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The Modal Logic of 'All and Only'

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1 The modal logics of 'All' and 'Only' We work with the customary language of modal propositional logic, in which formulas are built in the usual way by application of some functionally complete set of truth-functional primitive connectives alongside the singulary connective '$\Box$', from a stock of sentence letters (or 'propositional variables'), of which there are taken to be countably many. For further background and terminology not explained here, see [6]. This language is interpreted by means of frames $\langle W, R\rangle$ and models $\langle W, R, V\rangle$ thereon, with various options being open for the definition of truth of a formula A at a point x in such a model, notated '$\mathfrak{M} \models_x A$' (where $\mathfrak{M} = \langle W, R, V\rangle$ and $x \in W$). We consider only variations on the clause governing $\Box$-formulas in the otherwise standard inductive definition of the $\models$-relation. The contrast between the following pair of clauses, of which the first figures in the standard definition, is quite interesting:

[All] $\mathfrak{M} \models_x \Box A$ iff for all $y \in W$, if xRy then $\mathfrak{M} \models_y A$
[Only] $\mathfrak{M} \models_x \Box A$ iff for all $y \in W$, if $\mathfrak{M} \models_y A$ then xRy

The weakest logic on the 'all' semantics—the system, that is, which is determined by the class of all frames when truth at a point in a model is as dictated by [All]—is of course the system K, while the logic occupying a similarly minimal position when the 'only' semantics is in force is the system of Karmo in [4], called Anti-K. Recapitulating the details relevant to our present purposes, we recall that K may be axiomatized by closing the class of substitution instances of nonmodal tautologies under modus ponens and the rule:

$$\text{[K]} \quad \frac{(A_1 \wedge \ldots \wedge A_n) \rightarrow B}{(\Box A_1 \wedge \ldots \wedge \Box A_n) \rightarrow \Box B},$$

while for Anti-K this rule is replaced by:

$$\text{[Anti-}K\text{]} \quad \frac{A \rightarrow (B_1 \vee \ldots \vee B_n)}{(\Box B_1 \wedge \ldots \wedge \Box B_n) \rightarrow \Box A}.$$

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In these schematic presentations of the rules, $n \geq 0$; for the $n = 0$ case, we note that an implication with an empty conjunction as antecedent is identified with its consequent, and one with an empty disjunction as consequent, with the negation of its antecedent. (At most, one of these cases arises for any given implication in the application of the above rules.) Evidently the rules preserve the property of being valid on a given frame when truth is defined with [All] or [Only] respectively, from which the soundness part of the above determination claims follows. Completeness is proved by a consideration of the 'canonical' model $\langle W, R, V \rangle$ for the system in question, whose W consists of all maximal consistent (for the given system) sets of formulas and whose R is defined in the case of the completeness proof for K with respect to (w.r.t.) the 'all' semantics by:

(1) xRy iff for all $\Box B \in x$, $B \in y$

and for Anti-K w.r.t. the 'only' semantics by:

(2) xRy iff for some $\Box B \in x$, $B \in y$.

The valuation V is then defined so as to secure that for atomic A we have $\langle W, R, V \rangle \models_x A$ iff $A \in x$, a property which extends automatically upward through the various modes of truth-functional composition (thanks to x's being maximal consistent w.r.t. a system extending truth-functional logic) leaving the inductive step for '$\Box$' which is handled in the 'membership-to-truth' direction via definition (1) for K and (2) for anti-K (together with the inductive hypothesis), and in the 'truth-to-membership' definition by an appeal to the distinguishing rule in the above axiomatizations of the two systems. The credit, incidentally, for the key novelty here—the definition of R by (2) for the canonical model for Anti-K—should go to Vander Nat, who put it to a somewhat similar use at p. 634 of [7].

