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On Certain Normalizable Natural Deduction Formulations of Some Propositional Intermediate Logics

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1 Introduction As was mentioned in [1], the sequence-conclusion approach to natural deduction enables us easily to get natural deduction formulations of some intermediate and modal logics from the corresponding sequent calculi. In this paper, using the method described in [1], pp. 360–366, and the mapping f, defined in the same paper, pp. 371–375, from the class of proofs in a (cut-free) sequent calculus into the class of derivations of the corresponding sequence-conclusion natural deduction system, we will present several normalizable formulations of some intermediate logics.

Our starting point will be some of the known cut-free Gentzen-type formulations of certain intermediate logics.

2 Sequence-conclusion natural deduction First of all, let us say a few words about the sequence-conclusion approach to natural deduction. It is a simple generalization of the well-known Gentzen natural deduction system (see [18]), which was developed by Prawitz (see [11], [12]) and by many subsequent authors (see [10] and [21]), and generalized in different directions, by, e.g., Shoesmith and Smiley [17], Schroeder-Heister [15], Segerberg [16], and so on.

We suppose that the premises and the conclusion of any inference rule are finite sequences of formulas. So, for instance, the rules for the introduction and elimination of implication will be as follows:

$$(\mathrm{I}\to)\qquad \frac{\begin{array}{c}[A]\\ \Delta, B\end{array}}{\Delta, A \to B}$$

$$(\mathrm{E}\to)\qquad \frac{\Delta, A \qquad \Lambda, A \to B}{\Delta, \Lambda, B}.$$

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If Δ is an empty sequence of formulas, then the rule (I $\to$) is intuitionistically admissible, but an arbitrary Δ leads us to classical logic.

Also, we suppose that instead of the rule

$$(\mathrm{E}\vee) \qquad \frac{\Delta, A \vee B}{\Delta, A, B}$$

we have the rule

$$(\mathrm{E}_1\vee) \qquad \frac{\Delta, A \vee B \quad \begin{matrix}[A] \\ \Lambda\end{matrix} \quad \begin{matrix}[B] \\ \Lambda\end{matrix}}{\Delta, \Lambda}.$$

3 Gentzen-type formulations From the works of Maehara [9] and Umezawa [22], it is known that there is an alternative sequent calculus to Gentzen's *LJ*, extending *LJ* and corresponding to Heyting's logic too, in which the cut elimination theorem holds (see [4] or [19]). This system is obtained from *LJ* by allowing finite sequences of formulas in the succedent of any inference rule except the rules for introducing implication and negation (and universal quantifier, in the case of the predicate calculus) in the succedent. In other words, it is a subsystem of Gentzen's *LK* obtained by the restriction $\Delta = \varnothing$ in the rules

$$(\mathrm{R}\to) \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta,\ A \to B}$$

$$(\mathrm{R}\neg) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta,\ \neg A}.$$

Therefore, all the inference rules of *LK*, except the two mentioned above, are admissible in *LJ*. Different variants of these rules give sequent calculi for intermediate logics.

Now we will review a few sequent calculi corresponding to some propositional intermediate logics. By $\Delta(M)$, S_n $(1 \le n < \omega)$, and S_ω, we denote the extensions of the Heyting propositional calculus by the new axiom(s):

$\Delta(M) \qquad ((P \to A) \to P) \to P$

where A is any formula provable in an arbitrary propositional logic M and P a propositional variable not contained in A,

S_n $(1 \le n < \omega) \qquad (A \to B) \vee (B \to A)$ and A_n

where the sequence A_n is defined inductively by

$$A_1 = ((P_1 \to P_0) \to P_1) \to P_1$$
$$A_n = ((P_n \to A_{n-1}) \to P_n) \to P_n,$$

and

$S_\omega \qquad (A \to B) \vee (B \to A).$

The Gentzen-type formulations are obtainable as follows.

