Notre Dame Journal of Formal Logic
Volume 24, Number 1, January 1983

Frege's Definition of Number

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I will interpret Frege on the main issues about his definition of numbers as extensions at [8], §68: its point, its correctness, and its implications for the nature of number. My view, in short, is that Frege understood this definition (henceforth *D*) as a partly free construction of the numbers. His escape from the subjectivism this seems to entail was to relegate differences between equally correct constructions to a Kantian realm of appearance.

As background for the problems I want to discuss, let me note a few of the relatively clear facts about *D* (cf. [1]). Frege's interest in defining number is subordinate to a dominant goal of proving the arithmetical laws. Many of these, of course, had already been established in number theory, but in Frege's view the axioms that number theorists simply assume should be proved as well (§§1-4).[1] In fact, two relatively independent points of view motivate this demand. For the mathematician it is simply a matter of proving whatever can be proved. Even if the axioms are entirely certain, their proof will advance mathematics by revealing mathematically interesting connections between propositions. But these connections also bear on philosophical questions about the analyticity or apriority of arithmetic. On Frege's understanding of analyticity, for example, analyticity will be shown only by a derivation of the axioms from purely logical laws. A search for such derivations is therefore the

*The first drafts of this paper and [1] were roughly simultaneous (1976). Although I have departed substantially from Benacerraf's perspective, interacting with him then was an enormous help. My more recent aid and comfort was the appearance of [18] when I was developing the main ideas of Section 1. Sluga's outstanding book gave my view of Frege's relation to Kant a broad historical foundation, led me to think about Frege's theory of judgment, and made lengthy stage-setting unnecessary in my own exposition. Although [18] is by no means beyond reproach, I think it will come to be seen as the watershed of Frege scholarship. I also thank a referee of this *Journal*.

Received August 24, 1981; revised January 8, 1982

common business of philosophy and mathematics. Now apparently axioms are the rock bottom of arithmetic. But Frege observes that if definitions could transform axioms of arithmetic into theorems of something else, their proof would be possible after all:

> . . . we very soon come to propositions which cannot be proved so long as we do not succeed in analyzing concepts which occur in them into simpler ones or in reducing them to something more general. Now here it is above all number which must be defined or recognized as indefinable. This shall be the task of this book (§4).

It therefore seems fair to describe Frege as aiming to prove the laws of number within a definitional extension of an appropriate formalized theory. In this light [8] must be considered a preliminary investigation. Frege is occupied with motivating the attempt at proof, clarifying the concept of number, and arguing that the underlying theory should be a system of *logic,* not psychology or (more plausibly) geometry. All this restricts him to sketches of the formal proofs to follow in [9]. Nonetheless, his project throughout is to produce a certain piece of mathematics.

What this description of [8] leaves open are above all the epistemology and semantics of definition. Crucial to Frege is that when we "define number" within a certain theory T, it is indeed *number* that we are defining. Let us suppose that T is a standard set theory. Then it is easy enough to introduce a predicate 'N' to apply, say, to ϕ, $\{\phi\}$, $\{\{\phi\}\}$, . . ., but this would not seem to facilitate the proof of a single arithmetical law unless the sets in this sequence are also numbers. If they are not, the propositions about them cannot belong to arithmetic, so the proofs thereof establish arithmetical laws in name only. Now mathematics as such does not tell us whether sets are numbers, or whether propositions of arithmetic occur in set theory. These questions are philosophical. Hence, philosophy must judge the success of Frege's mathematical project. It is his philosophical account of the relation between sets (extensions) and numbers that I will try to explain. I will start with the central question why D is correct.

1 Some of Frege's most distinguished critics have suggested that if D is not incorrect in Frege's own eyes, it is at least in tension or even inconsistent with his principles (see, e.g., [5], [6], [17], [18]). Although I could hardly disagree more, I want to approach the correctness problem by considering this view. It deals directly with several puzzles of concern to us, and even anticipates my theory somewhat.

The question of Frege's attitude towards D arises because of his seeming preference for another form of definition. Prominent in Frege's accounts of extensions (Law V of [9]), various geometrical notions ([8], §64), and all sorts of number (§§55, 60, 63, 104) are equations between propositions apparently different in structure and subject matter. These equations are formally like contextual definitions: $f(a) = f(b) \equiv aRb$, where R is an equivalence relation and f the function we are trying to introduce. *Considered* as definitions they fail, but, as Frege evidently appreciates, they are very natural attempts to specify our mathematical concepts. In comparison, all D seems to offer is a technical success for which Frege himself provides conspicuously little motivation or defense. It thus becomes easy to suspect that Frege actually favors the contextual introduction of concepts or terms, in spite of the formal obstacles he encounters. Reflection on Frege's context principle ("only in the context of a proposition do words mean anything") then strengthens this impression. First, the fundamental importance of the context principle seems to lie in the justification of Frege's contextual transformations. This principle seems to express a holistic view of meaning which allows the equivalence of propositions constructed from quite different elements, as $f(a) = f(b)$ and aRb are. But if that is its role, Frege must (allegedly) prefer these equivalences as definitions, or their justifying principle would not itself be critical. Second, the context principle might even explain such a preference. The primacy of propositions may mean that definitions *must* pair propositional wholes, not words or phrases, so that explicit definitions such as D are ruled out. Of course such a strong reading of the context principle is at least not obvious. But in any case, contextual definition is seen as Frege's first choice, and the context principle at least justifies this style of definition. From this viewpoint Frege's use of D must then be reluctant or anyway not fully warranted within the framework of [8].

Perhaps none of Frege's critics would endorse the foregoing reasoning just as it stands. But no account even roughly along these lines is credible, because Frege could hardly have advanced the central definition of [8] in awareness of an obvious conflict with his principles. Even a merely apparent conflict is out of the question unless Frege is supposed to have acquiesced in its appearance at the arguably most important point in the book. We might add that if Frege's basic aim is to prove the laws of number, a definition which does make proof possible should be entirely acceptable, whatever its form. Let us therefore try to see how the standard critical view of D has gone wrong.

