

Material Implication and Entailment

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The paradoxes of material implication have been called “paradoxes” of a sort because they seem to allow truth to material implications where the antecedents and consequents, respectively, have no relevance to each other. We shall show, however, that in cases of true material implications, their antecedents and consequents, respectively, have some relevance to each other.

“ $p \rightarrow q$ ” is true in three cases: where “ $p \cdot q$ ”, “ $\sim p \cdot q$ ” and “ $\sim p \cdot \sim q$ ” are respectively true. Let us import these conjunctions alternately as explicit antecedents of “ $p \rightarrow q$ ” and construct the truth tables of the resulting formulas, as follows:

$$(1) \quad \frac{(p \cdot q) \rightarrow (p \rightarrow q)}{}$$

T	T	T
F	T	F
F	T	T
F	T	T

$$(2) \quad \frac{(\sim p \cdot q) \rightarrow (p \rightarrow q)}{}$$

F	T	T
F	T	F
T	T	T
F	T	T

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$$(3) \quad \frac{(\sim p \cdot \sim q) \rightarrow (p \rightarrow q)}{\quad}$$

F	T	T
F	T	F
F	T	T
T	T	T

(1), (2), and (3) are tautologies.

(1), (2), and (3) are summarized by the following rule for material implication: *A material implication is true if either the consequent is true or the antecedent is false.* This rule may be symbolized by two formulas:

$$(4) \quad q \rightarrow (p \rightarrow q)$$

and

$$(5) \quad \sim p \rightarrow (p \rightarrow q).$$

(4) and (5) state the so-called *paradoxes of material implication* (see [1], p. 10). Performing exportation on (4), we get:

$$(6) \quad (q \cdot p) \rightarrow q$$

and commuting “ $q \cdot p$ ” in (6) we get:

$$(7) \quad (p \cdot q) \rightarrow q.$$

Note that in (7), which gives the formula for a material implication with a true consequent, we have imported “ q ” into the antecedent. This is essential in order to express what has so far been merely implicit, namely, that a material implication with a true consequent has in effect the truth of such a consequent implicit in the antecedent. In (7) this becomes clear: It is not the antecedent “ p ” alone that implies the consequent “ q ” but the “ q ” implicit in “ p ” that implies (in fact, entails) itself in the consequent. We may, therefore, say that a material implication with a true consequent has the truth of such a consequent implicitly contained in the antecedent. And it is this implicit “ q ” in the antecedent “ p ” that really implies the “ q ” in the consequent. Another way of saying this is that the antecedent “ p ”, together with the implicit “ q ”, entails the consequent “ q ”. This is the reason that a true consequent is implied by any antecedent, whether true or not. For in assuming the truth of the consequent, we in effect include such an assumption in the antecedent [as in (4), (6), (7)], although in most cases, we are not aware of this, since such an assumption is not usually articulated in “ $p \rightarrow q$ ”. However, when we articulate the assumption of the truth of “ q ” in the antecedent of “ $p \rightarrow q$ ”, as in (4), (6), and (7), the material implication no longer seems paradoxical. This is further shown by the following statement form, which is a tautology:

$$(8) \quad \frac{q \rightarrow \{(p \rightarrow q) \leftrightarrow [(p \cdot q) \rightarrow q]\}}{\quad}$$

T T	T T T	T	T	T T
F T	T F F	F	F	T F
T T	F T T	T	F	T T
F T	F T F	T	F	T F

(8) reads: If “ q ” is true, then “ $p \rightarrow q$ ” is equivalent to “ $(p \cdot q) \rightarrow q$ ”. In fact, although I have encountered some disagreement on this interpretation with some colleagues, I wish to maintain that “ $p \rightarrow q$ ” and “ $(p \cdot q) \rightarrow q$ ” are logically equivalent when “ q ” is true; for they both have the truth value “true” in the two rows of the above truth table (the first and the third) where “ q ” is true. In other words, the equivalence relation cannot be false when “ q ” is true. Note that the equivalence relation is also true in the fourth row, although here the truth value of “ q ” is false; but that is because in this row, “ p ” is false, the equivalence relation being also true in all rows where “ p ” is false. Also, with the truth of “ q ” assumed, each side of the above equivalence relation expressed in the consequent of (8) may be derived from the other. It should be noted that whatever we say in this paper concerning material implications also holds true *mutatis mutandis* of material biconditionals. Hence, if our thesis is correct that in material implications the antecedent entails the consequent when the material implication is true [cf. (43), (44), and (45)], then material equivalences must be regarded as contingent logical equivalences on any condition that makes the material equivalence true.

