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The Logic of Free Acts and the Powers of God

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In this paper I provide a formalization of the logic of free acts for which I obtain a decision procedure by means of a reduction to $S5$. Although this formalization has independent interest, I provide it chiefly for the sake of an application to philosophical theology, namely an argument to show that a certain characterization of the powers of God is coherent, and, moreover, compatible with God's giving to some other agent a power to act freely. (Here, note, I am not relying on any purported reconciliation of free will and predestination: I believe that free acts are not predestined.)

1 Preliminaries A formal language is designed to represent the interesting features of some class of propositions. In this paper I shall consider propositions about the *free acts* and the *powers* of agents. Let us represent

> X acts freely at time t, and a consequence of X's action is that α

as:

$$X\Delta_t\alpha \ .$$

And let us represent:

> X has the power at time t to act freely in such a way that a consequence of X's act is that α

as:

$$X\Pi_t\alpha \ .$$

Notice that I am not only concerned with the intended consequences of acts. By a consequence I mean anything brought about by the act. For that reason, my formalization might be too crude for an application to deontic logic.

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In addition I shall use the concept of time-dependent necessity (necessity *per accidens*). So let us represent:

It is necessary at time t that θ

as

$$L_t\theta \ .$$

I shall use 'M_t' as an abbreviation for '$\sim L_t \sim$'. So $M_t\theta$ represents:

It is possible at time t that θ.

And I shall abbreviate 'L_{t_i}' to 'L_i', 'M_{t_i}' to 'M_i'.

The conceptual role played by time-dependent necessity is that no agent, not even God, can bring it about that θ at time t, if θ is either necessarily true or necessarily false at time t.

It is widely assumed that if some process, event, or act occurs before t then it is necessary at t. Nowhere in this paper do I assume this, which, in fact, I am inclined to reject. But if the reader believes it and it helps him or her give content to the notion of necessity at time t, it will do no harm for the purposes of this paper.

If a proposition is necessary at all times I say it is *eternally necessary*. Presumably, all logical truths are eternally necessary.[1] I, like most theists, would claim that the existence and attributes of God are eternally necessary, even if, as I am inclined to believe,[2] they are logically contingent. The atheist might claim that the existence of laws of nature and the nonexistence of God are eternally necessary.

Nothing I have said so far would prevent the combination of the logic of free acts with tensed logic. But in this paper I shall restrict my attention to propositions considered to be true or false independently of time.

How suitable is this formalization to the problems raised by God's acts? At one stage I thought it was not suitable. For is not God thought of as *eternal*, in some sense? However, if by His[3] eternity you merely mean—as I do—that He undergoes no *real* change,[4] then His eternity is compatible with His acting at a time, provided he knew before He acted everything He knew afterwards. Alternatively, suppose you mean by God's eternity that even His acts have no temporal location. In that case, you should, I suggest, rely on the following method. First examine any problems raised by God's powers on the assumption that He does act in time. Then see in what ways, if any, God's eternity is considered relevant to the discussion about God's powers. The justification for this method is that it is extremely hard to think at all about acts out of time. So we should, as a first approximation, consider God as acting in time, even if He does not.

We may now write down some axiom-schemata for a logic of free acts. (I shall use Γ as a variable ranging over Δ and Π.)

AS1 *If θ is a wff obtained by uniform substitution from a thesis of S5, then*:

$$\vdash \theta \ .$$

AS2 *If* $t_2 > t_1$, *then*:

$$\vdash L_1\theta \supset L_2\theta \ .$$

AS3 $\vdash (L_t(\xi \supset \eta) \ \& \ X\Gamma_t\xi) \supset X\Gamma_t \eta \vee L_t\eta$.

AS4 $\vdash X\Delta_t\theta \supset \theta$.

AS5 $\vdash X\Delta_t\theta \supset X\Pi_t\theta$.

AS6 $\vdash X\Pi_t\theta \supset M_tX\Delta_t\theta$.

AS7 $\vdash X\Pi_t\theta \supset M_t(\sim\theta)$.

AS8 *If* $X \neq Y$, *then*:

$$\vdash \sim (X\Delta_1\theta \ \& \ Y\Delta_2\theta) \ .$$

AS9 *If* $X \neq Y$, *then*:

$$\vdash \sim X\Delta_1(Y\Delta_2\Phi) \ .$$

AS10 $\vdash X\Delta_1(\sim Y\Delta_2\theta) \supset X\Delta_1(\sim Y\Pi_2\theta)$.

AS11i *If* $t_2 > t_1$, *then*:

$$\vdash X\Delta_1\theta \supset L_2(X\Delta_1\theta) \ .$$

AS11ii *Unless* $t_2 > t_1$,

$$\vdash \sim L_2(X\Delta_1\theta) \ .$$

AS12 $\vdash X\Pi_t\theta \supset L_tX\Pi_t\theta$.

All these axiom-schemata are intuitively plausible except AS8, which I shall discuss below. However, the plausibility of AS7, AS10, and AS11ii depends on a libertarian (anti-reconciliationist) understanding of the concept of a free act.

Many instances of AS8 represent falsehoods, for X and Y could act simultaneously and overdetermine the truth of θ. Nonetheless I include AS8 because I am primarily concerned with what is *consistent* with our intuitions about free acts, not with what follows from them. As a consequence, if I have to choose between a system that is too strong and one that is too weak, I should err on the side of excessive strength. It is intuitively plausible that no two agents can perform acts whose *immediate consequences* are identical. And any wff consistent in the system I develop will, *a fortiori*, represent a proposition consistent with that intuition.

The formal system I would obtain using those axiom-schemata I call *TLA* (time-dependent logic of free acts). However, *TLA* is hard to work with, for it involves a proliferation of modal operators. Even assuming, as I would rather not assume, that time necessarily has the structure of the real line, the proliferation of operators leads to difficulties. (The reader who is skeptical of this should try to prove analogues of the results of this paper for *TLA*.)

As a consequence I shall consider two formal systems (*LA* and *LA**) which although they involve messier axiom-schemata are easier to handle at the meta-level. I shall replace $X\Gamma_t\xi$ by $X\Gamma(\theta, \xi)$ where θ is a suitable description of what *is* necessary at time t. I shall keep, however, one modal operator L, representing eternal necessity. Now one might think that θ would be a suitable description of what is necessary at time t just in case

(i) $L_t\theta$

and

(ii) If $L_t\phi$ then $L(\theta \supset \phi)$.

However that would involve X's power to act at time t as part of the description of what is necessary at time t, namely θ, where the power in question is now represented as $X\Pi(\theta, \xi)$. Such circularity should be avoided. Consequently when describing what is necessary at time t, for the purposes of describing X's power or X's act at time t, we should leave out X's power to act. We then describe what else is necessary at that time (up to eternal necessity).

Similarly, if we are to avoid circularity we should not include X's act as part of the description of the consequences of X's act. So in the description $X\Delta(\theta, \xi)$, the formula $X\Delta(\theta, \xi)$ should neither be ξ nor a conjunct of ξ. (The failure to exclude such formulas would result in infinitely long wff, so they are automatically excluded in the formal system I develop.)

In the next section I shall develop the formal systems LA and LA^*, based on my use of $X\Gamma(\theta, \xi)$ instead of $X\Gamma_t\xi$. The axiom-schemata are motivated by the corresponding ones for TLA. Indeed the sole purpose of introducing TLA in the first place was to motivate the systems LA and LA^*.