What interest, the reader may wonder, attaches to deriving the completeness result for Anti-K as above, with a construction parallelling that involved for the canonical model completeness proof for K, when the result could be obtained instead by reducing the case of Anti-K to that of K? The obvious reduction proceeds by considering the variation on the clause [All] got by replacing 'xRy' on its right-hand side with 'not xRy'. Call this: [$\overline{\text{All}}$]. What is the minimal system on the [$\overline{\text{All}}$] semantics? Clearly none other than K itself, since the complement of a binary relation is again a binary relation, and every binary relation is the complement of some binary relation. (Evaluation at x in $\langle W, R, V \rangle$ by [$\overline{\text{All}}$] is equivalent to evaluation at x in $\langle W, \bar{R}, V \rangle$ by [All], where the overlining indicates complementation. The fact that $\bar{\bar{R}} = R$ is not needed for this argument.) To digress for a moment: although [$\overline{\text{All}}$] thus brings by itself no new logic, some interest may attach to the question of the weakest logic (or indeed its extensions) in a bimodal language whose two operators are interpreted, one with the aid of [All], the other with [$\overline{\text{All}}$], since axiomatic account must then be taken of their interaction. This topic was treated in [2], where the operators concerned were written as $\Box$ and $\blacksquare$ respectively. In what follows, however, we shall be concerned with the monomodal language, though we shall have occasion, in Section 2, to reach for bimodal frames for its interpretation. Digression over. To resume our discussion of reducing the completeness question for Anti-K to that for K, we now observe that if a formula $\Box{\sim}A$ is evaluated in accordance with $[\overline{\text{All}}]$, the effect is precisely the same as if $\Box A$ were evaluated in accordance with [Only], so that (since the weakest logic for [All] is K) the class of formulas valid under [Only] is that consisting of K-theorems with '$\Box$' replaced everywhere by '$\Box\sim$'. (This is why the rule [Anti-K] has the form it has: think of its application as the result of contraposing, De-Morganing, and then applying the rule [K] but attaching '$\Box\sim$' instead of '$\Box$'. This 'reduction' theme is taken up in [4] with Karmo's discussion of the translation $*$; see also the top of p. 347 in [2].) Now, apart from the fact that this urge to reduce the one case to the other (as opposed to merely relating the two) seems incorrectly to suggest a conceptual priority of [All] over [Only], as well as to require (as noted in [4]) for its execution the presence of negation in the language, and so not to adapt to more expressively austere settings in which the direct canonical model proof is quite at home, the interest for us of what I have just called the direct approach (using definition (2), etc.) is that it provides the ingredients for the solution of another completeness question, and one that will occupy us for the remainder of this article.

Our purpose, taking up a matter raised in note 2 of [2], will be to find the weakest logic for a clause on '$\Box$' which is something of a composite version of [All] and [Only]. While they required, for the truth of a $\Box$-formula at a point, that the immediate subformula be true at all accessible points, or at only accessible points, respectively, we now consider the requirement of truth at *precisely* the accessible points:

[All-&-Only] $\mathfrak{M} \models_x \Box A$ iff for all $y \in W$, $\mathfrak{M} \models_y A$ iff xRy.

We work our way toward this end, achieved in Section 4, via two weaker logics in Sections 2 and 3, the second an extension of the first and a subsystem of the final system. Of course, in the language of [2], as described above, we can get the force of such an operator by writing instead '$\Box A \wedge \blacksquare{\sim}A$'. But it is another matter to provide a complete axiomatization in the present monomodal language, in which what is thus conjunctively represented is taken as an indissoluble unit. While on the subject of such possibilities of definition, it may be worth noting that if the standard modal language (with the [All] semantics), whose primitive modal operator I shall here write 'L' to avoid confusion with '$\Box$' for which the interpretation given by [All-&-Only] is intended, is equipped with propositional quantifiers ranging over arbitrary sets of points in the models and a propositional identity connective '$=$' (with $A = B$ true at a point if the truth-sets of A and B in the given model coincide), then one could define $\Box A$ by: $\forall q(Lq \leftrightarrow (A = (A \wedge q)))$. The proposition expressed by A is being said by the definition to be included in a proposition q iff the proposition q is necessary in the usual (i.e., [All]) sense. It does not appear to be possible to effect such a definition without recourse to propositional identity or something interdefinable therewith. (This connective '$=$' is of course itself definable in the language of [2] by: $\Box(A \leftrightarrow B) \wedge \blacksquare(A \leftrightarrow B)$.)

2 *The system AO* Our route toward the minimal logic for [All-&-Only] is one which involves replacing that clause by something more general, because it proves easier to examine the more general version and then re-specialize. We consider bimodal frames $\langle W, R_1, R_2\rangle$ even though only one modal operator is present in the language (cf. [7]). Let $\mathfrak{M}$ be a model on such a frame; put:

[All$_1$-&-Only$_2$] $\mathfrak{M} \models_x \Box A$ iff for all y such that xR_1y, $\mathfrak{M} \models_y A$, and for all y such that $\mathfrak{M} \models_y A$, xR_2y.

The idea is that once we have the logic determined by the class of all such frames with this definition of truth in force, the provision of which will be the main business of the present section, we proceed to seek the extension thereof which is determined by the class of all those frames $\langle W, R_1, R_2\rangle$ satisfying the further condition that $R_1 = R_2$; clearly this is the minimal logic for the all-and-only semantics introduced at the end of Section 1. The background provided in that section enables us to despatch fairly promptly the question of the system determined by the class of all bimodal frames (no special conditions on the R_i) with the clause [All$_1$-&-Only$_2$] in place in the definition of $\models$, as it will be for the remainder of this article.

