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Equivalent Versions of a Weak Form of the Axiom of Choice

GARY P. SHANNON

We make the following definition:

Definition Let $(Q,\leq)$ be a quasi-order (i.e., $\leq$ is reflexive and transitive). Two elements x,y of Q are said to be incompatible if there does not exist $z \in Q$ such that $z \leq x$ and $z \leq y$. A subset I of Q is said to be an incompatible set if any two elements of I are incompatible. For each $x \in Q$, let $l(x)$ denote the set of lower bounds of x; and let $c(x)$ denote the set of elements of Q that are compatible with x. "Countable" is used here to mean "countably infinite".

Let $U^f_{\aleph_0}$ ($U^{\aleph_0}_{\aleph_0}$) denote the statement that the union of a countable collection of pairwise disjoint nonempty finite (countable) sets is countable.

Let $AC^f_{\aleph_0}(AC^{\aleph_0}_{\aleph_0})$ denote the statement that there exists a choice function for any countable family of finite (countable) nonempty sets.

Let $ACS^f_{\aleph_0}(ACS^{\aleph_0}_{\aleph_0})$ denote the statement that for any countable family of finite (countable) nonempty sets there exists a countable subfamily for which a choice function exists.

It is known that in ZF, $U^f_{\aleph_0}$ is equivalent to $AC^f_{\aleph_0}$. (Let $\mathcal{C} = \{C_n\colon n \in \mathbb{N}\}$ be a countable collection of pairwise disjoint nonempty finite sets. Let $|C_n| = m_n$; and, by $AC^f_{\aleph_0}$, choose for each $n \in \mathbb{N}$ a function $f_n\colon C_n \to m_n \times \{n\}$ such that f_n is 1-1 and onto. Then $\bigcup_{n\in\mathbb{N}} f_n$ is 1-1; $\bigcup_{n\in\mathbb{N}} f_n\left[\bigcup_{n\in\mathbb{N}} C_n\right]$ is an infinite subset of $\mathbb{N} \times \mathbb{N}$ (which is countable)—and therefore $\bigcup_{n\in\mathbb{N}} C_n$ is countable. Conversely, if $\mathcal{C} = \{C_n\colon n \in \mathbb{N}\}$ is a countable collection of finite nonempty sets then $\mathfrak{D} = \{C_n \times \{n\}\colon n \in \mathbb{N}\}$ is a countable collection of pairwise disjoint nonempty finite sets and hence, by $U^f_{\aleph_0}$, $\bigcup_{n\in\mathbb{N}} (C_n \times \{n\})$ is countable. Therefore there exists a $g\colon \bigcup_{n\in\mathbb{N}} (C_n \times \{n\}) \to \mathbb{N}$, such that g is 1-1 and onto. A choice

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function for $\mathcal{C}$ is then given by $f(C_n)$ = the element x of C_n such that $g(x,n)$ is the least element of $g[C_n \times \{n\}]$.)

It is also known that in *ZF* $U_{\aleph_0}^{\aleph_0}$ implies $AC_{\aleph_0}^{\aleph_0}$, but it is not known if $AC_{\aleph_0}^{\aleph_0}$ implies $U_{\aleph_0}^{\aleph_0}$ ([7], pp. 203, 324).

$AC_{\aleph_0}^{f}$ is also equivalent to König's Infinity Lemma ([5], p. 298; [1], pp. 202, 203), Ramsey's Theorem [6], and a limited version of Tychonoff's Theorem [4]. It is clear that $AC_{\aleph_0}^{f}$ implies $ACS_{\aleph_0}^{f}$, and it will be shown that $ACS_{\aleph_0}^{f}$ implies $AC_{\aleph_0}^{f}$.

Let $Q_{\aleph_0}(P_{\aleph_0})$ denote the statement that if $(Q,\leq)$ is a countable quasi-order (partial-order) that contains incompatible sets of arbitrarily large finite cardinality then Q contains a countable incompatible set. ($P_{\aleph_0}$ follows from an old result of Erdös and Tarski [2].)

