspectives on these systems. And, of course, the absence of so apparently central a property—or more precisely, of the rules or schemata required to guarantee this property—may well indicate one respect in which the C-systems do not adequately meet condition (II).

In the following section, we investigate more precisely the failure of the C-systems to enjoy the property of intersubstitutivity of provable equivalents, and whether these systems can be extended so as to secure this property.

2 The C-systems and the property of intersubstitutivity of provable equivalents The postulates (axiom schemata and rule) of the base system, C_ω, are as follows:

(1) $A \supset (B \supset A)$
(2) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$

(3) $\dfrac{A \qquad A \supset B}{B}$

(4) $(A \,\&\, B) \supset A$

(5) $(A \,\&\, B) \supset B$

(6) $A \supset (B \supset (A \,\&\, B))$

(7) $A \supset (A \vee B)$

(8) $B \supset (A \vee B)$

(9) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$

(10) $A \vee \neg A$

(11) $\neg\neg A \supset A$.

Postulates (1) to (9) axiomatize positive intuitionistic logic. From this basis, C_ω is neatly constructed by adding (10) and (11), rather than by adding their intuitionistic "duals", $\neg(A \,\&\, \neg A)$ and $A \supset \neg\neg A$.

We note that, by Theorem 1 of [6], sufficient to collapse C_ω into C_0 (classical logic) is the addition of the *reductio* schema, $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$. The remaining systems $C_n (1 \leq n < \omega)$ extend C_ω by adding this schema, though in a qualified fashion rather than *simpliciter*. For each system, the schema is qualified by a formula which can be interpreted as expressing the proposition that B is not paradoxical, or "behaves classically". For C_1, the qualification is the formula B°, which is defined as $\neg(B \,\&\, \neg B)$. In addition, compounding principles ensure that compounds of "classical" formulas are themselves "classical".

The postulates of C_1 are those of C_ω together with the following:

(12)° $B^\circ \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$

(13)° $(A^\circ \,\&\, B^\circ) \supset (A \,\&\, B)^\circ$

(14)° $(A^\circ \,\&\, B^\circ) \supset (A \vee B)^\circ$

(15)° $(A^\circ \,\&\, B^\circ) \supset (A \supset B)^\circ$

(16)° $B^\circ \supset (\neg B)^\circ$.

For each remaining $C_n (1 \leq n < \omega)$, B° is replaced by $B^{(n)}$, which is the conjunction $B^n \,\&\, B^{n-1} \,\&\ldots\&\, B^1$, where $B^1 = B^\circ$ and $B^i = B^{0 \ldots 0}_{i\,times}$. (For completeness, $B^{(1)}$ is defined to be B^1.)

The postulates of $C_n (1 < n < \omega)$ are those of C_1, except that (12)° to (16)° are replaced by the following:

(12)$^{(n)}$ $B^{(n)} \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$

(13)$^{(n)}$ $(A^{(n)} \,\&\, B^{(n)}) \supset (A \,\&\, B)^{(n)}$

(14)$^{(n)}$ $(A^{(n)} \,\&\, B^{(n)}) \supset (A \vee B)^{(n)}$

(15)$^{(n)}$ $(A^{(n)} \,\&\, B^{(n)}) \supset (A \supset B)^{(n)}$

(16)$^{(n)}$ $B^{(n)} \supset (\neg B)^{(n)}$.

By Theorem 9 of [4], the systems $C_0, C_1, C_2, \ldots, C_\omega$ are all distinct and form a linear hierarchy with strongest member C_0 and weakest member C_ω.

We now turn to the property of intersubstitutivity of provable equivalents (hereafter, SE). For axiomatic systems like the C-systems, it is most natural to define two formulas B and C to be *provably equivalent* just in case $(B \supset C) \,\&\, (C \supset B)$, abbreviated by $B \equiv C$, is derivable. (We note that this is not the only sort of equivalence definable in these systems; other definitions are given in [8].)

SE then is the property that, where B is a subformula of a theorem A, and C is a formula which is provably equivalent to B, the result of substituting C for some or all occurrences of B in A is also a theorem.

Because the C-systems incorporate positive intuitionistic logic, they at least enjoy the property of intersubstitutivity of provable equivalents in negation-free contexts (SE^+). It is where negation is involved that intersubstitutivity fails, as the following result shows.

Theorem 1 *The systems* $C_n (1 \leq n \leq \omega)$ *do not enjoy* SE.

Proof: Easily derived in C_ω, and therefore in each $C_n (1 \leq n \leq \omega)$, are the schemata $A \equiv (A \,\&\, A)$ and $\neg A \supset \neg A$. If $C_n (1 \leq n \leq \omega)$ enjoyed SE, then $\neg A \supset \neg (A \,\&\, A)$, the result of substituting $A \,\&\, A$ for (one occurrence of) A in $\neg A \supset \neg A$, would also be derivable. However, the following matrices show that $\neg A \supset \neg (A \,\&\, A)$ is not derivable in $C_n (1 \leq n \leq \omega)$, for they validate the postulates of these systems, but invalidate this schema when A is assigned the value 1.