One source of difficulty has been a confusion between the context principle proper and Frege's more general theory of the priority of propositions over terms. (For both the confusion and the theory see [18], esp. pp. 90-95.) Frege's remarks on the context principle suggest just a single use: allowing Frege to define number by specifying the senses of numerical propositions (p. X; §§60, 62, 106). If this seems insufficiently important, we should remember that Frege's major opposition (see [18]) would expect associations with ideas or intuitions to give the meanings of number words. Frege's reply is, roughly, that the meaning of a word is its contribution to the meanings of the propositions in which it occurs, so that making such a contribution is sufficient (and necessary) for meaningfulness even in the absence of anything like an idea. Thus the context principle is certainly a foundation of Frege's approach to number. All this is reasonably familiar ground, and I will omit further details.[2] The point is that if this natural view of the context principle is even approximately right, that principle is much weaker than is often thought. Read literally, it will not remotely justify equivalence claims for relational propositions (aRb) and the corresponding identities ($f(a) = f(b)$). (No wonder Frege does *not* invoke his principle to support those claims.) Nor will it conflict with the use of D as a definition, because D plainly does show the contribution of a number word to any containing context. (Thus D also fixes the senses of identities and other numerical propositions as §62 demands, *pace* [18].)

But if the context principle does not justify the equivalences of §63 et al., then the consistency of the context principle with the use of explicit definitions does not yet show that Frege can accept an explicit definition of number. Perhaps the somehow holistic views on meaning that do underlie equivalence claims for very differently constructed propositions also bar explicit definitions. We cannot go into these views here—regrettably, since they are very much part of the background for the account of Frege I myself will offer. But there is no reason to suspect that explicit definition is problematic for Frege, because such definition can be so interpreted as not to favor words over larger semantic units. An explicit definition is just an identity (or a biconditional); identities are propositions as much as anything else is; and Frege can regard any definition as one proposition which fixes the logical properties of a range of others. Viewed in this way, explicit definition should harmonize with any degree of semantic holism Frege might espouse.

There remains the possibility of a Fregean objection specifically to the explicit definition of *number.* The two prominent suggestions in this area are that Frege prefers the contextual definition of terms for *abstract* objects as a way of easing ontological worries [5], and that the inaccessibility of *logical* objects to intuition recommends a contextual procedure in their case [18]. But I believe that neither of these readings can be justified. Although I will not try to refute them directly, it should be clear by the end of this section that the abstractness or logical character of numbers in no way determines or limits the form their definition may take.

The sort of view I have been criticizing could hardly have persisted, given the objections that can be raised, if it were not a way of getting at something important. Since the basic error is to suppose that something is wrong with D, we might try to subtract it and see what is left. I think it would be something like this: that the (would-be) definitions of §§55 and 63 have an important sort of primacy, that they are unequal to D in status, and that these points are tied to the difference between contextual and explicit definition. I would agree on all counts. What I now want to develop is a viewpoint capable of accommodating and clarifying these insights.

While this will be a major task, the first step is easy and already quite explanatory. We must recognize that D is *arbitrary* within the limits set by its contextual predecessors.[3] The latter express our existing arithmetical concepts without themselves amounting to a definition. Frege's task is therefore to define number by a choice which is partway arbitrary, but which must conform to the constraints uncovered by the analysis of our concepts. Just this conformity will make the definition correct. Observe that even the preceding few remarks (which are no more than a hint) show the importance of the contextual definitions. Also, some of the puzzles about D are solved at once. *D is* discontinuous with the preceding text, but not due to any failure on Frege's part. Rather, it represents a shift from analysis to stipulation. The paucity of motivation provided is inevitable given the arbitrariness of Frege's choice. *Now* his defense of D looks adequate. Less needs to be shown; instead of arguing that D is *the* truth about number, Frege must only demonstrate agreement with the conditions on number previously laid down. Just this task occupies him throughout §§69-83. Finding D arbitrary is thus a major step towards making overall sense of [8].

As indicated, the arbitrariness of D is not the whole story. But let us stay just with this part of the story for a moment. It immediately leads to a familiar, fundamental difficulty (cf. [2], [15]). If D is even somewhat arbitrary, there must be a number n and two objects a and b such that $n = a$ and $n = b$ are both acceptable definitions. (Adjust for use and mention.) Yet $a \neq b$, so at least one of these definitions must be false. So either Frege must hold to the uniqueness of D after all, or he must allow false identities in mathematics. Since the latter is impossible, we seem to have a decisive argument for uniqueness. The only escape would apparently be to admit a, b, and n as a mysterious triad (?) of objects, two distinct from each other and identical to a third.

If it were not for this argument, one would certainly hope for universal agreement on the arbitrariness of D. The relevant evidence is impressive and lies mostly on the surface.[4] What is lacking is not evidence but a way to avoid the incoherence to which the claim of arbitrariness seems to commit Frege.

Frege does have a way out. Unquestionably, his theory has serious problems, but it is a subtle resource in a vexing situation. To understand it we must first see how the definition of number is supposed to solve a problem Frege takes over from Kant. This background will lead us to a deeper appreciation of the importance of D; we will then be able to retrace Frege's steps towards his definition in a way that ties his views on contextual definition, arbitrary choice, and the nature of number into a single theory.

Frege's life work is in large measure an answer to a single question: "How do we grasp logical objects, in particular the numbers? Through what are we justified in regarding numbers as objects?" ([9], II, p. 265; cf. §147, op. cit.; cf. [8], §62). The problem here, as Sluga has emphasized, is thoroughly Kantian.[5] Kant, after introducing a general, formal notion of an object or individual, describes thought as the application of concepts to objects the mind has grasped. Corresponding to the general notion of an object is a general notion of intuition as the direct apprehension which provides the material for thought. But in practice this generality is much restricted by the limits of the human mind. Our intuitions are exclusively *sensible*, even though other forms of apprehension may be possible for other beings (notably God). This view of our faculties, of course, is the basis for Kant's striking views on the limits to what we can think or say. Now Frege's relation to this position is somewhat obscured by a question about intuition. While he adopts Kant's notion of an object, he seems to connect sensibility with the *meaning* of 'intuition', not just with the form intuition takes in human beings (e.g., §89). But in any case Frege accepts the need for a way of grasping any object of thought. As he says, any such object must somehow be "given" to us (e.g., §62). If Frege, like Kant, leaves the exact force of this idea somewhat unclear, his resulting problem about arithmetic is nonetheless apparent. Since Frege has argued *both* that numbers are objects *and* that nothing like sensible intuition is available for them, a new way to apprehend them must be found.

For Frege this issue also has a linguistic dimension. The use of a singular term is legitimate only once we have been given its bearer (§62). Rather broadly speaking then, some sort of cognitive process must exhibit the numbers to us before we may use numerical terms. This has major consequences.

