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Pointwise Definable Substructures of Models of Peano Arithmetic

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Let PA be Peano arithmetic formalized in a first-order language $L(PA)$ with $0, S, +, \cdot$ as nonlogical symbols and based on the usual Peano axioms with the axiom scheme of induction. Let M be a model of PA. Since we have in PA definable Skolem functions, $Def(M) \prec M$ where $Def(M)$ is the substructure of M with the universe consisting of elements definable in M without parameters. If M is a nonstandard model, then we have in M nonstandard formulas. Therefore we can consider substructures of M analogous to $Def(M)$ with universes consisting of points definable by certain nonstandard formulas and initial segments of M generated by such pointwise definable substructures.

After recalling some basic information on satisfaction classes we give the precise definition of pointwise definable substructures. We distinguish two cases: (a) definability without parameters bigger than the defining formulas and (b) definability with a parameter bigger than the defining formulas. We consider properties of such substructures and of their families.

1 Introduction A serious approach to the possibility of nonabsoluteness of the finite (and so of the logical syntax too) was realized first by Robinson in [15] where he has also shown that nonstandard languages have no uniquely determined semantics. Krajewski (in [11]) has explicitly introduced and has studied the notion of a satisfaction class.

Recall that if M is a nonstandard model of PA and $F\!m$ is a formula of $L(PA)$ strongly representing in PA the recursive set of Gödel numbers of formulas of $L(PA)$ (cf., e.g., [1] and [16]) then we have in M nonstandard objects a such that $M \models F\!m[a]$. We call them nonstandard formulas. They determine a nonstandard language which we denote by $Form(M)$. To speak about its

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semantics we need the notion of a satisfaction class. For the convenience of the reader recall here Krajewski's definition (cf. [11], see also [5]).

To avoid notational complexity we shall resign being pedantic and we shall not distinguish between logical connectives and quantifiers on the one hand and their counterparts in the arithmetization of the language on the other. Hence we shall write for example $\neg\phi$, $\phi \,\&\, \psi$, $(Ex_k)\phi$ (where ϕ is a (possibly nonstandard) formula in the sense of the model M) instead of $\underset{\cdot}{neg}^M(\phi)$, $\underset{\cdot}{con}^M(\phi,\psi)$, $\underset{\cdot}{q}^M(k,\phi)$ where $\underset{\cdot}{neg}$, $\underset{\cdot}{con}$, and $\underset{\cdot}{q}$ are terms of $L(PA)$ strongly representing in PA the recursive functions *neg*, *con*, *q*, respectively, such that:

$$\begin{aligned} neg(\ulcorner\phi\urcorner) &= \ulcorner\neg\phi\urcorner \\ con(\ulcorner\phi\urcorner,\ulcorner\psi\urcorner) &= \ulcorner\phi \,\&\, \psi\urcorner \\ q(k,\ulcorner\phi\urcorner) &= \ulcorner(Ex_k)\phi\urcorner \end{aligned}$$

($\ulcorner\phi\urcorner$ denotes here the Gödel number of the formula ϕ).

We say that $\Phi \subseteq Form(M)$ is closed under immediate subformulas iff whenever any of the formulas $\neg\phi$, $(Ex_k)\phi$ is in Φ then ϕ is in Φ and whenever $\phi \,\&\, \psi$ is in Φ then so are ϕ and ψ.

Satisfaction classes on M are certain sets of pairs of the form $\langle\phi,a\rangle$ where $\phi \in Form(M)$ and a is a valuation for ϕ; i.e., a is a sequence of elements of M with domain corresponding to the set of free variables of ϕ. Using a coding of finite sequences we can treat satisfaction classes as subsets of M.

Definition 1.1 If M is a model of PA then a subset S of M is said to be a satisfaction class on M iff:

(a) if $x \in S$ then $x = \langle\phi,a\rangle$ for some $\phi \in Form(M)$ and some valuation a for ϕ

(b) the class $\Phi(S) = \{\phi \in Form(M)\colon (Ea)(\langle\phi,a\rangle \in S) \vee (a)\ [a$ is a valuation for $\phi \to \langle\neg\phi,a\rangle \in S]$ is closed under immediate subformulas

(c) if $M \models \phi[a]$ then $\langle\ulcorner\phi\urcorner,a\rangle \in S$

(d) if $\neg\phi \in \Phi(S)$ and a is a valuation for ϕ then $\langle\neg\phi,a\rangle \in S \equiv \langle\phi,a\rangle \notin S$

(e) if $\phi \,\&\, \psi \in \Phi(S)$ and a is a valuation for $\phi \,\&\, \psi$ then $\langle\phi \,\&\, \psi,a\rangle \in S \equiv \langle\phi,a'\rangle \in S \,\&\, \langle\psi,a''\rangle \in S$ where a' and a'' are suitable valuations for ϕ and ψ, respectively, obtained from a

(f) if $(Ex_k)\phi \in \Phi(S)$ then $\langle(Ex_k)\phi,a\rangle \in S \equiv [(x_k$ is a free variable of ϕ and $(Eb)(\langle\phi,a^\frown b\rangle \in S))$ or $(x_k$ is not a free variable of ϕ and $\langle\phi,a\rangle \in S)]$ where $a^\frown b$ is a suitable valuation for ϕ obtained from a and b.

We shall often write simply $S(\phi;a)$ or $S(\phi(a))$ instead of $\langle\phi,a\rangle \in S$.