$\supset$	0	1	2	3	4	$\neg$
*0	0	1	2	3	4	4
*1	0	0	2	3	4	3
2	0	0	0	3	3	3
3	0	0	2	0	2	2
4	0	0	0	0	0	0

&	0	1	2	3	4
0	0	1	2	3	4
1	1	0	2	3	4
2	2	2	2	4	4
3	3	3	4	3	4
4	4	4	4	4	4

$\vee$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	2	0	2
3	0	1	0	3	3
4	0	1	2	3	4

(Only the values 0 and 1 are designated.)

A much sharper result can be obtained for C_ω:

Theorem 2 *In* C_ω, *the schema* $\neg B \equiv \neg C$ *is derivable if and only if* B *and* C *are the same formula.*

Proof: The proof proceeds by considering the Gentzen-style system WG_ω of [10], which is stronger than C_ω in that, while for every formula $A \supset B$ derivable in C_ω the sequent $A \to B$ is derivable in WG_ω, the converse does not hold. (The terminology used in this proof is that of [7].)

We note first that the inference figure Cut is proved in [10] to be eliminable from WG_ω, and that the system is shown to be not finitely trivializable, i.e., there is no formula B such that $B \to C$ is derivable for an arbitrary formula C. From this it follows that WG_ω has no derivable sequent of the form $\to \neg B$. For WG_ω has no initial sequents of this form; hence, such a sequent could only be derived by the inference figure $\neg$-IS from the sequent $B \to$. But from the latter $B \to C$ follows by Thinning in the succedent for an arbitrary formula C, contradicting the fact that WG_ω is not finitely trivializable. Hence, WG_ω has no derivable sequent of the form $\to \neg B$.

We now show by induction on the length of derivation in WG_ω that, if $\Gamma \to \neg B$ is a derivable sequent, then $\neg B$ is a (possibly improper) subformula of some member of Γ.

Base case: In this case, $\Gamma \to \neg B$ is an initial sequent of the form $\neg B \to \neg B$. Of course, $\neg B$ is a subformula of itself.

Inductive step: In this case, $\Gamma \rightarrow \neg B$ is derived by application of some inference figure. The only candidates are the structural figures (Thinning, Contraction, and Interchange), the -IA figures ($\supset$-IA, &-IA, and $\vee$-IA), and $\neg$-IS.

However, $\neg$-IS is not a possibility. For if $\Gamma \rightarrow \neg B$ were derived from B, $\Gamma \rightarrow$ by $\neg$-IS, then the sequent B & $(\&\Gamma) \rightarrow$ would also be derivable (where $(\&\Gamma)$ represents the conjunction of all of the members of Γ), and hence so would B & $(\&\Gamma) \rightarrow C$ for arbitrary C by Thinning, contradicting the fact that WG_ω is not finitely trivializable. A similar argument shows that $\Gamma \rightarrow \neg B$ cannot be the result of Thinning in the succedent.

This leaves only those figures in which the principal formula occurs in the antecedent. We consider only Thinning and $\supset$-IA; the remaining figures can be dealt with similarly.

If $\Gamma \rightarrow \neg B$ is derived by Thinning in the antecedent, then Γ is a sequence of the form C, Γ', and the upper sequent of the figure is $\Gamma' \rightarrow \neg B$. The observation that $\rightarrow \neg B$ cannot be derived in WG_ω ensures that Γ' is not empty. By the inductive hypothesis, then, $\neg B$ is a subformula of some member of Γ', and hence also of some member of Γ.

If $\Gamma \rightarrow \neg B$ is derived by $\supset$-IA, then Γ is a sequence of the form $C \supset D, \Gamma_1, \Gamma_2$, and the upper sequents of the figure are $\Gamma_1 \rightarrow C$ and $D, \Gamma_2 \rightarrow \neg B$. By the inductive hypothesis, $\neg B$ is a subformula of D or of some member of Γ_2. But then it must also be a subformula of some member of Γ.

We have shown, then, that for every sequent of the form $\Gamma \rightarrow \neg B$ derivable in WG_ω, $\neg B$ must be a subformula of some member of Γ. To complete the proof of Theorem 2 it suffices to note that, if $\neg B \equiv \neg C$ is a theorem of C_ω, then $\neg B \rightarrow \neg C$ and $\neg C \rightarrow \neg B$ are both derivable sequents in WG_ω, and therefore $\neg B$ and $\neg C$ must each be a subformula of the other, from which it follows that they are in fact the same formula.