My description of [8] as an attempt to give (or outline) formal proofs was consistent with a reading far less interesting than Frege's own. Before we can *prove* any laws we must be able to *state* them, and Frege's intention is to describe the mental processes which make a statement possible. The situation Frege imagines at the start of his investigation is not one in which we decide to seek proofs of propositions we have simply believed without proof so far. Rather, we have linguistic forms without the definite meanings which a grasp of the subject matter would provide (cf. pp. I-II, VII-VIII). Our first achievement in carrying out the reasoning of the book is to come to *understand* arithmetical language. Only when we have formulated what we could not formulate before can we proceed to proof.

To see how all of this follows, notice first that Frege's cognitive and definitional projects coincide. We know that to use a singular term we must have grasped its bearer, but it is also obvious that Frege takes the formally adequate definition of numerical terms to allow their subsequent use. Definition must therefore be capable of giving us its object. For the numbers, it seems to be *the* way left open once intuition has been eliminated (§§62, 104). But now, if definition is how we are given the numbers, then we cannot talk about them before defining them. Hence we cannot yet express or think propositions which contain arithmetical terms.

Not incidentally, the novelty of terms denoting numbers is confirmed by our recent reflections on arbitrariness. Again, let $n = a$ define n, and let $n = b$ be a correct alternative. Because all identities in our initial language have truth values (cf. §62), $a \neq b$ must have already held before we defined n. So if a (or, by parallel reasoning, any other term) already referred to n, b must have referred to something definitely distinct from n, and $n = b$ could not have been a correct definition. Reference to numbers is therefore impossible in our initial language.

If definition is concept formation in [8], then it is natural to regard the text preceding §68 as a model for the process of developing the number concept. An inspection seems to bear this out. Roughly speaking, we find Frege beginning with the criticism of more or less ordinary views about number. In this stage the perception and intuition of numbers are dismissed as possibilities, and Frege proposes to derive arithmetic from pure logic. Working within logic, we initially lay down the contextual definitions; of course these do not actually define number, but in effect they delimit a range of acceptable definitions. Once this is done we can see how to make a free choice which will give us the number series. Under other circumstances, the first, critical stage of Frege's procedure might be unnecessary. Since logic (as Frege sees it) is a priori, and since empirical knowledge depends on arithmetic but not vice versa (§10), one should be able to develop arithmetic out of logic without preliminaries—as soon as one has begun to think. But Frege imagines us to have in all probability picked up misguided opinions before we turn to serious thought about number. They must be corrected before we can take the right path.

These last remarks place [8] in the tradition of "analytic" expositions: presentations of new results in the order of idealized discovery or cognitive advance. Perhaps the best-known example is the *Meditations,* and although Frege generally shows no signs of particularly Cartesian influence, I think the resemblance of [8] to the *Meditations* reflects, if not direct influence, at least certain fundamental points of agreement. But I cannot pursue this comparison. Now that we have a framework, our next task is to explain in detail how the concept of number is to be acquired. I will begin with the contextual definition of §63.[6]

Frege proposes an equivalence between

(A) There is a 1-1 correlation between F and G

and

(B) The number of F = the number of G.

As Frege explains (§66; cf. §56) this does not define number. The failure is obvious to a modern reader, but before we can consider its implications three, more subtle aspects of Frege's equivalence deserve comment.

First, (A) is *psychologically prior* to (B). We become able to make judgments in the form (B) by noticing the equivalence of propositions of this sort to their previously grasped counterparts. This is no accident of psychology or of our circumstances. The step from (A) to (B) is irreversible and a necessary part of attaining the concept of number. If this is not obvious from §63, it is made clear by Frege's more extended discussion of direction in §64.

Like (A) and (B),

(A') a is parallel to b

and

(B') The direction of a = the direction of b

are equivalent. Thus, parallelism and direction may each be defined with reference to the other. But Frege says that going from (B') to (A') "reverses the true order of things". Why? It is surely no *mathematical* error. One can readily imagine a mathematician defining parallelism in terms of direction within the context of a certain problem or theory. Frege, however, is thinking of definitions as conceptual steps forward. When we conceive direction for the *first* time, it is in terms of parallelism—a concept grasped earlier without help from a notion of direction we could not yet have. Frege's argument for the priority of (A') shows that this is his point. He observes that the intuitions from which geometrical concepts must originate can exhibit parallel lines but never directions, so that we must somehow obtain the latter notion from the former. It is rather uninteresting that once we have both concepts we may find them to be interdefinable, because then definition no longer has anything to do with discovery. So it is in discovery that (A') is prior—and we may add that this order is fixed. It is plainly determined (for Frege) by the very nature of intuition and its objects. The corresponding moral for number is clear. Judgments of the form (A) are made as part of an initial logical endowment which precedes any understanding of number. Progress towards such understanding must then proceed via the conversion of (A) to (B). (Compare the usually rather obscure claim that Frege made equinumerosity prior to number.)

Second, it is important that (B) contains neither the predicate 'number' nor any terms for particular numbers. (Read 'F' and 'G' as free variables.) In Frege's view, we must begin by grasping the *number of* function on concepts. Then we can define the number concept as applying to its range; the individual numbers are values of specific arguments. This is Frege's way of acknowledging the connection between number and counting. Intuitively, grasping a particular collection of objects which *happen* to be the numbers should not amount to having a notion of number at all. Only someone who matches objects in this collection with concepts according to cardinality is regarding the numbers *as* numbers; here, of course, "according to cardinality" means giving F and G the same number just in case (A) obtains. Frege rules out any notion of number independent of cardinality by making the function come first. Only in terms of a "counting function," as one might say, can anyone define number. (Indeed, the counting function must itself be grasped *as* a counting function. In some sense there may be an abstract possibility of grasping *number of* without seeing that it gives the same values for equinumerous arguments, but no one could do so. The first step towards *number of* is always to rewrite (A) as (B).) Recognizing the equivalence of (A) and (B) is, therefore, the exact beginning of arithmetical thought.

Third, what I have called equivalence is more specifically *identity of content* (§§63-64). Since this doctrine is bound up with some of the most obscure, difficult ideas in all of Frege, no full analysis is possible here, but certain aspects of it are crucial to the theory of number. I will now try briefly to explain as much as we need.[7]

The content of (A) and (B) is itself an entity prior to these propositions and more objective. Contents are *unstructured* bearers of truth values, and propositions (as I am now using the word) are structures which the mind formulates by way of representing contents to itself. This representation is the basis of thought. To think is to take attitudes toward contents, yet we cannot simply think contents as they are found. Since our reason can work only with structured entities, we must structure a content in the process of grasping it. We do so, according to Frege, by representing contents as definite arrangements of concepts and terms. In this sense, grasping is a process of analysis. Now as (A) and (B) show, a single content may admit radically different analyses. Thus there is no asking whether a particular concept or term appears in the content c, although this question makes sense for any of the propositions that express c. It is for this reason that contents are more objective. Each propositional structure is the mind's contribution, introduced as one among several which could equally well permit a grasp of something intrinsically unstructured.