2 The formal systems LA and LA*   In this section I present the formal systems LA and LA^*. LA^* might seem capable of representing a richer variety of propositions than LA. However, in Section 4 I prove a result which enables us to restrict our attention to the simpler system LA, by showing that every wff in LA^* is equivalent to one in LA.

The atomic wffs are the letters p, q, and r, with subscripts as required. In addition to the basis and definitions of $S5$, using the modal operator L, there are extra primitive symbols, formation rules and axioms.

The extra primitive symbols are:

(i) A, B, C, with subscripts as required, and G. (These are to be thought of as the names of agents, there being only one name for each agent. G is reserved for use as a name for God.)

The following metalogical variables are used:

(i) Lower case Greek letters, ranging over all wffs.
(ii) X and Y, ranging over A, B, C (with or without subscripts) and G.
(iii) Γ ranging over Δ and Π.
(iv) K ranging over L and M.

Subscripts are used, as required, to provide extra metalogical variables.

The extra formation rule for LA is:*

FR* $X\Gamma(\theta, \xi)$ is a wff.

The extra formation rule for LA is:

FR If θ and ϕ contain no modal operator, L or M, then $X\Gamma(\theta, \xi)$ is a wff.

The degree of a wff is defined as follows:

(1) Atomic wffs have degree zero.
(2) Any wff formed out of wffs of degree at most m, one of which is of degree m, without using FR or FR* is also of degree m.
(3) If θ and ϕ are of degree m and n respectively, the degree of $X\Gamma(\theta, \phi)$ is one plus the greater of m and n.

The extra axioms are given by the following schemata:

AS1 *If θ is a wff obtained by uniform substitution in a thesis of S5, then:*

$$\vdash \theta \ .$$

AS2 $\vdash (L(\theta \equiv \phi) \ \& \ X\Gamma(\theta, \xi)) \supset X\Gamma(\phi, \xi)$.
AS3 $\vdash (L(\xi \supset \eta) \ \& \ X\Gamma(\theta, \xi)) \supset (X\Gamma(\theta, \eta) \vee L((\theta \ \& \ X\Pi(\theta, \xi)) \supset \eta))$.
AS4 $\vdash X\Delta(\theta, \xi) \supset \theta \ \& \ \xi$.
AS5 $\vdash X\Delta(\theta, \xi) \supset X\Pi(\theta, \xi)$.
AS6 $\vdash X\Pi(\theta, \xi) \supset MX\Delta(\theta, \xi)$.
AS7 $\vdash X\Pi(\theta, \xi) \supset M(X\Pi(\theta, \xi) \ \& \ \theta \ \& \ {\sim}\xi)$.
AS8 *If $X \neq Y$, then*:

$$\vdash {\sim}(X\Delta(\theta, \xi) \ \& \ Y\Delta(\phi, \xi)) \ .$$

AS9 *If $X \neq Y$, then*:

$$\vdash (X\Delta(\theta, \xi) \ \& \ L(X\Delta(\theta, \xi) \supset Y\Delta(\phi, \eta))) \supset L(\theta \ \& \ X\Pi(\theta, \xi) \supset Y\Delta(\phi, \eta)) \ .$$

AS10 *If $X \neq Y$, then*:

$$\vdash (X\Delta(\theta, \xi) \ \& \ L(X\Delta(\theta, \xi) \supset {\sim}Y\Delta(\phi, \eta)) \supset (L(\theta \ \& \ X\Pi(\theta, \xi) \supset {\sim}Y\Delta(\phi, \eta)) \vee L(\xi \supset {\sim}Y\Pi(\phi, \eta))) \ .$$

The transformation rules are the same as for S5, namely:

MP (Modus Ponens) If $\vdash \phi \supset \psi$ and $\vdash \phi$ then $\vdash \psi$.
RN (Rule of Necessitation) If $\vdash \phi$ then $\vdash L_t\phi$.

3 Some additional remarks on the axioms

(1) I have not included $\vdash X\Pi(\theta, \xi) \supset \theta$. Likewise I have not included an analogue of schema (12) for *TLA*. This enables me to interpret $X\Pi(\theta, \xi)$ as 'X *would* have the power . . .'.
(2) In AS3 the second disjunct of the consequent is $L((\theta \ \& \ X\Pi(\theta, \xi) \supset \eta))$, not $L(\theta \supset \eta)$. That is because I have said that θ, the description of what is necessary at the time of the act, must not include $X\Pi(\theta, \xi)$ itself.
(3) The intuition behind the schema (IIi) of *TLA* is not expressible in *LA**. But that intuition may instead be used to motivate the following *definition* of a temporal ordering of acts:

$$X\Delta(\theta, \xi) < Y\Delta(\phi, \eta)$$

just in case $L(\phi \supset X\Delta(\theta, \xi))$.

(4) The analogues of schema (9) and schema (10) of *TLA* are, respectively:

If $X \neq Y$, *then*:

$$\vdash \sim X\Delta(\theta,\ Y\Delta(\phi,\ \eta))\ .$$

If $X \neq Y$, *then*:

$$\vdash Y\Delta(\theta,\ \sim Y\Delta(\phi,\ \eta) \supset X\Delta(\theta,\ \sim Y\Pi(\phi,\ \eta)\ .$$

Using AS1 to AS7, these are interderivable with AS9 and AS10. I prefer AS9 and AS10 because they are only of degree one.

4 Tautologies, substitution of equivalents, and normal forms

Tautologies: If θ is obtained by uniform substitution in a tautology of the Sentential Calculus, then θ is said to be a tautology. We can test whether a wff is a tautology in the usual way.

Substitution of Equivalents: The usual rule for substitution of equivalents holds. One has to check that if $\vdash \theta \equiv \phi$, then:

(1) $\vdash \sim\theta \equiv \sim\phi$, $\vdash \theta \vee \sigma \equiv \phi \vee \sigma$, $\vdash \sigma \vee \theta \equiv \sigma \vee \phi$
(2) $\vdash L\theta \equiv L\phi$
(3) $\vdash X\Gamma(\theta,\ \xi) \equiv X\Gamma(\phi,\ \xi)$, $\vdash X\Gamma(\sigma,\ \theta) \equiv X\Gamma(\sigma,\ \phi)$.

Normal Forms: Any wff of *S5* can be reduced to a modal conjunctive normal form (see [4]), that is, a conjunction of disjunctions of wffs of the form β, $L\sigma$ and $M\delta$, where β, σ, and δ contain no modal operators. Hence by AS1, every wff in *LA* can be reduced to Modal Conjunctive Normal Form. The following theorem shows that any wff in *LA** can be reduced to Modal Conjunctive Normal form and justifies considering *LA* rather than *LA**.

Theorem 1 *If* α *is any wff in LA* then there is a wff* β *in LA such that* $\alpha \equiv \beta$ *is a thesis of LA**.

Proof: Four lemmas are required.

Lemma 1 *If K is L or M*, $\vdash K\psi \supset L((\theta\ \&\ (\phi \vee K\psi)) \equiv \theta)$.

Proof-sketch: $\alpha \supset ((\theta\ \&\ (\phi \vee \alpha)) \equiv \theta)$ is a tautology and, hence, by AS1,

(1) $\vdash K\psi \supset ((\theta\ \&\ (\phi \vee K\psi)) \equiv \theta)$.

By (1) and the standard result that if $\vdash \alpha \supset \beta$ then $\vdash L\alpha \supset L\beta$,

(2) $\vdash LK\psi \supset L((\theta\ \&\ (\phi \vee K\psi)) \equiv \theta)$.