Theorem 2 shows that, despite the incorporation of postulates (10) and (11), C_ω is a very weak system with respect to negation: no two (different) negated formulas are provably equivalent in this system. We turn now to the stronger C-systems. For simplicity, we initially restrict our attention to C_1.

Particular interest attaches to the question of which formulas are provably equivalent in C_1 to the schema B°, because of the special role which this formula plays. It might be expected that such trivial variants as $\neg(\neg B \;\&\; B)$ or $\neg((B \;\&\; \neg B) \;\&\; (B \;\&\; \neg B))$ could be proved equivalent in C_1 to B°, and hence equally capable of expressing the proposition that B "behaves classically". But in the absence of SE there is no guarantee of a uniform argument to this effect, for although $(B \;\&\; \neg B) \equiv (\neg B \;\&\; B)$ and $(B \;\&\; \neg B) \equiv ((B \;\&\; \neg B) \;\&\; (B \;\&\; \neg B))$ are easily derived in C_1, it does not follow that $\neg(B \;\&\; \neg B) \equiv \neg(\neg B \;\&\; B)$ and $\neg(B \;\&\; \neg B) \equiv \neg((B \;\&\; \neg B) \;\&\; (B \;\&\; \neg B))$ are also derivable. These cases must therefore be considered individually.

A first result is promising:

Theorem 3 *In* C_1, *the schema* $\neg(B \;\&\; \neg B) \equiv \neg((B \;\&\; \neg B) \;\&\; (B \;\&\; \neg B))$ *is derivable.*

Note In this and subsequent proofs we will make use of the following rules and schemata, easily shown to be derivable in positive intuitionistic logic and therefore in all of the C-systems.

Transitivity: $\dfrac{C \supset D \quad D \supset E}{C \supset E}$

Permutation of antecedents: $\dfrac{C \supset (D \supset E)}{D \supset (C \supset E)}$

Importation: $\dfrac{C \supset (D \supset E)}{(C \,\&\, D) \supset E}$

Exportation: $\dfrac{(C \,\&\, D) \supset E}{C \supset (D \supset E)}$

Prefixing: $(D \supset E) \supset ((C \supset D) \supset (C \supset E))$

Suffixing: $(C \supset D) \supset ((D \supset E) \supset (C \supset E))$

Distribution: $(C \,\&\, (D \vee E)) \equiv ((C \,\&\, D) \vee (C \,\&\, E))$
$(C \vee (D \,\&\, E)) \equiv ((C \vee D) \,\&\, (C \vee E))$.

In addition, we will use the following schemata, easily shown to be derivable in C_ω with the assistance of (9) and (10):

C_ω*-reductio:* $(C \supset \neg C) \supset \neg C$
$(\neg C \supset C) \supset C$.

Proof of Theorem 3: The derivation of $\neg(B \,\&\, \neg B) \supset \neg((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B))$ is as follows. By postulate (4), we have both

$$((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B)) \supset (B \,\&\, \neg B) \text{ and } (B \,\&\, \neg B) \supset B,$$

from which follows, by transitivity,

$$((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B)) \supset B.$$

A similar argument, using (5) also, yields

$$((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B)) \supset \neg B.$$

Substituting $((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B))$ for A in (12)° and permuting antecedents yields

$$(((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B)) \supset B) \supset ((((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B)) \supset \neg B) \supset (B^\circ \supset \neg((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B)))).$$

But the first two antecedents have been shown to be derivable, hence by two applications of (3) we obtain the desired

$$B^\circ \supset \neg((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B)).$$

The converse derivation is as follows. We show first that $(B \,\&\, \neg B)^\circ$ is derivable in C_1. Substituting $(B \,\&\, \neg B) \,\&\, \neg(B \,\&\, \neg B)$ for A in (12)° and permuting antecedents yields

$$(((B \,\&\, \neg B) \,\&\, \neg(B \,\&\, \neg B)) \supset B) \supset (((B \,\&\, \neg B) \,\&\, \neg(B \,\&\, \neg B)) \supset \neg B) \supset (B^\circ \supset \neg((B \,\&\, \neg B) \,\&\, \neg(B \,\&\, \neg B))).$$

The first two antecedents are easily derived, leaving

$$B^\circ \supset \neg((B \,\&\, \neg B) \,\&\, \neg(B \,\&\, \neg B)).$$

But an instance of (5) is

$$((B \,\&\, \neg B) \,\&\, \neg(B \,\&\, \neg B)) \supset B^\circ,$$

hence by transitivity we obtain

$$((B \,\&\, \neg B) \,\&\, \neg(B \,\&\, \neg B)) \supset \neg((B \,\&\, \neg B) \,\&\, \neg(B \,\&\, \neg B)).$$

From this follows, by C_ω-*reductio* and (3),

$$\neg((B \,\&\, \neg B) \,\&\, \neg(B \,\&\, \neg B)),$$

which is by definition $(B \,\&\, \neg B)^\circ$.