But even if we do not accept the hypothesis suggested here that “ $p \rightarrow q$ ” and “ $(p \cdot q) \rightarrow q$ ” are logically equivalent on the condition that “ q ” is true, nevertheless, if we articulate the assumption that “ q ” in “ $p \rightarrow q$ ” is true, we obtain (4) and ultimately (7), in which the antecedent entails the consequent.

Let us now consider the following examples:

- (9) If the moon is made of green cheese, then Ronald Reagan is President of the United States.

(9) is true because its consequent is true, and this seems paradoxical because there is no relevance at all between its antecedent, “the moon is made of green cheese”, and its consequent, “Ronald Reagan is President of the United States”. However, if we make explicit our assumption of the truth of the consequent in (9), the paradox disappears, to wit:

- (10) Because Ronald Reagan is President of the United States, (even) if the moon is made of green cheese, Ronald Reagan is President of the United States.

Here is another example, this time one with a true antecedent:

- (11) Because Ronald Reagan is President of the United States, (even) if Mikhail Gorbachev is Premier of the Soviet Union, Ronald Reagan is President of the United States.

Note that since a true statement is implied by *any* statement, it is implied by contradictory statements. Hence, the following is also true:

- (12) Whether or not the moon is made of green cheese, Ronald Reagan is President of the United States.

This, however, is logically equivalent to:

- (13) Ronald Reagan is President of the United States.

for the following biconditional is a tautology:

(14)
$$\frac{[(p \vee \sim p) \rightarrow q] \leftrightarrow q}{\begin{array}{ccc} T & T T & T T \\ T & F F & T F \\ T & T T & T T \\ T & F F & T F \end{array}}$$

In fact, (12) is an emphatic way of saying (13).

Let us in turn perform exportation on (5). Here we get

(15) $(\sim p \cdot p) \rightarrow q.$

Note that in (14) the antecedent is now explicitly self-contradictory. At this point, let us remind ourselves that the proposition “*p*” is logically equivalent to the proposition “*p* is true”, as is shown by the following truth table:

(16)
$$\frac{p \leftrightarrow \text{“}p \text{ is true”}}{\begin{array}{cc} T & T \\ F & T \end{array}} \quad \begin{array}{c} T \\ F \end{array}$$

Hence, the material implication “*p* → *q*” may equivalently be read: “If ‘*p*’ is true, then ‘*q*’ is true.” (15), therefore, may be read: “If ‘ $\sim p \cdot p$ ’ is true, then ‘*q*’ is true”. This, too, is not paradoxical. For if a false proposition is true, then every proposition would likewise be true, whether the proposition in question is true, false or self-contradictory. True propositions, of course, would be true on their own account, and false propositions would also be true on the hypothesis that a false proposition is true [see (5) and (15)]. However, since no false proposition can be true at the same time (“ $\sim p \cdot p$ ”, of course, is necessarily false and cannot be true under any condition), we may regard a material implication which is true only because its antecedent is false as degenerate. Consider the following statement:

(17) If the United States is in Asia, then there are Martians.

Since the antecedent of (17) is false, (17) is true on account of the paradox. But the following statement with the same antecedent but whose consequent is the contradictory of the consequent of (17) is also true for the same reason:

(18) If the United States is in Asia, then there are no Martians.