We axiomatize a system AO (for '<u>A</u>ll$_1$ and <u>O</u>nly$_2$') as we axiomatized K and anti-K in Section 1, except that instead of [K] or [Anti-K], the modal rule is the following amalgamation of those two rules:

$$\text{[AO]} \quad \frac{(B_1 \wedge \ldots \wedge B_m) \to C \quad C \to (A_1 \vee \ldots \vee A_n)}{(\Box A_1 \wedge \ldots \wedge \Box A_n \wedge \Box B_1 \wedge \ldots \wedge \Box B_m) \to C}$$

That this two-premise rule preserves validity on any frame according to the semantics just introduced involves a check that may be left to the reader, settling the matter of AO's soundness (on this semantics) with respect to the class of all frames. For completeness with respect to this same class, thereby qualifying AO as the minimal logic in the present semantic setting, we use, not surprisingly, a canonical model argument (close to that of [7], in fact). The canonical model for AO is to be $\mathfrak{M} = \langle W, R_1, R_2, V\rangle$, with W and V as usual, and R_1 and R_2 defined by definitions (1) and (2), respectively, of R in Section 1. In showing that truth and membership coincide, we find that these definitions render automatic the membership-to-truth direction for $\Box$-formulas, as in the cases discussed in Section 1. The converse implication is established by appeal to [AO]. In more detail: suppose $\mathfrak{M} \models_x \Box C$. Then

(i) for all $y \in W$, xR_1y implies $\mathfrak{M} \models_y C$, and so implies (induction hypothesis) $C \in y$, for each such y. This, by the definition of R_1, means that the set $\{B\colon \Box B \in x\} \cup \{{\sim}C\}$ is AO-inconsistent, so that for some m, where the B_i are drawn from the left-hand term of this union, the formula $(B_1 \wedge \ldots \wedge B_m) \to C$ is provable in AO;

and also:

(ii) for all $y \in W$, $\mathfrak{M} \models_y C$ implies xR_2y, and so (induction hypothesis) $C \in y$ implies xR_2y. This means, by the definition of R_2 that the set $\{{\sim}A\colon \Box A \in x\} \cup \{C\}$ is AO-inconsistent, so that for some n, where the A_i are drawn from the left term of this union, the formula $C \to (A_1 \vee \ldots \vee A_n)$ is provable in AO.

The formulas we concluded under (i) and (ii) to be provable in AO then provide the premises for an application of the rule [AO], which gives as conclusion the formula $(\Box A_1 \wedge \ldots \wedge \Box A_n \wedge \Box B_1 \wedge \ldots \wedge \Box B_m) \to \Box C$, and since each conjunct in the antecedent belongs to x, we infer that $\Box C \in x$ also. This is as much as needs to be given by way of proof for

Theorem 1 *The system* AO *is determined by the class of all frames (on the [All_1-&-$Only_2$] semantics).*

The remainder of this section will be taken up with adapting two familiar results from the [All] tradition so that we can use them in subsequent sections, as well as stating one novel lemma of little independent interest except that it will be needed in Section 3. First, we consider generated models (cf. [6]).

Definition If $\langle W, R_1, R_2, V\rangle$ is a model with $x \in W$, then the *submodel of* $\langle W, R_1, R_2, V\rangle$ *generated by* x is to be the structure $\langle W^x, R_1^x, R_2^x, V^x\rangle$ in which W^x is the set of all points from W to which x bears the ancestral (= reflexive transitive closure) of $R_1 \cup \bar{R}_2$; R_1^x, R_2^x and V^x are the restrictions to W^x of R_1, R_2 and V.

(Point to note here: although it is the restriction of R_2 itself that figures as the second accessibility relation in the reduced structure, it is the complement of that relation which is used (in union with R_1) to generate from x the set of worlds on which the reduced structure is based.)

Generation Lemma *If* $\mathfrak{M}^* = \langle W^x, R_1^x, R_2^x, V^x\rangle$ *is the submodel of* $\mathfrak{M} = \langle W, R_1, R_2, V\rangle$ *generated by some element* $x \in W$, *then for all* $y \in W^x$:

$$\mathfrak{M}^* \models_y A \textit{ iff } \mathfrak{M} \models_y A, \textit{ for all formulas } A.$$

Proof: by induction on formula complexity.

