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# Inconsistent Number Systems

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*1 Introduction* In a previous paper ([8]), it was shown that there are finite inconsistent arithmetics which are extensions of consistent Peano arithmetic formulated with a base of relevant logic, and also of the set of truths of the classical standard model of arithmetic. In the present paper, the study of the operations of inconsistent number-theoretic structures, especially finite structures, is continued. The interest is particularly in displaying inconsistent theories and associated finite structures which extend standard classical structures, in the sense that all truths of the latter hold also in the former. The principal thesis to be argued on that basis is that classical mathematics is a *special case* of inconsistent mathematics.

The view of mathematics, as based on classical two-valued logic as a deductive tool, has it that from inconsistency all propositions are deducible. Hence, inconsistency-toleration is achieved in the present paper by use of a logic with a weaker deductive relation $\vdash$, the three-valued logic RM3, the third value of which has a natural interpretation, 'both true and false' (cf. Section 2). It should not be thought, however, that theories in which a weaker $\vdash$ is used inevitably lead to sacrifice of some classical propositions. It is one purpose of this paper to demonstrate this, by displaying inconsistent theories which contain various well-known classical consistent complete subtheories.

Aside from its capacity for contradiction containment, RM3 is chosen for two reasons. First, being three valued it is reasonably easy to deal with, particularly in yielding a rich model theory. Second, every RM3-theory displayed is also a theory of all the usual relevant logics such as E and R, which have an independently natural motivation. The interest of those logics for mathematics may be judged accordingly. Indeed, since every classical theory is an RM3-theory and thus also an E- or R-theory, the "special case" thesis above has another dimension: just as consistent mathematics is a special case (under the assumption of consistency or closure under classical deducibility) of inconsistent mathematics, so classical logic is a special case (in which closure under classical deducibility, for instance the rule of Disjunctive Syllogism, holds over a limited

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subject matter) of inconsistency-tolerating logic. Again, no sacrifice of mathematical richness is envisaged; on the contrary, the hope is to show that further mathematically rich structures are to be uncovered by the expanded perspective.

In Sections 2 and 3, the basic model-theoretic framework is set up. In Section 3, this is used to study inconsistent theories which include the classical theories of various rings and fields—that is, the standard theories of addition, multiplication, subtraction, and division. In that section the notion of a model with identity is defined, and it is argued that the existence of inconsistent models with identity supports the thesis that inconsistent mathematics can be seen as extensional in a perfectly standard sense of that term, as a study whose subject matter can be viewed as objects with inconsistent properties. An important outcome of this section is that it is not so easy to develop an inconsistent theory of fields even with an inconsistency-tolerating logic; the problems seem to be deeper, to do with identity and functionality. In Section 4, order is studied; and in Section 5, order is put together with the arithmetical operations to study ordered rings and fields.

A model-theoretic framework is employed, but I suggest that this is a consequence of the fact that intuitive inconsistent thinking is undeveloped (though not entirely absent) among mathematicians and logicians. It is to be hoped that its development will not prove ultimately impossible, but in its absence it is necessary to demonstrate that control of the deductive consequences of contradictions is possible. Thus, it is certainly not being claimed that the 'natural logic' of mathematicians is nonclassical, a disputed question in recent debates within philosophical logic (see e.g., [3]–[6], [10], [11], [14]). Mathematicians do seem to be habitual consistentizers. But if there is any way to expand this perspective, it must proceed by demonstrating the existence of rich mathematical structures which are nonetheless inconsistent. Conversely, the paraconsistency movement has somewhat shirked its duties in calling for inconsistency-tolerating logics but omitting to demonstrate the existence of rich inconsistent mathematical theories (e.g., [14]). If, say, there were no particularly interesting inconsistent theory of fields, perhaps because of problems about the desired functional properties of division and subtraction (as in the light of Section 3 may well turn out to be the case), then it is no use calling for a paraconsistent logic if it is not much use when you get it. Nonetheless, this paper aims to show that an expanded perspective is available. There is, I suggest, a 'seamless web' between classical consistent structures with a limitingly simple logic (tacit or not), and structures in which less deductive power leads to increased richness and freedom.

One direction in which this paper could be extended is toward nonstandard models of various number theories. This is done in sequels ([12] and [13]).

***2 Basic definitions and the extendability lemma*** We begin with a general notion of an *assignment* which has minimal semantic features and then work toward semantic features of *models*. The point of the exercise is to pick apart some simple model-theoretic concepts which coincide classically, taking advantage of the greater freedom afforded by weakened background logic. We will see that some of the remaining connections are invariant with respect to broad variations in background logic, while others are specific to RM3. The eventual aim is to establish conditions, in Section 3, under which the resulting structures look more like extensional theories of inconsistent objects.

We consider various sublanguages of the language $L$ consisting of simple terms (names), one for each real number; function symbols $+$, $\times$, $-$, $\div$; atomic predicates $=$, $<$, $\in$ (the latter is used only briefly in Section 5); variables $x$, $y$, $z, \ldots$; and operators $\neg$, &, $\forall$ (the latter also written $()$). Complex terms, wffs, and sentences are defined in the usual way, as are $\supset$, $\vee$, $\equiv$, and $\exists$. We regard sentences of the form $t_1 = t_2$, $t_1 < t_2$, $t_1 \in t_2$ with no occurrences of $\neg$, &, $\forall$ as atomic, irrespective of whether the terms contain occurrences of function symbols. Only theories whose theorems contain no free variables are considered, and, for simplicity, no term is a variable. An RM3-*assignment* (abbreviated to 'assignment') is a function $I$ assigning to the wffs of $L$, or the appropriate sublanguage of $L$ under investigation at the time, values from the set $\{T,N,F\}$ in accordance with:

(1) For any atomic wff with terms $t_1$, $t_2$, we have $I(t_1 = t_2)$, $I(t_1 < t_2)$ and $I(t_1 \in t_2)$ all belong to $\{T,N,F\}$, (read 'true, neuter, false').[1]
(2) $I(\neg A)$ and $I(A \,\&\, B)$ are given by the RM3-matrices:

| & | T | N | F | $\neg$ |
|---|---|---|---|---|
| *T | T | N | F | F |
| *N | N | N | F | N |
| F | F | F | F | T |

(3) $I((x)A) = min\{y$: for some term $t$, $I(A(t|x)) = y\}$, where *min* is relative to the ordering: false < neuter < true. A sentence $A$ *holds in an assignment* $I$ iff $I(A) \in \{\mathrm{T,N}\}$.