This theory is plainly Kantian, not only in its view of the bearers of truth values as being prior to their elements (which Sluga emphasizes), but in its claim that the mind imposes form to grasp a realm of formless things. But a detailed comparison with Kant is best saved for later. The present interest of Frege's theory of judgment is that it is at once his theory of concept acquisition.

According to Frege, we are not equipped with prior concepts, but rather gain them in the process of analyzing contents. Any concept of ours must have originated in some act of analysis. Since functions are concepts for Frege, *number of* must also be grasped in this way. Unfortunately, Frege leaves the mechanisms of analysis extremely vague, but for number anyway, analysis and definition must apparently be equivalent. If we need a definition to acquire the concept of number, and if the theory of judgment has us gain concepts by analysis, then analyzing a content c in a way that gives us number must somehow amount to defining the new element, number, that our propositional analysis of c should contain. Analysis cannot always work like this, because when we acquire our first concepts we have nothing with which to define them, but if it is unclear *why* number is a special case, that is still what Frege's treatment of (A) and (B) suggests. Frege evidently does not take himself to have succeeded in re-analyzing the content of (A) as (B), because if he could state (B) he would have the concept of number as early as §63. The inadequacy of (A) ≡ (B) as a definition signals a failure of analysis. We may infer that having grasped a concent c which admits analysis into a numerical proposition q is not enough. Apparently the equivalence of q to the proposition p in which form c is first grasped must actually define number.

There is much more to be said about this theory of judgment, but I believe we now know enough to assess the situation after §63. At first sight Frege has made no progress. As just observed, (B) has not yet been grasped, so that what I have called the first step toward arithmetical thought is still absent. But reflection shows that we must be missing something. (B) *obviously presupposes* a concept of number, so how could we have thought that formulating it would help us to get one? Well, (B) need not be read as having numerical subject matter. What (A) ≡ (B) successfully defines is an *unstructured* binary predicate (cf. §56). 'The number of ______ = the number of . . .' appears to be built up from identity and terms, but since the terms are undefined this composition is illusory and cannot be found in (B). The semantical units in (B) are 'F', 'G' and a simple predicate. On this parsing, (B) is perfectly graspable. But now the whole exercise looks pointless. If all we wanted was an unstructured predicate, we could simply have understood 'there is a 1-1 correlation between ______ and . . .' in that way. Since (A), so construed (call it (A*)), can hardly be different from (B) as just parsed, (A) ≡ (A*) would do every bit as well as (A) ≡ (B). It is, however, clear that fusing parts of (A) to obtain (A*) is absolutely no help in gaining a concept of number, so the same goes for obtaining the fused version of (B). The attempt to avoid a reading of (B) that we cannot have seems to have driven us to one that is simply useless.

Frege does not clearly recognize these difficulties. But fortunately, the basis of a reply is there. If we ask for the intuitive significance of (B), the answer might be that it at least suggests the possibility of an equivalent to (A) with numerical subject matter. It shows that if we were to have number terms, we could use them to restructure the apparently nonarithmetical content of (A). What (B) provides, then, is a dummy structure which shows how genuine number terms would function. This idea can be expressed via Frege's notion of a second-level concept (cf. [10], p. 277). (A) ≡ (B) implicitly defines the second-level concept of a *cardinality function*: f is such a function if letting 'number of' in (B) stand for f yields an equivalent to (A). For each cardinality function, I would add, there is the concept of its range, so that in virtue of §63 we also have the concept of a cardinality concept. These second-level concepts are crucial. They allow Frege to solve the correctness problem—to say why the extensions in a certain sequence are really numbers, not impostors. His solution is that the objects a genuine definition of number picks out will satisfy a cardinality concept. He also needs second-level talk to avoid an incoherent description of his definitional project. Strictly, someone who lacks a concept of number cannot intend to define number. The proposition that number be defined already contains the very concept to be acquired by definition. For Frege to describe the sort of definition he is after without reference to the objects it will specify, he must refer to the *type* of concept his definition should provide. Thus it is exactly right to ask for "*a* sharply delimited concept of number" (§68), rather than for a definition of the numbers.[8] The concept of a number concept is just what he already has. The (half-completed) transformation of (A) into (B) provides it. So that step is a foundation of his enterprise after all.

To be accurate, we have not established that §63 completely determines the second-level number concept. Further conditions might eliminate the ranges of some so-called cardinality functions as numbers. But (Note 3 aside) Frege finds no other restrictions. What more than the equivalence of (A) to (B) would a definition of number have to confirm? If no other propositions are available, then our present definition of a cardinality function exactly captures the concept. What next? Because there are many cardinality functions, gaining this concept yields no grasp of any particular one. But since *any* cardinality function will do, Frege must, it seems, only think of one and number can be defined. In a way this is just what happens at §68. But from the viewpoint of the theory of judgment the process is not so simple. Recall that a definition of number should also be the analysis of a content, and that such an analysis should apparently be represented by a biconditional linking a numerical to a nonnumerical proposition. In effect, however, I have just suggested that there are no such (true) biconditionals besides $(A) \equiv (B)$. If there were any, they would represent information about number which our concept of a cardinality concept would have to include; if $(A) \equiv (B)$ captures that concept, no further conditions on number can obtain. This means that there is no content from which we can extract a concept of number by analysis. Hence there is apparently no way for us to form such a concept. Our attempted advance to a definite number concept starting from the second level has run out of material.

Frege's strategy is implicit in the foregoing remarks: we must use a *new* content. How? We no longer need a biconditional. Most simply, the new content will be grasped and asserted as an identity between a number term and a term from our previous vocabulary. The truth of the identity, plus the known value of the old term, will let us solve for the value of the number term, just as the truth of $(A) \equiv (B)$ would have let us use known concepts to solve for number if that definition had succeeded. D, I claim, is Frege's new content, or better its propositional representation. Now from the viewpoint of the preceding paragraph this content, c, must merely be one not previously grasped, which leaves it open that c was really there all along. But our apprehension of c is not mere discovery, by an extension of the argument that confirmed the novelty of number terms. If c existed before our definition, it was true, since contents have (unchanging) truth values.