To continue, it follows straightforwardly from the above result and $(13)^\circ$ that $((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B))^\circ$ is also derivable in C_1. We now consider the result of substituting $B \,\&\, \neg B$ and $(B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B)$ for, respectively, A and B in $(12)^\circ$. As just observed, the first antecedent is derivable in C_1, leaving

$$((B \,\&\, \neg B) \supset ((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B))) \supset$$
$$(((B \,\&\, \neg B) \supset \neg((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B))) \supset \neg(B \,\&\, \neg B)).$$

The antecedent of the above schema is also easily derived, using (6) and (2), leaving

$$((B \,\&\, \neg B) \supset \neg((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B))) \supset \neg(B \,\&\, \neg B).$$

But an instance of (1) is

$$\neg((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B)) \supset ((B \,\&\, \neg B) \supset \neg((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B))),$$

whence by transitivity we obtain the desired

$$\neg((B \,\&\, \neg B) \,\&\, (B \,\&\, \neg B)) \supset \neg(B \,\&\, \neg B).$$

We have shown, then, that one of the formulas put forward as a trivial variant of B° is in fact provably equivalent to it in C_1. Unfortunately, the same cannot be said for the second formula, $\neg(\neg B \,\&\, B)$.

Theorem 4 *In C_1, the schema $\neg(B \,\&\, \neg B) \equiv \neg(\neg B \,\&\, B)$ is not derivable.*

Proof: One half of this schema, $\neg(B \,\&\, \neg B) \supset \neg(\neg B \,\&\, B)$ is in fact derivable in C_1 as follows. Instances of postulates (5) and (4) respectively are $(\neg B \,\&\, B) \supset B$ and $(\neg B \,\&\, B) \supset \neg B$. Substituting $\neg B \,\&\, B$ for A in $(12)^\circ$ and permuting antecedents yields $((\neg B \,\&\, B) \supset B) \supset (((\neg B \,\&\, B) \supset \neg B) \supset (B^\circ \supset \neg(\neg B \,\&\, B)))$. But the first two antecedents have been shown to be derivable; hence by two applications of (3), we obtain $B^\circ \supset \neg(\neg B \,\&\, B)$, which is just the desired $\neg(B \,\&\, \neg B) \supset \neg(\neg B \,\&\, B)$.

The converse, however, is not derivable. This is shown by the following matrices, which validate the postulates of C_1 but invalidate this schema when B is assigned the value 1.

$\supset$	0	1	2	3	4	5	$\neg$
*0	0	0	0	0	0	5	5
*1	0	0	0	0	0	5	2
*2	0	0	0	0	0	5	3
*3	0	0	0	0	0	5	4
*4	0	0	0	0	0	5	5
5	0	0	0	0	0	0	4

&	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	4	3	4	5
2	2	3	2	4	4	5
3	3	3	3	3	4	5
4	4	4	4	4	4	5
5	5	5	5	5	5	5

$\vee$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	0	1	1	1
2	0	0	2	0	2	2
3	0	1	2	3	3	3
4	0	1	2	3	4	4
5	0	1	2	3	4	5

(Only the value 5 is not designated.)

That $\neg(B \,\&\, \neg B)$ and $\neg(\neg B \,\&\, B)$ are not provably equivalent in C_1 is certainly curious, if not anomalous. For, as is argued with respect to the similarly deficient system J_1 in [13], such fine discrimination between what would ordinarily be regarded as mere syntactic variants demands some sort of justification. But the motivating considerations for C_1, namely, conditions (I) to (IV), not only do not support such discrimination but militate against it. For although $\neg(B \,\&\, \neg B) \equiv \neg(\neg B \,\&\, B)$ is not derivable in C_1, the matrices in the proof of Theorem 1 show that this schema can be added to C_1 without compromising its satisfaction of condition (I). For the C-systems, satisfaction of (I) is equivalent to the underivability of the schema $(A \,\&\, \neg A) \supset B$. But the matrices in question invalidate this schema (when A is assigned the value 1 and B is assigned the value 2, for example) while at the same time validating not only the postulates of C_1 but also $\neg(B \,\&\, \neg B) \equiv \neg(\neg B \,\&\, B)$.

The absence of the above schema from C_1, then, presents itself as not only anomalous in its own right, but in contravention of at least the spirit of condition (II). These considerations univocally suggest that C_1 should be extended to include this schema.