The second technique to be adapted is taken from Sahlqvist [5], and consists in deriving from a generated model an equivalent ('unravelled') model with certain convenient properties to be exploited below (Observation 1). Let $\mathfrak{M} = \langle W, R_1, R_2, V\rangle$ be a model generated by x (that is, let it be the submodel generated by x of some model containing x):

Definition $\tilde{\mathfrak{M}} = \langle \tilde{W}, \tilde{R}_1, \tilde{R}_2, \tilde{V}\rangle$ is the *unravelling of* $\mathfrak{M}$ *from* x when:

W is the set of all $(R_1 \cup \bar{R}_2)$-chains from x, i.e., those sequences $\langle u_1, \ldots, u_n\rangle$, $n \geq 1$, with $u_1 = x$ and either $u_i R_1 u_{i+1}$ or $u_i \bar{R}_2 u_{i+1}$. We use 'a', 'b' as variables over such sequences, with '$\dot{a}$' to denote the last element in a, and '$a^\frown u$' ($u \in W$) for the sequence $\langle u_1, \ldots, u_n, u\rangle$ where $a = \langle u_1, \ldots, u_n\rangle$.

We define $a\tilde{R}_1 b$ to hold iff for some $u \in W$, $b = a^\frown u$ and $\dot{a}R_1 u$, and $a\tilde{R}_2 b$ to hold iff for all $u \in W$, if $b = a^\frown u$, then $\dot{a}R_2 u$.

$\tilde{V}(p_i) = \{a \in \tilde{W}: \dot{a} \in V(p_i)\}$.

Then we have, as an induction on formula-complexity will again reveal, the

Unravelling Lemma *With* $\mathfrak{M}$ *and* $\tilde{\mathfrak{M}}$ *as in the above definition, for all* $a \in W$:

$$\tilde{\mathfrak{M}} \models_a A \textit{ iff } \mathfrak{M} \models_{\dot{a}} A, \textit{ for all formulas } A.$$

The convenient feature of such unravelled models (or more accurately, the frames thereof) to which we shall have occasion to appeal in Section 3 we state here as:

Observation 1: In an unravelled model $\langle W, R_1, R_2, V\rangle$ where $S, T \in \{R_1, \bar{R}_2\}$ if both aSb and $a'Tb$, then $a = a'$.

The final tool we shall need for our discussion of a special modal rule in Section 3 involves a sequence of formulas.

Definition For any formula B we define the following sequence of formulas:

$$\Phi_0(B) = \Box B$$
$$\Phi_{n+1}(B) = \Box(B \leftrightarrow \Phi_n(B)).$$

Φ-Sequence Lemma *Let* $\mathfrak{M} = \langle W, R_1, R_2, V\rangle$ be a model generated by $x \in W$. Then $\mathfrak{M} \models_x \Phi_i(B)$ *for all* $i \in Nat$ *iff for all* $y \in W$, $\mathfrak{M} \models_y \Box B$.

Proof: Show by induction on n that $\mathfrak{M} \models_x \bigwedge_{i=0}^{n} \Phi_i(B)$ iff for all y such that $d(x, y) \leq n$, $\mathfrak{M} \models_y \Box B$, where the '$\bigwedge$' notation is for iterated conjunction and $d(x, y)$ is the least k such that $x(R_1 \cup \bar{R}_2)^k\, y$.

3 The condition $R_1 \subseteq R_2$ In this section, our quarry is the logic determined by the condition on frames $\langle W, R_1, R_2\rangle$ that $R_1 \subseteq R_2$. The converse inclusion will be added in Section 4, completing our task. To avoid undue proliferation of subscripts, we shall call the following system $AO_{1\subseteq 2}$ (deleting the 'R's, in other words). We add then to the basis given for AO in Section 2 a rule somewhat in the style of Gabbay [1]:

[Φ-rule] $$\frac{A \rightarrow {\sim}(\Phi_0(q) \wedge \ldots \wedge \Phi_n(q))}{{\sim}A}$$ Provided the variable q does not appear in the formula A.

(Strictly, we have an infinite collection of rules here, one for each $n \in Nat$.)

We devote this section to an extended proof, with some informal discussion, of:

Theorem 2 *The system* $AO_{1\subseteq 2}$ *is determined by the class of all frames* $\langle W, R_1, R_2\rangle$ *in which* $R_1 \subseteq R_2$.

The idea behind the Φ-rule can best be explained if we begin by noting the following immediate consequence of the [All_1-&-$Only_2$] truth-definition:

Observation 2: For any model $\mathfrak{M} \models \langle W, R_1, R_2, V\rangle$: $R_1 \subseteq R_2$ if for each $y \in W$ there is some formula B such that $\mathfrak{M} \models_y \Box B$.