Since (17) and (18) are both true and are relevant to each other, they should be considered as a whole, rather than separately. In fact, I believe that considering (17) and (18) and similar mutually relevant propositions separately has been the cause of much of the confusion concerning the logical significance of the paradoxes of material implication. This practice, in fact, has some analogy to the informal fallacy called “special pleading”. (17) and (18) together are logically equivalent to the following proposition:

(19) If the United States is in Asia, then it is both true that there are Martians and there are no Martians.

This in turn is logically equivalent to:

(20) The United States is not in Asia.

(17) to (20) can be summarized symbolically as follows:

$$(21) \quad \frac{[p \rightarrow (p \cdot \sim p)] \leftrightarrow \sim p}{\begin{array}{cccc} T & F & F & T & F \\ T & F & F & T & F \\ F & T & F & T & T \\ F & T & F & T & T \end{array}}$$

which is a tautology. (21) tells us that a conditional whose antecedent implies contradictory consequents is logically equivalent to the categorical statement that the antecedent is false. Every conditional with a false antecedent has a logically relevant pair whose consequent is contradictory to its own. We suggest that in view of their mutual relevance, they should be taken together. And if we do so, their ultimate logical import is simply the statement that the antecedent is false as in (21), which is what we began with. (The same things *mutatis mutandis* may be said of a conditional true only because its consequent is true [cf. (14)].)

Consider the following example:

(22) Anything is soluble in water, iff in case it is placed in water at t , it dissolves therein at t .

(22) has a logically relevant pair, namely:

(23) Anything is insoluble in water, iff in case it is placed in water at t , it does not dissolve therein at t .

(22) and (23) may be symbolized, respectively, as follows:

$$(24) \quad S \leftrightarrow (W \rightarrow D)$$

$$(25) \quad \sim S \leftrightarrow (W \rightarrow \sim D).$$

Let us regard (24) and (25) as premises in a polyargument and derive what significant propositions we can out of them. It is clear that if we assume “ $\sim W$ ”, we can obtain both “ S ” and “ $\sim S$ ”, a contradiction, since “ $\sim W$ ” implies both “ $W \rightarrow D$ ” and “ $W \rightarrow \sim D$ ”, which are materially equivalent, respectively, to “ S ” and “ $\sim S$ ”. Therefore, (24) and (25) cannot both be true without contradiction if “ $\sim W$ ” is true. They are thus both true only if “ W ” is true. We can also obtain from (24) and (25), respectively:

$$(26) \quad S \rightarrow (W \rightarrow D)$$

and

$$(27) \quad \sim S \rightarrow (W \rightarrow \sim D).$$

From (26) and (27) we can easily derive:

$$(28) \quad W \rightarrow (S \leftrightarrow D)$$

which reads: “If anything is placed in water at t , then it is soluble in water iff it dissolves therein at t ”. (28) and similar formulas have been used to avoid the paradox to which (22), (23), and the like are separately subject. We have, however, shown that (22) and (23) together are not subject to the paradox, since they are true without contradiction only if “ W ” is true.

A conditional with a false antecedent, however (or a true consequent,

regardless of what its antecedent may be), may have use for logic; at least, for instance, as a step in the proof of some arguments. Consider the following example:

- (29) If in case Ben leaves, Jonas will take his place, then the job can be completed in due time. However, Ben does not leave. Therefore, the job can be completed in due time.

Using *B*, *J*, and *C*, respectively to symbolize the three simple propositions of (29), and constructing a formal proof of validity for it, we have the following:

- (30) 1. $(B \rightarrow J) \rightarrow C$
 2. $\sim B / \therefore C$
 3. $\sim B \vee J$ 2 addition
 4. $B \rightarrow J$ 3 material implication
 5. C 1,4 modus ponens

(30.2-4) in effect show that “ $\sim B$ ” does what “ $B \rightarrow J$ ” could have done to obtain the conclusion “*C*”. Of course, “*J*” could have done the same thing, too, if this were the second premise in (30.2), instead of “ $\sim B$ ”. We may say, therefore, that the truth of “ $B \rightarrow J$ ” in (29.4) is degenerate. However, (30.2-4) help us prove “*C*” in (30) on the basis of its premises and our rules of inference.

