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The Nature of Reflexive Paradoxes: Part II

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A number of unsolved problems were left hanging in Part I. In particular, there is the problem of finding precise formulations of the general condition for a formula to be a reflexive contradiction C_1 and of the corresponding general condition for a resolution R_1. Secondly there is the question of what exactly a resolution amounts to since the removal of the contradictions cannot change paradoxical items such as the Russell class, the barber, the catalogue, etc. from inconsistent concepts to consistent concepts. Thirdly there is the question of what would count as a minimal resolution and whether such a resolution is possible. These are the main points, though there are a number of related subsidiary questions. I shall first take up these questions in a general way and then apply the conclusions to the familiar paradoxes and to the problem of constructing a consistent set theory.

1 The general criterion for a reflexive contradiction It was argued in Part I, Section 2.4, that a formula QA of quantification theory is a reflexive contradiction in case A entails an inequality condition and the quantifier arrangement is such that permissible instantiation cases of QA presuppose the denial of that condition. In the special case where QA contains just two distinct variables and is of the form $(\exists x)(y)A(y,x)$, the criterion can be given precisely: the formula is a reflexive contradiction in case $A \supset (y \neq x)$ is a thesis (condition C). In the general case, however, not only is the intuitive criterion C_1 imprecise,

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but it has the disadvantage that since A is not precluded from containing bound variables, or free variables which are not bound by the quantifiers in Q, the inequality condition may contain a choice operator (Part I, Section 2.5). This last difficulty is easily met, however, by formulating the criterion in terms of formulas in prenex or Skolem normal form; and once this is done, the first problem, of stating the criterion in a precise form, can also be met. It is simpler to begin with the Skolem normal form (Snf).

1.1 Consider a formula QA in Snf; i.e., let QA be of the form

$$(\exists x_1) \ldots (\exists x_n)(y_1) \ldots (y_m)A(x_1, \ldots, x_n, y_1, \ldots, y_m, c_1, \ldots, c_r),$$

where $c_1, \ldots, c_r$ are individual constants. QA may contain predicate parameters or constants and sentential variables or constants, but no schematic letters; i.e., QA is a formula not a schema.

By an extension of the two-variable case, QA will be a reflexive contradiction if A entails an inequality between a variable which is existentially quantified in QA (an e-variable), say x_i, and one which is universally quantified (a u-variable), say y_j; i.e., if $A \supset (x_i \neq y_j)$ is a thesis. For whatever instantiation value is given to x_i, the same value is available as an instantiation for y_j. Hence a permissible instantiation case of QA presupposes the equality $x_i = y_j$ for some value of x_i, y_j, while A entails $x_i \neq y_j$ for all values of x_i, y_j. So one condition for QA to be a reflexive contradiction is:

(i) $A \supset (x_i \neq y_j)$ for any i, j.

By a similar argument, QA will be a reflexive contradiction if A entails an inequality between any two distinct u-variables, i.e., if

(ii) $A \supset (y_i \neq y_j)$ for any i, j such that $i \neq j$.

However, the same is not the case if A entails an inequality between two e-variables. For given that x_i has been instantiated by some value x_i^0, then by the ordinary restrictions on an existential instantiation, this value is not then available as an instantiation value for any other e-variable. Hence, even if $A \supset (x_i \neq x_j)$ holds there is no incompatibility involved since the equality $x_i = x_j$ is not presupposed in permissible instantiation cases of QA.

Suppose now QA contains individual constants $c_1, \ldots, c_r$. Since c_j occurs in A, it is not available as an instantiation value for any e-variable. Hence, even if A entails $x_i \neq c_j$ for some i, j, this is not incompatible with permissible instantiation values for x_i. On the other hand, if A entails $y_i \neq c_j$, where y_i is a u-variable, then there is an incompatibility since c_j is available as a value for y_i; i.e., a permissible instantiation presupposes $y_i = c_j$ for some value of y_i. So,

(iii) $A \supset (y_i \neq c_j)$ for any i, j.

Consider now cases in which $A \supset (a \neq a)$, where a is any variable or constant, i.e.,

(iv) $A \supset (y_i \neq y_i)$ for any i
(v) $A \supset (x_i \neq x_i)$ for any i
(vi) $A \supset (c_i \neq c_i)$ for any i.