Let $QU_{\aleph_0}^{f}(QU_{\aleph_0}^{\aleph_0})$ denote the statement that if $(Q,\leq)$ is a quasi-order that contains incompatible sets of arbitrarily large finite cardinality, and if Q can be written as a countable union of finite (countable) sets, then Q contains a countable incompatible set. Let $PU_{\aleph_0}^{f}, PU_{\aleph_0}^{\aleph_0}$ denote the analogous statements for partial orders.

Let $QM_{\aleph_0}^{f}(QM_{\aleph_0}^{\aleph_0})$ denote the statement that if $(Q,\leq)$ is a quasi-order that contains incompatible sets of arbitrarily large finite cardinality, and if Q can be written as a countable union of finite (countable) sets, then Q contains a maximal countable incompatible set. Let $PM_{\aleph_0}^{f}, PM_{\aleph_0}^{\aleph_0}$ denote the analogous statements for partial orders.

It will be shown that in *ZF*, $QU_{\aleph_0}^{f}$, $PU_{\aleph_0}^{f}$, $QM_{\aleph_0}^{f}$, and $PM_{\aleph_0}^{f}$ are equivalent to each other and to $AC_{\aleph_0}^{f}$.

Theorem 1 *$AC_{\aleph_0}^{f}$ is a theorem of $ZF \cup \{ACS_{\aleph_0}^{f}\}$.*

Proof: It will be shown that $U_{\aleph_0}^{f}$ is a theorem of $ZF \cup \{ACS_{\aleph_0}^{f}\}$.

Let $\mathcal{C}$ be a countable collection of pairwise disjoint nonempty finite sets; then $\mathcal{C}$ can be written as $\mathcal{C} = \{C_n: n \in \mathbb{N}\}$. Each C_n is finite, and the union of a finite collection of finite sets is finite, so for each $n \in \mathbb{N}$ there exists a natural number k_n such that $\left|\bigcup_{i=0}^{n} C_i\right| = k_n$. Therefore for each $n \in \mathbb{N}$ there exists a 1-1 function from $\bigcup_{i=0}^{n} C_i$ onto $k_n \times \{n\}$. For each $n \in \mathbb{N}$ there exist finitely many such functions; and therefore, by $ACS_{\aleph_0}^{f}$, there exists a collection $\{f_{t_n}: n \in \mathbb{N}\}$ such that f_{t_n} is a 1-1 map of $\bigcup_{i=0}^{t_n} C_i$ onto $k_{t_n} \times \{t_n\}$. Let $g_0 = f_{t_0}$. For each $n, n \geq 1$, let g_n denote the restriction of f_{t_n} to $\bigcup_{i=t_{n-1}+1}^{t_n} C_i$. Then g_n is a 1-1 map of $\bigcup_{i=t_{n-1}+1}^{t_n} C_i$ into $k_{t_n} \times \{t_n\}$, and therefore $g = \bigcup_{n\in\mathbb{N}} g_n$ is a 1-1 map of $\bigcup_{n\in N} C_n$ into $\mathbb{N} \times \mathbb{N}$. Since the g_n's have pairwise disjoint ranges, $g\left(\bigcup_{n\in\mathbb{N}} C_n\right)$ is a countable $\left(([3], \text{pp. } 79, 81)\text{—since } g\left(\bigcup_{n\in\mathbb{N}} C_n\right) \text{ is infinite}\right)$ subset of $\mathbb{N} \times \mathbb{N}$.

Therefore (since g is 1-1) $\bigcup_{n\in\mathbb{N}} C_n$ is countable, and $U_{\aleph_0}^{f}$ is a theorem of

$ZF \cup \{ACS^f_{\aleph_0}\}$. Thus $AC^f_{\aleph_0}$ is a theorem of $ZF \cup \{ACS^f_{\aleph_0}\}$. (Theorem 1 can also be proved by showing that König's Infinity Lemma is a theorem of $ZF \cup \{ACS^f_{\aleph_0}\}$, but the proof of this is more involved than the proof given above.)

Theorem 2 *$Q_{\aleph_0}$ is a theorem of $ZF \cup \{AC^f_{\aleph_0}\}$.*

Proof: Assume that $(Q, \leq)$ is a countable quasi-order that contains incompatible sets of arbitrarily large finite cardinality. Let $Q = \{q_n: n \in \mathbb{N}\}$.