There is, however, a case when the truth of a material conditional with a false antecedent is not degenerate, and that is when its consequent is a well-known falsehood. In this case, we may treat such a conditional as an enthymeme with the conditional as the first premise and the denial of its consequent (which is a well-known falsehood) as the second premise, and by modus tollens, we obtain the denial of the antecedent in question, thus:

- (31) $p \rightarrow q$
 $\sim q / \therefore \sim p.$

For example:

- (32) If Walter Mondale is the President of the United States, then the United States is in Asia.

The consequent of (32) is a well-known falsehood. We can make (32) an expressed premise of an enthymeme. Then we may state the denial of this consequent as the second premise of the implied argument, thus:

- (33) The United States is not in Asia.

We may then obtain the denial of the antecedent of (33) as the implicit conclusion of the enthymeme, to wit:

- (34) Walter Mondale is not the President of the United States.

Notice that in (34) the material conditional (32) has disappeared, and in its place we have (34), which is a statement in declarative form. In fact, (32) is an ironic way of stating (34).

However, a material conditional with a false antecedent and a true consequent seems truly paradoxical at first sight. Consider the following example:

- (35) If Walter Mondale is President of the United States, then Nancy Reagan is the First Lady of the United States.

Here the antecedent is false and the consequent is true. Since the convention in the United States is that the wife of the President is the First Lady, and Nancy Reagan is not the wife of Walter Mondale, it would seem that on that basis (35) is false. But under the current interpretation of material implication (34) is true. Hence, (35) is both true and false on the prevailing interpretations: true on its own account because of the “paradox”, but false under the current American socio-political convention. However, (35) should be treated as the conclusion of an enthymeme with two implied premises, namely:

(36) It is false that Walter Mondale is President of the United States.

and

(37) Nancy Reagan is the First Lady of the United States.

For simplicity’s sake, let us take each premise in turn and analyze its logical import on (35). Symbolizing (35) and (36) with appropriate symbols, we have:

- (38) 1. $\sim W \therefore W \rightarrow N$
 2. $W \therefore N$ conditional proof
 3. $W \vee N$ 2 addition
 4. N 1,3 disjunctive syllogism.

It is clear in (38.1 and .2) that the enthymeme implied in (35) and which is fully expressed in the premises and conclusion of (37) involves a contradiction.

Let us now analyze (35) with (37) as its premise:

- (39) 1. $N \therefore W \rightarrow N$
 2. $W \therefore N$ conditional proof
 3. $N \cdot W$ 1,2 conjunction
 4. N 3 simplification.

It is clear also in (39) that it is not “ W ” that implies “ N ” but the unexpressed enthymematic premise “ N ”. So there is nothing paradoxical about (35), either on account of the fact that its antecedent is false, or on the ground that its consequent is true. We should remember that the premises of an argument are assumed to be true. If we have contradictory premises, therefore, we are in effect assuming that contradictory propositions are both true. If this is so, then any proposition, be it true, false or self-contradictory, would likewise be true. Hence, any conclusion would be permissible in an argument with contradictory premises.

That a true material conditional with a false antecedent also satisfies the logical form “ $(p \cdot q) \rightarrow q$ ” can be seen from the fact that “ $\sim p$ ” also implies the equivalence of “ $(p \rightarrow q)$ ” and “ $[(p \cdot q) \rightarrow q]$ ” as we have suggested in (8), to wit:

(40) $\sim p \rightarrow \{(p \rightarrow q) \leftrightarrow [(p \cdot q) \rightarrow q]\}$

F	T	T	T	T	T	T	T
F	T	T	F	F	F	F	T
T	T	F	T	T	F	T	T
T	T	F	T	F	T	F	T

(40) shows that a true material implication with a false antecedent also satisfies the logical form “ $(p \cdot q) \rightarrow q$ ”, for the equivalence between “ $p \rightarrow q$ ” and

“($p \cdot q$) \rightarrow q ” is true in both rows (the third and the fourth) where “ $\sim p$ ” is true. Hence, we may say that the consequent of a true material conditional is implicit in its antecedent even when the latter is false.