By (2), AS1 and MP,

(3) $\vdash K\psi \supset L((\theta\ \&\ (\phi \vee K\psi)) \equiv \theta)$.

Lemma 2 *If K is L or M*, $\vdash \sim K\psi \supset L((\theta\ \&\ (\phi \vee K\psi)) \equiv (\theta\ \&\ \phi))$.

Proof-sketch: $\sim\alpha \supset ((\theta\ \&\ (\phi \vee \alpha)) \equiv (\theta\ \&\ \phi))$ is a tautology: the proof continues as for Lemma 1.

Lemma 3 *If Γ is Δ or Π and if K is L or M,*

$$\vdash X\Gamma(\theta \,\&\, (\phi \vee K\psi), \xi) \equiv ((X\Gamma(\theta, \xi) \,\&\, K\psi) \vee (X\Gamma(\theta \,\&\, \phi, \xi) \,\&\, \sim K\psi)) \;.$$

Proof-sketch: By AS2,

(1) $\vdash L(\alpha \equiv \beta) \supset (X\Gamma(\alpha, \xi) \equiv X\Gamma(\beta, \xi))$.

By (1), Lemma 1, AS1, and MP,

(2) $\vdash K\psi \supset (X\Gamma(\theta \,\&\, (\phi \vee K\psi), \xi) \equiv X\Gamma(\theta, \xi))$.

By (1), Lemma 2, AS1, and MP,

(3) $\vdash \sim K\psi \supset (X\Gamma(\theta \,\&\, (\phi \vee K\psi), \xi) \equiv X\Gamma(\theta \,\&\, \phi, \xi))$.

By (2), (3), AS1, and MP,

(4) $\vdash X\Gamma(\theta \,\&\, (\phi \vee K\psi), \xi) \equiv ((X\Gamma(\theta, \xi) \,\&\, K\psi) \vee (X\Gamma(\theta \,\&\, \phi, \xi) \,\&\, \sim K\psi)$.

Lemma 4 *If Γ is Δ or Π and if K is L or M,*

$$\vdash X\Gamma(\xi, \theta \,\&\, (\phi \vee K\psi)) \equiv ((X\Gamma(\xi, \theta) \,\&\, K\psi) \vee (X\Gamma(\xi, \theta \,\&\, \phi) \,\&\, \sim K\psi)) \;.$$

Proof-sketch: By AS1, AS5, AS7, and MP,

(1) $\vdash X\Gamma(\theta, \xi) \supset \sim L(\theta \supset \xi)$.

By (1), AS1, AS2, and MP,

(2) $\vdash L(\xi \equiv \eta) \supset (X\Gamma(\theta, \xi) \equiv X\Gamma(\theta, \eta))$.

The proof now continues as for Lemma 3.

Proof of Theorem: The proof is by induction on the degree of the wff α. The wffs in LA^* of degree zero are all wffs in LA. Assume that the theorem holds for all wffs of degree at most k and that α in LA^* is of degree $k + 1$. By the Principle of Substitution of Equivalents it suffices to prove the required result for wff $X\Gamma(\theta, \xi)$ of degree $k + 1$. Since θ and ξ are of degree at most k, one can assume that there are wffs λ, ζ in LA such that $\vdash \theta \equiv \lambda$ and $\vdash \xi \equiv \zeta$. Hence by the Principle of Substitution of Equivalents, $\vdash X\Gamma(\theta, \xi) \equiv X\Gamma(\lambda, \zeta)$, so it suffices to prove the required result for wff $X\Gamma(\lambda, \zeta)$, where λ and ζ are in LA and hence can be reduced to Modal Conjunctive Normal Form. Hence by the Principle of Substitution of Equivalents, it suffices to prove the required result for wff $X\Gamma(\mu, \eta)$ where μ and η are the conjunctions of disjunctions of wffs of the form β, $L\gamma$, and $M\delta$, where β, γ, and δ contain no modal operators. The required result now follows from repeated applications of Lemmas 3 and 4.

5 Closed sets and skeletons Having presented the systems LA and LA^* and having shown that we need not consider LA^*, in the next four sections I discuss methods for establishing the *consistency* of wffs. These methods involve the *semantics* for LA in the sense of a theory of models for LA. I do not consider this semantics to provide even an approximation to a theory of *meaning* for propositions about acts and powers. Consequently, the completeness of LA is of no intrinsic interest, I merely prove it in order to convert the decision procedure for validity ("truth" at all worlds in all models) into a decision procedure for consistency.

When discussing the semantics of LA it is convenient to have models not just for the whole of LA but also for suitable sets of wffs in LA. These sets I call closed sets.

Definition A set $\bar{S}$ of wffs in LA is said to be a *closed set* if it has the following six properties:

CS1 α is in $\bar{S}$ iff $\sim\alpha$ is in $\bar{S}$.
CS2 $\alpha \vee \beta$ is in $\bar{S}$ iff both α and β are in $\bar{S}$.
CS3 $L\alpha$ is in $\bar{S}$ iff α is in $\bar{S}$.
CS4 $X\Delta(\alpha, \beta)$ is in $\bar{S}$ iff $X\Pi(\alpha, \beta)$ is in $\bar{S}$.
CS5 If $X\Gamma(\alpha, \beta)$ is in $\bar{S}$, α and β are in $\bar{S}$.
CS6 If $X\Gamma(\alpha, \beta)$ and $Y\Gamma(\gamma, \delta)$ are in $\bar{S}$, then $X\Gamma(\alpha \,\&\, \gamma, \beta \vee \delta)$ and $Y\Gamma(\alpha \,\&\, \gamma, \beta \vee \delta)$ are in $\bar{S}$.

CS1 and CS5 ensure that if a wff is in $\bar{S}$ enough relevant wffs are in $\bar{S}$ for one to have sufficiently many axioms in $\bar{S}$. CS6 is required for technical reasons which will be apparent in the proof of Theorem 3.

Definition If S is any set of wffs in LA, the *closure* $\bar{S}$ of S is the intersection of all closed sets containing S.

Notice that the empty set, the set of all wffs, and the set of all wffs of degree less than m, are all closed sets. Notice also that any intersection of closed sets is closed. Hence every set has a closure and the closure is closed.

Closed sets are handled by means of their skeletons, and the associated proper axioms.

Definition A subset R of a closed set $\bar{S}$ is said to be a *skeleton* of $\bar{S}$ if it has the following two properties.

SK1 R contains every atomic wff in $\bar{S}$.
SK2 For every $X\Gamma(\alpha, \gamma)$ in $\bar{S}$ there is one and only one pair of wffs $\langle\beta, \delta\rangle$ such that $X\Gamma(\beta, \delta)$ is in R and such that $\alpha \equiv \beta$ and $\gamma \equiv \delta$ are tautologies.

Definition If R is a skeleton of $\bar{S}$, the set of instances of AS2 to AS12 formed out of R using $\sim$, $\vee$, and L but without using FR form the set of *proper axioms associated* with R.

6 Models for closed sets If $\bar{S}$ is a closed set, a wff α in $\bar{S}$ is said to be an $\bar{S}$-thesis if there is a derivation of α the lines of which are wffs in $\bar{S}$, that is a derivation from axioms in $\bar{S}$ using MP and RN. In this way one obtains a formal system $LA(\bar{S})$. Similarly, if one restricts oneself to the axiom-schema AS1, one obtains a formal system $S5(\bar{S})$. The Semantics of $LA(\bar{S})$ is obtained by considering some of the models for $S5(\bar{S})$.