Definition 1.2 A satisfaction class S on M is called full iff for every $\phi \in Form(M)$ and every valuation a for ϕ we have that $\langle\phi,a\rangle \in S$ or $\langle\neg\phi,a\rangle \in S$.

Let L_S be the language $L(PA)$ with an additional predicate symbol S and let $PA(S)$ be the theory in the language L_S based on the following axioms: the axioms of PA, the induction schema for all formulas of L_S, and a set of L_S sentences stating that S is a satisfaction class.

Definition 1.3 (cf. [13], [4]) A satisfaction class S on a model M is said to be substitutable iff $(M,S) \models PA(S)$.

There are a number of interesting results on satisfaction classes. They have been used for example to characterize the resplendency of models of PA (cf. [10] and [12]) or the recursive saturation of them (cf. [13] and [4]).

From now on we make the following general assumption:

Assumption M is a countable model of PA and S is a fixed full substitutable satisfaction class on M.

In fact in many results which will follow we do not need such strong assumptions (in particular in many cases we shall not need the full substitutivity of S), but to avoid the complications in formulating theorems it is convenient to assume that S is full and substitutable.

To define pointwise definable structures we shall need the following notion:

Definition 1.4 An initial segment $I \subseteq_e M$ is said to be closed under logical operations (shortly: closed) iff for any $\phi, \psi \in I$, $k \in I$ if $M \models \dot{F}m(\phi)$ & $\dot{F}m(\psi)$ then $\neg\phi \in I$, ϕ & $\psi \in I$, $(Ex_k)\phi \in I$.

It can easily be seen that the following proposition holds ($I\Sigma_1$ denotes here the subtheory of PA with the axiom scheme of induction restricted to Σ_1 formulas only, similarly $I\Delta_0$, cf. [14]).

Proposition 1.5
(a) *If $I \models I\Sigma_1$ then I is closed.*
(b) *If $I \models I\Delta_0$ and I is closed under exponentiation then I is closed.*
(c) *If I is closed under $+$ and $\cdot$ or if I is closed under exponentiation and $2 \in I$ then I is closed.*

Theorem 1.6 *The family of all initial segments $I \subseteq_e M$ such that I is closed is of the order type of the Cantor set 2^ω with its lexicographical ordering*:

$$b^1 < b^2 \equiv (En)_\omega(b^1_n = 0 \ \& \ b^2_n = 1 \ \& \ (m)_{<n}(b^1_m = b^2_m)).$$

In the proof of this theorem we shall use the following lemma.

Lemma 1.7 (cf. [9]) *Let $(X,<)$ be a complete linear ordering. Then X is isomorphic to the Cantor set if X has a subset W such that*
(a) *the order type of W is $1 + \eta$ (η being the order type of rationals)*
(b) $(x)_{X-W} x = sup\{w \in W\colon w < x\}$
(c) $(x)_W\, x > sup\{w \in W\colon w < x\}$
(in (b) *and* (c) *suprema are in the sense of X rather than W).*

Proof of Theorem 1.6: For an $a \in M$ define

$$\begin{aligned} J(a) = sup\{x \in M\colon\ & (En)_\omega(Ey)\{seq(y) \ \& \ lh(y) = n \\ & \& \ (i)_{\le n}[(Ez)_{<a}((y)_i = z) \vee (Ej,k)_{<i}((y)_i = neg((y)_j) \\ & \vee (y)_i = con((y)_j,(y)_k) \vee (y)_i = q(j,(y)_k))] \\ & \& \ x = (y)_n)\}\}, \end{aligned}$$

where for $X \subseteq M$ we have

$$sup\ X = \{y \in M\colon (Ex)_X(y \le x)\}.$$

Hence $J(a)$ is the supremum of all (truly) finite iterations of logical operations applied to formulas (whose Gödel numbers are) $< a$.

We claim now that the family $\{J(a): a \in M\}$ has the smallest element, has no greatest element, and is densely ordered by inclusion. In fact, the smallest element is the set ω of standard natural numbers. There is no greatest element because for any $a \in M$, $J(a) \neq M$, which follows from the fact that M is the model of PA and $J(a) \nvDash I\Sigma_1$ because in $J(a)$ we can define ω by the following Σ_1 formula:

$$\begin{aligned} n \in \omega \equiv (Ex)(Ey)\{seq(y) \;\&\; lh(y) = n \;\&\; (i)_{\leq n}[&(Ez)_{<a}((y)_i = z) \\ &\vee (Ej,k)_{<i}((y)_i = neg((y)_j) \\ &\vee (y)_i = con((y)_j,(y)_k) \\ &\vee (y)_i = q(j,(y)_k))] \;\&\; x = (y)_n)\}. \end{aligned}$$

To prove the density assume that $J(a) < J(b)$. For any $n \in \omega$ we have

$$(Ec)\,[a \in J_n(c) < b],$$

where $J_n(c)$ is the supremum of all n-fold iterations of logical operations applied to formulas (whose Gödel numbers are) $< c$. By overspill there exists a nonstandard $u > \omega$ such that

$$(Ec)\,[a \in J_u(c) < b].$$

Hence in particular there exists a c such that

$$a \in J(c) < b.$$

Now we use Lemma 1.7. So take as X the family $\{I \subseteq_e M: I \text{ is closed}\}$, and as W the family $\{J(a): a \in M\}$. It can be easily seen that conditions (a)–(c) of the lemma are satisfied. Hence the family of initial segments of M which are closed is of the order type of the Cantor set 2^ω.