A subset $S$ of $L$ is an L-*semitheory* (relative to Logic L) iff if $A \in S$ and $A \vdash_{\mathrm{L}} B$ then $B \in S$. $S$ is an L-*theory* iff $S$ is an L-semitheory and in addition if $A \in S$ and $B \in S$ then $A \,\&\, B \in S$. Where no confusion will result, we often drop the 'L–' when L = RM3. A set $S$ of sentences is *determined by an* RM3-*assignment* $I$ iff ($A \in S$ iff $A$ holds in $I$). A set $S$ of sentences is *consistent* iff for all closed wffs $A$, not both $A \in S$ and $\neg A \in S$; otherwise *inconsistent*. $S$ is *trivial* (or *absolutely inconsistent*) iff $S = L$; otherwise *nontrivial*. $S$ is *complete* iff for all closed wffs $A$, either $A \in S$ or $\neg A \in S$; otherwise *incomplete*. If $S$ is determined by an RM3-assignment, then $S$ is a complete RM3-theory; but not every RM3-theory is determined by an RM3-assignment, since not every RM3-theory is complete (not every classical theory is complete, and every classical theory is an RM3-theory).

A basic result is the following:

**Proposition 1** (Extendability Lemma) *Let* $I$, $I^1$ *be* RM3-*assignments with the same sets of terms. If the atomic sentences holding in* $I$ *are a subset of the atomic sentences holding in* $I^1$ *and if in addition the negations of atomic sentences holding in* $I$ *are a subset of the negations of atomic sentences holding in* $I^1$, *then the theory determined by* $I$ *is a subset of the theory determined by* $I^1$.

*Proof:* By induction on the complexity of sentences. We observe first that the hypothesis of the proposition is equivalent to the following: if $A$ is atomic then: (i) if $I(A) = \mathrm{T}$ then $I^1(A) \in \{\mathrm{T,N}\}$; (ii) if $I(A) = \mathrm{N}$ then $I^1(A) = \mathrm{N}$; and (iii) if $I(A) = \mathrm{F}$ then $I^1(A) \in \{\mathrm{N,F}\}$. The induction proves that (i)-(iii) hold of all formulas. The base clause is already proved. The $\neg$ and & clauses are straightforward from the $\neg$ and & table. If $A$ is of the form $(x)\,B$ then either (i) $I((x)B) = \mathrm{T}$, whence $I(B(t/x)) = \mathrm{T}$ for all terms $t$. So for all terms $t$ (same terms, by hypothesis) $I^1(B(t/x)) \in \{\mathrm{T,N}\}$; whence $I^1((x)B) \in \{\mathrm{T,N}\}$. The two other alternatives (ii) and (iii), where $I((x)B) \in \{\mathrm{N,F}\}$, are similar.

Note one consequence of this. If a theory determined by an RM3-assignment is consistent and complete, then it is in fact a theory of classical first-order logic, since in the absence of the value Neuter, RM3-assignments are just classical models. Hence we can begin with any model from classical model theory (provided that it is equipped with appropriate names) and extend it by adding additional atomic sentences to make it inconsistent, evaluating all complex sentences as in RM3-assignments. The Extendability Lemma then ensures that the resulting theory is a supertheory of the classical theory commenced with. Furthermore, this extension is controlled by the assignment to atomic sentences, so to speak, so that if even one atomic sentence or its denial remains with the value False in the supertheory, it is nontrivial (absolutely consistent). There are two related desiderata with this strategy which will come out later: the substitutivity of identity, and the functionality of $+$, $\times$, $-$, $\div$. Setting these aside here, the general strategy so described for producing inconsistent extensions of classical theories (particularly determined by finite models) is a basic concern in what follows.

***3 Identity, with applications to arithmetical operations*** Consider first the *classical standard model of the natural numbers*, equipped with names for the natural numbers. In view of the Extendability Lemma, the set of sentences holding therein can be extended by adding any collection of sentences of the form $\neg n = n$ and evaluating in an RM3-assignment. Note that the contradiction does not spread to other sentences of the form $\neg m = m$. Similarly, collections of sentences of the form $n = m$ for distinct $n$, $m$, may be added with the same result.

This raises the following question. If we add, say, $0 = 2$ to the standard model of the natural numbers, then, in virtue of the substitutivity of identity and the fact that $\neg 0 = 2$ also holds, have we not imported the further sentence $\neg 0 = 0$? The answer is no, and it illustrates the generality of the Extendability Lemma. The rule of substitutivity of identity (SI) in the form *if* $t_1 = t_2$ *holds, then* $Ft_1$ *holds iff* $Ft_2$ *holds* (all terms $t_1$, $t_2$, with $t_2$ replacing $t_1$ in $F$ in at least one place) does not always hold in our assignments. What is the case, if the sentences holding in an RM3-assignment include those holding in the standard model of the natural numbers, is that $(t_1 = t_2 \;\&\; Ft_1) \supset Ft_2$ holds, since it holds in the standard model. But it is not in general true that if $A \supset B$ holds and $A$ holds then $B$ holds. In particular, $((A \;\&\; \neg A) \supset B) \;\&\; (A \;\&\; \neg A)$ might hold while $B$ does not. However, this leads to no loss of information from classical arithmetic, since we do have that if $(A \supset B) \;\&\; A$ holds, *and if moreover* $(A \supset B) \;\&\; A$ *holds back in the standard model for arithmetic*, then $B$ holds (trivial). A special case of interest is this: if $t_1 = t_2 \;\&\; Ft_1$ holds and *if moreover* $\neg t_1 =$

*$t_2$ and $\neg Ft_1$ both do not hold*, then $Ft_2$ holds. (Reason: for then $t_1 = t_2$ & $Ft_1$ holds back in the classical complete subtheory, wherein $Ft_2$ could be detached.)