At this point, it is pertinent to ask what logical difference there is, if any, between what we have called degenerate material conditionals and nondegenerate ones. The degenerate ones are those which are true merely because their antecedents are false, or else merely because their consequents are true. We can say that the first class of degenerate material implications cannot, without contradiction, be the conditional premise in modus ponens, because an implicit premise of this kind of material implication is that the antecedent is false. Since in modus ponens one asserts in the categorical premise the truth of the antecedent, a contradiction results, as can be seen in the following demonstration:

- (41)
- | | |
|-----------------------|----------------------------|
| 1. $\sim p$ | |
| 2. $p \rightarrow q$ | premise in modus ponens |
| 3. p | premise in modus ponens |
| 4. q | conclusion in modus ponens |
| 5. $\sim p \cdot p$. | 1,4 conjunction |

Since in (41) a contradiction is involved in the use of a material conditional with a false antecedent in modus ponens, we may also derive from such “ $p \rightarrow q$ ” and “ p ” any other proposition “ k ” and thus substitute the same for “ q ” in (41.4). On the other hand, a material conditional true only because its consequent is true also cannot without contradiction be the conditional premise in modus tollens, to wit:

- (42)
- | | |
|-----------------------|-----------------------------|
| 1. q | |
| 2. $p \rightarrow q$ | premise in modus tollens |
| 3. $\sim q$ | premise in modus tollens |
| 4. $\sim p$ | conclusion in modus tollens |
| 5. $q \cdot \sim q$. | 1,3 conjunction |

Since (42) also involves a contradiction as seen in (42.5), we may also substitute any other proposition “ k ” for “ $\sim p$ ” in (42.4).

Of course, a material conditional whose antecedent is false and whose consequent is true cannot without contradiction be the conditional premise of either modus ponens or modus tollens.

We may say, therefore, that a nondegenerate material conditional must be one which, although asserted to be true, does not presuppose beforehand determinate truth values of its antecedent or its consequent. This in fact is the reason it is called a hypothetical. Its truth is perhaps determined, *among other ways*, empirically from the constant conjunction of states of affairs described by the antecedent and the consequent or else is derived from more general truths. The categorical premise either asserts in modus ponens that its antecedent is true, in which case its consequent is likewise true, or in modus tollens, denies truth to its consequent, in which case the truth of its antecedent is also denied. A material conditional is false if its antecedent is true and its consequent false.

That the consequent of a true material conditional regardless of the truth values of its component statements is implicit in its antecedent is also shown by the following biconditional, which is a tautology:

$$(43) \quad \frac{(p \rightarrow q) \leftrightarrow \{(p \rightarrow q) \leftrightarrow [(p \cdot q) \rightarrow q]\}^1}{}$$

T	T	T	T	T	T	T
F	T	F	F	F	T	F
T	T	T	T	F	T	T
T	T	T	T	F	T	F

(43) may be read “ $p \rightarrow q$ is true, iff ‘ $(p \rightarrow q) \leftrightarrow (p \cdot q) \rightarrow q$ ’ is likewise true”. Let us call the left side of (43) “(43a)” and the right side “(43b)”. Let us in turn call the left side of (43b) “(43c)” and the right side “(43d)”. As we can see, (43b) is true in all and only those rows of (43) where “ $p \rightarrow q$ ” [(43a) and (43c)] is true. (43d) asserts an entailment relation between the antecedent “ $p \cdot q$ ” and its consequent “ q ”, “ q ” being deducible from “ $p \cdot q$ ” by simplification. Since (43c) may replace (43d) in all but only those rows of (43) where (43a) is true, we may, therefore, say that (43c) [as well as (43a)] also expresses an entailment relation in all those rows where (43) is true, but not in those rows where it is false; that is, where “ p ” is true and “ q ” is false. The reason “ $p \rightarrow q$ ” by itself does not express an entailment relation between its antecedent and its consequent is that “ $p \rightarrow q$ ” can be false, and this is when “ p ” is true and “ q ” is false. But “ q ” is implicit in its antecedent “ p ”, when “ $p \rightarrow q$ ” is true. In fact, although (43b) *by itself* may be regarded as a material biconditional, (43c) is logically equivalent to (43d), given the hypothesis that (43a) is true; since in this case (43c) is true iff (43d) is also true, as we have suggested in (8). The biconditional “ $(p \rightarrow q) \leftrightarrow [(p \cdot q) \rightarrow q]$ ” cannot be false whenever “ $p \rightarrow q$ ” is true.