Corollary 1.8 *The cardinality of the family* $\{I \subseteq_e M: I \text{ is closed}\}$ *is* $2^{\aleph_0}$.

2 Pointwise definability without parameters

Definition 2.1 Let $I \subseteq_e M$ be a closed initial segment. We define substructures of M with the following universes:

$$\begin{aligned} D(I) &= \{x \in M: (E\phi)_I(E\bar{a})_I(M,S) \models [Fm(\phi) \\ &\qquad \& \; S((E!x)\phi;\bar{a}) \;\&\; S(\phi;x,\bar{a})]\}, \\ M^{D(I)} &= sup\; D(I) = \{x \in M: (Ey)_{D(I)}(x \leq y)\}. \end{aligned}$$

Remarks:

1. Being pedantic we ought to write $D(M,S,I)$ and $M^{D(M,S,I)}$ but since M and S are fixed we simplify the notation.
2. The structure $D(I)$ is simply a submodel of M consisting of elements definable by nonstandard formulas belonging to I with parameters from I and $M^{D(I)}$ is an initial segment of M generated by $D(I)$.

Proposition 2.2
(a) $D(I) \prec M$
(b) $M^{D(I)} \prec_e M$.

Proof: Case (a) follows from the fact that S is substitutable and hence we have definable Skolem functions for L_S. For the proof of case (b) assume that $M \models$

$(Ex)\phi(x,a)$ where $a \in M^{D(I)}$ and ϕ is a standard formula. It suffices to show that there is a $b \in M^{D(I)}$ such that $M \models \phi(b,a)$. So let $x_0 \in D(I)$ be such that $a \leq x_0$ and let $\psi \in I$ be the definition of x_0. Consider the formula:

$$\chi(y) \equiv seq(y) \;\&\; lh(y) = x_0 \;\&\; (i)_{<lh(y)}[\phi((y)_i,i) \;\&\; (z)_{<(y)_i}\neg\phi(z,i)].$$

One can eliminate here x_0 by substituting its definition ψ. Hence $\chi \in I$. Observe that $(M,S) \models (Ey)S(\chi,y)$. Hence there is a $y_0 \in D(I)$ such that $(M,S) \models S(\chi,y_0)$. We have $lh(y_0) = x_0 \geq a$ and

$$(M,S) \models \phi((y_0)_a,a)$$

and $(y_0)_a \in M^{D(I)}$ since $(y_0)_a < y_0$.

Proposition 2.3 *The following inclusions hold:*

$$N_0 \subseteq Def(M) \subsetneq D(I) \subsetneq D(M) = M \text{ and } I \subseteq D(I)$$

where N_0 is the standard model of PA and $N_0 \subsetneq_e I \subsetneq_e M$ and I is closed.

Proof: It is obvious. We shall show only that $D(I) \neq D(M)$ for $I \subsetneq_e M$. Consider the following function

$$f(u) = \mu x\text{: "}x \text{ is not definable by formulas } \leq u\text{"};$$

i.e.,

$$f(u) = \mu x\text{: } (\phi)_{\leq u}[Fm(\phi) \;\&\; S(\phi;x) \rightarrow (Ez)_{<x}S(\phi;z)].$$

Since S is substitutable (in fact it is enough to have here only $(M,S) \models L\Sigma_0(S)$; cf. [14]) hence $(M,S) \models$ "f is a function". Let now $a \in M - I$. Then $f(a) \in D(M) - D(I)$.

Proposition 2.4 *If $I \not\equiv M$ then $I \subsetneq D(I)$.*

Proof: Assume $I = D(I)$. Then since $D(I) \prec M$ we would have $I \equiv M$ which contradicts the assumption.

Proposition 2.5 *Let f be the following function (defined in (M,S)):*

$$f(x) = \begin{cases} \mu y\; S(x;y), & \text{if } Fm(x) \;\&\; S((E!y)x;\, \emptyset), \\ 0, & \text{otherwise.} \end{cases}$$

Then
(a) *if I is closed under f then $D(I) = I$ and $M^{D(I)} = I$,*
(b) *if $(I, S \cap I) \prec_3 (M,S)$ then $D(I) = I$ and $M^{D(I)} = I$, where $\mathfrak{A} \prec_3 \mathfrak{B}$ means that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ with respect to Σ_3^0 formulas.*

Proof: (a) is obvious. (b) follows from the fact that the formula "$f(x) = y$" is $\Sigma_2^0(S)$.

Remark: Observe that if $(I, S \cap I) \prec_1 (M,S)$ then $I \prec M$ and similarly if I is closed under f then $I \prec M$. This follows from the fact that in both cases $S \cap I$ is a satisfaction class on I and if $S \cap I$ is a satisfaction class on I then for any standard formula ϕ and any sequence $\bar{a} \in I$ we have

$$M \models \phi[\bar{a}] \equiv S(\phi;\bar{a}) \equiv (S \cap I)(\phi;\bar{a}) \equiv I \models \phi[\bar{a}].$$

Hence $I \prec M$.

We can ask if $D(I)$ can be an initial segment of M. We have only the following negative result.