Of course, it is not to be expected that the mere addition of $\neg(B \,\&\, \neg B) \equiv \neg(\neg B \,\&\, B)$ will remedy any other deficiencies of C_1. For, as Theorems 3 and 4 show, the presence of a schema stating the equivalence of B° to one syntactic variant is not sufficient to guarantee its equivalence to any other. And even if sufficiently many such schemata could be added to secure the equivalence of B° to all such variants there is no reason to believe that the deficiencies exhibited in Theorem 1 would not remain, for it is shown there that, in general, $\neg A \equiv \neg(A \,\&\, A)$ is not derivable in C_1, notwithstanding that the particular instance of this schema obtained by substituting $B \,\&\, \neg B$ for A is shown to be derivable in Theorem 3.

Rather than attempting to treat individually the symptoms of the failure of the C-systems to enjoy SE, a more obviously systematic strategy is to attempt to secure this property directly. In [12], two methods of extending the C-systems in order to secure SE are proposed:

(i) addition of the rule RC: $\dfrac{C \supset D}{\neg D \supset \neg C}$

(ii) addition of the (weaker) rule EC: $\dfrac{C \equiv D}{\neg D \supset \neg C}$.

Because the C-systems enjoy SE^+, and lack only the property of intersubstitutivity of provable equivalents in negated contexts, it is evident that the addition of either RC or EC is sufficient to guarantee SE in full. Moreover, the admissibility of EC (in any extension of a C-system) is also a necessary condition for SE.

3 The RC-systems We first investigate the result of adding RC to the C-systems. For each $C_n (1 \leq n \leq \omega)$, the result of adding RC will be called RC_n. (We note that this diverges from the nomenclature of [12], in which the resulting systems are called $CC_n (1 \leq n \leq \omega)$.)

The following initial result and its proof are taken directly from [12]:

Theorem 5 $RC_\omega \neq C_0$ *(classical logic). In particular, the schema* $(A \mathbin{\&} \neg A) \supset B$ *is not derivable in* RC_ω.

Proof: The following matrices validate the postulates of RC_ω but invalidate the above schema when A is assigned the value 1 and B is assigned the value 2.

$\supset$	0	1	2	$\neg$
*0	0	1	2	2
1	0	0	2	0
2	0	0	0	0

&	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\vee$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

(Only the value 0 is designated.)

Thus, the addition of RC to C_ω does not result in any compromise of condition (I), and the system so obtained certainly does not suffer from the deficiencies exhibited in Theorem 2. Unfortunately, the same is not true of the stronger C-systems. The following result is proved in [12], but we employ a rather simpler proof below.

Theorem 6 *For* $1 \leq n < \omega$, $RC_n = C_0$.

Proof: It suffices to show that, for any $1 \leq n < \omega$, the formula $B^{(n)}$ is derivable in RC_ω. Since this is a subsystem of each $RC_n (1 \leq n < \omega)$, it follows that, in each such RC_n, the formula qualifying the *reductio* schema in $(12)^{(n)}$ is derivable, and hence so is unqualified *reductio*. As noted earlier this suffices to collapse C_ω, and therefore every $RC_n (1 \leq n < \omega)$, into classical logic.

We first show that B° is derivable in RC_ω for any formula B. An instance of postulate (4) is $(B \mathbin{\&} \neg B) \supset B$. Applying RC yields $\neg B \supset \neg(B \mathbin{\&} \neg B)$. Similarly, (5) and RC yield $\neg\neg B \supset \neg(B \mathbin{\&} \neg B)$. These, together with (9), yield $(\neg B \vee \neg\neg B) \supset \neg(B \mathbin{\&} \neg B)$. But the antecedent is an instance of (10), so (3) yields $\neg(B \mathbin{\&} \neg B)$, which is B° by definition.

From this it follows straightforwardly that B^n is derivable in RC_ω for $1 < n < \omega$, since each such B^n is itself of the form $(B^{n-1})^\circ$. A simple inductive argument then shows that the conjunction $B^n \mathbin{\&} B^{n-1} \mathbin{\&} \ldots \mathbin{\&} B^\circ$, which is by definition $B^{(n)}$, is also derivable.

Thus, the addition of RC to the C-systems collapses all but the weakest system, C_ω, into classical logic. We turn instead to EC, in the hope that the addition of this rule will not have such drastic consequences.

4 The EC-systems For each $C_n(1 \leq n \leq \omega)$, the result of adding EC is called EC_n. (This is as in [12].)

Theorem 7 $EC_\omega \neq C_0$.

Proof: This follows from Theorem 5 and the fact the EC is derivable from RC in any extension of C_ω; hence, EC_ω is a subsystem of RC_ω.

In fact, EC_ω is a proper subsystem of RC_ω.

Theorem 8 $EC_\omega \neq RC_\omega$.

Proof: The following matrices validate the postulates of EC_ω, but invalidate the schema $\neg(B \,\&\, \neg B)$, shown to be derivable in RC_ω in the proof of Theorem 6, when B is assigned the value 1.