A *model* for $S5(\bar{S})$ consists of a set W called the set of *possible worlds* and a valuation V which assigns at each world in W a truth-value T or F to each wff in $\bar{S}$. The usual rules must hold:

V~ *V assigns T to α at w iff V assigns F to ~α at w.*
V∨ *V assigns T to α ∨ β at w iff either V assigns T to α at w or V assigns T to β at w.*
VL *V assigns T to Lα at w iff V assigns T to α at all worlds in W.*

These rules ensure that all instances of AS1 in $\bar{S}$ are T at all worlds, and that if $\phi \supset \psi$ and ϕ are T at some world w then ψ is T at w.

A model for $S5(\bar{S})$ is also a *model for* $LA(\bar{S})$ if, in addition, V assigns the value T at all worlds to every instance of AS2 to AS12 in $\bar{S}$. It follows that every thesis in $LA(\bar{S})$ is $LA(\bar{S})$-valid, that is, T at all worlds in all models for $LA(\bar{S})$.

The semantics of LA can be handled by using the strategy of first constructing models for $LA(\bar{S})$, where $\bar{S}$ is some suitably small closed set, and then extending these models to models for the whole of LA. The following theorem is a collection of a number of results useful in the construction of models for closed sets.

Theorem 2
(1) *Let R be a skeleton of the closed set* $\bar{S}$. *If W is a set of worlds and if U is an assignment of truth-values at members of W to members of R, then there is a unique model for* $S5(\bar{S})$ *with set of worlds W and valuation V such that V agrees with U on R and such that if* $\alpha \equiv \beta$ *and* $\gamma \equiv \delta$ *are tautologies V assigns the same truth-values to the wff* $X\Gamma(\alpha, \gamma)$ *and* $X\Gamma(\beta, \delta)$, *assuming that they are both in* $\bar{S}$.
(2) *Suppose U is the assignment of truth-values mentioned above and V its extension to* $\bar{S}$. *If V assigns the value T to every proper axiom associated with R at every world in W, the pair* $\langle W, V\rangle$ *is also a model for* $LA(\bar{S})$.
(3) *If S is a finite set of wffs, its closure* $\bar{S}$ *has a finite skeleton and there is a finite set of associated proper axioms. Moreover there are rules for writing down, in a finite number of steps, a skeleton of R and the set of associated proper axioms.*

Proof:

(1) Let Q be the union of the set of all atomic wffs in $\bar{S}$ with the set of all wffs in $\bar{S}$ of the form $X\Gamma(\alpha, \gamma)$. By SK2, U extends to a unique valuation U^* of Q which assigns the same truth-values to $X\Gamma(\alpha, \gamma)$ and $X\Gamma(\beta, \delta)$ if $\alpha \equiv \gamma$ and $\beta \equiv \delta$ are tautologies. Every wff in $\bar{S}$ is formed out of members of Q in precisely one way using ~, ∨, and L. Therefore there is a unique extension V of U^* to the whole of $\bar{S}$ satisfying the rules V~, V∨ and VL.

(2) If ϕ is an instance of axiom-schema AS2 to AS12 in $\bar{S}$, one can obtain a wff $\phi^\dagger$ in the set of proper axioms associated with R by replacing every occurrence of $X\Gamma(\alpha, \gamma)$ in ϕ by the unique $X\Gamma(\beta, \delta)$ in R such that $\alpha \equiv \beta$ and $\gamma \equiv \delta$ are tautologies. V assigns the same truth-values to ϕ and $\phi^\dagger$, so ϕ is T at all worlds.

(3) The following procedure can be used to obtain a skeleton R of $\bar{S}$, for any given finite set S:

Step 1: Write down all the atomic wffs and all the wffs of the form $X\Gamma(\alpha, \beta)$ which occur in wffs in S.

Step 2: If $X\Gamma(\alpha, \gamma)$ and $X\Gamma(\beta, \delta)$ both appear in the list and $\alpha \equiv \beta$ and $\gamma \equiv \delta$ are tautologies, delete the second of $X\Gamma(\alpha, \gamma)$ and $X\Gamma(\beta, \delta)$.

Step 3: If $X\Delta(\alpha, \beta)$ is in the list but $X\Pi(\alpha, \beta)$ is not in the list write down $X\Pi(\alpha, \beta)$ and conversely.

Step 4: If, for some integer $n \geq 2$, $X_1\Pi(\alpha_1, \beta_1)$, $X_2\Pi(\alpha_2, \beta_2), \ldots,$ and $X_n\Pi(\alpha_n, \beta_n)$ are already in the list before Step 4, but not necessarily in that order, write down $X_1\Pi(\alpha_1 \;\&\; \alpha_2 \;\&\ldots\&\; \alpha_n, \beta_1 \vee \beta_2 \vee \ldots \vee \beta_n)$, if it is not already in the list.

Step 5: Repeat Step 3.

Step 6: Repeat Step 2.

In this way one obtains a finite set of wffs. It is easy to check that this set is a skeleton for $\bar{S}$. The set of associated proper axioms is obtained in the obvious way.

Example: If S consists of the single wff $A\Delta(q \vee \sim q, p \;\&\; B\Pi(p, q))$, a skeleton of $\bar{S}$ consists of the following ten wffs:

p; q; $B\Pi(p, q)$
$A\Delta(q \vee \sim q, p \;\&\; B\Pi(p, q))$
$B\Delta(p, q)$
$A\Pi(q \vee \sim q, p \;\&\; B\Pi(p, q))$
$B\Pi(p \;\&\; (q \vee \sim q), q \vee (p \;\&\; B\Pi(p, q)))$
$B\Delta(p \;\&\; (q \vee \sim q), q \vee (p \;\&\; B\Pi(p, q)))$
$A\Pi((q \vee \sim q) \;\&\; p, (p \;\&\; B\Pi(p, q)) \vee q)$
$A\Delta((q \vee \sim q) \;\&\; p, (p \;\&\; B\Pi(p, q)) \vee q)$.

Let W consist of four worlds, w_1, w_2, w_3, and w_4. The following describes a model for $LA(\bar{S})$.

V assigns T to p at w_1 and w_2
V assigns T to q at w_1 and w_3
V assigns T to $B\Pi(p, q)$ at w_1 and w_2
V assigns T to $A\Delta(q \vee \sim q, p \;\&\; B\Pi(p, q))$ at w_1 and w_2
V assigns T to $B\Delta(p, q)$ at w_1
V assigns T to $A\Pi(q \vee \sim q, p \;\&\; B\Pi(p, q))$ at all worlds.
Otherwise V assigns F to each wff in the skeleton.

I leave it to the reader to check that all the associated proper axioms are T at all worlds. Notice also that A is *maxipotent* at all worlds; that is, $M(A\Delta(\theta, \xi)) \supset A\Pi(\theta, \xi)$ is true at every world for every θ and ξ.

7 The extension theorem The following theorem shows that if one has a model for $\bar{S}$ using a *finite* set of possible worlds, one can extend the valuation from $\bar{S}$ to the set of all wffs in LA.