So the rule SI does not hold in all RM3-assignments. This is by no means catastrophic. Intentional theories, for instance modal theories, in which SI fails have been extensively investigated. Many philosophers have taken the failure of SI as the mark of the intensional. Even so, it is obvious that a central role will be played by those models for which SI does hold. In fact, it is useful to use a more semantically based notion which ensures SI. We call an assignment an *assignment with identity* iff for all terms $t_1$, $t_2$, if $t_1 = t_2$ holds then for all predicates $F$, $Ft_1$ holds iff $Ft_2$ holds; where $Ft_2$ is like $Ft_1$, except that $t_2$ replaces $t_1$ in at least one place. This is evidently a generalization of the corresponding classical notion which nevertheless remains within its spirit. We also say that an assignment is *reflexive* iff $t = t$ holds for all terms $t$. Now the idea of an assignment with identity does not determine much by itself, but coupled with reflexivity it determines a lot, as the following proposition shows. First, some definitions: an assignment is *functional* iff for all terms $t_1$, $t_2$, if $t_1 = t_2$ holds then $f(t_1) = f(t_2)$ holds provided that both the latter are defined, and both are undefined otherwise. An assignment is *symmetric* iff $t_1 = t_2$ holds iff $t_2 = t_1$ holds, and *transitive* iff if $t_1 = t_2$ holds and $t_2 = t_3$ holds, then $t_1 = t_3$ holds (all $t_1$, $t_2$, $t_3$). An assignment which is reflexive, symmetric, and transitive is *normal*. Now we have necessary and sufficient conditions for a model with identity.

**Proposition 2** (1) *$I$ is an assignment with identity iff for all terms $t_1$, $t_2$, if $t_1 = t_2$ holds then for all atomic $F$, $I(Ft_1) = I(Ft_2)$.* (2) *If $I$ is a reflexive assignment with identity, then $I$ is normal and functional.* (3) *If $I$ is reflexive and $=$ is the only predicate of the language, then $I$ is an assignment with identity iff $I$ is functional and for all $t_1$, $t_2$, if $t_1 = t_2$ holds then for all $t_3$, $I(t_1 = t_3) = I(t_2 = t_3)$ and $I(t_3 = t_1) = I(t_3 = t_2)$.*

*Proof:* (1) $R \to L$ follows by a straightforward induction on the complexity of terms. $L \to R$: Let $I(Ft_1) \neq I(Ft_2)$ for some atomic $F$ while $t_1 = t_2$ holds. If one of $Ft_1$, $Ft_2$, does not hold then the other does, so that $I$ is not an assignment with identity. Otherwise, if both $Ft_1$, $Ft_2$ hold, then one of $\neg Ft_1$, $\neg Ft_2$ does not hold while the other does, again incompatible with identity.

(2) Symmetry: Let $t_1 = t_2$ hold. By identity, if $t_1 = t_2$ holds, then $t_1 = t_1$ holds iff $t_2 = t_1$ holds. By reflexivity, $t_1 = t_1$ holds. Hence $t_2 = t_1$ holds. Transitivity: Let $t_1 = t_2$ and $t_2 = t_3$ hold. By identity, if $t_1 = t_2$ holds then $t_1 = t_3$ holds iff $t_2 = t_3$ holds. Hence $t_1 = t_3$ holds. Functionality: Let $t_1 = t_2$ hold. By identity, $f(t_1) = f(t_1)$ holds iff $f(t_1) = f(t_2)$ holds. By reflexivity, $f(t_1) = f(t_1)$ holds. Therefore, $f(t_1) = f(t_2)$ holds.

(3) $L \to R$ follows from (1) and (2). $R \to L$: From (1), we need only prove that if $t_1 = t_2$ holds, then for atomic $F$, $I(Ft_1) = I(Ft_2)$. Clearly, atomic $F$ have one of four forms: $t_1 = t_3$, $t_3 = t_1$, $f(t_1) = t_3$, or $t_3 = f(t_1)$. In the first two cases, the conditions of the theorem ensure what we want. In the third case, we have to prove that if $t_1 = t_2$ holds, then for any $t_3$, $I(f(t_1) = t_3) = I(f(t_2) = t_3)$. But if $t_1 = t_2$ holds, then by functionality, $f(t_1) = f(t_2)$ holds; hence by the conditions of the theorem, for any $t_3$, $I(f(t_1) = t_3) = I(f(t_2) = t_3)$ as required. The fourth case is similar. This completes the proof.

Note that all of (1)–(3) are true over a broad class of logics, since the inductions needed for (1) and (3) will work provided that $I$ assigns values in a Lindenbaum algebra, and (2) and the remainder of (3) need only minimal properties for 'holds'. Proposition 2 is thus a general result for model theory based on many different logics.

The conditions for an assignment with identity can be made more semantically based, so the idea of an assignment is now strengthened to that of a model.

An RM3-*model* is a pair $\langle D,I\rangle$ where $D$ is a domain and $I$ is a function which is an RM3-assignment and which in addition has the following four properties: (1) $I$ assigns to every *simple* term a member of $D$, and $I$ is onto $D$; so that every object is named. This has the effect that our substitutional quantification becomes objectual. (2) $I$ assigns to every $n$-ary functional expression an $n$-ary partial function on $D$. (3) The assignment to complex terms is given by $I(f(t_1 \ldots t_n)) = I(f)(I(t_1) \ldots I(t_n))$, provided that these are defined. (4) $I$ satisfies: $t_1 = t_2$ holds iff $I(t_1) = I(t_2)$. These have the effects that $I$ is normal and functional.

A model is *infinite* iff $\bar{\bar{D}} \geq \aleph_0$, otherwise *finite*. If $\langle D,I\rangle$ is a model and $I$ is an assignment with identity, then $\langle D,I\rangle$ is a *model with identity*. Thus, if $\langle D,I\rangle$ is a model, then its semantical features ensure that $I$ is normal and functional. Further, then, a necessary and sufficient condition for a model for an equational theory to be a model with identity is that if $t_1 = t_2$ holds, then for any $t_3$, $I(t_1 = t_3) = I(t_2 = t_3)$ and $I(t_3 = t_1) = I(t_3 = t_2)$. We could introduce further semantical conditions on the domain to ensure models with identity: the obvious maneuver is to introduce for each $n$-ary relational symbol a truth extension and a falsity extension, the intersection of which would be the neuter extension. But we do not consider that here, since the aim is less model theoretic than it is to establish the model theory as a convenient device for studying inconsistent mathematical objects and demonstrating that the inconsistency is under control.