$\supset$	0	1	2	3	4	5	6	7	$\neg$
*0	0	1	2	3	4	5	6	7	7
1	0	0	2	3	2	3	6	6	2
2	0	1	0	3	1	5	3	5	5
3	0	1	2	0	4	1	2	4	4
4	0	0	0	3	0	3	3	3	3
5	0	0	2	0	2	0	2	2	2
6	0	1	0	0	1	1	0	1	0
7	0	0	0	0	0	0	0	0	0

&	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	4	5	4	5	7	7
2	2	4	2	6	4	7	6	7
3	3	5	6	3	7	5	6	7
4	4	4	4	7	4	7	7	7
5	5	5	7	5	7	5	7	7
6	6	7	6	6	7	7	6	7
7	7	7	7	7	7	7	7	7

$\vee$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	0	1	1	0	1
2	0	0	2	0	2	0	2	2
3	0	0	0	3	0	3	3	3
4	0	1	2	0	4	1	2	4
5	0	1	0	3	1	5	3	5
6	0	0	2	3	2	3	6	6
7	0	1	2	3	4	5	6	7

(Only the value 0 is designated.)

Thus, EC_ω is distinct from RC_ω. We now determine whether $EC_n(1 \leq n < \omega)$ are distinct from $RC_n(1 \leq n < \omega)$. Several lemmas are required.

In the following lemmas, F_n is defined for each system $C_n(1 \leq n < \omega)$ to be the schema $B^{(n)} \,\&\, (B \,\&\, \neg B)$.

Lemma 1 *In each $C_n(1 \leq n < \omega)$ the schema $F_n \supset A$ is derivable.*

Proof: An instance of postulate $(12)^{(n)}$ of each $C_n(1 \leq n < \omega)$ is $B^{(n)} \supset ((\neg A \supset B) \supset ((\neg A \supset \neg B) \supset \neg\neg A))$. Permuting antecedents yields $(\neg A \supset B) \supset ((\neg A \supset \neg B) \supset (B^{(n)} \supset \neg\neg A))$. An instance of (1) is $B \supset (\neg A \supset B)$, so transitivity delivers $B \supset ((\neg A \supset \neg B) \supset (B^{(n)} \supset \neg\neg A))$. Permutation yields $(\neg A \supset \neg B) \supset$

$(B \supset (B^{(n)} \supset \neg\neg A))$, which by (1) and transitivity is further reduced to $\neg B \supset (B \supset (B^{(n)} \supset \neg\neg A))$. Permutation and importation transform this into $(B^{(n)} \,\&\, (B \,\&\, \neg B)) \supset \neg\neg A$. By (11) and transitivity, this yields $(B^{(n)} \,\&\, (B \,\&\, \neg B)) \supset A$, which is, by definition, $F_n \supset A$.

Lemma 2 *In C_ω the schema $A \vee A^{(i)}$ is derivable, for $1 \leq i < \omega$.*

Proof: We first consider the schema $\neg A^i$. For $1 < i < \omega$, $\neg A^i$ is by definition $\neg\neg(A^{i-1} \,\&\, \neg A^{i-1})$. By (11), we have $\neg\neg(A^{i-1} \,\&\, \neg A^{i-1}) \supset (A^{i-1} \,\&\, \neg A^{i-1})$, and by (5), $(A^{i-1} \,\&\, \neg A^{i-1}) \supset \neg A^{i-1}$; hence, by transitivity, we obtain $\neg A^i \supset \neg A^{i-1}$. A straightforward inductive argument shows that we therefore have $\neg A^i \supset \neg A^\circ$ for all $1 < i < \omega$, and thus for all $1 \leq i < \omega$, since $\neg A^1$ is just $\neg A^\circ$. But $\neg A^\circ$ is by definition $\neg\neg(A \,\&\, \neg A)$. By (11), we have $\neg\neg(A \,\&\, \neg A) \supset (A \,\&\, \neg A)$, and by (4), $(A \,\&\, \neg A) \supset A$; whence by transitivity again we obtain $\neg A^\circ \supset A$. This, together with the above-derived $\neg A^i \supset \neg A^\circ$, yields $\neg A^i \supset A$, for $1 \leq i < \omega$.

To continue, an instance of postulate (10) of C_ω is $A^i \vee \neg A^i$, which, together with the schema $\neg A^i \supset A$ of the preceding paragraph, quickly leads to $A \vee A^i$, for all $1 \leq i < \omega$. Explicitly, we have $A \vee A^i, A \vee A^{i-1}, \ldots, A \vee A^\circ$, which can be conjoined to yield $(A \vee A^i) \,\&\, (A \vee A^{i-1}) \,\&\ldots\&\, (A \vee A^\circ)$. An appropriate number of applications of distribution and (3) transform this schema into $A \vee (A^i \,\&\, A^{i-1} \,\&\ldots\&\, A^\circ)$, which is by definition the desired $A \vee A^{(i)}$.