Theorem 3 (The Extension Theorem) *Suppose that $\bar{S}$ is a closed set, suppose that W is a* finite *set of possible worlds and V is a valuation such that the pair $\langle W, V\rangle$ is a model for $LA(S)$. Then there is a valuation $V^\dagger$ which agrees with V on S, such that the pair $\langle W, V^\dagger\rangle$ is a model for LA itself.*

Proof: The proof is in five parts. In part 1 a sequence $\bar{S}_0$, $\bar{S}_1$, $\bar{S}_2$, etc., of closed sets is defined whose union is the set of all wffs in LA. In part 2, the valuation V is extended from $\bar{S}$ to $\bar{S}_0$. In part 3, I describe the rule for extending the valuation from $\bar{S}_k$ to $\bar{S}_{k+1}$. Part 4 is a lemma for Part 5. In Part 5, I show that by using the rule stated in Part 3 one obtains models for $LA(\bar{S}_1)$, $LA(\bar{S}_2)$, etc. Since the valuation V_{k+1} for $LA(\bar{S}_{k+1})$ agrees with the valuation V_k for $LA(\bar{S}_k)$ on $\bar{S}_k$, one can then define $V^\dagger$ by: $V^\dagger$ assigns the same truth-values as V_k to members of $\bar{S}_k$. Since for any integer k the pair $\langle W, V_k \rangle$ is a model for $LA(\bar{S}_k)$, the pair $\langle W, V^\dagger \rangle$ is a model for LA itself, as required.

Part 1. The Closed Sets $\bar{S}_k$: $\bar{S}_0$ is the closure of S_0, where S_0 is the union of $\bar{S}$ and the set of all atomic wffs in LA. $\bar{S}_{k+1}$ is the closure of S_{k+1}, where S_{k+1} is the set of all wffs $X\Gamma(\alpha, \beta)$ such that α and β are members of $\bar{S}_k$. Since all wffs of degree at most k are in $\bar{S}_k$, the union of the $\bar{S}_k$ is the set of all wffs in LA.

Part 2. The Extension to $\bar{S}_0$: V_0, the valuation for $\bar{S}_0$, is defined thus:

(1) V_0 agrees with V on $\bar{S}$
(2) V_0 assigns F at every world to every atomic wff not in $\bar{S}$
(3) V_0 satisfies the rules V$\sim$, V$\vee$ and VL.

It is easy to check that V_0 is unambiguous and the pair $\langle W, V_0 \rangle$ is a model for $S5(\bar{S}_0)$. The only axioms in $\bar{S}_0$ which are not in $\bar{S}$ itself are instances of AS1. Hence all instances of AS2 to AS12 in $\bar{S}_0$ are T at all worlds, so the pair $\langle W, V_0 \rangle$ is a model for $LA(\bar{S}_0)$.

Part 3. The Extension from $\bar{S}_k$ *to* $\bar{S}_{k+1}$: Assume that the pair $\langle W, V_k \rangle$ is a model for $\bar{S}_k$. In order to construct the valuation V_{k+1}, it is convenient to have the following definition.

Definition If $X\Gamma(\gamma, \delta)$ is in $\bar{S}_k$ and $X\Gamma(\alpha, \beta)$ is in $\bar{S}_{k+1}$, $X\Gamma(\gamma, \delta)$ is said to *dominate* $X\Gamma(\alpha, \beta)$ if the following two conditions hold:

DOM 1 V_k assigns the value F at some world in W to $(\gamma \,\&\, X\Pi(\gamma, \delta)) \supset \beta$.
DOM 2 V_k assigns the value T to all worlds in W to $\alpha \equiv \gamma$ and $\delta \supset \beta$.

Note: *Since* V_k assigns T at all worlds to instances of AS7 in $\bar{S}_k$, if $X\Pi(\gamma, \delta)$ is in $\bar{S}_k$ it dominates itself unless V_k assigns to it F at all worlds.

The *extended valuation*, V_{k+1}, is obtained by assigning T at a world w to $X\Gamma(\alpha, \beta)$ in $\bar{S}_{k+1}$ if and only if there are γ, δ such that $X\Gamma(\gamma, \delta)$ is in $\bar{S}_k$, $X\Gamma(\gamma, \delta)$ dominates $X\Pi(\alpha, \beta)$ and $X\Gamma(\gamma, \delta)$ is assigned T at w by V_k. Truth-values are then assigned to all wffs in $\bar{S}_{k+1}$ in accordance with the rules V$\sim$, V$\vee$ and VL. It is easy to check that V_{k+1} is unambiguous and that the pair $\langle W, V_{k+1} \rangle$ is a model for $S5(\bar{S}_{k+1})$.

Part 4. A Lemma:

Lemma *If* $X\Gamma(\alpha, \beta)$ *is in* $\bar{S}_{k+1}$, *either* $X\Delta(\alpha, \beta)$ *and* $X\Pi(\alpha, \beta)$ *are assigned* F *at all worlds in* W *by* V_{k+1} *or there are* γ, δ *such that* $X\Gamma(\gamma, \delta)$ *is in* $\bar{S}_k$ *and* V_{k+1} *assigns the value* T *at all worlds in* W *to*:
(1) $\alpha \equiv \gamma$ *and* $\delta \supset \beta$; *and*

(2) *$X\Delta(\gamma, \delta) \equiv X\Delta(\alpha, \beta)$ and $X\Pi(\gamma, \delta) \equiv X\Pi(\alpha, \beta)$.*
Also $X\Gamma(\gamma, \delta)$ dominates $X\Gamma(\alpha, \beta)$.

Proof: By the construction of V_{k+1}, unless both $X\Delta(\alpha, \beta)$ and $X\Pi(\alpha, \beta)$ are F at all worlds, there will be at least one $X\Gamma(\gamma, \delta)$ in $\bar{S}_k$ dominating $X\Gamma(\alpha, \beta)$. Since W is *finite*, one can find finitely many wffs $X\Pi(\gamma_i, \delta_i)$, $i = 1 \ldots m$, in $\bar{S}_k$ such that: if w is in W, $X\Gamma(\alpha, \beta)$ is T at w iff one of the $X\Gamma(\gamma_i, \delta_i)$ is T at w.

Let γ be γ_1 & γ_2 &...& γ_m, let δ be $\delta_1 \vee \delta_2 \vee \ldots \vee \delta_m$. Then by CS6 $X\Gamma(\gamma, \delta)$ is in $\bar{S}_k$. One now obtains the following six sublemmas.

Sublemma 1 *For $i = 1 \ldots m$, V_k assigns F to $(\gamma_i \,\&\, X\Pi(\gamma_i, \delta_i)) \supset \delta$ and to $(\gamma_i \,\&\, X\Delta(\gamma_i, \delta_i)) \supset \delta$ at some world in w.*

Proof: Since V_k assigns T at all worlds to instances of AS5 in $\bar{S}_k$, it suffices to prove that $(\gamma_i \,\&\, X\Pi(\gamma_i, \delta_i)) \supset \delta$ is F at some world. By DOM 1, since $X\Gamma(\gamma_i, \delta_i)$ dominates $X\Gamma(\alpha, \beta)$, $(\gamma_i \,\&\, X\Pi(\gamma_i, \delta_i)) \supset \beta$ is F at some world w, hence $\gamma_i \,\&\, X\Pi(\gamma_i, \delta_i)$ is T at w and β is F at w. By DOM 2, $\delta_i \supset \beta$ is T at all worlds, so, since β is F at w, δ_i is F at w for $i = 1 \ldots m$. Hence δ is F at w and so $(\gamma_i \,\&\, X\Pi(\gamma_i, \delta_i)) \supset \delta$ is F at w.

