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A Note on Conway Multiplication of Ordinals

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We denote by 'ω' the first transfinite ordinal, and by '$\alpha + \beta$', '$\alpha\beta$', and 'α^β', respectively, the usual ordinal sum, product, and exponentiation of an ordinal α by an ordinal β. We assume that the reader is familiar with Cantor's ω-normal form theorem, which uniquely represents a nonzero ordinal as a sum of powers of ω. Given a nonzero ordinal α, we let $\ell(\alpha)$ be the number of summands in the ω-normal form of α, and express this form as $\Sigma\{\omega^{e(\alpha,i)}c(\alpha,i); i < \ell(\alpha)\}$.

Regrettably, there seems to be no "nice" way of formulating an adequate definition of "natural" ordinal addition, and so we shall have to use the following not-so-nice way.

Let α, β be ordinals. If $\alpha\beta = 0$, set $\alpha \mathbin{\mathring{+}} \beta = \alpha + \beta$; otherwise set $\alpha \mathbin{\mathring{+}} \beta$ equal to the unique ordinal γ whose ω-normal form has the following properties:

(1) $\{e(\gamma,i); i < \ell(\gamma)\} = \{e(\alpha,j); j < \ell(\alpha)\} \cup \{e(\beta,k); k < \ell(\beta)\}$.

(2) (a) If $e(\gamma,i) = e(\alpha,j)$ for some $j < \ell(\alpha)$ but $e(\gamma,i) = e(\beta,k)$ for no $k < \ell(\beta)$, then $c(\gamma,i) = c(\alpha,j)$.

(b) If $e(\gamma,i) = e(\beta,k)$ for some $k < \ell(\beta)$ but $e(\gamma,i) = e(\alpha,j)$ for no $j < \ell(\alpha)$, then $c(\gamma,i) = c(\beta,k)$.

(c) If $e(\gamma,i) = e(\alpha,j) = e(\beta,k)$ for some $j < \ell(\alpha)$ and $k < \ell(\beta)$, then $c(\gamma,i) = c(\alpha,j) + c(\beta,k)$.

We can now define Conway multiplication, denoted by '$\times$', which was introduced by Gonshor in [1] and attributed by him to Conway.

If $\alpha\beta = 0$, then we set $\alpha \times \beta = 0$; otherwise we set $\alpha \times \beta = (\alpha \times \delta) \mathbin{\mathring{+}} \alpha$ if $\beta = \delta + 1$ for some δ, and $\alpha \times \beta = \sup\{\alpha \times \delta; \delta < \beta\}$ if β is a limit ordinal.

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What in general does $\alpha \times \beta$ look like, compared to $\alpha\beta$? If β is finite, then simple iteration shows that the normal form of $\alpha \times \beta$ is derived from that of α by multiplying each $c(\alpha, i)$ by β. Thus Conway multiplication by an integer has no effect on the exponents of the normal form of the multiplicand. By using this fact, together with a straightforward inductive argument, we obtain Gonshor's assertion that if β is a limit ordinal, then $\alpha \times \beta = \alpha\beta$.

Thus if we denote the finite and infinite parts of an ordinal γ by '$F(\gamma)$', '$I(\gamma)$', then we see that $\alpha \times \beta = \alpha I(\beta) \mathring{+} \alpha \times F(\beta)$. In fact, $\alpha I(\beta) \mathring{+} \alpha \times F(\beta)$ can be replaced by $\alpha I(\beta) + \alpha \times F(\beta)$. If $I(\beta) = 0$, then this is clear; and if $I(\beta) \neq 0$, then each exponent in the normal form of $\alpha I(\beta)$ is greater than any exponent in the normal form of $\alpha \times F(\beta)$, and the result again follows.

One of the more difficult proofs in the classical theory of ordinal numbers is that which shows that for any infinite α, β, $\alpha\beta = \beta\alpha$ if and only if $\alpha^n = \beta^m$ for some positive integers m, n. We wish to show that a similar result holds for Conway multiplication; for any infinite α, β we have $\alpha \times \beta = \beta \times \alpha$ if and only if $\alpha \uparrow n = \beta \uparrow m$ for some positive integers m, n, where for any ordinal γ and positive integer k we set $\gamma \uparrow k = \gamma \times \gamma \times \ldots \times \gamma$ (k products).

Clearly if $\alpha \uparrow n = \beta \uparrow m$, then $\alpha \times \beta = \beta \times \alpha$; and so henceforth we assume that α, β are infinite ordinals such that $\alpha \times \beta = \beta \times \alpha$.

If α, β are both limit ordinals then we can replace Conway multiplication by ordinary multiplication, and the result follows from the classical theory.

The case in which exactly one of α, β is a limit ordinal is not possible. For suppose that α is limit and β is successor. Then we would have $\ell(\alpha \times \beta) = \ell(\beta) + \ell(\alpha) - 1$ and $\ell(\beta \times \alpha) = \ell(\alpha)$; and since we must have $\ell(\beta) \geqslant 2$, the two right-hand sides cannot be equal.

Thus we are left with the case in which both α and β are infinite successor ordinals. In order to deal with this case, we introduce the concept of regularizing an ordinal.

Let γ be any nonzero ordinal. We define the regularization $\theta(\gamma)$ of γ by setting $\ell(\theta(\gamma)) = \ell(\gamma)$, $e(\theta(\gamma), i) = e(\gamma, i)$ for $i < \ell(\gamma)$, and $c(\theta(\gamma), i) = 1$ for $i < \ell(\gamma)$. We claim that $\theta(\alpha) \times \theta(\beta) = \theta(\beta) \times \theta(\alpha)$.