Now suppose that we could instead have defined number by D', a proposition inconsistent with D. If the content of D exists and is true before we assert D, the same should go for D', since D and D' are equal alternatives. That is, if we were to assert D', we would be asserting a proposition the content (c') of which was a pre-existing truth. But then the truth c' exists even if we do not assert D', and since D' *would* be inconsistent with D, c' *is* inconsistent with c. There cannot, however, be two true, jointly inconsistent contents. Therefore c' could not precede an assertion of D', and by symmetry c cannot exist before we give D. The arbitrariness of D establishes the novelty of its content. We *make* a content, somehow, by asserting D. Moreover, our assertion makes it *true,* and we *know* that we have created a truth. (If our new content might, as far as we know, be false, we would still not know what the numbers are.) Have we created the numbers themselves? It is more apt to say that we have given certain things a new property. Since D identifies numbers with objects previously given, we seem to have added numberhood to extensions, not to have made entirely new objects. But since extensions themselves are literally to be created before we define number (see Section 2), numbers too are ultimately of our making.

Let us review. The definition of number serves a chain of goals for Frege. To prove the propositions of arithmetic we must state them; stating a proposition requires the grasp of a referent for each contained term; and numerical referents must be provided by a definition. The definition of number is an act of constructing a new truth which makes numbers of certain extensions. The relation between numbers and these extensions is thus genuine identity, although not an identity fixed in advance. We might have constructed numbers as something other than equivalence classes under cardinality. This shows that in a sense the essence of arithmetic lies in the concepts at the second level. The result of our alternative construction would still have been *a* number sequence, described by a set of theorems constituting *an* arithmetic. Doing arithmetic really means doing some arithmetic or other—whichever theory corresponds to the number sequence of one's choice.

Of the many questions that might now be raised, I want to deal with one that constitutes a specific objection to my interpretation. By way of wrapping up a loose end, however, let me first offer a remark on contextual and explicit definition. It is tempting to suppose that Frege would have us gain concepts from old contents just by contextual definition, reserving the explicit style for the introduction of new contents. This is strictly false. The content of a contextual definition could clearly be new, and explicit versions of analyses of old contents may be found. But I think my discussion shows why it would be *natural* for Frege to use explicit definition when creating a content and contextual definition when acquiring a concept by reconstrual. To this extent the critics who have seen Frege's recourse to D as an admission of failure are right. Explicit definition signals an inability to obtain number from the contents already given to us. But since Frege is perfectly willing to go beyond those contents, this failure is not defeat.

Now for the objection: how can Frege call four the same for everyone who deals with it (§61), if my four is (say) the class of all quadruples while yours is something else? In explaining Frege's tolerance of arbitrary definition we seem to have run afoul of his fundamental belief in the objectivity of number. But in fact objectivity as Frege understands it does not depend on uniform choices of objects. It is secured by the structural agreement between all arithmetics. This is the point of the longest section of [8] (§26), a remarkable passage which at once answers the deepest questions about Frege's definition of number and reveals its historical meaning.

What Frege does is to transfer Kant's theory of intuition to the objects of reason. Recall that for Kant, the things we find in space are intrinsically non-spatial. But our sensibility requires a spatial presentation, and we cannot think of objects except as they are given to sensibility. Our mental activity must therefore contribute the space in which objects appear, as well as their particular geometrical structures. On Frege's understanding (§26), the construction of space is somewhat arbitrary. Two subjects *a* and *b* may so construct their own spaces that what appears in one way to *a* appears very differently to *b*. Yet, Frege continues, the same underlying things may be presented to each. In this sense, space belongs *merely* to appearance. All this is clearly reflected in the theory of number. Where Kant is considering how objects may be given to sensibility, Frege's concern is how contents can be given to reason. Further, the specific problem is that although contents are intrinsically unstructured, our reason deals only with things in structural guise, and the solution, again, is that we supply the structure. For the contents of arithmetical propositions, this involves the mental construction of numbers. Numbers are thus, like space, auxiliary objects introduced so that we may grasp a further realm of things; a realm which we can, although it exists independently, comprehend only as it appears to us (cf. §26, *ad fin.*).[9] Number and space are also alike in their arbitrariness. As there are different, adequate intuitions of space, so various correct definitions can give us numbers. But here, too, the alternatives provide access to the very same things.

We now have the basis for a sweeping reconsideration of Frege's views on logic. Frege confines logic, like Kantian geometry, to the phenomena: since logical relations depend on logical structure, they hold not among contents in themselves but only among appearances. The logical grammar of the *Begriffsschrift* now seems to stand to thought much as geometry stands to perception in Kant: it specifies the possible structures under which we can think contents. (Logical space, one might say, is the form of reason's quasi-intuitive grasping.) Most importantly for our present purposes, Frege has the structure of the laws of logic determine the *objective* meanings of logical words, hence of words in arithmetic. Again, Frege begins by considering geometry. Geometrical terms, he observes, may seem to lack objective (universally shared) meanings because of the variations in how points, lines, planes, and so forth can appear to intuition. Since planes in *a*'s space, for example, may be entirely unlike the planes represented to *b*, two rational beings may agree verbally on the geometrical description of an object that *looks* very different to each of them. But Frege claims that communication in geometry is possible across differences in the subjective presentations of objects. There is an important sense of meaning in which 'plane' is synonymous for all rational beings. This follows, actually, from the Kantian claim that the nature of mind determines a *single* geometry for everyone's appearances: if *a* and *b* *agree* on geometry, 'plane' must have one meaning for them. Frege indicates that this meaning is given by the role of 'plane' in the system of geometrical axioms. (Here [18], pp. 130-134 is important.) As we would now put it, any set of appearances must model these axioms, hence every rational being intuits things that play the role of planes within her space. This role fixes an objective meaning for 'plane', even though subjective planes may be quite distinct. To turn now to the question of number, various subjects may certainly construct different numbers.[10] But a unique set of arithmetical laws (including the laws of counting (§§55, 63) as well as the usual axioms) derives from the laws of thought (§14). The respective roles of 'number', 'four', 'plus', and such are the same in the principles accepted by every arithmetizing creature. As in geometry, this makes arithmetical words univocal. No matter how *a* and *b* define number—no matter which "numbers" they define—there is a basic sense in which they hold common axioms, or in which they can agree that four numbers the leaves on a lucky clover (cf. the botanist of §26). It is just in this sense that four is the same for everyone.