Sublemma 2 *For $i = 1 \ldots m$, V_{k+1} assigns T at all worlds in W to $X\Gamma(\gamma_i, \delta_i) \supset X\Gamma(\gamma, \delta)$.*

Proof: By DOM 2, $\alpha \equiv \gamma_i$ is T at all worlds. Therefore $\gamma_i \equiv \gamma$ is T at all worlds. Also $\delta_i \supset \delta$ is T at all worlds. Hence

(1) $L(\gamma_i \equiv \gamma)$ and $L(\delta_i \supset \delta)$ are T at all worlds.

By Sublemma 1,

(2) $L((\gamma_i \,\&\, X\Gamma(\gamma_i, \delta_i)) \supset \delta)$ is F at all worlds.

Since all instances of AS2 and AS3 in $\bar{S}_k$ are T at all worlds the required result follows from (1) and (2).

Sublemma 3 *V_{k+1} assigns T at all worlds to $X\Gamma(\alpha, \beta) \supset X\Gamma(\gamma, \delta)$.*

Proof: If $X\Gamma(\alpha, \beta)$ is T at a world w, there is some i such that $X\Gamma(\gamma_i, \delta_i)$ is T at w. By Sublemma 2, $X\Gamma(\gamma_i, \delta_i) \supset X\Gamma(\gamma, \delta)$ is T at w, so $X\Gamma(\gamma, \delta)$ is T at w. Hence at any world w, $X\Gamma(\alpha, \beta) \supset X\Gamma(\gamma, \delta)$ is T.

Sublemma 4 *V_k assigns F at some world to $(\gamma \,\&\, A\Pi(\gamma, \delta)) \supset \beta$.*

Proof: $X\Gamma(\gamma_i, \delta_i)$ dominates $X\Gamma(\alpha, \beta)$, so by DOM 1

(1) $(\gamma_i \,\&\, X\Pi(\gamma_i, \delta_i)) \supset \beta$ is F at some world.

By Sublemma 2,

(2) $X\Pi(\gamma_i, \delta_i) \supset X\Pi(\gamma, \delta)$ is T at all worlds.

By DOM 2, $\alpha \equiv \gamma_i$ is T at all worlds so

(3) $\gamma_i \equiv \gamma$ is T at all worlds.

By (1), (2), and (3),

(4) $(\gamma \; \& \; X\Pi(\gamma, \delta)) \supset \beta$ is F at some world.

Sublemma 5 *$X\Gamma(\gamma, \delta)$ dominates $X\Gamma(\alpha, \beta)$.*

Proof: By DOM 2, $\alpha \equiv \gamma_i$ and $\delta_i \supset \beta$ are T at all worlds, hence $\alpha \equiv \gamma$ and $\delta \supset \beta$ are T at all worlds. The result now follows from Sublemma 4 and the definition of dominance.

Sublemma 6 *V_{k+1} assigns T at all worlds to $\alpha \equiv \gamma$, $\delta \supset \beta$, and $X\Gamma(\gamma, \delta) \supset X\Gamma(\alpha, \beta)$.*

Proof: Since by Sublemma 5, $X\Gamma(\gamma, \delta)$ dominates $X\Gamma(\alpha, \beta)$, this result follows from the construction of V_{k+1}.

The lemma now follows from Sublemmas 3, 5, and 6.

Part 5. The pair $\langle W, V_{k+1}\rangle$ is a model for $LA(\bar{S}_{k+1})$.

Proof: One has to prove that V_{k+1} assigns T to all instances of AS2 to AS12 at every world in w. I shall provide the proofs for AS3, AS7, and AS8, and leave the other proofs, which are either similar or straightforward consequences of the lemma, to the reader.

Proof for AS3: Suppose that:

(1) $\xi \supset \eta$ is T at all worlds in W; and
(2) $X\Gamma(\theta, \xi)$ in $\bar{S}_{k+1}$ is T at some world w in W; and
(3) $(\theta \; \& \; X\Pi(\theta, \xi)) \supset \eta$ is F at some world in W.

I shall show that $X\Gamma(\theta, \eta)$ is T at w.

By (2) and the lemma, there is some $X\Gamma(\gamma, \delta)$ in $\bar{S}_k$ such that

(4) $\theta \equiv \gamma$ and $\delta \supset \xi$ are T at all worlds, and
(5) $X\Gamma(\gamma, \delta) \equiv X\Gamma(\theta, \xi)$ is T at all worlds.

By (1) and (2),

(6) $\delta \supset \eta$ is T at all worlds.

By (3), (4), (5), and (6),

(7) $(\gamma \; \& \; X\Gamma(\gamma, \delta)) \supset \eta$ is F at some world in W.

By (4), (6), and (7),

(8) $X\Gamma(\gamma, \delta)$ dominates $X\Gamma(\theta, \eta)$.

By (2) and (5),

(9) $X\Gamma(\gamma, \delta)$ is T at w.

By (8), (9) and the construction of V_{k+1}

(10) $X\Gamma(\theta, \eta)$ is T at w.

Proof for AS7: Suppose that:

(1) $X\Pi(\theta, \xi)$ in $\bar{S}_{k+1}$ is T at a world w in W; and
(2) $X\Pi(\theta, \xi)$ & θ & $\sim\xi$ is F at all worlds.

I obtain a contradiction as follows: By (1) and the lemma there is some $X\Pi(\gamma, \delta)$ in $\bar{S}_k$, *dominating* $X\Pi(\theta, \xi)$, such that:

(3) $\theta \equiv \gamma$ and $\delta \supset \xi$ are T at all worlds; and
(4) $X\Pi(\theta, \xi) \equiv X\Pi(\gamma, \delta)$ is T at all worlds.

By (2), (3) and (4),

(5) $(\gamma \,\&\, X\Pi(\gamma, \delta)) \supset \xi$ is T at all worlds.

But (5) contradicts the dominance of $X\Pi(\gamma, \delta)$ over $X\Pi(\theta, \xi)$.

Proof for AS8: Suppose that:

(1) $X\Delta(\theta, \xi)$ and $Y\Delta(\phi, \xi)$ are both in $\bar{S}_{k+1}$ and T at w.

I obtain a contradiction as follows: By (1) and the lemma there are
$X\Delta(\gamma, \delta)$ and $Y\Delta(\zeta, \eta)$ *dominating*
$X\Delta(\theta, \xi)$ and $Y\Delta(\phi, \xi)$ respectively, such that:

(2) $\theta \equiv \gamma$, $\delta \supset \xi$, $\phi \equiv \zeta$ and $\eta \supset \xi$ are all T at all worlds; and

(3) $X\Delta(\theta, \xi) \equiv X\Delta(\gamma, \delta)$ and
$Y\Delta(\phi, \xi) \equiv Y\Delta(\zeta, \eta)$ are T at all worlds.

By CS6, $X\Delta(\gamma \,\&\, \gamma, \delta \vee \eta)$ and $Y\Delta(\zeta \,\&\, \zeta, \delta \vee \eta)$ are in $\bar{S}_k$, and since every instance of AS8 is $\bar{S}_k$ in T at all worlds,

(4) $X\Delta(\gamma \,\&\, \gamma, \delta \vee \eta) \,\&\, Y\Delta(\zeta \,\&\, \zeta, \delta \vee \eta)$ is F at all worlds.

Assume that:

(5) $X\Delta(\gamma \,\&\, \gamma, \delta \vee \eta)$ is F at w.