It is sometimes thought that contradiction-toleration is a matter of the use of theories of intensional logics, or perhaps that it is a matter of "mere syntax". To the contrary, it is argued here that the study is extensional in at least two senses. It is syntactically extensional, in dealing only with the connectives $\neg$, &, $\forall$; and it is extensional in dealing with models with identity. In this sense, it can usefully be viewed as dealing with mathematical objects which have inconsistent properties, especially when models which inconsistently extend various consistent classical standard theories of classes of mathematical objects are considered.

As an example, consider the following class of inconsistent finite models with identity in which all sentences of the classical standard model for the arithmetic of $(+,\times)$ hold (investigated in [8]). There are names (i.e., simple terms) for all the nonnegative integers, with the domain being the integers modulo $m$, i.e., $\{0,1,\ldots,m-1\}$; $+$, $\times$, are interpreted as addition and multiplication in arithmetic modulo $m$. Set $I(n)$, for every name $n$, to be $n(mod\ m)$. With $I(+,\times)$ this determines $I(t)$ for every term $t$. And finally set $I(t_1 = t_2) = N$ iff $t_1(mod\ m) = t_2(mod\ m)$, i.e., iff $I(t_1) = I(t_2)$; and $I(t_1 = t_2) = F$ otherwise. In [8], these are called $\mathrm{RM3}^{\mathrm{m}}$, and it is proved that they are inconsistent, nontrivial, complete, $\omega$-inconsistent, $\omega$-complete, and decidable. In [8], the interest in these structures is that they are extensions of the axiomatic arithmetic $R^{\#}$, and show that Gödel's Second Incompleteness Theorem can be escaped after a fashion in inconsistent and relevant mathematics. Here, the interest is that they are models with identity and determine finite inconsistent extensions of the classical standard theory of arithmetic.

A simple development of these results can be obtained from the well-known fact that the algebra of the integers modulo $m$ enables a natural definition of 'minus $n$' and thereby subtraction. This can be exploited to display finite inconsistent extensions of the classical theory, with names, of the full ring of integers $Z$ (positive and negative). Take names for all the integers. The domain is the integers modulo $m$; $+$ and $\times$ are, as before, $+(mod\ m)$ and $\times(mod\ m)$. The additive inverse $(-n)$ modulo $m$ of a number $n$ is given classically by $m-$ ($n$ $mod\ m$) if $n\ mod\ m \neq 0$, and 0 otherwise; and then subtraction $mod\ m$ is given by $(k-(mod\ m)n =_{df} k\ mod\ m + (mod\ m)(-n) mod\ m.)$ So here we interpret '$-$' to be '$-mod\ m$'. This determines $I(t)$ for all terms $t$. Set $I(t_1 = t_2) = \mathrm{N}$ iff $I(t_1) = I(t_2)$, i.e., iff $t_1\ mod\ m = t_2\ mod\ m$; and set $I(t_1 = t_2) = \mathrm{F}$ otherwise. Clearly, the condition of Propositions 2 and 3 for a model with identity is satisfied. Also every true identity of the classical theory of integers holds, since if classically $t_1 = t_2$ then $t_1\ mod\ m = t_2\ mod\ m$. So, by the Extendability Lemma, we have

**Proposition 3** *There are finite inconsistent models with identity in which every sentence of the classical theory of the ring of integers Z holds.*

A useful and obvious result is the Term Elimination Lemma. The above models have finite domains and infinite numbers of simple terms, the latter being necessary if we are to have extensions of the various classical theories with names. But, as might be expected, the simple terms can be cut down to just one per member of the domain, while preserving the assignments to all terms, and preserving the values of all sentences in the weaker vocabulary. In particular, the term-free quantified theory remains identical. It needs models with identity to make this work, so that is another use for the notion. Let $\langle D, I\rangle$ be a model with identity. Select only one term from each set $\{t\text{: } (\exists x)(I(t) = x \in D)\}$, and let $I^1$ assign to it the same value it is assigned by $I$. Functional expressions are assigned the same partial functions on the domain as before, but functional terms only in the weaker language are assigned values. Atomic sentences in the weaker language are given the same values by $I^1$ as by $I$. This evidently ensures the base clause of an induction to prove the following.

**Proposition 4** (Term Elimination Lemma) *A sentence in the cut-down language has exactly the value in $\langle D, I\rangle$ that it has in $\langle D, I^1\rangle$.*

*Proof* (Inductive Clause): The $\neg$ and & clauses are straightforward. For the $\forall$ clause (a.i) if $(x)Fx$ is T in $\langle D, I\rangle$ then $I(Ft) = \mathrm{T}$ for all terms of the vocabulary of $I$, so $I^1(Ft) = \mathrm{T}$ for all terms of the vocabulary of $I^1$, so $(x)Fx$ is T in $I^1$. (a.ii) If $I(x)Fx) = \mathrm{N}$ then $I(Ft) = \mathrm{T}$ or N for all terms $t$, and N for at least one. So $I^1(Ft) = \mathrm{T}$ or N for all terms $t$ of the vocabulary of $I^1$. But also by the construction of $I^1$, for some $t^*$ we must have $I^1(Ft^*) = \mathrm{N}$. Hence $I^1((x)Fx) = \mathrm{N}$. (a.iii) The F clause is similar, with 'F' replacing 'N'. Conversely (b.i) If $I^1((x)Fx) = \mathrm{T}$ then $I^1(Ft) = \mathrm{T}$ for all $t$ in weaker vocabulary. But for every term $t^*$ of $I$, we have $I(t^*) = I(t)$ for one of these $t$; so that, since $M$ is a model with identity, $Ft^*$ agrees with some $Ft$ of $I^1$. But all of the latter are T, so also every $I(Ft^*)$ must be. Hence $I((x)Fx) = \text{T}$. (b.ii) and (b.iii) the N and F cases are similar.