It may seem wrong for Frege to say that four is the same for *a* and *b* in spite of differences in definition. But if 'four' has a single meaning, how can *a* and *b* not be talking about the same thing? Perhaps this could be taken to show that they do not agree in meaning either, but we must remember that in a sense numbers are not the subject matter of arithmetical discourse. If corresponding propositions in different arithmetics share objective meaning, they plainly share content in Frege's technical sense (notice, for example, that (*A*) will have the same content as *any* (*B*)), so that in entertaining such propositions *a* and *b* are grasping one entity. Our numbers belong merely to the appearances of the contents we need to apprehend. In Frege's view, saying that everyone (who grasps numbers) has the same ones is rather like saying that the color-blind man can refer to our red and green (§26). When we discuss numbers or hues we are in a sense referring to subjective items, but in a deeper sense not.

We can now make our earlier remarks about the second level more precise. If arithmetical words get their objective meanings from the laws of number, these meanings can be given by second-level conditions referring to the structure of number theory. To illustrate, the meaning 'number' receives from a certain system *Z* of propositions is just: concept which, when assigned to the word 'number', makes all of *Z* true. As long as *a* and *b* agree on *second*-level number concepts—and as rational beings they must—their arithmetical statements can have the same contents. At bottom they will be saying and thinking the same things. (There is obviously a connection between making meaning a matter of systemic role and giving primacy to second-level concepts. Although I will not explore this issue any further here, I expect it to be a fruitful subject for future research. It bears, for example, on Frege's attitudes toward model theory, on his controversy with Hilbert, and on post-Fregean issues about meaning.)

Since my aims are historical, I will not try to evaluate Frege's position overall. Obviously, the theory of content is both metaphysically and cognitively suspect, but to say exactly what is wrong and what, if anything, can be salvaged would take too long. I do, however, wish to pursue one natural line of criticism in a way that should allow a broader grasp of what Frege is up to.

The entire project of [8] must look very odd to a mathematician. Frege, remember, holds that earlier mathematicians proved no propositions of arithmetic because they understood none. The point of constructing numbers is, in the first place, to allow us even to grasp arithmetical contents. A mathematician, however, might well question both Frege's diagnosis and his proposed remedy. To begin with, there must be a sense in which mathematics as it stands is already correct. Since Frege would not suggest that we start looking for a greatest prime number, or for a natural number that is not the sum of four squares, he himself must regard past number theorists as having shown something. For similar reasons, ordinary arithmetic cannot *simply* lack content. There is no assimilating the theorems of Gauss to Carrollian nonsense or worse. And even if Frege is somehow right about content, a construction of numerical objects is not the only way out. Instead of arbitrarily picking a number sequence to talk about, Frege could use definite propositions at the second level to express what all arithmetics have in common. Let us assume for simplicity that cardinality functions are defined on sets. Then Frege can replace '$0 < 1$' by the statement that for any cardinality function f, $R_f(f(\phi), f(\{\phi\}))$, where R_f is the obvious ordering of the range of f. There is no need to choose a particular one and zero. In effect, we work within a general theory of ω-sequences, one which does without numerical objects. At least, this move is open to anyone with second-level arithmetical concepts, which Frege apparently does not regard as having previously been unavailable.

Now as far as the question of constructing numbers versus staying at the second level is concerned, I think Frege is in no difficulty. He is not committed to saying that *only* a construction of numbers can give us arithmetical contents. It is simply *a* way which, for several reasons, is superior from Frege's point of view. First, it is conservative. On Frege's approach, not a single line of arithmetic needs to be rewritten. The only change is that existing lines are now properly meaningful. If, however, we decide to do ω-sequence theory, restatements of all the old theorems will be required. Underlying this orthographic conservation is a deeper respect for mathematical belief. Mathematicians have traditionally believed in numbers and taken arithmetic to describe them; now, at last, they can rightly maintain their view. Second, ω-sequence talk is conspicuously cumbersome. Given the choice between adopting this new language and providing a numerical subject matter, the latter is *far* easier as long as construction is indeed within our powers. Third, the right statements about all ω-sequences come out *false* only if at least one ω-sequence exists. We must therefore exhibit such a sequence in any case; then it is particularly natural just to let it *be* the numbers. But the (excellent) reasons for Frege's choice do not rule out the alternative, and he gives no evidence of having thought otherwise.

What about the defectiveness of earlier arithmetic? Here the dispute seems to be partly verbal. On the one hand, Frege would be the first to grant the achievements of earlier mathematicians. Without going into detail, we may simply say that from his point of view, their pseudoarithmetic shows us which propositions to assert once number is defined. On the other hand, mathematicians who falsely believe they are describing numbers *are* wrong. It is tempting to call the error harmless, but that really just means that certain methods will not detect it; that it will not, for example, lead to inconsistencies or undermine the usual applications of arithmetic. This Frege would regard as a discovery about the limits of our practice (mathematicians' practice included), not as a full vindication. Mere shells of arithmetical propositions may suffice for certain purposes, but to rest content with them is to remain in error. In short, the insistence that arithmetic is already in order amounts, for Frege, to a refusal to step out of a convenient, but limited point of view.

It seems to me that this far Frege is right about standard mathematics. I would even roughly endorse his reasons, arguing that since numbers are in no sense perceptible, and since nothing we already believe specifies referents for number words, these words do not denote—which is to say that there are no numbers. This can obviously not be shown here (see, however, [20]). I mention it only to suggest that a Fregean critique of mathematics may be quite appropriate. If mathematics is, as one might well say, "mathematically" in order, it may yet be semantically ("philosophically?") defective in a way that should interest any reflective mind. To be sure, I have so far only been concerned with failures of *reference* in mathematics. The charge that mathematics lacks content is much more problematic, and quite inadequately developed by Frege. I would therefore only comment that something of it may be salvageable. I think one can argue that arithmetic as it stands lacks *fully determinate* content, for reasons related to the failures of reference. Since Frege himself could on reflection hardly maintain the charge of complete absence of content (but see [9], II, §56), and would therefore be forced to admit something like the *partial* grasping of contents, we may suggest that, even here, his position is close to one that must still be taken seriously.

It should now be clear that Frege's theory of number was not just, in its own time, a remarkable feat of philosophical imagination, but that at least some of its underlying ideas transcend its immediate and now perhaps quite foreign context. If other, central elements in Frege's thought do seem flatly unacceptable now, I would hasten to point out that subsequent essays in the metaphysics of logicism have not been manifestly more successful (e.g., [2], [15]). In the space remaining I want to extend our view of Frege's place in the constructivist tradition.