From this observation, we could secure the completeness result we are seeking if we could show that each $AO_{1\subseteq 2}$-consistent formula can be verified at a point in a model each point of which verified some $\Box$-formula. The Φ-rule will enable us to derive the $\exists\forall$-form of this $\forall\exists$-statement: we pick a $\Box$-formula, depending only on the form of the consistent formula in question, which formula will be true at every point of some model housing a point at which the consistent formula is true. Elaborating this a little: suppose A is a formula whose negation is unprovable in $\mathrm{AO}_{1\subseteq 2}$, and q is a variable absent from A. We extend not simply $\{A\}$ but rather the set $\{A, \Phi_0(q), \ldots, \Phi_n(q), \ldots\}$ to a set which is maximal consistent with respect to the system. But how do we know—as is presumed here for the invoking of Lindenbaum—that this infinite set is consistent to begin with? Because if it is not, then the Φ-rule (for some choice of n) will apply to give the conclusion that A's negation was provable in the system, contradicting the assumption we made about A. Let x be a point in the canonical model for the system that our infinite set, just seen to be consistent, maximally extends to, and consider the submodel of that model generated by x. By the Φ-Sequence Lemma ('only if' direction) each point in the generated submodel verifies $\Box q$, so, by Observation 2, $R_1^x \subseteq R_2^x$, if we may so refer to the relations of the generated model. Since we have now shown how to verify an arbitrary $\mathrm{AO}\text{-}_{1\subseteq 2}$-consistent formula in a model on a frame meeting the inclusion condition figuring in the title of this section, our proof of completeness is concluded.

The soundness half of the claim made by Theorem 2, however, requires more attention than usual. To prove that the Φ-rule preserves validity on each frame $\langle W, R_1, R_2\rangle$ in which $R_1 \subseteq R_2$—and for brevity we refer to the class of such frames as C—let us suppose otherwise, i.e., that we have some frame $\langle W, R_1, R_2\rangle \in C$ on which a premise for some application of the rule is valid but the conclusion is not, so that for some valuation V, and some $x \in W$, $\langle W, R_1, R_2, V\rangle \models_x A$, even though for all V' and all $y \in W$, $\langle W, R_1, R_2, V'\rangle \models_y A \rightarrow {\sim}(\Phi_0(q) \wedge \ldots \wedge \Phi_n(q))$. Call the unravelling from x of the submodel of $\langle W, R_1, R_2, V\rangle$ generated by x, $\langle \tilde{W}, \tilde{R}_1, \tilde{R}_2, \tilde{V}\rangle$. By the Generation and Unravelling Lemmas, we have:

(i) $A \rightarrow {\sim}(\Phi_0(q) \wedge \ldots \wedge \Phi_n(q))$ is valid on the frame $\langle \tilde{W}, \tilde{R}_1, \tilde{R}_2\rangle$
(ii) $\langle \tilde{W}, \tilde{R}_1, \tilde{R}_2, \tilde{V}\rangle \models_{\langle x\rangle} A$.

Notice that if (as we are assuming) $\langle W, R_1, R_2\rangle \in C$, the definitions of the processes of generation and unravelling imply:

(iii) $\langle \tilde{W}, \tilde{R}_1, \tilde{R}_2\rangle \in C$ (i.e., $\tilde{R}_1 \subseteq \tilde{R}_2$).

Now consider that valuation $\tilde{V}'$ which is like $\tilde{V}$ on each variable except for the variable q featured here in the application of the Φ-rule, and which is such that $\tilde{V}'(q) = \{v \in \tilde{W}\colon u\tilde{R}_1 v \text{ for some } u \in \tilde{W}\}$. Since q does not appear in the formula A, we have $\langle \tilde{W}, \tilde{R}_1, \tilde{R}_2, \tilde{V}'\rangle \models_x A$, by (ii). We now claim that the formula $\Box q$ is true at every point in this new model. How could the formula be false at a point $w \in \tilde{W}$? One possibility would be because w bears $\tilde{R}_1$ to some point z at which q is false: but this is ruled out by the way $\tilde{V}'$ was defined. The other possibility would be for w to fail to bear $\tilde{R}_2$ to a point z at which q is true. But to have $z \in V'(q)$ there must be some point in $\tilde{W}$ bearing $\tilde{R}_1$ to z, and since we are in an unravelled model and do not have $w\tilde{R}_2 z$, this point can only be w itself (Observation 1, from Section 2). This means w bears $\tilde{R}_1$ to z without bearing $\tilde{R}_2$ to z, contradicting the fact (iii) that the frame of the model belongs to C. (This is the only point at which the condition that $R_1 \subseteq R_2$ is appealed to in the whole argument.) $\Box q$ is, then, true at each point in the model $\langle \tilde{W}, \tilde{R}_1, \tilde{R}_2, \tilde{V}'\rangle$, so by the Φ-Sequence lemma ('if' direction) each of the formulas $\Phi_0(q), \ldots, \Phi_n(q)$ is true at x in $\langle \tilde{W}, \tilde{R}_1, \tilde{R}_2, \tilde{V}'\rangle$. Since the conjunction of these formulas is true at x at that point, and so also is the formula A, we have reached a contradiction with (i).