The effect of this lemma is that the two classes of models previously considered now yield inconsistent models with cut-down languages (finite numbers of simple terms, exactly one for each member of the domain), with the same sentences in the weaker language, including the term-free language, holding. These cease to be inconsistent extensions of, e.g., the classical theory of $Z$ with names, but remain inconsistent extensions of the finite consistent arithmetics modulo $m$.

We now bring in division, and thus the theory of fields. It turns out that the interaction between subtraction and division is not smooth sailing. The following are a set of postulates adequate for the classical theory of fields (see [15], p. 130)

(1) $(x,y,z)(x + (y + z) = (x + y) + z)$
(2) $(x,y)(x + y = y + x)$
(3) $(x)(x + 0 = x)$
(4) $(x)(x + (-x) = 0)$
(5) $(x,y,z)(x \times (y \times z) = (x \times y) \times z)$
(6) $(x,y)(x \times y = y \times x)$
(7) $(x)(x \times 1 = x)$
(8) $(x)(\neg x = 0 \supset x \times x^{-1} = 1)$
(9) $(x,y,z)(x \times (y + z) = (x \times y) + (x \times z))$
(10) $\neg 0 = 1$.

First, there are certainly finite inconsistent fields because (as is well known) there are finite consistent fields. The finite arithmetics modulo $p$, $\{0,1,\ldots,p-1\}$, where $p$ is *prime*, permit a definition of a unique multiplicative inverse $n^{-1}$ for any $n \in \{1,2,\ldots,p-1\}$ though not for $n = 0$ (see, e.g., [2], p. 40). Therefore, if we take names only for $\{0,1,\ldots,p-1\}$, interpret $+$, $\times$, $-$, $\div$ as in arithmetic modulo $p$, and set $I(t_1 = t_2) = \text{T}$ iff $I(t_1) = I(t_2)$, and F otherwise, we have the classical consistent theory of fields. Thus, setting instead $I(t_1 = t_2) = \text{N}$ for $I(t)$ and F otherwise, we have by the Extendability Lemma:

**Proposition 5** *There are finite inconsistent models with identity in which every sentence of the classical theory of fields holds.*

It would be desirable to see finite inconsistent extensions of the full theory with all names of the field of rationals $Q$. But it is not clear how to do this with these methods, because the interpretation function $I(n)$ assigning to all names of rationals members of the domain $\{0,1,\ldots,p-1\}$ would seem to assign infinitely many nonzero rationals to 0, as it does in the case of the integers. But then for these, an inverse $n^{-1}$ is not defined, while it is in the full theory of $Q$.

A useful general result can be obtained as a consequence of the Extendability Lemma.

**Proposition 6** *Let $A$ be an algebra $\langle D,0_1,\ldots,0_n\rangle$ where $D$ is a set and $0_1,\ldots 0_n$ are relations on $D$. Let $h$ be a homomorphism from $A$ to a subalgebra*

*$A^1$ with $D^1 = h(D)$ and operations the restriction of $0_1, \ldots, 0_n$ to $D^1$. Then the classical equational theory of A with names for all elements of D can be inconsistently extended to an RM3-model with identity using the assignment $I(t) = h(t)$, $I(t_1 = t_2) = N$ iff $I(t_1) = I(t_2)$ and $I(t_1 = t_2) = F$ otherwise.*

*Proof:* Certainly the assignment $I$ is a model: $I(t)$ is defined on domain $D^1$, and if $I(t_1) = I(t_2)$ then evidently $O_i(t_1) = O_i(t_2)$. Also, it plainly satisfies the condition for a model with identity.

An application of this is that whenever classically one can partition an algebra into equivalence classes via a homomorphism onto a subalgebra, one may instead literally inconsistently identify distinct elements in the larger algebra, thereby obtaining an inconsistent extension of it. That is, of course, precisely what the modulo arithmetics are doing, with the *caveat* about division noted before. Another example is as follows. There are finite models inconsistently extending the classical $(+, \times, \div)$ theory of the nonnegative rationals $Q^+$ with names (see also Section 5 for order). Consider the following subalgebra of that structure:

$D = \{0,1\}$ with the operations

| $-$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

| $\times$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

| $\div$ | 0 | 1 |
|---|---|---|
| 0 | U | 0 |
| 1 | U | 1 |

U = undefined

The homomorphism $h$ is given by $h(0) = 0$, $h(n) = 1$ for all $n > 0$.

Thus there is a finite model with identity in which the classical $(+, \times, \div)$ theory of the nonnegative rationals $Q^+$ with names holds. Notice how introducing the negative rationals and thereby subtraction would wreck this model: we want some element to function as an additive inverse $-n$ for each $n$, but if we identify more than one rational $n_1$, $n_2$ with a given element, they have the same additive universe, so that $n_1 - n_2 = n_1 - n_1 = 0$; and so division by $n_1 - n_2$ is (improperly) undefined. Thus, the prospects for a sensitive inconsistent theory of arithmetical fields look bleak, not for reasons of propositional logic, but because of the functional interaction of $-$ and $\div$. Relevant logic has hitherto not taken proper cognizance of the fact that a good inconsistent mathematics might be difficult to obtain for reasons beyond the purely sentential.

There are, needless to say, infinite inconsistent extensions of the theory of $Q$, even models with identity, e.g., set $I(n = n) = N$ for every rational $n$, and $F$ otherwise. We encounter some of these in later sections, when order is introduced.

One interest in such inconsistent theories of division, both finite and infinite, is that they permit a solution to the following problem (raised by Graham Priest). Ordinarily one wants postulates such as the Cancellation Law ([2], p. 2) to hold when extending the theory of rings to that of integral domains and fields:

$$(x)(\neg x = 0 \supset (y,z)(x \times y = x \times z \supset y = z)).$$

But in inconsistent theories such as those of this section (see also Section 5) $\neg 0 = 0$ and $(x)(0 \times x = 0)$ hold, and one does not want to detach the consequent to get $y = z$ for all $y$, $z$; yet one also does not want to forbid detachment for those $x$ which are classically not identical with zero. However in the inconsistent finite fields modulo prime $p$ above, while both $\neg 0 = 0$ and $(x)(0 \times x = 0)$ hold, we cannot detach the consequent (because patently we do not have $(y,z)(y = z)$ holding). But on the other hand, the fact that they really are fields means that for those $x$ of the model which are "really" not identical with zero, i.e. for which $x = 0$ has value $F$ in the model, we can detach because we do have that $x \times y = x \times z \supset y = z$, even that if $x \times y = x \times z$ holds then $y = z$ holds. Problems: Are there any finite inconsistent models with identity of the full classical theory of $Q$ with names? Is the above two-element model the only finite inconsistent model for $Q^+$? Is the addition of $\neg n = n$ to any model with identity still a model with identity?