Proposition 2.6 *If M is such that $N_0 \subsetneq Def(M)$ and N_0 is strong in M then there are I such that $D(I)$ is not an initial segment of M.*

Proof: Let $b \in Def(M)$ be nonstandard and let $a \in M$ be a nonstandard element such that $a \notin Def(M)$ and $a < b$ (such an a exists since N_0 is strong in M; cf. [3]). Hence

$$(n)_\omega (M,S) \models \neg (E\phi)_{<n} \text{ [}\phi \text{ is a formula \& } \phi \text{ is a definition of } a \text{ with parameters } <n\text{]}.$$

By overspill

$$(En_0)_{>\omega} (M,S) \models \neg (E\phi)_{<n_0} \text{ [}\phi \text{ is a formula \& } \phi \text{ is a definition of } a \text{ with parameters } <n_0\text{]}.$$

Take k = maximum of such n_0's. Now let I be a closed initial segment of M such that $N_0 \subseteq I < k$. Such segments exist since for example initial segments being models of PA lie arbitrarily low in M; cf. [4], [13]. We have now that $a \notin D(I)$ but $b \in D(I)$. Hence $D(I)$ is not an initial segment of M.

Remark: Observe that if $N_0 \not\equiv M$ then $N_0 \subsetneq Def(M)$ and vice versa.

Before proving the next theorem recall the following definition:

Definition 2.7 Let $I \subseteq_e M$. We say that ω codes I in M iff there exists a function $f \in M$ (i.e., coded in M) such that all standard natural numbers are in $dom(f)$ and

$$(x)_m (x \in I \equiv (En)_\omega\, M \models x < f(n)).$$

Theorem 2.8 *If ω noncodes I in M then $D(I)$ and $M^{D(I)}$ are recursively saturated.*

Proof: Let $\Phi(x,b)$ be a consistent recursive type in $D(I)$ with a parameter $b \in D(I)$. Let $\phi_0 \in I$ be a definition of b. We have

$$(n)_\omega (M,S) \models (Ex)(\phi)_{<n}[\underset{\cdot}{\Phi}(\phi) \to S(\phi;b,x)]$$

and

$$(n)_\omega (M,S) \models (Ex)(\phi)_{<n}[\underset{\cdot}{\Phi}(\phi) \to (Ez)(S(\phi_0;z) \;\&\; S(\phi;z,x))]$$

where $\underset{\cdot}{\Phi}$ is a formula of $L(PA)$ strongly representing in PA the recursive set Φ. By overspill there exists a nonstandard $n_0 > \omega$ such that

$$(M,S) \models (Ex)(\phi)_{<n_0}[\underset{\cdot}{\Phi}(\phi) \to (Ez)(S(\phi_0;z) \;\&\; S(\phi;z,x))].$$

Hence the type $\Phi(x,b)$ is realized in M. It is enough to show that the realizing element can be found already in $D(I)$. So let Θ be a recursive function enumerating Φ and $\underset{\cdot}{\Theta}$ its representation in $L(PA)$. Let

$$\delta'(x,k) \equiv (y)_{<k}(Ez)[S(\phi_0;z) \;\&\; S(\underset{\cdot}{\Theta}(y);z,x)],$$
$$\delta(x,k) \equiv \delta'(x,k) \;\&\; (v)[\delta'(v,k) \to x \leq v],$$
$$f(k) = \ulcorner \delta(x,k) \urcorner.$$

The function f is coded in (M,S) and hence in M and $(k)_\omega[f(k) \in I]$. By overspill there exists $n_0 > \omega$ such that $f(n_0) \in I$. Indeed, otherwise $f''(\omega)$ would be cofinal with I and ω would code I via f, which contradicts the assumption. But $f(n_0)$ is the (Gödel number of) the definition of an element realizing Φ. Hence Φ is realized in $D(I)$.

The recursive saturation of $M^{D(I)}$ follows from the recursive saturation of $D(I)$ and the fact that $D(I) \subseteq_{cf} M^{D(I)}$ (cf. [7], [17]).

Observe that we used in the proof only the fact that ω noncodes I in M via a particular function f. Hence it follows that our condition, though sufficient, is not necessary.

Observe also that $card\{I \subseteq_e M\colon \omega \text{ codes } I\} = \aleph_0$ and hence $card\{I \subseteq_e M\colon \neg(\omega \text{ codes } I)\} = 2^{\aleph_0}$. By cardinality argument we have:

Proposition 2.9 *There are $2^{\aleph_0}$ closed initial segments $I \subseteq_e M$ such that $\neg(\omega$ codes $I)$.*

Recall the following definition:

Definition 2.10 ([2], [3]) Let Q_1 and Q_2 be two families of initial segments of the model M. We say that Q_1 is symbiotic with Q_2 iff for any $a,b \in M$, $a < b$ we have

$$(\mathrm{E}I)_{Q_1}(a \in I < b) \quad \text{iff} \quad (\mathrm{E}J)_{Q_2}(a \in J < b).$$

Proposition 2.11 *The family $\{I \subseteq_e M\colon I$ closed and $\neg(\omega$ codes $I)\}$ is symbiotic with $\{I \subseteq_e M\colon I$ closed$\}$.*

Proof: Let $a,b \in M$, $a < b$ and assume that there is a closed initial segment I such that $a \in I < b$. Since we can construct an indicator for this family (cf. [2] and [3] for the definition of an indicator), using standard tricks one can show that there are $2^{\aleph_0}$ such segments between a and b. But there are only countably many segments I such that ω codes I. Hence between a and b there must be at least one (in fact $2^{\aleph_0}$) segments I such that I is closed and $\neg(\omega$ codes $I)$.