By the semantics of $S5$

(6) $L(\gamma \equiv (\gamma \,\&\, \gamma))$ is T at w.

By the semantics of $S5$,

(7) $L(\delta \supset \delta \vee \eta)$ is T at w.

By (6), (7) and the truth of all instances of AS2 and AS3 in $\bar{S}_{k+1}$ at w,

(8) $X\Delta(\gamma, \delta) \supset (X\Delta(\gamma \,\&\, \gamma, \delta \vee \eta) \vee L((\gamma \,\&\, X\Pi(\gamma, \delta)) \supset \delta \vee \eta))$ is T at w.

By (1) and (3),

(9) $X\Delta(\gamma, \delta)$ is T at w.

By (5), (8) and (9), and the semantics of $S5$,

(10) $(\gamma \,\&\, X\Pi(\gamma, \delta)) \supset \delta \vee \eta$ is T at all worlds

By (2),

(11) $(\delta \vee \eta) \supset \xi$ is T at all worlds.

By (10) and (11),

(12) $(\gamma \,\&\, X\Pi(\gamma, \delta)) \supset \xi$ is T at all worlds.

(12) contradicts the dominance of $X\Pi(\gamma, \delta)$ over $X\Pi(\theta, \xi)$. Similarly if one assumes that:

(13) $Y\Delta(\zeta \,\&\, \zeta, \delta \vee \eta)$ is F at w, one obtains a contradiction.

The required result follows from (4) and the two contradictions.

8 Decision procedures The Extension Theorem enables one to show that some wffs, for example, $A\Delta(q \vee {\sim}q, p \,\&\, B\Pi(p, q))$, are consistent. For example, in Section 7 I showed that if $\bar{S}$ was the closure of the set consisting of $A\Delta(q \vee {\sim}q, p \,\&\, B\Pi(p, q))$, there was a model for $LA(\bar{S})$ such that at some world $A\Delta(q \vee {\sim}q, p \,\&\, B\Pi(p, q))$ was T. By the Extension Theorem this model can be extended to a model for LA and so $A\Delta(q \vee {\sim}q, p \,\&\, B\Pi(p, q))$ is consistent.

I shall now describe a general method for deciding whether a wff θ is LA-valid. I shall also prove that LA is complete, so this is a decision procedure for consistency also.

By Theorem 2, if $S = \{\theta\}$, $\bar{S}$ has a finite skeleton $\{\beta_1, \ldots, \beta_k\}$ and a finite set of associated proper axioms, $\{\alpha_1, \ldots, \alpha_m\}$. One can define *S5-transforms* of wffs in $\bar{S}$ as follows, denoting the transform of ϕ by $\phi^\dagger$:

Rule 1. The transform of β_i in R is p_i. In particular if ϕ is an atomic wff in $\bar{S}$, by SK1, ϕ belong to R and so $\phi^\dagger$ has been defined.
Rule 2 If $X\Gamma(\gamma, \delta)$ is in $\bar{S}$, by SK2, there is a unique $X\Gamma(\xi, \eta)$ in R such that $\gamma \equiv \xi$ and $\delta \equiv \eta$ are tautologies; then $(X\Gamma(\gamma, \delta))^\dagger = (X\Gamma(\xi, \eta))^\dagger$.
Rule 3. $({\sim}\phi)^\dagger = {\sim}(\phi)^\dagger$.
Rule 4. $(\phi \vee \psi)^\dagger = \phi^\dagger \vee \psi^\dagger$.
Rule 5. $(L\phi)^\dagger = L(\phi^\dagger)$.

These rules define unambiguously the $S5$-transform of any wff in $\bar{S}$. The decision procedure is now a consequence of the following theorem.

Theorem 4 *If θ is any wff, let $\bar{S}$ be the closure of $\{\theta\}$ and define S5-transforms as above. Then, if $\alpha_1, \ldots, \alpha_m$ are the associated proper axioms, θ is $LA(\bar{S})$-valid iff*

$$(L(\alpha_1^\dagger) \,\&\, L(\alpha_2^\dagger) \,\&\ldots\&\, L(\alpha_m^\dagger)) \supset \theta^\dagger \textit{ is S5-valid}\ .$$

Proof: If θ is not $LA(\bar{S})$-valid, there is a model $\langle W, V\rangle$ for S such that V assigns the value F to θ at some world. One can construct a model $\langle W, V^\dagger\rangle$ for $S5$ where:

(1) If ϕ is in $\bar{S}$, $V^\dagger$ assigns the same values to $\phi^\dagger$ as V assigns to ϕ; and

(2) $V^\dagger$ assigns F to any atomic wff other than $p_1, \ldots, p_k$. Now $\alpha_1^\dagger, \ldots, \alpha_m^\dagger$ are T at all worlds, but $\theta^\dagger$ is F at some world, hence $(L(\alpha_1^\dagger) \,\&\, L(\alpha_2^\dagger) \,\&\ldots\&\, L(\alpha_m^\dagger)) \supset \theta^\dagger$ is F at some world and so is not $S5$-valid.

Conversely if $(L(\alpha_1^\dagger)\ \&\ L(\alpha_2^\dagger)\ \&\ldots\&\ L(\alpha_m^\dagger)) \supset \theta^\dagger$ is not $S5$-valid, there is a model $\langle W, V^\dagger\rangle$ for $S5$ such that

$$(L(\alpha_1^\dagger)\ \&\ L(\alpha_2^\dagger)\ \&\ldots\&\ L(\alpha_m^\dagger)) \supset \theta^\dagger$$

is F at some world. One can construct a model $\langle W, V\rangle$ for $S5(\bar{S})$ where V assigns the same values to ϕ as $V^\dagger$ assigns to $\phi^\dagger$. It follows that: $(L\alpha_1\ \&\ L\alpha_2\ \&\ldots\&\ L\alpha_m) \supset \theta$ is F at some world, hence $L\alpha_1, L\alpha_1, \ldots, L\alpha_m$ are T at some world and θ is F at some world. But since $L\alpha_1, \ldots, L\alpha_m$ are T at some world, all the associated proper axioms $\alpha_1, \ldots, \alpha_m$ are T at all worlds; hence, by Theorem 2, $\langle W, V\rangle$ is a model for $LA(\bar{S})$. Therefore θ is not $LA(\bar{S})$-valid.

I say that a wff is *finitely valid* if it is T at all worlds in all *finite* models (that is, models in which there occur only finitely many worlds). I now obtain:

Corollary to Theorem 4 *The result stated in Theorem 4 remains correct if we replace validity by finite validity.*

(The proof is similar to the proof of Theorem 4.)

Using Theorem 3 (The Extension Theorem), Theorem 4, and its corollary, we obtain the following sequence of results:

Theorem 5 *Suppose that $\bar{S}$ is the closure of $\{\theta\}$. Then θ is $LA(\bar{S})$-valid iff θ is $LA(\bar{S})$-finitely valid.*

Proof: $S5$ has the *finite model property* (see [1] or [4]). So a wff is $S5$-finitely valid iff it is $S5$-valid. Theorem 5 now follows from Theorem 4 and its corollary.