By equating the normal forms of $\alpha \times \beta$ and $\beta \times \alpha$, we obtain two systems of equations, (A) and (B):

(A) In this system, each equation has one of four forms: $\Sigma\{e(\alpha, i_k); k < r\} = \Sigma\{e(\beta, j_k); k < s\}$, where r, s take values from the set $\{1, 2\}$ and the i_k, j_k take values from the sets $\{0, \ldots, \ell(\alpha) - 1\}$, $\{0, \ldots, \ell(\beta) - 1\}$, respectively.

(B) In this system, each equation again has one of four forms: $\Pi\{c(\alpha, i_k); k < r\} = \Pi\{c(\beta, j_k); k < s\}$, where r, s, i_k, j_k are subject to the same restrictions as in (A).

Now since the lengths and exponents of the normal forms of $\theta(\alpha)$, $\theta(\beta)$ are the same as those of α, β, respectively, we see that system (A) is satisfied by $\theta(\alpha)$ and $\theta(\beta)$. Furthermore, since all coefficients of the normal forms of $\theta(\alpha)$ and $\theta(\beta)$ are 1, system (B) is trivially satisfied. Thus we must have $\theta(\alpha) \times \theta(\beta) = \theta(\beta) \times \theta(\alpha)$.

But $F(\theta(\alpha)) = F(\theta(\beta)) = 1$ and so, as is clearly seen from our general representation of $\times$, we have $\theta(\alpha) \times \theta(\beta) = \theta(\alpha)\theta(\beta)$ and $\theta(\beta) \times \theta(\alpha) = \theta(\beta)\theta(\alpha)$.

Since $\theta(\alpha), \theta(\beta)$ are both infinite successor ordinals, we know from classical theory that $\theta(\alpha) = \xi^m$, $\theta(\beta) = \xi^n$ for some ordinal ξ and some integers $m, n > 0$.

Clearly we may assume that m, n are coprime, for otherwise we could replace ξ, m, n by $\xi^d, m/d, n/d$, respectively, where $d = gcd(m, n)$.

Set $p = \ell(\xi) - 1$; then $\ell(\alpha) - 1 = mp$ and $\ell(\beta) - 1 = np$. We wish to show that for some ordinal μ of length $p + 1$ we have $\alpha = \mu \uparrow m$, $\beta = \mu \uparrow n$. We may assume without loss of generality that $n \geqslant m$, and we set $n = sm + r$, with $r < m$. We note that because of our assumption that m, n are coprime, either $r > 0$ and m, r are coprime, or else $r = 0$ and $m = 1$.

In either case, by equating the normal forms of $\alpha \times \beta$ and $\beta \times \alpha$ we obtain:

(1) $c(\beta, i) = c(\alpha, i)$ for $i < mp$
(2) $c(\beta, mp + i) = c(\beta, i)F(\alpha)$ for $i < (n - m)p$
(3) $c(\beta, (n - m)p + i)F(\alpha) = c(\alpha, i)F(\beta)$ for $i < mp$

and another set of equations relating the exponents. These however we do not need to worry about, because the exponents of α and β are just those of $\theta(\alpha)$ and $\theta(\beta)$, and so from the established relations $\theta(\alpha) = \xi^m$, $\theta(\beta) = \xi^n$ we know that the exponents satisfy the required conditions.

If $m = 1$ and $r = 0$, then from (2) we conclude immediately that $c(\beta, kp + i) = c(\beta, i)F(\alpha)^k$ for $k < n$, $i < p$; and then by applying (3) we obtain $F(\beta) = F(\alpha)^n$. But these are just the equations we need for the identity $\beta = \alpha \uparrow n$.

Hence we may assume that $r \neq 0$. In this case it is straightforward to show that $c(\alpha, i)F(\beta) = c(\alpha, rp + i)F(\alpha)^s$ for $i < (m - r)p$, and that $c(\alpha, i)F(\beta) = c(\alpha, i - (m - r)p)F(\alpha)^{s+1}$ for $(m - r)p \leqslant i < mp$. Therefore, because m, r are coprime, we have $c(\alpha, i)F(\beta)^{mp} = c(\alpha, i)F(\alpha)^{(m-r)ps}F(\alpha)^{rp(s+1)} = c(\alpha, i)F(\alpha)^{np}$ for each i; hence $F(\beta)^m = F(\alpha)^n$ and so $F(\alpha) = c^m$, $F(\beta) = c^n$ for some integer c.

Now let u, v be positive integers such that $um - vr = 1$. Then we have $c(\alpha, jp + i)F(\beta)^v = c(\alpha, (j - 1)p + i)F(\alpha)^{vs+u}$ for $0 < j < m$, $i < p$. That is, $c(\alpha, jp + i) = c(\alpha, (j - 1)p + i)c^{m(vs+u)-v(sm+r)} = c(\alpha, (j - 1)p + i)c$. Once again these are just the relations we require for the identity $\alpha = \mu \uparrow m$ for some infinite successor ordinal μ with $\ell(\mu) = p + 1$ and $F(\mu) = c$.

If we now apply (1), (2) to our representation of α, we obtain a corresponding representation for β, and we can thus conclude that $\beta = \nu \uparrow n$ for some infinite successor ordinal ν with $\ell(\nu) = p + 1$ and $F(\nu) = c$.

But the first p terms in the normal forms of α and β are equal by (1). Thus $\mu = \nu$.

REFERENCE

[1] Gonshor, H., "Number theory for the ordinals with a new definition for multiplication," *Notre Dame Journal of Formal Logic*, vol. 21 (1980), pp. 708-710.

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