Having now shown that the system $\mathrm{AO}_{1\subseteq 2}$ is determined by the class of all frames in C, I must apologize for the heavy reliance on the somewhat unwieldy Φ-rule: it was the only way I could think of to get out the proof of completeness. The possibility remains, nonetheless, that the systems AO and $\mathrm{AO}_{1\subseteq 2}$ do not differ as to theorems, and that this rule can be shown to be admissible in AO (like a Gabbay-style irreflexivity rule for the modal system K). I have no further information on this subject, however.

4 The condition $R_1 = R_2$ It remains to extend the system considered in the previous section to that determined by the class of all $\langle W, R_1, R_2\rangle$ in which $R_1 = R_2$; as noted in Section 1, this will give us the weakest logic for the [All-&-Only] (as opposed to [All_1-&-Only_2]) semantics. As in that section, the axiomatization leaves something to be desired with respect to simplicity, though a few abbreviated definitions will make the presentation of the proof-theory rather less painful. We assume in giving these definitions that $\Diamond A$ is itself defined in the usual way, as $\sim\Box\sim A$, the upshot for the truth-conditions of $\Diamond$-formulas being that $\Diamond A$ is true at a point iff either some R_1-related point verifies A or else some R_2-unrelated point falsifies A. We now introduce four binary connectives by the definitions:

$$A \ \nabla^+ \ B =_{df} \Box A \wedge \Diamond(A \rightarrow B)$$
$$A \ \nabla^- \ B =_{df} \Box A \wedge \Diamond\sim(A \vee B)$$
$$A \ \Delta^+ \ B =_{df} \Box A \rightarrow \Box(A \wedge B)$$
$$A \ \Delta^- \ B =_{df} \Box A \rightarrow \Box(A \vee \sim B).$$

With these four connectives, we shall be able to simulate, in a restricted way, quantification over R_1-related points as well as over R_2-unrelated points. While we cannot unrestrictedly express these concepts (e.g., find, for any given formula B, a formula true at a point iff some point R_1-related to that point has B true at it), some pencil and paper checking will reveal that, for any formulas A and B, *if* $\Box A$ is true at a point x in some model $\langle W, R_1, R_2, V\rangle$, *then*:

> A ∇^+ B (A ∇^- B) is true at x iff B is true at some point R_1-related ($\bar{R}_2$-related) to x in the model.

And:

> A Δ^+ B (A Δ^- B) is true at x iff B is true at all points R_1-related ($\bar{R}_2$-related) to x in the model.

These are the truth-conditions the formulas listed receive straight from the force of [All_1-&-Only_2], without any special notice being taken of the conditions $R_1 \subseteq R_2$ or $R_1 = R_2$ of the previous or the present section. Thus, they could have been stated in Section 2, had there been any point in doing so. (I have stated them here because they will be exploited in both halves of the proof of Theorem 3.) Of course if attention is restricted to frames meeting the latter condition—and, as we introduced 'C' for Section 3's class of frames, we refer to the collection of all such frames as C' for brevity—then the truth-conditions listed can be further simplified as the relations R_1 and R_2 need no longer be distinguished. Before proceeding to the axiomatization, we make a further observation on the basis of the [All_1-&-Only_2] clause, as it applies in the $R_1 = R_2$ setting:

Observation 3: In a model on a frame in C', if $\Box A \wedge \Box B$ is true at any point, then A and B are true at precisely the same points.

The reason is that the conjuncts say that the truth-set of A coincides with the set of points related to the given point, and so does the truth-set of B. Note that here we do not fuss about R_1- vs. R_2-relatedness, since the distinction has collapsed. (And Observation 3, of course, holds for the original [All-&-Only] semantics of Section 1.)