Let $\mathcal{C}$ be the collection of incompatible subsets, I, of Q such that $|I| \geq 2$ and such that there exists $y \in I$ such that $l(y)$ contains incompatible sets of arbitrarily large finite cardinality.

If $\mathcal{C} = \emptyset$ then let $\mathfrak{D}$ be the collection of all incompatible subsets of Q. If $\mathfrak{D}$ is not countable then Q contains a countable incompatible set. Assume that $\mathfrak{D}$ is countable; then $\mathfrak{D}$ can be written as $\mathfrak{D} = \{D_n: n \in \mathbb{N}\}$. Let g_1 be the least natural number such that D_1 is a proper subset of D_{g_1} $\Big($$g_1$ exists since D_1 is finite and for each $y \in D_1$ there exists a maximal finite incompatible set $I(y) \subseteq l(y)$ (since $\mathcal{C} = \emptyset$). Therefore the largest incompatible subset of $\bigcup_{y \in D_1} c(y)$ has cardinality $\sum_{y \in D_1} |I(y)|$, and hence $Q \neq \bigcup_{y \in D_1} c(y)$. Let m be the least natural number such that $q_m \in Q - \bigcup_{y \in D_1} c(y)$. Then $D_1 \cup \{q_m\} \in \mathfrak{D}$$\Big)$. For each $n > 1$ define, by induction, g_{n+1} to be the least natural number such that D_{g_n} is a proper subset of $D_{g_{n+1}}$. Then $\bigcup_{n \in \mathbb{N}} D_{g_n} \subseteq Q$ is a countable incompatible set, therefore Q contains a countable incompatible set.

Assume $\mathcal{C} \neq \emptyset$. If $\mathcal{C}$ is not finite and not countable then Q contains a countable incompatible set. Assume that $\mathcal{C}$ is countable (the proof for the case that $\mathcal{C}$ is finite is similar); then $\mathcal{C}$ can be written as $\mathcal{C} = \{I_n: n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, choose (by $AC^f_{\aleph_0}$) $p_n \in I_n$ such that $l(p_n)$ contains incompatible sets of arbitrarily large finite cardinality; and for each $n \in \mathbb{N}$, let $\mathcal{C}_n = \{I \in \mathcal{C}: I \subseteq l(p_n)\}$. If $\mathcal{C}_n = \emptyset$ for any n, then (by an argument similar to that given above for $\mathcal{C} = \emptyset$) Q contains a countable incompatible set. Therefore assume that $\mathcal{C}_n \neq \emptyset$ for all $n \in \mathbb{N}$. Choose $r_1 \in I_1 - \{p_1\}$ and let $\lambda(1)$ be the least natural number such that $I_{\lambda(1)} \in \mathcal{C}_1$. Choose $r_2 \in I_{\lambda(1)} - \{p_{\lambda(1)}\}$ and let $\lambda(2)$ be the least natural number such that $I_{\lambda(2)} \in \mathcal{C}_{\lambda(1)}$. Assume that r_n is defined $(n \geq 2)$, $r_n \in I_{\lambda(n-1)} - \{p_{\lambda(n-1)}\}$. Let $\lambda(n)$ be the least natural number such that $I_{\lambda(n)} \in \mathcal{C}_{\lambda(n-1)}$. Then choose (by $AC^f_{\aleph_0}$) $r_{n+1} \in I_{\lambda(n)} - \{p_{\lambda(n)}\}$. By construction, $\ldots \leq p_{\lambda(3)} \leq p_{\lambda(2)} \leq p_{\lambda(1)} \leq p_1$, and $r_2 \leq p_1, r_3 \leq p_{\lambda(1)}, r_4 \leq p_{\lambda(2)}, \ldots$. Thus it follows that for each $m < n \in \mathbb{N}$, $n > 0$, r_m and r_n are incompatible. Then $\{r_n: n \in \mathbb{N}\}$ is a countable incompatible subset of Q.

Therefore $Q_{\aleph_0}$ is a theorem of $ZF \cup \{AC^f_{\aleph_0}\}$.