Theorem 6 *Let $\bar{S}$ be any closed set containing θ. Then θ is LA-valid iff it is $LA(\bar{S})$-valid.*

Proof: It suffices to show that θ *is* F at some world in some LA-model iff θ is F at some world in some $LA(\bar{S})$-model. We may restrict any LA-model to $\bar{S}$, providing an $LA(\bar{S})$-model. So it is obvious that if θ is F at some world in some LA-model then θ is F at some world in some $LA(\bar{S})$-model. Conversely, suppose θ is F at some world in some $LA(\bar{S})$-model. Then let $\bar{S}^*$ be the closure of $\{\theta\}$. We may restrict any $LA(\bar{S})$-model to $\bar{S}^*$. So θ is F at some world in some $LA(\bar{S}^*)$-model. Therefore, by Theorem 5 θ is F at some world in some finite $LA(\bar{S}^*)$-model. By the Extension Theorem we may extend the valuation of that model from $\bar{S}^*$ to LA itself, which shows that θ is F at some world in some LA-model.

Theorem 7 *LA is complete.*

Proof: By Theorem 6, θ is $LA(\bar{S})$-valid iff θ is LA-valid. Hence, by Theorem 4, θ is LA-valid iff $(L(\alpha_1^\dagger)\ \&\ L(\alpha_2^\dagger)\ \&\ldots\&\ L(\alpha_m^\dagger)) \supset \theta^\dagger$ is $S5$-valid. But $S5$ is complete. Therefore $(L(\alpha_1^\dagger)\ \&\ L(\alpha_2^\dagger)\ \&\ldots\&\ L(\alpha_m^\dagger)) \supset \theta^\dagger$ is $S5$-valid iff $(L(\alpha_1^\dagger)\ \&\ L(\alpha_2^\dagger)\ \&\ldots\&\ L(\alpha_m^\dagger)) \supset \theta^\dagger$ is a thesis of $S5$. But if $(L(\alpha_1^\dagger)\ \&\ L(\alpha_2^\dagger)\ \&\ldots\&\ L(\alpha_m^\dagger)) \supset \theta^\dagger$ is a thesis of $S5$, then $(L\alpha_1\ \&\ L\alpha_2\ \&\ldots\&\ L\alpha_m) \supset \theta$ is, by AS1, a thesis of $LA(\bar{S})$. Also $\alpha_1 \ldots \alpha_m$ are theses of LA and so, by repeated uses of AS1, RN, and MP, $L\alpha_1\ \&\ L\alpha_2\ \&\ldots\&\ L\alpha_m$ is a thesis of $LA(\bar{S})$. Hence by MP, θ is a thesis of $LA(\bar{S})$ and *a fortiori* a thesis of LA. Thus if θ is LA-valid θ is a thesis of LA; so LA is complete.

Corollary to Theorem 7 *If $\bar{S}$ is a closed set containing θ, then θ is a thesis of $LA(\bar{S})$ iff θ is a thesis of LA.*

Proof: By a proof similar to that for Theorem 7 we can show that $LA(\bar{S})$ is complete. So, θ is a thesis of $LA(\bar{S})$ iff θ is $LA(\bar{S})$-valid. Therefore, by Theorem 6, θ is a thesis of $LA(\bar{S})$ iff θ is LA-valid. But LA is complete, so θ is a thesis of $LA(\bar{S})$ iff θ is a thesis of LA.

9 The powers of God I now turn to the problem of characterizing the powers of God. (What I say owes much to recent discussions of the Stone Paradox (see, among others, [5], p. 210; [6], pp. 221–223; [9], pp. 74–79; and [7], pp. 163–173).)

First note the failure of attempts to characterize God's powers which are not relative to what is necessary at the time or to what other acts have been performed[5]: For example, suppose we said:

$$G\Pi_t\theta \text{ unless it is logically impossible that } G\Delta_t\theta \ .$$

Then we would have the following absurdity: Suppose God freely acts at time t_1 and the immediate effect of this act is that a Big Bang occurs at a later time t_2. (Such a time gap is not, I take it, logically impossible.) Then it is not the case that God has the power to produce the Big Bang at time t_2, He has already produced it. But it is logically possible that He produce the Big Bang at time t_2 because it is logically possible that He had not earlier done so. This contradicts the proposed characterization of God's powers.

For a nonabsurd characterization, then, we require something like:

$$G\Pi_t\xi \text{ unless } L_t \sim G\Delta_t\xi \ .$$

This is not stated in terms of logical necessity at all, and so is not, I suppose, an explication of *omnipotence.* However, using the formalization LA we can produce (in the metalanguage) a nonabsurd explication of omnipotence:

$$G\Pi(\theta, \xi) \text{ unless it is logically impossible that } G\Delta(\theta, \xi) \ .$$

We also have a characterization using eternal necessity, analogous to '$G\Pi_t\xi$ unless $L_t \sim G\Delta_t\xi$': I say G is *maxipotent* just in case we have the truth-schema:

$$MG\Delta(\theta, \xi) \supset G\Pi(\theta, \xi) \ .$$

The omnipotence characterization might seem to ascribe *more* power to God than maxipotence. But the converse of maxipotence, namely

$$G\Pi(\theta, \xi) \supset MG\Delta(\theta, \xi)$$

is a subschema of AS6. So omnipotence and maxipotence coincide as descriptions of God's powers, provided we already know which truths are eternally necessary. The difference between the two characterizations is that, because of AS6, the omnipotence characterization puts a constraint on which truths are eternally necessary. For instance, but for the omnipotence characterization, I think we would say that God's existence is eternally necessary, so He cannot destroy himself. But if, as is believed by many, God's existence is *logically* contingent, it seems plausible that His continued existence is logically contingent, so, relying on the omnipotence characterization, He would have to be able to destroy Himself. Thus His continued existence would not be eternally necessary. The omnipotence constraint strikes me as a back-to-front way of discovering what is eternally necessary. It seems more straightforward first to consider what is eternally necessary and then characterize God's powers using maxipotence. I suggest that the theist construct a *hypothesis* about what is eternally necessary, based on the principle that God and His attributes are eternally necessary. There is then a *presumption* against introducing eternal necessities not entailed by God's existence and attributes.

I have reasons, then, for preferring a characterization of God's powers in terms of maxipotence. But if you reject my reasons, we could *interpret* eternal necessity as logical necessity. In that case, maxipotence and omnipotence are identified. So in either case it is of interest whether or not maxipotence is coherent. Furthermore, for those of us who reject predestination, it is important to determine whether or not God's maxipotence is compatible with the freedom of other agents. The results of this paper provide an argument to show they are compatible. For, using the example in Section 7, we can show that the maxipotence of G is consistent with our axioms and with $G\Delta(q \vee {\sim}q, p \,\&\, B\Pi(p, q))$. So unless there are relevant intuitions about free acts not captured by the axioms, we have shown that the maxipotence of God is compatible with His granting some other agent (B) the power to act freely. Now, of course, I cannot be *sure* there is not some further relevant intuition. So I conclude my paper by challenging anyone who doubts the coherence of my characterization of God's powers to provide a relevant intuition.

NOTES

1. I do not here distinguish metaphysical necessity (truth at all worlds) from logical necessity.
2. For an elegant argument which seems to show this see [3].
3. My use of the pronoun 'He' for God is, I claim, correct English. But it might seem to suggest what I do not want to suggest, namely that God is masculine rather than feminine in some analogous sense. The alternatives such as 'He or She', 'She', and 'It' would also have unwanted apparent implications.
4. By a *real* change I mean coming to have or to lose a nonrelational property.
5. That God's powers should be characterized in a time-relative fashion has also been argued recently by several authors. See [2], [8], and [10]. One notable difference between my approach and that of these authors is that I characterize God's powers as relative to what is *necessary* at a given time, rather than as relative to the *history* up to that time.

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