***4 Order*** The aim in this section is to introduce order, and in the next section to study the inconsistent interplay between order and arithmetical operations, particularly the theory of ordered fields. In this section, we look at $=$ and $<$ alone. Among other things, it is shown that a standard result of model theory, namely that the theory of dense order with no first and last elements is $\aleph_0$-categorical, breaks down given a suitable extension of that concept to cover the more general inconsistent case.

The following postulates suffice for the standard classical theory of dense order without endpoints (e.g., [1], p. 324; [7], pp. 78, 90):

(i) Irreflexivity $(x)(\neg x < x)$
(ii) Asymmetry $(x,y)(x < y \supset \neg y < x)$
(iii) Transitivity $(x,y,z)(x < y \supset .\ y < z \supset x < z)$
(iv) Comparability $(x,y)(\neg x = y \supset .\ \neg x < y \supset y < x)$
(v) Exclusiveness $(x,y)((x = y \supset .\ \neg x < y\ \&\ \neg y < x)\ \&\ (x < y \supset \neg x = y))$
(vi) No endpoints $(x)(\exists y,z)(x < y\ \&\ z < x)$
(vii) Denseness $(x,y)(x < y \supset (\exists z)(x < z\ \&\ z < y))$
(viii) Mixing $(x,y,z)(z = y \supset .\ (y < z \supset x < z)\ \&\ (z < y \supset z < x))$.

These postulates hold in the classical (and RM3-) models with identity whose domain is the rational numbers, which we may also take as terms naming themselves; with $I(t = t) = \mathrm{T}$ and $I(t_1 = t_2) = \mathrm{F}$ otherwise, and $I(t_1 < t_2) = \mathrm{T}$ iff $t_1 < t_2$, and F otherwise. It is a standard result that all classical models of (i)–(viii) of cardinality $\aleph_0$ are isomorphic. Now in the case where every element of the domain has a name, the following version of isomorphism lends itself to natural generalization. Two models $\langle D,I\rangle$, $\langle D^1,I^1\rangle$ are isomorphic iff there is a 1 to 1 correspondence $f$: $D \to D^1$ such that for all atomic terms $t_1,\ldots,t_n$, $t_1^1,\ldots,t_n^1$; if $I^1(t_1^1) = f(I(t_1)),\ldots,I^1(t_n^1) = f(I(t_n))$, then for all atomic $F$, $Ft_1\ldots t_n$ holds in $I$ iff $Ft_1^1\ldots t_n^1$ holds in $I^1$.

Now extend the model of the previous paragraph to an inconsistent RM3-model as follows. Take the rationals as simple terms as before, but domain $D =$ the integers $Z$. For each rational $n$, set $I(n) =$ the integral part of $n$. Set $I(n = m) = \mathrm{N}$ iff $I(n) = I(m)$, and F otherwise; and set $I(n < m) = \mathrm{N}$ iff $I(n) \leq$

$I(m)$, and F otherwise. By the Extendability Lemma, every sentence of the classical theory of $Q$ continues to hold, and hence (i)–(viii) hold. Furthermore, it is a model with identity; since if $n = m$ holds, i.e. $I(n) = I(m)$, then clearly for all atomic $F$, $I(Fn) = I(Fm)$. Note in passing that the discreteness postulate $(x)(\exists y)(x < y \,\&\, (z)(x < z \supset . \neg y < z \supset y = z))$ also holds in this model; so that both discreteness and denseness postulates can be inconsistently satisfied. But there is no 1 to 1 correspondence which preserves atomic sentences between the domain of this model and that of the previous model: a 1 to 1 correspondence $f$ from $Z$ to $Q$ must eventually reverse the order on some of the elements of $Q$, so that while $I(n) \leq I(m)$ and thus $n < m$ holds in the inconsistent model, $f(I(m)) < f(I(n))$ in the classical model. Thus

**Proposition 7** *There are non-isomorphic* RM3*-models with identity, of cardinality* $\aleph_0$, *in which every sentence of the classical theory of dense order without endpoints holds.*

Indeed, the Term Elimination Lemma may be used on this model to dispense with all names except names for the integers, and the same result applies to this model. Again, a similar result can be simply obtained using a finite model, which can also be used to show that the order theories of $R$, $Q$, and $Z$ have a common inconsistent extension. Take domain $D = \{0,1\}$, and do three constructions corresponding to three sets of simple names, those of $R$, $Q$, and $Z$. In each case, set $I(n) = 0$ if $n \leq 0$ and $I(n) = 1$ if $n > 0$; set $I(n = m) =$ N if $I(n) = I(m)$, and F otherwise; and set $I(N < m) =$ N if $I(n) \leq I(m)$, and F otherwise. The three cases are inconsistent extensions of the order theories of $R$, $Q$, and $Z$ respectively, by the Extendability Lemma; and the conditions for being models with identity are satisfied. The case of $Q$ evidently provides an example of a finite-domain model in which all sentences of the theory of dense order without endpoints hold. But also, the Term Elimination Lemma can be applied to each of these constructions, to give that the set of term-free sentences of each of the order theories of $R$, $Q$, $Z$ holds in the same (two-element) model, the term-free sentences of which are thus a common inconsistent extension of them all. Problem: Is there a way to extend the theory of $R$ directly to that of $Q$?