It follows from Theorem 2 that $QU^f_{\aleph_0}$ is a theorem of $ZF \cup \{AC^f_{\aleph_0}\}$ (since $AC^f_{\aleph_0}$ implies $U^f_{\aleph_0}$).

Theorem 3 *$AC^f_{\aleph_0}$ is a theorem of $ZF \cup \{QU^f_{\aleph_0}\}$.*

Proof: To prove $AC^f_{\aleph_0}$ it suffices (by Theorem 1) to prove $ACS^f_{\aleph_0}$.

Let $\mathcal{C}$ be a countable family of finite sets. It can be assumed (without loss of generality) that the sets of $\mathcal{C}$ are pairwise disjoint. Define $\leq$ on $Q = \bigcup_{A \in \mathcal{C}} A$

by $x \leq y$ iff there exists $A \in \mathcal{C}$ such that $x \in A$ and $y \in A$. Then $\leq$ is a quasi-order on Q; and two elements u, v are incompatible iff there exist $A, B \in \mathcal{C}$, $A \neq B$, with $u \in A$ and $v \in B$.

Let $n \in \mathbb{N}$, and let $A_1, \ldots, A_n$ be elements of $\mathcal{C}$. Choose $x_1 \in A_1$, $x_2 \in A_2, \ldots, x_n \in A_n$ (this can be done in *ZF*). Then $\{x_1, \ldots, x_n\}$ is an incompatible subset of Q. Therefore $(Q,\leq)$ contains incompatible sets of arbitrarily large finite cardinality, and hence (by $QU^f_{\aleph_0}$) $(Q,\leq)$ contains a countable incompatible set, I.

Let $\mathfrak{D} = \{A \in \mathcal{C}: A \cap I \neq \varnothing\}$, and define f on $\mathfrak{D}$ by $f(A) = A \cap I$. Then $\mathfrak{D}$ is a countable subfamily of $\mathcal{C}$ for which a choice function exists.

Therefore $ACS^f_{\aleph_0}$ is a theorem of $ZF \cup \{QU^f_{\aleph_0}\}$ – and hence $AC^f_{\aleph_0}$ is a theorem of $ZF \cup \{QU^f_{\aleph_0}\}$.

Therefore $QU^f_{\aleph_0}$ is equivalent to $AC^f_{\aleph_0}$.

It is clear that $Q_{\aleph_0}$ implies $P_{\aleph_0}$. The converse is given in the following:

Claim 1 *$Q_{\aleph_0}$ is a theorem of $ZF \cup \{P_{\aleph_0}\}$.*

Proof: Let $(Q,\leq)$ be a countable quasi-order that has incompatible subsets of arbitrarily large finite cardinality. Then Q can be written as $Q = \{q_n: n \in \mathbb{N}\}$.

For each $n \in \mathbb{N}$, let $A_n = \{q_i \in Q: q_i \leq q_n \text{ and } q_n \leq q_i\}$, and let $P = \{A_n: n \in \mathbb{N}\}$. Define $\lesssim$ on P by $A_i \lesssim A_j$ iff $q_i \leq q_j$. Then $\lesssim$ is a partial-order on P.

Since $(Q,\leq)$ has incompatible subsets of arbitrarily large finite cardinality so does $(P,\lesssim)$, and hence (by $P_{\aleph_0}$), P contains a countable incompatible set, I.

Let J be the set of natural numbers, n, such that $I = \{A_n: n \in J\}$ and such that for any $m \in \mathbb{N}$, if $m < n$ then $A_m \neq A_n$.

Then $\{q_i: i \in J\}$ is a countable incompatible subset of Q; and thus $Q_{\aleph_0}$ is a theorem of $ZF \cup \{P_{\aleph_0}\}$.

Note that the statement: "If $(R,\leq)$ is a quasi-order that contains incompatible sets of arbitrarily large finite cardinality, then R contains a countable incompatible set" is not a theorem of $ZF \cup \{AC^f_{\aleph_0}\}$ – since from this statement it follows that any infinite set contains a countable subset (simply define $\leq$ on an infinite set R by $x \leq y$ iff $x = y$) – but this result requires a stronger form of the Axiom of Choice than $AC^f_{\aleph_0}$ ([7], pp. 322, 323).