The cut-free sequent calculi $G\Delta(M)$ of the class of logics $\Delta(M)$ are described in [4]. The calculus $G\Delta(M)$ is obtained from the classical logic *LK* by the restrictions $\Gamma, A \vdash_M B$ and $\Gamma, A \vdash_M$ on the rules

$$(\mathrm{R}\!\rightarrow)\qquad \frac{\Gamma,\Pi,A \vdash \Delta,B}{\Gamma,\Pi \vdash \Delta,\ A \rightarrow B}$$

$$(\mathrm{R}\neg)\qquad \frac{\Gamma,\Pi,A \vdash \Delta}{\Gamma,\Pi \vdash \Delta,\neg A}$$

respectively.

Sonobe obtains the cut-free gentzenizations GS_n and GS_ω of the logics S_n and S_ω in [20] in the following way: let the formula A_i be of the form $B_i \rightarrow C_i$ or $\neg B_i$ $(1 \leq i \leq m)$, $\Lambda = A_1, \ldots, A_m$ and $\Lambda_i = A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_m$. If A_i is $\neg B_i$, then C_i is an empty expression. Then the inference rule characterizing the system GS_n $(2 \leq n < \omega)$ is

$$\frac{\Gamma, B_i \vdash C_i,\Lambda_i,\Delta \quad (1 \leq i \leq m)}{\Gamma \vdash \Lambda, \Delta}$$

provided that the sequents $\Gamma, B_i \vdash C_i, \Lambda_i$ $(1 \leq i \leq m)$ are provable in GS_{n-1}. Note that S_1 is classical logic with the sequent calculus LK as GS_1.

The system GS_ω, corresponding to S_ω, i.e., to Dummett's well-known LC (see [3]), which is the limit of the sequence S_n, is characterized by the rule

$$\frac{\Gamma, B_i \vdash C_i,\Lambda_i \quad (1 \leq i \leq m).}{\Gamma \vdash \Lambda}$$

Unfortunately, it is not so clear how one can get a normalizable natural deduction system from GS_ω, and we will use a modification of GS_ω. We will characterize our system, denoted by GS'_ω, by the rule

$$\frac{\Gamma, B_i \vdash C_i,\Lambda_i,\Delta \quad (1 \leq i \leq m)}{\Gamma \vdash \Lambda, \Delta}$$

provided that the sequents $\Gamma, B_i \vdash C_i, \Lambda_i$ $(1 \leq i \leq m)$ are provable in GS_ω. It is possible to show that for such an extension of GS_ω the cut elimination theorem holds too, and that GS'_ω is equivalent to GS_ω. This formulation has a shortcoming: provability in GS'_ω is defined by means of provability in GS_ω, and both GS'_ω and GS_ω correspond to the same logic.

Some interesting considerations on the problem of gentzenization of different intermediate systems can be found in [2], [5], [8], [13], and [14].

4 Natural deduction formulations In this section, we will transform the systems $G\Delta(M)$, GS_n, and GS'_ω into the corresponding natural deduction systems $N\Delta(M)$, NS_n, and NS_ω. There are two papers by López-Escobar ([6] and [7]) related to natural deductions in intermediate logics.

The above-mentioned systems will be obtained from the system NC for classical logic, introduced in [1], by the corresponding restrictions on the inference rules for introduction of implication and negation.

These rules, in the case of the systems $N\Delta(M)$, will be as follows

$$\frac{\begin{matrix}[A]\\ \Delta, B\end{matrix}}{\Delta, A \rightarrow B} \qquad \text{and} \qquad \frac{\begin{matrix}[A]\\ \Delta\end{matrix}}{\Delta, \neg A}$$

provided that there is a subset Γ of hypotheses of the given derivation such that $\Gamma, A \vdash_M B$ for the first rule, i.e., $\Gamma, A \vdash_M$ for the second one.

For the systems NS_n $(2 \leq n \leq \omega)$ these rules look like

$$\frac{\begin{array}{c}[B_i]\\ C_i, \Lambda_i, \Delta \;\; (1 \leq i \leq m)\end{array}}{\Lambda, \Delta}$$

where B_i, C_i, Λ_i, and Λ are as in the corresponding rules in sequent calculi, provided that there is a subset Γ of the hypotheses of our derivation such that the sequents $\Gamma, B_i \vdash C_i, \Lambda_i$ $(1 \leq i \leq m)$ are provable in GS_{n-1}. (Note that $GS_{\omega-1}$ is GS_ω.)