2 I have not discussed Frege's demand for truth values for numerical identity statements (§§56, 66), although it is perhaps the most familiar issue about D. Frege has been much criticized on this point because identities pairing number words with nonmathematical (or even just nonarithmetical) terms are widely held to lack truth values. The charge of asking for excess determinacy, however, is doubly misdirected. First, the view of identity attributed to Frege looks like common sense. If Caesar and two are objects, mustn't they either be alike or different? What else are we to think? While I do not know how to argue that *entities*—not fictions, possible fat men in doorways, or other creatures of extravagant imaginations—must either be identical to one another or distinct, I think argument is hardly necessary. Second, the attribution is inaccurate because Frege does allow us to identify the numbers without determining Caesar's numberhood. I admit that this looks plainly wrong. It seems to conflict not just with the remarks on identity in [8], but also with the requirement that functions be totally defined. Since concepts are functions from objects (or *n*-tuples thereof) to truth values, that requirement means that the extension of any predicate must be fixed over *all* objects ([9], II, §§56-65). But what we now know about Frege calls this condition ("completeness") into question. Zero, for example, could not have been arbitrarily defined if it had in advance satisfied '_____ = the extension of *not self-identical*' or its negation, so completeness appears to be violated.

The constructivist resolution of this dilemma emerges most clearly from the early sections of [9], where Frege introduces his formal language. At first glance a modern reader is likely to find in this procedure only the leisurely development of an interpreted syntax. Functional notation, negation, identity, extensional abstraction, a description operator, and other signs are set out with accompanying elucidations. But Frege is not just specifying functions and the rest within a given domain. Along with the language, *the universe itself* is being expanded in stages. At first the universe of objects contains just the truth values. Extensions are added in §§9-10. The absence of other objects is indicated by Frege's handling of identity questions about extensions. Having used Law V to define the identity relation among extensions, he remarks:

> Since up to now we have introduced only the truth values and courses of values [extensions] as objects, [the only remaining question is] whether one of the truth values can perhaps be a course of values (§10).

Nor has he added *all* extensions at this point. Extensions are extensions of concepts, and he has only those concepts which can in some sense be obtained from the truth functions, quantifiers, and the *extension of* operator. I do not want to say just how these basic concepts yield the totality available; that is a difficult question whose difficulties are closely related to the inconsistency of Frege's system. But clearly some such restriction on the concepts we have is in force, because the very next section (§11) introduces yet another concept—which in turn increases the supply of extensions. In this context the completeness condition needs a special reading[11]: the values of a function must be specified for every object *given so far.* '_____ = the extension of *not self-identical*' is, for example, undefined for Caesar as argument because he is no object of discourse yet.[12]

In fact, the objects and concepts Frege *has* are shadowy, their only features being their relations to each other:

> With this [the arbitrary identification of truth values with their unit sets] we have determined the courses of values so far as is here possible. As soon as there is a further question of introducing a function that is not completely reducible to functions known already, we can stipulate what value it is to have for courses of values as arguments; and this can be regarded as much as a further determination of the courses of values as of that function (§10).

Of course this makes it obscure how anything can ever have definite features at all, since any new item will only be specified with reference to the indefinite things we have. However that may be, Frege clearly does not envision the usual procedure of assigning words to things in a finished, already grasped ontology. Language and subject matter are concurrently expanding, and growth increases determinacy.

Our look at [9] has confirmed what we learned in Section 1: Frege regards mathematicizing as a series of (partly) free mental acts, some of which give our present objects of discourse new properties while others provide new objects, the nature of which may then be further determined. This makes obvious his affinity to intuitionism. I do not think that the alleged realist manifesto at [8], §96, undermines this comparison. ("The mathematician cannot create things at will, any more than the geographer; he too can only discover what is there and give it a name.") *In its context* it is a denial of *unrestricted* creative power (cf. "at will"), as would be needed to construct, say, sums of divergent series (Frege's example). Beyond that its implications are just unclear. We have ample evidence that simple discovery is not Frege's model for mathematical progress in [8]. The relation between naming and reality is quite unlike what is found in geography. Yet our naming (e.g., of numbers) *is* in some sense a means of grasping "what is there". For these reasons I believe that §96 lacks a definite upshot. When Frege returns to the same question in [9] (II, §§139-147) he is more precise and subdued. His objection to various constructivists is conspicuously not the impossibility of making new objects or attaching new properties (see especially §143). It is rather that one cannot construct at will. Simply postulating an object with certain properties provides no guarantee of consistency, and inconsistent constructions really construct nothing at all. This, Frege argues, makes the power to construct seem useless, since one needs a consistency proof to establish the success of one's construction—and mathematicians prove conditions consistent just by exhibiting objects that satisfy them, which was the original point of the construction. The only *useful* way to construct new objects (Frege continues) is demonstrated by his own introduction of the extensions in [9], I. According to Frege, Law V shows how to construct without the crippling restrictions imposed by the need to prove one's stipulations consistent ([9], II, §§146-147).[13]

Frege's constructivism (or its extent) can come as a surprise only if we overlook its historical context. His principal disputes with his contemporaries took place *within* a broadly constructivist framework which—presumably due to Kant's influence—dominated German mathematics in his time. (The polemics of [9] provide a good sense of Frege's opposition.) Typically, constructivism took forms very different from Frege's. Formalists, for example, tended to locate the essence of mathematics in our construction of *inscriptions.* But it is important to see that even the formalists Frege so vehemently denounced were fellow constructivists; and his own brand of constructivism was not entirely uncommon (cf. [8], p. X). To mention one prominent case, Dedekind concurred with Frege on all of the following points: that the fundamental laws of number should be proved; that the primary motive for proof is to reduce our stock of unproved propositions; that proof requires saying what the numbers are; that the concept of number flows from the laws of pure thought, not from ideas or intuitions; and that numbers and sets are free creations of the human mind ([4], *passim*; [3], especially pp. 5, 11ff). Agreement on such points—which might of course only be partial or partially verbal in various instances—defined a special movement within the current of constructivist thought. It was the origin of logicism. Far from being opposed to constructivism, first-generation logicists wanted to found arithmetic on the constructive powers of pure reason.