Theorem 2.12 *Let $I \subsetneq_e J \subsetneq_e M$ be closed. Then $D(I) \subsetneq D(J)$.*

Proof: For $n \in \omega$ let Tr_n be the natural truth definition for Σ_n formulas. We define the following functions F_n in PA (cf. [8]):

$$\begin{aligned} F_n(0) &= \ulcorner v_2 = v_1 + 1 \urcorner, \\ F_n(x+1) &= \mu w\colon (\phi)_{\le F_n(x)}(u)_{\le F_n(x)}[\phi \in \Sigma_n \\ &\qquad \& \ Tr_n((Ez)\phi; u) \to (Ez)_{<w} Tr_n(\phi; u, z)]. \end{aligned}$$

The formula "$y = F_n(x)$" is of the class Σ_{n+1}. Let $j \in J - I$. We can arithmetize the syntax in such a way that $Tr_j \in J$. Hence $\ulcorner y = F_j(x) \urcorner \in J$. Consider the formula

$$\phi \equiv \text{“}x = F_j(j)\text{”}.$$

One can see that $\phi \in J$ and $\phi \in \Sigma_{j+1}$. Hence the element x_0 defined by ϕ is in $D(J)$. We claim that $x_0 \notin D(I)$. Indeed if $x_0 \in D(I)$ then there would be a formula $\psi \in I$ defining x_0. Hence $\psi < j$. So $\psi \in \Sigma_j$. But $PA \vdash (x)(x < F_a(x))$ for any a. Hence $\psi < j < F_j(j)$ and $x_0 < x_0$, a contradiction. Hence $x_0 \notin D(I)$.

It follows from the proof that $x_0 > D(I)$ and hence $x_0 > M^{D(I)}$. So we have $M^{D(J)} \supseteq D(J) \supsetneqq M^{D(I)}$ and the following corollary (of the proof) holds:

Corollary 2.13 *Let $I \subsetneqq_e J \subsetneqq_e M$ be closed. Then $M^{D(I)} \subsetneqq_e M^{D(J)}$.*

Corollary 2.14 *If $I \subsetneqq_e J \subsetneqq_e M$ are closed then $D(I)$ is not a cofinal substructure of $D(J)$.*

Remark: In the proof of Theorem 2.12 it suffices to assume that S is Δ_0-substitutable. Hence to have the situation that $D(I) = D(J)$ for closed $I \subsetneqq_e J \subsetneqq_e M$ we should take a rather pathological satisfaction class S on M; i.e., being even not Δ_0-substitutable.

Definition 2.15

$$\mathcal{A} = \{D(I)\colon I \subseteq_e M,\ I \text{ closed}\},$$
$$\mathcal{B} = \{M^{D(I)}\colon I \subseteq_e M,\ I \text{ closed}\}.$$

We ask now how big the families $\mathcal{A}$ and $\mathcal{B}$ are. We shall answer this question using various types of measures.

Theorem 2.16 *The families $\mathcal{A}$ and $\mathcal{B}$ are of the order type of the Cantor set 2^ω with its lexicographical ordering. Hence card $\mathcal{A}$ = card $\mathcal{B} = 2^{\aleph_0}$.*

Proof: It follows from Theorem 1.6, Theorem 2.12, and Corollary 2.13.

Following [6] we shall denote

$$Y = \{N \subseteq_e M\colon N \prec M\},$$
$$Y_1 = \{N \subseteq_e M\colon N \prec M \ \&\ N \text{ is not recursively saturated}\}.$$

Kotlarski has shown in [6] that

(1) *card* $Y_1 = \aleph_0$,
(2) Y is symbiotic with Y_1 and Y_1 is symbiotic with $Y - Y_1$.

We can ask now if the family $\mathcal{B}$ is symbiotic with Y. The answer gives the following

Theorem 2.17 *The family $\mathcal{B}$ is not symbiotic with Y.*

Proof: Let $a \in M$ be any nonstandard element and let $e > \omega$. We shall find an element $b \in M$ such that $(EN)_Y(a \in N < b)$ but $\neg(EN)_{\mathcal{B}}(a \in N < b)$. We have

$$\begin{aligned}(n)_\omega(M,S) \models (k)_{\le n}(E\phi)_{>k}(F\dot{m}(\phi)\ \&\ \phi < e\ \&\ S((E!x)\phi;a)\\ \&\ (\psi)_{<\phi}(F\dot{m}(\psi)\ \&\ S((E!z)\psi;a)\\ \rightarrow (x)(z)(S(\phi;x,a)\ \&\ S(\psi;z,a) \rightarrow x \neq z))).\end{aligned}$$

By overspill there is an $n_0 > \omega$, $n_0 < a$ such that this same holds in (M,S) for every $k \le n_0$. Let now b be the maximum of elements of M which are defined by formulas $<n_0$ with the parameter a. First we show that b itself is defined by a formula $<n_0$. In fact let $\Gamma(\phi)$ be a formula

$$\begin{aligned}F\dot{m}(\phi)\ \&\ (Ey)S([(E!x)\phi(x,a)\ \&\ \phi(y,a)]\\ \vee\ [\neg(E!x)\phi(x,a)\ \&\ y = 0]).\end{aligned}$$

The formula Γ is of the form $(Ey)\Delta(\phi,y)$ where Δ is a bounded formula of the language L_S. We have

$$(M,S) \models (\phi)_{<n_0}(F\dot{m}(\phi) \to (Ey)\Delta(\phi,y)).$$

Hence by collection (cf. [14])

(*) $\qquad (M,S) \models (Ez)(\phi)_{<n_0}(Ey)_{\leq z}(F\dot{m}(\phi) \to \Delta(\phi,y)).$

Let z_0 be the smallest z with this property. Of course $z_0 \neq 0$. We claim that $(M,S) \models (E\phi)_{<n_0}\Delta(\phi,z_0)$. If not then $z_0 - 1$ would also have the property (*) contradicting the choice of z_0. It can be easily seen that z_0 is our b and that the smallest formula $\phi < n_0$ such that $(M,S) \models \Delta(\phi,z_0)$ is the definition of b.