We now axiomatize the final system, to be called $\text{AO}_{1=2}$, by extending the basis given for $\text{AO}_{1\subseteq 2}$ in Section 3 with all instances of the following schema:

(*) $C_1\ \nabla_1\ (C_2\ \nabla_2\ (\ldots(C_m\ \nabla_m\ (\Box A \wedge \Box B))\ldots)) \to D_1\ \Delta_1\ (D_2\ \Delta_2\ (\ldots (D_n\ \Delta_n\ (A \to B))\ldots))$

where each ∇_i (Δ_i) is either ∇^+ or ∇^- (either Δ^+ or Δ^-).

Then we have:

Theorem 3 $\text{AO}_{1=2}$ *is determined by* C'.

Proof: We begin with a proof of soundness. To see that no instance of the schema (*) is false at any point in a model on a frame in C', note that the truth of the antecedent at a point x requires there to be some $(R_1 \cup \bar{R}_2)$-chain of length m from x to some point at which $\Box A \wedge \Box B$ is true. As Observation 3 reminds us, this is a sufficient condition for A and B to have the same truth-value at all points in the model, and so for $A \to B$ to be true at each point, and therefore, in particular, true at each point reachable by an $(R_1 \cup \bar{R}_2)$-chain of length n from x, which suffices for the truth (at x) of the consequent.

Turning to completeness, suppose A is a formula consistent with $\text{AO}_{1=2}$. We will show how to verify A at a point in a model on a frame in C'. We know from Section 3 that we can form a generated submodel of the canonical model for any extension of AO closed under the Φ-rule in which the given consistent formula A is true at the generating point, with the formula $\Box q$ true at every point of the submodel for some atomic formula q not occurring in A, and whose relations R_1 and R_2 satisfy the condition $R_1 \subseteq R_2$ of that section. Accordingly, let x be a point of the canonical model (the R_i defined as in Section 2) for $\text{AO}_{1=2}$ at which the formula A is true, and let the model generated by x, meeting this inclusion condition and throughout which $\Box q$ is true, be $\langle W, R_1, R_2, V\rangle$. Our proof will be complete if we can show that the converse inclusion, $R_2 \subseteq R_1$ also holds for this model. So suppose otherwise: that there are points y and z in W with yR_2z and not yR_1z. This means that there is a formula $\Box A \in y$ with $A \in z$, and also a formula $\Box B \in y$ with $B \notin z$. Thus (i) $\Box A \wedge \Box B \in y$, and (ii) $A \to B \notin z$. Since we are in a model generated by the point x, there are $(R_1 \cup \bar{R}_2)$-chains from x to all elements of W; we may take $\langle u_1, \ldots, u_m\rangle$ and $\langle v_1, \ldots, v_n\rangle$ to be such chains with $u_1 = v_1 = x$ and $u_m = y$, $v_n = z$. Now consider that instance of (*) with $C_i = D_j = q$ for $1 \leq i \leq m$, $1 \leq j \leq n$, which chooses, in the antecedent, ∇_i to be ∇^+ if $u_iR_1u_{i+1}$ and to be ∇^-

if $u_i\bar{R}_2u_{i+1}$ (note that the possibilities are mutually exclusive), and in the consequent, selects Δ_i as Δ^+ if $v_iR_1v_{i+1}$, and as Δ^- if $v_i\bar{R}_2v_{i+1}$. The location of y as u_m, together with (i), forces the antecedent of this instance of (*) to belong to x. Since x is a point taken from the canonical model for $\mathrm{AO}_{1=2}$, it contains the instance in question of (*), and therefore, the consequent belongs to x: but this gives a contradiction in view of the location of z as v_n and fact (ii).

It would be of interest if the schema (*) could be replaced by some schema of the form:

$$(\Box A \wedge \Box B) \rightarrow \ldots\ldots\ldots\ldots\ldots$$

My original attempts at finding an axiomatization of the logic of 'all and only' aimed at a simpler schema of this form (perhaps as the sole axiom, to be subjoined to the smallest classical—or 'congruential'—modal logic), but they were unsuccessful. One possible filling for the . . . , if we were working in a richer language, would be: $A = B$, where '$=$' is the connective of propositional identity mentioned in Section 1. The closest one can come to this in the present language would be:

(**) $(\Box A \wedge \Box B) \rightarrow (C \leftrightarrow C')$ where C' is any formula differing from C in having zero or more occurrences of A in C replaced by B.

The idea would be to investigate the force of this axiom-schema in terms of the semantic framework suggested by the fact that the background logic is the smallest classical system (called E in [6]): neighborhood semantics. A special case of (**) shows the sort of effect one can achieve:

(***) $(\Box A \wedge \Box B) \rightarrow (A \rightarrow B)$.