It follows from Claim 1 that $PU^f_{\aleph_0}$ is equivalent to $QU^f_{\aleph_0}$.

Theorem 4 *In ZF, $QM^f_{\aleph_0}$ is equivalent to $AC^f_{\aleph_0}$.*

Proof: $QM^f_{\aleph_0}$ clearly implies $QU^f_{\aleph_0}$, thus $AC^f_{\aleph_0}$ is a theorem of $ZF \cup \{QM^f_{\aleph_0}\}$.

Assume that $(Q,\leq)$ is a quasi-order that contains incompatible sets of arbitrarily large finite cardinality, and that Q can be written as a countable union of finite sets. Then (assuming $AC^f_{\aleph_0}$), Q is countable and hence Q can be written as $Q = \{q_n: n \in \mathbb{N}\}$. By Theorem 2, Q contains a countable incompatible set, I.

If $Q = \bigcup_{x \in I} c(x)$ then I is a maximal countable incompatible set.

Suppose that $Q \neq \bigcup_{x \in I} c(x)$. Let $d(1)$ be the least natural number such that $q_{d(1)} \in Q - \bigcup_{x \in I} c(x)$, and let $I_1 = I \cup \{q_{d(1)}\}$. For $n > 1$ define, by induction, $d(n)$ and I_n as follows: Let $d(n)$ be the least natural number such that

$q_{d(n)} \in Q - \bigcup_{x \in I_{n-1}} c(x)$ $\left(\text{if } Q \neq \bigcup_{x \in I_{n-1}} c(x)\right)$ and let $I_n = I_{n-1} \cup \{q_{d(n)}\}$. Either $Q = \bigcup_{x \in I_k} c(x)$ for some k and then I_k is a maximal countable incompatible set; or $Q \neq \bigcup_{x \in I_k} c(x)$ for all k and then $\bigcup_{k \in \mathbb{N}} I_k$ is a maximal countable incompatible set.

Therefore $QM^f_{\aleph_0}$ is a theorem of $ZF \cup \{AC^f_{\aleph_0}\}$.

By an argument similar to that of Claim 1 it follows that $QM^f_{\aleph_0}$ is equivalent to $PM^f_{\aleph_0}$.

A summary of the implications is that in ZF:

$$\begin{tikzcd}
PU^f_{\aleph_0} \arrow[r, Leftrightarrow] & QU^f_{\aleph_0} \arrow[d, Rightarrow] & AC^f_{\aleph_0} \arrow[l, Rightarrow] \arrow[r, Leftrightarrow] \arrow[dl, Rightarrow] \arrow[d, Leftrightarrow] \arrow[dr, Rightarrow] & QM^f_{\aleph_0} \arrow[r, Leftrightarrow] & PM^f_{\aleph_0} \\
 & ACS^f_{\aleph_0} \arrow[r, Rightarrow] & U^f_{\aleph_0} & Q_{\aleph_0} \arrow[r, Leftrightarrow] & P_{\aleph_0}.
\end{tikzcd}$$

Note that all of the arrows except possibly $AC^f_{\aleph_0} \Rightarrow Q_{\aleph_0}$ are reversible. I do not know if in ZF $Q_{\aleph_0}$ is equivalent to $AC^f_{\aleph_0}$.

Some of the ideas for families of finite sets extend to families of countable sets; but some do not, and some of the implications are not known. A summary of the implications is that in ZF:

$$\begin{tikzcd}
PU^{\aleph_0}_{\aleph_0} & QU^{\aleph_0}_{\aleph_0} \arrow[l, Rightarrow] \arrow[d, Rightarrow] & AC^{\aleph_0}_{\aleph_0} \arrow[dl, Rightarrow] & QM^{\aleph_0}_{\aleph_0} \arrow[l, Rightarrow] \arrow[r, Rightarrow] & PM^{\aleph_0}_{\aleph_0} \\
 & ACS^{\aleph_0}_{\aleph_0} & U^{\aleph_0}_{\aleph_0}. \arrow[u, Rightarrow] \arrow[ur, Rightarrow] & &
\end{tikzcd}$$

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Department of Mathematics and Statistics
California State University, Sacramento
Sacramento, California 95819