By the construction there exists an initial segment $N \in Y$ such that $a \in N < b$.

Let now $J \subseteq_e M$ be a closed initial segment such that $a \in D(J)$. Let $\psi \in J$ be the definition of a and ϕ the definition of b. Consider the formula

$$\chi(v) \equiv (z)[\psi(z) \to \phi(v,z)].$$

Of course $\chi \in J$ and defines b. Hence $b \in D(J)$. Consequently we have shown that there are $a,b \in M$, $a < b$ such that

(1) $(EN)_Y(a \in N < b)$.
(2) for every closed $J \subseteq_e M$ if $a \in D(J)$ then $b \in D(J)$.

From this it follows also that

(3) for every closed $J \subseteq_e M$ if $a \in M^{D(J)}$ then $b \in M^{D(J)}$.

Hence $\neg(EN)_{\mathcal{B}}(a \in N < b)$.

Corollary 2.18 *The family $\mathcal{B}$ is symbiotic neither with Y_1 nor with $Y - Y_1$.*

Proposition 2.19 *The family $Y - \mathcal{B}$ is of the cardinality $2^{\aleph_0}$.*

Proof: By Theorem 2.17 there are $a,b \in M$, $a < b$ such that $(EN)_Y(a \in N < b)$ but $\neg(EN)_{\mathcal{B}}(a \in N < b)$. Since we can construct an indicator (in the language L_S) for the family Y (cf. [13], [4]) by standard tricks we get that between a and b there are $2^{\aleph_0}$ initial segments belonging to Y. Consequently $card(Y - \mathcal{B}) = 2^{\aleph_0}$.

Proposition 2.20 *The family $\mathcal{B} \cap \{M^{D(I)}: I \subseteq_e M, I\ closed, I \neq M\}$ contains no semiregular initial segment.*

Proof: It follows from the easy observation that for any N from our family $cf(N) = I < N$ and for a semiregular N we have $cf(N) = N$.

To formulate the next theorem recall the following definition.

Definition 2.21

(a) A function F: $M \to M$ is normal iff F is definable in (M,S) and is strictly increasing.
(b) A set $A \subseteq Y$ is normal iff for some normal function F

$$A = \{N \in Y: (x)_N[F(x) \in N]\}.$$

(c) A set $B \subseteq Y$ is stationary iff for all normal sets $A \subseteq Y$, $A \cap B \neq \emptyset$.

Theorem 2.22
(a) *Let* $X \subseteq Y$ *be stationary. Then* X *contains arbitrarily large initial segments.*
(b) *The family* $\mathcal{B}$ *is not stationary.*

The theorem will follow from the more general Theorem 3.15.

3 Pointwise definability with a parameter

Definition 3.1 Let $I \subseteq_e M$ be closed, $a \in M$ and $a > I$. We define substructures of M with the following universes:

$$D(I,a) = \{x \in M\colon (E\phi)_I(E\bar{b})_I(M,S) \models [Fm(\phi) \ \& \ S((E!x)\phi;a,\bar{b}) \ \& \ S(\phi;x,a,\bar{b})]\},$$
$$M^{D(I,a)} = sup\ D(I,a) = \{x \in M\colon (Ey)_{D(I,a)}(x \leq y)\}.$$

As before we ought to write $D(M,S,I,a)$ and $M^{D(M,S,I,a)}$ but since M and S are fixed we simplify the notation.

Proposition 3.2
(a) $D(I,a) \prec M$,
(b) $M^{D(I,a)} \prec_e M$.

Proposition 3.3
(a) *If* $I \subsetneq_e M$ *is closed and* $a > I$ *then the following inclusions hold*:

$$N_0 \subseteq D(\omega,a) \subsetneq D(I,a) \subsetneq D(M,a) = M$$

and $a \in D(\omega,a)$, $I \subseteq D(I,a)$.

(b) *If* $I \not\equiv M$ *then* $I \subsetneq D(I,a)$.

Theorem 3.4 *If* ω *noncodes* I *in* M *then* $D(I,a)$ *and* $M^{D(I,a)}$ *are recursively saturated.*

Proof: Is similar to the proof of Theorem 2.8.

Theorem 3.5 (Kotlarski [6]) *If* $N \prec_e M$ *and* N *is not recursively saturated then there exists* $a \in M$ *such that* $N = M^{D(\omega,a)}$.

Corollary 3.6
(a) *If* $N = M^{D(I)}$ *for some closed* $I \subseteq_e M$ *and* N *is not recursively saturated then there exists an* $a \in M$ *such that* $N = M^{D(\omega,a)}$.
(b) *If* $N = M^{D(I,a)}$ *for some closed* $I \subseteq_e M$ *and* $a > I$ *is not recursively saturated then there exists a* $b \in M$ *such that* $N = M^{D(\omega,b)}$.