Now we will modify the definition of the mapping f given in [1]. For all our natural deduction systems, instead of the clause involving (E$\vee$) we will have:

$$\frac{\begin{array}{c}\vdots\\ \Gamma, A \overset{d'}{\vdash} \Delta\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma, B \overset{d''}{\vdash} \Delta\end{array}}{\Gamma, A \vee B \vdash \Delta}\, d \qquad f(d) = \frac{A \vee B \qquad \begin{array}{c}\Gamma[A]\\ f(d')\\ \Delta\end{array} \qquad \begin{array}{c}\Gamma[B]\\ f(d'')\\ \Delta\end{array}}{\Delta}.$$

Instead of the clause involving (I$\rightarrow$) and (I$\neg$) we will have the following:

In the case of $\Delta(M)$:

$$\frac{\begin{array}{c}\vdots\\ \Gamma, \Pi, A \overset{d'}{\vdash} \Delta, B\end{array}}{\Gamma, \Pi \vdash \Delta, A \rightarrow B}\, d \qquad f(d) = \frac{\begin{array}{c}\Gamma[A]\Pi\\ f(d')\\ \Delta, B\end{array}}{\Delta, A \rightarrow B}$$

where $\Gamma, A \not\vdash_M B$,

$$\frac{\begin{array}{c}\vdots\\ \Gamma, \Pi, A \overset{d'}{\vdash} \Delta\end{array}}{\Gamma, \Pi \vdash \Delta, \neg A}\, d \qquad f(d) = \frac{\begin{array}{c}\Gamma[A]\Pi\\ f(d')\\ \Delta\end{array}}{\Delta, \neg A}$$

where $\Gamma, A \not\vdash_M$,

In the case of S_n $(1 \leq n \leq \omega)$:

$$\frac{\begin{array}{c}\vdots\\ \Gamma, B_i \overset{d_i}{\vdash} C_i, \Lambda_i, \Delta \;\; (1 \leq i \leq m)\end{array}}{\Gamma \vdash \Lambda, \Delta}\, d \qquad f(d) = \frac{\begin{array}{c}\Gamma[B_i]\\ f(d_i)\\ C_i, \Lambda_i, \Delta \;\; (1 \leq i \leq m)\end{array}}{\Lambda, \Delta}$$

where $\Gamma, B_i \vdash C_i, \Lambda_i$ $(1 \leq i \leq m)$ are provable in GS_{n-1}.

Let L be any intermediate propositional calculus considered here, GL its Gentzen-type formulation, and NL the corresponding natural deduction system. By induction on the length of the proof in GL and in NL the following theorem is provable:

Theorem $\Gamma \vdash \Delta$ *is provable in GL iff* $\Gamma \vdash_{\overline{NL}} \Delta$.

By induction on the length of the proof d for $\Gamma \vdash \Delta$ in GL and the definition of f, we are able to prove the following statements:

Lemma *If d is a proof of the sequent $\Gamma \vdash \Delta$ in GL, then $\Gamma \vdash_{NL} \Delta$ by $f(d)$.*

Theorem *If the proof d of the sequent $\Gamma \vdash \Delta$ in GL is without applications of the cut rule, then $f(d)$ is a normal derivation in NL.*

Accordingly, as an immediate consequence of the cut elimination theorem for *GL* (see [4] and [20]) we have:

Normal Form Theorem *If $\Gamma \vdash_{NL} \Delta$ by a derivation d in NL, then there exists a normal derivation d' in NL by which $\Gamma \vdash_{NL} \Delta$.*

As a corollary of this theorem, the separability of the system *NL* can be obtained.

By double induction on the length of the considered A-maximal segment and the degree of the formula A, we can describe a normalization procedure and get the following:

Normalization Theorem *There is an effective procedure reducing every derivation in NL into a normal derivation.*

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