I do not mean to assimilate Frege to the mathematicians usually labeled constructivists. Brouwer and his followers, besides presenting a new theory of mathematical intuition, unambiguously made all mathematical objects creatures of our minds. If this shift to thorough subjectivism looks like a step in the wrong direction, it is an understandable response to the obscurities in such positions as Frege's. Another obvious change was the intuitionistic ontology, motivated by a sense of limits on what we can grasp. Frege (along with Dedekind, Cantor, and others) had accepted a Kantian equation of "objects" with *objects of thought,* but had not seriously considered what it would mean to think of each item in his ontology. It is not hard to see how a rigorous attempt to work out Frege's notion of an object might yield something like a restriction to intuitionistically constructible objects—but that would mean a radical departure from Frege.[14] Thus, it might be roughly accurate to describe Frege as being halfway to intuitionism. There is a significant overlap, but other elements in his thought are more at home in subsequent logicism. The sharp divergence of these two schools even as [9] was being completed is in part due to the tensions internal to such positions as Frege's. Of course many other factors (e.g., the empiricism of twentieth-century logicists) complicate this picture.

It would certainly be worthwhile to redraw the complex picture of Frege's relations to later foundational work. The result would, I think, be a richer, more interesting conception of Frege's influence than anything available now (as in, e.g., [16]). I hope that my discussion will encourage further research to that end. Indeed, I believe I have provided a basis for reconsidering Frege's relations to a range of important issues: unsayability; the limits to our referential powers (and the extension of these limits); analysis; the relation of ordinary language to its counterparts in theory; the nature of logic; ontological relativity. These connections, however, must remain implicit, as they go beyond the theory of number that has concerned us.

I think I have shown something new: that Frege has an important philosophy of number. For in my opinion, the naive, vague Platonism with which he is usually credited cannot count as that. And although it is pointless to argue over the extent to which the analysis of cardinality in [8] and the logicist construction of [9] are philosophy, these hardly, for all their brilliance, speak to a philosopher's characteristic concerns about knowledge, mind, and truth. But Frege's position as I have now explained it does. Nothing comparable is offered by his contemporaries or by any but the very best of his successors. Frege's stature as the first great figure in modern philosophy of mathematics—the philosophy responsive to logical and mathematical advances that are above all his own—is thus confirmed. If his theory now appears antiquated, any philosophy will look much the same in a proper historical light. Those that best survive illumination are distinguished by such depth and detail as we have found in Frege.

NOTES

1. Section references are to [8] unless otherwise indicated.

2. But it is worth adding that Frege seems to need an independent premise to the effect that propositional meanings are truth values or something determinative thereof.

3. See the discussion in Note 4. Actually, there are some further limits which I will generally leave unstated. Frege would (at this stage of his career) reject identifications of natural numbers with real numbers or geometrical objects.

4. See [1] and [19]; also [14]. Principal sources of evidence: (i) the arbitrary stipulation about extensions and truth values at [9], I, §10; (ii) the suggestion of possible alternatives to *D* at [8], §§68, 107. (Consider also Frege's tone at the opening of §68 and at the close of §10.)

   An additional, strangely neglected source is Frege's direct discussion of the correctness of *D* at §69. Frege recognizes that *D* will seem wrong because it gives numbers properties we do not think they have (such as having members or subsets); he replies not that these attributions are *true* but that "nothing stands in the way" of their acceptance. I see only one way to make sense of his discussion. We must assume a division of propositions into two classes, those the definition of number must preserve and those we may make true or false indifferently. In the former we find, for example, facts about one-one correlations and counting (cf. §§55, 63), while the latter contains set-theoretical (extension-theoretical) propositions about number. Since nothing stands in the way of our accepting *or* rejecting any given proposition of the second kind, there must be more than one correct definition of number.

   The foregoing is meant only as an indication of what is going on in a quite difficult passage. Although I will not further discuss §69, my understanding of it will be confirmed by the argument in the text.

5. See [18], especially ch. IV. Although I have tried to make my discussion of Frege and Kant more or less self-contained, my aim is to build on what Sluga has in my opinion already made clear. Besides [18], interested readers should consult [13].

6. Much of what I say about §63 also applies to or has reasonably obvious parallels for §55.

7. Here in particular I draw on [18] (pp. 90-95, 134-137). My principal complaint against Sluga is that he gives no hint of how problematic and unclear Frege's position is.

8. Frege's apparent references to number before §68 should (and can) be rewritten accordingly. Notice that our second-level concepts do not *generally* precede their instances. *Horse,* for example, is surely acquired before the concept of the concept horse. It is interesting to consider just when the second-level concepts would be prior for Frege.

9. To see how Kantian Frege's theory is even in detail, notice that if numbers are objects, and if we cannot grasp them by sensible intuition, then for a Kantian the faculty that apprehends them must be *intellectual intuition.* This faculty *originates* the very objects it grasps (*C.P.R.*, B 71ff), which are thus indeed reason's own creatures (§105).

10. While I can tell you my definition of number, what is peculiar to an intuition is incommunicable (§26). But this difference may be superficial. If numbers are given as extensions, which are in turn indefinable, there may be no telling whether your extensions are like mine. This uncertainty would transmit to our numbers.

11. So do Frege's remarks on piecemeal definition ([9], II, §56ff). Frege may later have found his procedure in [9] objectionably piecemeal after all—see [10], pp. 224-229 for this and other important changes of mind.

12. Contrast: "In [Frege's] system the quantifiers binding individual variables range over all objects . . . His universe is *the* universe . . . it consists of all that there is, and it is fixed" ([12], p. 325).

How does [9] square with the remarks on Caesar in [8]? The difference lies in the background language. In the earlier book Frege is more or less working within ordinary language, which quantifies over Caesar. Hence the definition of the numbers should settle their relation to him. [9], however, makes clear the independence of arithmetical thought from the recognition of any particular empirical objects. Arithmetic is developed directly from the pure logic with which thought in some sense begins. If we introduce Caesar at some point beyond [9], then we must of course determine who he is as fully as the language then to be spoken will allow.

13. Frege mentions the introduction of extensions, but not the definition of number, as a case of bringing in new things (§146). Since he also has no objection in principle to adding properties to objects already there (§143), my view that *D* gives extensions new properties seems reasonable.

Incidentally, Frege's word for "constructing" is *schaffen*, which straightforwardly means *making*. He uses it interchangeably with *schöpfen* (*creating*) at, e.g., §143. This is worth mentioning because 'construction' now has a colorless mathematical use.

14. To see how fuzzy Frege's notion of construction is, readers should struggle with the question just what concepts and extensions are supposed to be present early in [9].

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