That the consequent “ q ” of a material conditional “ $p \rightarrow q$ ” is implicit in its antecedent “ p ” is further shown in the following biconditional which is also a tautology:

$$(44) \quad \frac{(p \rightarrow q) \leftrightarrow [p \leftrightarrow (p \cdot q)]}{}$$

T	T	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	T	F	T	F
F	T	F	T	F	T	F

Let us call the left side of (44) “(44a)” and the right side “(44b)”. It is clear from (44) that (44a) and (44b) are logically equivalent. That (44a) and (44b) may replace each other for all logical purposes can be shown by the fact that (44b) can be used in the place of (44a) where (44a) is used in any logical operation, in modus ponens, modus tollens, and other logical operations.

Since under the hypothesis that “ $p \rightarrow q$ ” is true, (44b) cannot be false, we can say that under this hypothesis the relation between (44c) and (44d), that is, between “ p ” and “ $p \cdot q$ ” in (44) is stronger than that of a purely material biconditional as now conceived. Since, however, (44a) is logically equivalent (without condition) to (44b), we may also replace “ p ” in (44a) with “ $p \cdot q$ ” in (44d) in all those rows of (44) where (44a) is true, since in those cases (44b) is also true.

We have, therefore, in a true “ $p \rightarrow q$ ” a case of what may be called “conditional or contingent entailment”. For the entailment holds on the condition and only on the condition that “ $p \rightarrow q$ ” is true, but not when “ $p \rightarrow q$ ” is false.

With the foregoing considerations, I suggest that we may also define “ $p \rightarrow q$ ”, using (44), as follows:

$$(45) \quad p \rightarrow q =_{df} p \leftrightarrow (p \cdot q).$$

(45), I believe, makes the logical relation between the antecedent and the consequent of a material implication clearer than before. However, (44) may also be given as the general form for entailment. Nevertheless, we may distinguish material implication from entailment by characterizing material implication as “conditional or contingent entailment” as we suggested above; that is, “ $p \rightarrow q$ ” is true, iff “ $p \leftrightarrow (p \cdot q)$ ” is true, but false if “ p ” is true and “ q ” is false. On the other hand, “ $p \Rightarrow q$ ” (“ p entails q ”) is true, iff “ q ” cannot be false whenever “ p ” is true. In other words, p entails q , iff “ $p \leftrightarrow (p \cdot q)$ ” is necessarily true. Hence:

$$(46) \quad p \Rightarrow q =_{df} N[p \leftrightarrow (p \cdot q)].$$

For example, this obtains, as everyone knows, in the relation between the premises of a valid argument and its conclusion, among others.

The foregoing discussions are also relevant for the clarification of the consequences of the so-called paradoxes of material implication in other logical contexts and in the philosophy of science. But I shall leave this for other papers (not necessarily mine) to discuss.

It should be said that the foregoing interpretation of material implication and its “paradoxes” is not intended to replace, but to complement, other valid interpretations of the same.

NOTE

1. (44) may be generalized as follows:

$$(44.1) \quad p \leftrightarrow (p \leftrightarrow t)$$

where “ p ” is any proposition and “ t ” is any tautology. Nevertheless, since we have shown in (26) that “ $p \rightarrow q$ ” when true is logically equivalent to “ $(p \cdot q) \rightarrow q$ ”, which expresses an entailment relation and which can replace a true “ $p \rightarrow q$ ”, “ $p \rightarrow q$ ” has the force of an entailment when true.

REFERENCE

- [1] Copi, I. M., *Symbolic Logic*, 5th Ed., Macmillan, New York, 1979.

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