Lemma 3 *In each $C_n (1 \leq n < \omega)$, the schema $F_n^{(n)}$ is derivable.*

Proof: The following is an instance of (9):

$$(F_n \supset F_n^{(n)}) \supset ((F_n^{(n)} \supset F_n^{(n)}) \supset ((F_n \vee F_n^{(n)}) \supset F_n^{(n)})).$$

In each $C_n (1 \leq n < \omega)$ the first antecedent is derivable by Lemma 1. The second is easily derived in C_ω, and the third is also derivable in C_ω by Lemma 2. This leaves the desired $F_n^{(n)}$ as a schema derivable in each $C_n (1 \leq n < \omega)$.

Lemma 4 *In each $C_n (1 \leq n < \omega)$, the schema $(A \supset F_n) \supset A^{(n)}$ is derivable.*

Proof: The following is an instance of prefixing:

$$(F_n \supset A^{(n)}) \supset ((A \supset F_n) \supset (A \supset A^{(n)})).$$

The antecedent is derivable in each $C_n (1 \leq n < \omega)$ by Lemma 1, leaving

$$(A \supset F_n) \supset (A \supset A^{(n)}).$$

Permuting antecedents yields

$$A \supset ((A \supset F_n) \supset A^{(n)}).$$

An instance of postulate (1) is

$$A^{(n)} \supset ((A \supset F_n) \supset A^{(n)}),$$

which, together with the preceding schema and an instance of (9), yields

$$(A \vee A^{(n)}) \supset ((A \supset F_n) \supset A^{(n)}).$$

But the antecedent is derivable in C_ω by Lemma 2, leaving the desired

$$(A \supset F_n) \supset A^{(n)}.$$

Lemma 5 *In each* $C_n (1 \leq n < \omega)$, *the schema* $(A \supset F_n) \supset (A \supset F_n)^{(n)}$ *is derivable.*

Proof: An instance of postulate (6), with antecedents permuted, is

$$F_n^{(n)} \supset (A^{(n)} \supset (A^{(n)} \,\&\, F_n^{(n)})).$$

The antecedent is derivable in each $C_n (1 \leq n < \omega)$ by Lemma 3, leaving

$$A^{(n)} \supset (A^{(n)} \,\&\, F_n^{(n)}).$$

An instance of postulate $(15)^{(n)}$ of each $C_n (1 \leq n < \omega)$ is

$$(A^{(n)} \,\&\, F_n^{(n)}) \supset (A \supset F_n)^{(n)}.$$

By transitivity, the two preceding schemata yield

$$A^{(n)} \supset (A \supset F_n)^{(n)}.$$

But by Lemma 4, in each $C_n (1 \leq n < \omega)$, we have

$$(A \supset F_n) \supset A^{(n)},$$

whence, by transitivity again, we obtain the desired

$$(A \supset F_n) \supset (A \supset F_n)^{(n)}.$$

Lemma 6 *In each* $C_n (1 \leq n < \omega)$, *the schemata and deduction rules of positive classical logic are derivable.*

Proof: This is stated in various places, e.g. for C_1 in Theorem 3 of [4]. But it also follows fairly easily from Lemma 2 above. For to obtain an axiomatics for positive classical logic it suffices to add the schema $A \vee (A \supset B)$ to positive intuitionistic logic as axiomatized by postulates (1) to (9) of C_ω. By Lemma 2, we have $A \vee A^{(n)}$ in C_ω. Conjoined with postulate (10), this yields $(A \vee A^{(n)}) \,\&\, (A \vee \neg A)$. By distribution, this is equivalent to $A \vee (A^{(n)} \,\&\, \neg A)$. In each $C_n (1 \leq n < \omega)$, the schema $((A \,\&\, \neg A) \,\&\, A^{(n)}) \supset B$ follows straightforwardly from an instance of postulate $(12)^{(n)}$, and this reduces easily to $(A^{(n)} \,\&\, \neg A) \supset (A \supset B)$. Together with the preceding schema, and with the assistance of (9), this yields the desired $A \vee (A \supset B)$.