***5 Ordered rings and fields*** In this section, the question of putting together the arithmetical operations with the order relation is discussed. It is useful at this point to introduce a distinction. So far, models have been constructed in which, typically, all sentences of various classical theories hold; that is, inconsistent extensions of classical theories. We have thus been working implicitly with two desiderata for models: (i) that they make all sentences of the classical theory hold, and (ii) that they be models with identity. The interplay between arithmetic and order, however, tends to make this rather more difficult to achieve. So we consider a third, weaker desideratum: (iii) all members of a certain set of postulates (such as e.g., the order postulates of Section 4) hold. Classically, (iii) coincides with (i), but not necessarily in RM3. It should not be thought that this is inevitably a "defect" of RM3, of course, since many have argued that the deductive relationship of classical logic is too strong, precisely in its inability to provide contradiction containment. It will be seen in this section that there are occasions when (i) must be sacrificed while (iii) continues to hold. This can be amplified by a point from [8]. It is an interesting open question whether $R^{\#}$, i.e., Peano arithmetic formulated with a relevant $\rightarrow$ as its implication operator instead of $\supset$, contains all of classical Peano arithmetic $P^{\#}$. It is however a separate matter whether, if $R^{\#}$ does not contain all of $P^{\#}$, this would be a "defect" of $R^{\#}$, since it is arguable that *natural* arithmetic is formulated merely with "if . . . then", and relevant $\rightarrow$ is at least as good a candidate for that as $\supset$ is. Indeed, were we to discover that natural arithmetic suffered an inconsistency in virtue of some recondite feature, to do with the Godel sentence, say, it is by no means obvious that we would regard the contradiction as spreading uncontrollably and thus affecting our ability to calculate.

Begin with the integers $Z$. The following postulates classically suffice for its order theory (cf. Section 4): Irreflexivity, Asymmetry, Transitivity, Comparability, Exclusiveness, No First and Last Elements, Mixing, together with:

(ix) Discreteness $(x)(\exists y)(x < y \;\&\; (z)(x < z \supset .\, y < z \vee y = z))$
$\&\; (x)(\exists y)(y < x \;\&\; (z)(z < x \supset .\, z < y \vee z = y))$

(x) Sum Law $(x,y,z)(x < y \supset x + z < y + z)$

(xi) Product Law $(x,y,z)(x < y \supset .\, 0 < z \supset x \times z < y \times z)$.

First consider finite models. Take the finite $(+,\times,-)$ models modulo $m$ of Section 4 and add the atomic sentences $t_1 < t_2$ for all terms $t_1$, $t_2$ constructible from names for the integers. Set $I(t_1 < t_2) = \mathrm{N}$ iff $t_1 \leq t_2$, and F otherwise. By the Extendability Lemma, all classical consequences of the $(=,+,\times,-,<)$ theory of $Z$ with names hold. They are not, however, models with identity (Reason: $t_1 = t_2$ holds iff $t_1$ *mod* $m = t_2$ *mod* $m$, but $t_1$ *mod* $m = t_2$ *mod* $m$ together with $t_1 < t_3$ does not ensure $t_2 < t_3$; $t_2$ might be too large even though when collapsed modulo $m$ it is equal to $t_1$).

So there are finite models in which all sentences of the arithmetic and order theory of the integers holds, but which are not models with identity. Equally there are finite models with identity in which all the above order *postulates* hold. Take the above $(+,\times,-)$ models modulo $m$ with names for all of $Z$; and set $I(t_1 < t_2) = \mathrm{N}$ iff $t_1$ *mod* $m \leq t_2$ *mod* $m$, and F otherwise. To show that these are models with identity it suffices to consider atomic sentences of the form $t_1 < t_2$, since other atomic sentences have been dealt with earlier. But if $t_1 = t_2$ holds, then $t_1$ *mod* $m = t_2$ *mod* $m$; whence $I(Ft_1) = I(t_1 < t_3)$, say, $= \mathrm{N}$ iff $t_1$ *mod* $m \leq t_3$ *mod* $m$ iff $t_2$ *mod* $m \leq t_3$ *mod* $m$, so that $\mathrm{N} = I(t_2 < t_3) = I(Ft_2)$. Identity then follows from Proposition 2. However, not all classical consequences follow because the assignment to $<$ destroys the order on the integers (for example, in modulo *3*, *2* $<$ *4* is F, because *2 mod 3* $= 2 \leq$ *4 mod 3* $= 1$ does not hold).

There are, however, finite models with identity in which all classical sentences true in $Z$ hold. Take the $(+,\times,-)$ models as before, and set all sentences of the form $t_1 < t_2$ to be Neuter. Clearly the Extendability Lemma ensures that all sentences in the $(=,+,\times,-,<)$ theory of $Z$ continue to hold. That it is a model with identity follows from the fact that for atomic F of the form $t_1 < t_2$, trivially $I(Ft_1) = I(Ft_2)$, whether or not $t_1 = t_2$ holds. These models have the unsatisfactory feature that the order properties are rather insensitively ensured, in that all sentences of the form $t_1 < t_2$ are made to hold. Even so, it is still only possible to make this work by exploiting the inconsistency-toleration features of RM3, e.g., in making $(x)\neg x < x$ take the value $N$ and so hold also. To summarize these results:

**Proposition 8** *There are finite models both with and without identity of the classical* $(=,+,\times,-,<)$ *theory of the integers Z with names, and finite models with identity in which all classical* $(=,+,\times,-)$ *consequences hold and all order postulates hold as well.*

We turn to division. The problem is to see what can be made of the theory of ordered fields. In addition to standard field $(+,\times,-,\div)$ properties, order postulates are needed. Classically the previously mentioned postulates suffice: Irreflexivity, Asymmetry, Transitivity, Comparability, Exclusiveness, No First and Last Elements, Denseness, Mixing, Sum, and Product Laws.