Here I have taken C to be just the formula A itself, and dropped the '$\leftrightarrow$' to a '$\rightarrow$' since that is really no weakening. The neighborhood-semantical characterization of the smallest classical logic containing all instances of (***) among its theorems can be stated succintly if we permit ourselves the following terminology. Recalling that a neighborhood model is a triple $\langle W, N, V\rangle$ in which N is a function assigning to each $x \in W$ a collection of subsets of W (the so-called neighborhoods of x), and a formula $\Box A$ is counted true at a point x in such a model just in case the truth-set of A (in the model) belongs to $N(x)$, let us define a point y to be *internal* if y belongs to each set in $N(y)$, and to be *external* if y belongs to no set in $N(y)$. (Strictly speaking these concepts should be relativized to the frame $\langle W, N\rangle$ in question; note also that if $N(y)$ is empty, though not otherwise, y is both an external and an internal point.) Then it is easily seen that the (***)-logic alluded to is determined by the class of all neighborhood frames in which each point is either external or internal. I have not, on the other hand, obtained any similarly structural condition on frames appropriate for the full (**) logic of which this is just a subsystem.

The significance of the above line of thought for the logic of 'all and only' lies in the intimate connection between a certain class of neighborhood models and relational models under the [All-&-Only] truth-definition. Call a neighborhood frame $\langle W, N\rangle$ with the property that for all $x \in W$, $N(x)$ contains just one subset of W, a single-neighborhood frame. Then we may observe that every model on single-neighborhood frame is pointwise equivalent to some model (with the [All-&-Only] clause) on a relational frame, and conversely that each model of the latter type is pointwise equivalent to a model of the former sort. For, in the first case, given $\langle W, N, V\rangle$ with $N(x)$ a singleton for each x, defining Rxy to hold iff y belongs to the sole element of $N(x)$, yields: $\langle W, N, V\rangle \models_y A$ iff $\langle W, R, V\rangle \models_y A$ for all $y \in W$ and all formulas A (the second '$\models$' here being taken in the [All-&-Only] sense); and in the second case, given $\langle W, R, V\rangle$ we get the N of the equivalent single-neighborhood model by putting $N(x) = \{R(x)\}$, where $R(x)$ is itself understood as $\{y \in W\colon xRy\}$. This gives the following corollary to Theorem 3: the system $AO_{1=2}$ is determined by the class of all single-neighborhood frames. I mention the neighborhood-semantic perspective here to justify the hope of success for the route up from E through something along the lines of (**), as an alternative to the route followed here *via* the rule [AO].

The above digression on (**), brings out an additional point of interest which the (***) system shares with $AO_{1=2}$: Halldén-incompleteness. For note the following truth-functional readjustment of (***):

$$(\Box A \wedge A) \to (\Box B \to B)$$

and in particular the following instance thereof (with different variables for the schematic letters, and again, minor truth-functional manipulation):

$$(\Box p \to {\sim}p) \vee (\Box q \to q).$$

Neither disjunct is valid on every frame in the determining class, or indeed on any one frame therein containing both internal points (allowing the first disjunct, though not the second, to be falsified) and external points (at which the second disjunct, though not the first, can be falsified). The same example will do for $AO_{1=2}$, since (***) is after all just the $m = 0$, $n = 0$ case of our schema (*), though the diagnosis of the Halldén-incompleteness illustrated by the 'unreasonable' disjunction above might here be re-phrased in the relational terminology of the [All-&-Only] semantics, by describing the set of points at which the first disjunct is unfalsifiable as those which do not bear R to themselves, and the set of points at which the second is unfalsifiable as consisting of those which do bear R to themselves.

As a final observation, I remark that the [All-&-Only] semantics guarantees that for any formula A, *if* $\Box A$ is true at a point x in a model, then the truth set of A in the model is precisely $R(x)$: but we need the antecedent here—there is no formula B with the property that for any point x in any model, the truth-set of B in the model is precisely $R(x)$. Indeed, this idea only makes sense if we have somehow disallowed the possibility that $R(x) \neq R(y)$ for $x \neq y$. But if we could talk in terms of the truth set of a given formula, when considered from the perspective of a given point in the model, then the above antecedent could be removed. As the latter formulation suggests, such a possibility is opened up in the semantic framework of two-dimensional modal logic, and for a preliminary investigation of the use of sentential constants whose truth-set, relative to a given world, is precisely the set of worlds R-related to that world, [3] may be consulted.

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