This corollary indicates the fact that in the case of initial segments which are not recursively saturated we can replace the definability with the help of nonstandard formulas by definability with the help of standard formulas with an additional parameter (greater than the defining formulas). It shows the role of parameters and proves that they are more important in definability than the usage of nonstandard formulas. Compare it with the fact that Y_1 is symbiotic with Y but $\mathcal{B}$ is not symbiotic with Y. Hence there are $a, b \in M$, $a < b$ such that between them there exist structures of the form $M^{D(\omega,c)}$ for $c \in M$ but there is no structure of the type $M^{D(I)}$ for a closed $I \subseteq_e M$.

Theorem 3.7 *Let $I \subsetneq_e J \subsetneq_e M$ be closed, $a \in M$. Then $D(I,a) \subsetneq D(J,a)$ and $M^{D(I,a)} \subsetneq M^{D(J,a)}$.*

Proof: We consider three cases:
(a) $I \subsetneq_e J < a$
(b) $I < a \in J$
(c) $a \in I \subsetneq_e J$.

Case (a). We prove the theorem similarly to the proof of Theorem 2.12, but now in the definition of functions F_n we must add the parameter a.

Case (b). In this case $D(J,a) = D(J)$. We follow the proof of Theorem 2.12, but now we take $j \in J - I$ such that $a \leq F_j(j)$. Such j exists since the following claim holds:

Claim For every $a \in J$ there exists $j \in J$ such that $a \leq F_j(j)$.

This follows from the fact that if t is a Σ_n term then there exists a b such that $PA \vdash (c)_{>b}(t(c) < F_n(c))$ (cf. Lemma 3.4(iii) of [8]) and the fact that $PA \vdash (a)(F_n(a) < F_n(a+1))$ (cf. Lemma 3.4(i) of [8]).

Case (c). In this case $D(I,a) = D(I)$ and $D(J,a) = D(J)$ and we apply Theorem 2.12.

Definition 3.8

$$\mathfrak{A}' = \{D(I,a) \colon I \subseteq_e M,\ I \text{ closed},\ a > I\},$$
$$\mathfrak{B}' = \{M^{D(I,a)} \colon I \subseteq_e M,\ I \text{ closed},\ a > I\}.$$

Theorem 3.9 *The family $\mathfrak{B}'$ is symbiotic with Y.*

Proof: It follows immediately from the fact that Y_1 is equal to $\{M^{D(\omega,a)} \colon a \in M\}$ and that Y_1 is symbiotic with Y (cf. [6]).

Proposition 3.10 *The family $\mathfrak{B}'$ contains no semiregular initial segments.*

Hence no interesting (from the point of view of combinatorial properties, cf. [2], [3]) initial segments are generated by pointwise definable (even by nonstandard formulas with big parameters) substructures.

Consider now projections of the family $\mathfrak{B}'$.

Definition 3.11 For a fixed closed $I \subseteq_e M$ let

$$\mathfrak{B}'_I = \{M^{D(I,a)} \colon a \in M,\ a > I\}.$$

Similarly for a fixed $a \in M$:

$$\mathfrak{B}'_a = \{M^{D(I,a)} \colon I \subseteq_e M,\ I \text{ closed}\}.$$

In contrast with Theorem 3.9 we have the following theorem.

Theorem 3.12
(a) *For every $I \subseteq_e M$ closed, $I > \omega$, $\mathfrak{B}'_I$ is not symbiotic with Y (nor with Y_1).*
(b) *For any $a \in M$, $\mathfrak{B}'_a$ is not symbiotic with Y (nor with Y_1).*

Proof: (a) Fix a closed initial segment $I_0 \subseteq_e M$, $I_0 > \omega$. Take k such that $\omega < k \in I_0$ and take an initial segment $J \subseteq_e M$ such that $J > \omega$, J is closed and $J < k$ (it can be done since in M there are arbitrarily small initial segments being models of PA). Let $\omega < l \in J$, let a be any element of M such that $a > I_0$ and let

u = supremum of elements definable by formulas $<l$ with the parameter a
w = supremum of elements definable by formulas $<k$ with the parameter a.

There exists an initial segment $N < M$ such that $u \in N < w$. Indeed, take $N = M^{D(J,a)}$.

Assume that there is a b such that $u \in M^{D(I_0,b)} < w$. Then $b < a$. Indeed if $b \geq a$ then we would take

$$d = \max x\colon (Ez)_{\leq b}[S(\phi_0;z,x) \ \& \ (t)_{<x}\neg S(\phi_0;z,t)]$$

where ϕ_0 is the definition of w. Hence $d \geq w$ and is defined by the formula from I_0. So $w \in M^{D(I_0,b)}$ which gives a contradiction. Hence $b < a$.

Let now $\psi \in I_0$ be a formula defining some $y_0 \geq u$ from the parameter b; i.e., such that $(M,S) \models S((E!z)\psi;z,b) \ \& \ S(\psi;y_0,b)$. Consider the formula

$$\begin{aligned}\chi(z) \equiv (Ey)\{S(\psi;b,y) \ \& \ (t)_{<y}[S((E!x)\phi_0;t,x) \\ \rightarrow (x)(S(\phi_0;t,x) \rightarrow x < z)]\}.\end{aligned}$$

Let $z_0 = \mu z \chi(z)$. The definition of z_0 belongs to I_0. Hence z_0 is an element of $M^{D(I_0,b)}$ but $z_0 > w$ which is a contradiction.