We saw in Section 3 that bringing in division restricts rather drastically the possibilities for finite inconsistent models, or at any rate finite extensions of $Q$. We can, however, go further with a result of that section, namely that all classical consequences of the $(=,+,\times,\div)$ theory of the nonnegative rationals hold in a two-element model $D = \{0,1\}$ with operations as specified previously, and $I(0) = 0$, $I(n) = 1$ for all $n > 1$. To this we can add the ordering $I(n < m) =$ N iff $I(n) \leq I(m)$, and F otherwise. It now follows easily that

**Proposition 9** *There is a finite model with identity in which all classical consequences of the* $(=,+,\times,\div,<)$ *theory of* $Q^+$ *hold.*

Consider now the following model with identity: names for all real numbers $R$; $D = \{0,1,\ldots,p-1\}$; $I(n) = 0$ for $n \leq 0$ and $n > p-1$, and $I(n) =$ the nearest integer $\leq n$ for $0 < n \leq p-1$; $I(n = m) =$ N iff $I(n) = I(m)$, and F otherwise; $I(n < m) =$ N iff $I(n) \leq I(m)$, and F otherwise. It is as immediate that it is a model with Identity as it is that the conditions of the Extendability Lemma apply, so that the sentences holding therein include all classical consequences of the first-order $(=,<)$ theory of $R$, including the continuity schema ([15], p. 131).

$$(((\exists x)Fx \ \& \ (\exists y)((x)(Fx \supset x \leq y)) \supset (\exists z)(x)(Fx \supset x \leq z) \\ \& \ (y)((x)(Fx \supset x \leq y) \supset z \leq y)).$$

So the Term Elimination Lemma can be applied to this model to give the conclusion that the following is a model with identity: $D = \{0,1,\ldots p-1\}$ (naming themselves); $I(n = m) = N$ iff $n = m$ and $F$ otherwise; $I(n < m) =$ N iff $I(n) \leq I(m)$ and F otherwise. This satisfies all classical consequences of the $(=,<)$ theory of $R$ in this language, and in particular all universally quantified sentences containing no names. Now we exploit the fact that a classical field can be constructed on the above domain in the standard fashion. The model cannot be made wholly classical (sentences only T or F) since we already have that $I(t = t) =$ N for all $t$. But the construction of the field simply adds to the above by assigning $I(t_1 + t_2)$, $I(t_1 \times t_2)$, $I(t_1 - t_2)$ and $I(t_1 \div t_2)$ the values they standardly take in $\{0,\ldots,p-1\}$, and sets $I(t_1 = t_2)$ (for any terms $t_1$, $t_2$) $=$ N iff $I(t_1) = I(t_2)$, and F otherwise; and $I(t_1 < t_2) =$ N iff $I(t_1) \leq I(t_2)$, and F otherwise. The Extendability Lemma ensures that all classical consequences of the theory of fields hold, and it is straightforward to show that it remains a model with identity. That is,

**Proposition 10** *There exist inconsistent continuously ordered finite fields.*

A complication should be mentioned. It has *not* been proved that every classical consequence of the theory of real closed fields holds in this model. To see this, note that the ability to substitute field identities in the theory of continuous ordering does not ensure that all classical consequences hold; for instance the Sum and Product laws are not obtained this way. In fact, the Sum and Product laws hold in our model, but conceivably various of their classical consequences might not. We do have though, as might be expected, that standard systems of first-order postulates for complete ordered fields (e.g., [15], p. 130) hold.[2] This point can be amplified by considering a different continuous ordering for the finite fields. For any real $n$, $I(n) = 0$ for $n \leq 0$, $I(n) =$ the next whole number $\geq n$ for $0 < n \leq p - 2$, and $I(n) = p - 1$ otherwise. Then set $I(n < m) = \mathrm{T}$ if $I(n) < I(m)$, $I(n < m) = \mathrm{N}$ if $I(n) = I(m)$, and $I(n < m) =$ F otherwise; and $I(n = m) = \mathrm{N}$ iff $I(n) = I(m)$, and F otherwise. Again, this inconsistently extends the classical ordering on $R$, so every classical sentence true therein holds. Further, it is a model with identity. By the Term Elimination Lemma, this is true for the model restricted to the $p$ names $\{0,1,\ldots p-1\}$. Then we can add the $(+,\times,-,\div)$ theory of $p$-membered finite fields to this in the same fashion as before to get models with identity for different finite inconsistent continuously ordered fields. But now notice this: Sum and Product Laws fail here whereas they did not in the previous model. (Sum Law: In modulo $p$, $p - 2 < p - 1$ is T, but $p - 2 + 1 < (p - 1) + 1$ is F. Product Law: In modulo 3, $1 < 2$ & $2 < 0$ (or $2 \neq 0$) is T, but $1 \times 2 < 2 \times 2$ is $2 < 1$ which is F.) This latter argument works for all modulo primes $p \geq 3$, but not for modulo 2. So we can deduce a couple of RM3-independence results: Sum and Product Laws cannot be obtained from continuity ordering + field properties, and these together with the Product Law do not yield the Sum Law. The moral to draw, though, is that it would be incorrect to conclude from the previous model that finitude + continuity + field properties + model with identity give all the sentences of the classical theory of real closed fields. Problem: Is that conclusion nevertheless true?

The results of these two models suggest the following simple extension into set theory. Instead of the schema '$Fx$' in the continuity schema, replace it by '$x \in u$' and universally quantify the whole formula with respect to $u$, where $u$ ranges over subsets of the domain. Let $s, t, \ldots$ be names for these, so that $I(s) \in P(D)$. Then let $I(n \in s) = T$ iff $I(n) \in I(s)$, and $F$ otherwise. It is straightforward to verify that the set-theoretic continuity postulate holds in the above models which remain models with identity.

***6 Conclusion*** To amplify a point made at the beginning, the use of an explicit background logic to study mathematical structures is a mark of mathematical logic as opposed to "natural" mathematics. While natural mathematics does seem typically to proceed on a tacit consistency assumption, it is by no means obvious that this is essential. The test is to see whether the relaxation of that assumption leads to rich structures, and it is suggested that the evidence here that it does is initially promising. The assumption of consistency does not entail that natural logic is classical, and the case that natural logic is not classical has been extensively argued in recent times. Discoveries in semantics have shown that nonclassical logics of the paraconsistent kind must have inconsistent theories, so it would seem mandatory to display these. But it is a moot point the extent to which the deductive assumptions on which natural mathematics proceeds are logically necessary, or simply there because the history of the activity has made alternatives invisible. Only the investigation of such alternatives can determine that.

## NOTES

1. The terminology 'neuter' is perhaps a little misleading, since it suggests "neither true nor false", whereas in fact it is better construed as "both true and false", or perhaps "both it and its negation hold". We retain 'neuter' here on grounds of established practice.
2. Save $0^{-1} = 0$, concerning which opinion differs. [15], p. 130; [9], pp. 280, 286; [17].

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