Lemma 7 *In each* $C_n (1 \leq n < \omega)$, *the schema* $B \equiv ((B \supset F_n) \supset F_n)$ *is derivable.*

Proof: The following is an instance of postulate (9):

$$(F_n \supset A) \supset ((A \supset A) \supset ((F_n \vee A) \supset A)).$$

The first antecedent is derivable in each $C_n (1 \leq n < \omega)$ by Lemma 1, and the second is easily derived in C_ω, leaving

$$(F_n \vee A) \supset A.$$

Substituting $B \equiv ((B \supset F_n) \supset F_n)$ for A yields

$$(F_n \vee (B \equiv ((B \supset F_n) \supset F_n))) \supset (B \equiv ((B \supset F_n) \supset F_n)).$$

But the antecedent is an instance of the positive classical tautology

$$A \vee (B \equiv ((B \supset A) \supset A)),$$

and is therefore derivable in $C_n(1 \leq n < \omega)$ by Lemma 6, leaving the desired

$$B \equiv ((B \supset F_n) \supset F_n).$$

Lemma 8 *In each* $C_n(1 \leq n < \omega)$, *the schema* $((B \supset F_n) \supset F_n)^{(n)}$ *is derivable.*

Proof: Substituting $B \supset F_n$ for A in the schema of Lemma 5 yields

$$((B \supset F_n) \supset F_n) \supset ((B \supset F_n) \supset F_n)^{(n)}.$$

Substituting $(B \supset F_n) \supset F_n$ for A in the schema of Lemma 4 yields

$$(((B \supset F_n) \supset F_n) \supset F_n) \supset ((B \supset F_n) \supset F_n)^{(n)}.$$

These two schemata, with the assistance of (9), yield

$$(((B \supset F_n) \supset F_n) \vee (((B \supset F_n) \supset F_n) \supset F_n)) \supset ((B \supset F_n) \supset F_n)^{(n)}.$$

But the antecedent is an instance of the positive classical tautology $A \vee (A \supset B)$, and is therefore derivable in each $C_n(1 \leq n < \omega)$ by Lemma 6, leaving the desired

$$((B \supset F_n) \supset F_n)^{(n)}.$$

Finally, we are in a position to determine the result of adding the rule EC to the systems $C_n(1 \leq n < \omega)$.

Theorem 9 *For* $1 \leq n < \omega$, $EC_n = C_0$.

Proof: As noted at the end of Section 2, the addition of EC to $C_n(1 \leq n < \omega)$ suffices to guarantee the property SE. This permits the following very simple proof.

In each $EC_n(1 \leq n < \omega)$, B and $(B \supset F_n) \supset F_n$ are provably equivalent by Lemma 7, and the schema $((B \supset F_n) \supset F_n)^{(n)}$ is derivable by Lemma 8. Because they enjoy SE, it follows that $B^{(n)}$, which is the result of substituting B for $(B \supset F_n) \supset F_n$ in this schema, is also derivable in each $EC_n(1 \leq n < \omega)$. As in the proof of Theorem 6, this yields unqualified *reductio*, which suffices to collapse each $EC_n(1 \leq n < \omega)$ into classical logic.

It will be noted that the proof of Theorem 9 does not rely upon the actual derivability of the rule EC in $EC_n(1 \leq n < \omega)$; it is sufficient merely that this rule be admissible. But, as noted at the end of Section 2, the admissibility of EC in any extension of $C_n(1 \leq n \leq \omega)$ is a necessary condition for SE. We can therefore state the following more general result.

Theorem 10 *There is no extension of any* $C_n(1 \leq n < \omega)$ *which enjoys* SE *but which is weaker than classical logic.*

(We note that an alternative proof of Theorem 9 can be constructed using the schema $\neg\top$ in place of F_n, where $\top$ is defined to be an arbitrary theorem of C_ω. While this has the slight advantage that, unlike F_n, $\neg\top$ is not relative to each $C_n(1 \leq n < \omega)$, it has the disadvantage that Lemma 1 and those subsequent lemmas that rely upon it must be restated to apply to $EC_n(1 \leq n < \omega)$ rather than to $C_n(1 \leq n < \omega)$. This is because the schema $\neg\top \supset A$ is not derivable in the latter systems, but only in the former. This alternative proof, therefore, does not demonstrate as clearly that it is precisely because the C-systems fail to enjoy SE that they do not collapse into classical logic).

5 Conclusion Obtaining analogues of the C-systems which both enjoy SE and satisfy the paraconsistency conditions, then, is not to be achieved by extension but perhaps by some other method of variation. One possibility, which involves the least revision of the C-systems as they stand, is to retain all of the postulates of these systems but to redefine the schema B°. For there is nothing sacrosanct about the original definition of this schema as $\neg(B \,\&\, \neg B)$; in different contexts other candidates may well prove to be more adequate in expressing the proposition that B "behaves classically". A second and more radical possibility is to retain the method of constructing the higher C-systems but to change the base system. Among the possible alternatives to C_ω which suggest themselves are the systems NC_ω and OC_ω defined in [14]. A third possibility is to combine both of these approaches, which is essentially what is involved in the construction of the "intuitionistic" analogues of the C-systems in [3]. All of these possibilities will be considered in greater detail elsewhere.

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