Consequently there is no initial segment of the type $M^{D(I_0,b)}$ between u and w. Hence $\mathfrak{B}'_{I_0}$ is not symbiotic with Y (nor with Y_1).

(b) We prove it in a way similar to the proof of Theorem 2.17.

Theorem 3.12 could suggest that the crucial role is played here by definability by standard formulas with parameters. But this is not true since we have the following theorem.

Theorem 3.13 *The family $\mathfrak{B}' - Y_1$ is symbiotic with Y_1 (and hence with $Y - Y_1$ and Y).*

Proof: Let $a,b \in M$, $a < b$ be such that there exists an $N \in Y_1$ such that $a \in N < b$. We have then

$$(n)_\omega M \models (t)_{<n}[\underset{\cdot}{Term}(t) \rightarrow t(a) < b]$$

where $\underset{\cdot}{Term}$ is the formula of $L(PA)$ strongly representing in PA the set of (Gödel numbers of) terms of $L(PA)$. By overspill there is an $n_0 > \omega$ such that

$$(M,S) \models (t)_{<n_0}[\underset{\cdot}{Term}(t) \rightarrow S(t(a) < b;\ \emptyset)].$$

Take a closed initial segment $I_0 \subseteq_e M$ such that $N_0 \subsetneq I_0 < n_0$ and ω noncodes I_0. Then $a \in M^{D(I_0,a)} < b$. Hence there exists a segment $N' \in \mathfrak{B}' - Y_1$ such that $a \in N' < b$.

Theorem 3.14
(a) *For any closed initial segment $I \subseteq_e M$, the family $\mathfrak{B}'_I$ is of the order type η.*
(b) *For any $a \in M$, the family $\mathfrak{B}'_a$ is of the order type of the Cantor set 2^ω.*

Proof: (a) It is enough to show that $\mathfrak{B}'_{I_0}$ is densely ordered for a given closed initial segment $I_0 \subseteq_e M$. So let a and b be such that $M^{D(I_0,a)} < M^{D(I_0,b)}$. Take any $j \in I_0$. We have

$$\begin{aligned}(M,S) \models (Ec)(\phi)_{<j}\{&[\underset{\cdot}{Fm}(\phi) \ \& \ S((E!x)\phi;a) \rightarrow (x)(S(\phi;x,a) \rightarrow x < c)] \\ &\& \ [\underset{\cdot}{Fm}(\phi) \ \& \ S((E!x)\phi;c) \rightarrow (x)(S(\phi;x,c) \rightarrow x < b)]\}.\end{aligned}$$

Indeed we can find such an element c in $M^{D(I_0,a)}$. By overspill there is a $k > I_0$ such that

$$(M,S) \models (Ec)(\phi)_{<k}\{[Fm(\phi) \ \& \ S((E!x)\phi;a) \rightarrow (x)(S(\phi;x,a) \rightarrow x < c)] \\ \& \ [Fm(\phi) \ \& \ S((E!x)\phi;c) \rightarrow (x)(S(\phi;x,c) \rightarrow x < b)]\}.$$

Take such an element c. Then $M^{D(I_0,a)} < M^{D(I_0,c)} < M^{D(I_0,b)}$.
(b) Follows by Theorems 1.6 and 3.7.

Remark: The only dense subsets of the Cantor set 2^ω which can be distinguished by inner properties of it are sets E and D where

$$E = \{b \in 2^\omega\colon (Em)(n)_{>m}(b_n = 0)\},$$
$$D = \{b \in 2^\omega\colon (Em)(n)_{>m}(b_n = 1)\}.$$

Hence it is impossible to answer the following question: To which branches of the set of 2^ω are $\mathcal{B}', \mathcal{B}'_I, \mathcal{B}'_a$, respectively, isomorphic?

Remark: The family $\mathcal{B}'$ is not of the order type of the Cantor set 2^ω.

Theorem 3.15 *The family $\mathcal{B}'$ is not stationary.*

Proof (cf. [8]): We define in (M,S) the following function $F(a,i)$:

$$F(a,0) = \text{the value of the term } t_0 \text{ on } a, \text{ where } t_0 \text{ is the smallest term},$$
$$F(a,i+1) = \mu x\colon (j)_{\leq i+1}[Term(j) \rightarrow S(sub(j,F(a,i)) < x;\ \emptyset)].$$

(For the definition of the function sub see e.g. [16].) Define further

$$G(x) = \begin{cases} F(a,i), & \text{if } x = \langle a,i \rangle, \\ 0, & \text{otherwise} \end{cases}$$

and

$$H(0) = G(0),$$
$$H(i+1) = max(1 + H(i), G(i+1)).$$

The function H is normal but no initial segment $N \in \mathcal{B}'$ is closed under it. In fact let $N = M^{D(I,a)}$ for some $I \subseteq_e M$, I closed and $a > I$. Then $F(a,i) \notin N$ for $i > I$, $i \in N$. Hence $H(\langle a,i \rangle) \notin N$.

Corollary 3.16 *The families $\mathcal{B}$, $\mathcal{B}'_I$, and $\mathcal{B}'_a$ (for any closed $I \subseteq_e M$ and any $a \in M$) are not stationary.*

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