If these conditions are adopted, the formal concept of a reflexive contradiction is extended well beyond the intuitive concept. Intuitively, QA is a reflexive contradiction when two or more variables are instantiated with a common value and the resulting formula is self-contradictory, or when a u-variable is instantiated with a constant already in the formula (e.g., in the move from $(y)(f(y,x_0) \equiv \sim f(y,y))$ to $f(x_0,x_0) \equiv \sim f(x_0,x_0)$, where x_0 is a constant). It is the choice of a common value or of an existing constant which characterizes the reflexiveness. But there is also a second feature, namely that if the reflexive cases are excluded, no other instantiations lead to self-contradictory formulas. Where A contains predicate constants and no predicate parameters, these other instantiations may be rejected as being false by meaning (Part I, Section 2.1) but they are not formal contradictions, and indeed in the plausible cases all instantiations other than the reflexive ones are presumed to result in true sentences. Intuitively, therefore, a formula is a reflexive contradiction just in case its reflexive instantiation instances, but no others, are formal contradictions. Now consider a formula QA such that any one of (iv)–(vi) is satisfied. We conclude immediately $\vdash \sim A$, from which it follows that all instantiation cases of QA are self-contradictory. A trivial example is $(x)(fx \; \& \; \sim fx)$. In fact, if any of these conditions is satisfied by a wff of A which contains no quantifiers, then $\vdash \sim QA$, where Q is any arrangement of quantifiers which binds the variables in A. To classify such formulas as reflexive contradictions, therefore, is to obliterate the distinction between reflexive contradictions and others. On the other hand, there is some point in recognizing them as limiting cases since reflexive cases occur among their self-contradictory instantiation instances.

We have not yet exhausted the standard cases, however, for some reflexive contradictions depend on two or more successive instantiations–e.g., the cyclic paradoxes (Part I, Section 2.5). These will be covered if

(vii) $A \supset D$

where D is any disjunction of inequalities of the kind indicated in (i)–(iii), i.e., a disjunction formed from any inequalities of the form $x_i \neq y_j, y_i \neq y_j \; (i \neq j)$, $y_i \neq c_j$. Allowing D to stand also for single inequalities (one-termed disjunctions), (vii) includes (i)–(iii). The limiting cases (iv)–(vi) do not give rise to disjunctive conditions, however, for if, say, $x_i \neq x_i$ occurs in D, it can be deleted, and if there is a disjunction of several inequalities all of the form $a \neq a$, all but one can be deleted.

We can now state the general criterion in terms of Snfs as follows:

C_2 Where QA is as above and D is any single inequality or any disjunction of two or more inequalities in the set

$$\begin{aligned} \{&(x_i \neq y_j) \text{ all } i, j, \; 1 \leqslant i \leqslant n, \; 1 \leqslant j \leqslant m; \\ &(y_i \neq y_j), \; i \neq j \text{ and } 1 \leqslant i,j \leqslant m; \\ &(y_i \neq c_j), \text{ all } i, j, \; 1 \leqslant i \leqslant m, \; 1 \leqslant j \leqslant r\,\} \end{aligned}$$

then QA is a reflexive contradiction, and consequently $\sim QA$ is a thesis, if $A \supset D$ is a thesis for any such D.

Where D' is a single inequality of the form $a \neq a$, where a is any variable or constant, then QA is a limiting case of a reflexive contradiction if $A \supset D'$ is a thesis for any such D'.

Since $A \supset D$ and $A \supset D'$ are quantifier-free formulas, we can apply a decision procedure to test whether or not they are theses. Hence we can determine for any given formula in Snf whether or not it is a reflexive contradiction in terms of C_2.

It should be noticed that a given formula may satisfy more than one condition for being a reflexive contradiction. In fact, if A entails a single inequality, then it also entails every disjunction which contains it. But this is a trivial case. More important is the situation which arises if A separately entails two or more single inequalities, say $A \supset (x_i \neq y_j)$ and $A \supset (y_i \neq y_j)$, for this tells us that two distinct instantiation instances of QA are self-contradictory.

1.2 Consider now the formulation of the criterion in terms of prenex normal forms (pnfs). There are two advantages in using Snfs: the first is that QA contains no free variables; the second is that condition (i) above can be stated generally as a relationship between any e-variable and any u-variable since the order of the quantifiers in Q requires the e-variables to be instantiated first, and the values used are always available as values for the u-variables. With a pnf, of course, this is not so. If a u-variable is instantiated before an e-variable, the value chosen for the u-variable is not available as a permissible value for the e-variable. In this case there is no presupposed equality between the two values in permissible instantiation cases and therefore no incompatibility even if A entails an inequality between the two variables. On the other hand, since logical equivalence holds between an arbitrary formula and the pnf obtained from it, and not simply mutual provability as in the case of Snfs, it is advantageous to formulate the criterion in terms of pnfs.

To meet the second point, the only change required when QA is in pnf is that condition (i) be replaced by:

(i′) $A \supset (x_i \neq y_j)$ for any i,j such that the existential quantifier in which x_i occurs precedes the universal quantifier in which y_j occurs in Q.

The occurrence of free variables in A is more of a problem. In nonformal contexts it is unlikely that formulas containing free variables would be presented as paradox-generators because of their ambiguity, but such formulas can arise in formal contexts (e.g., $(\exists x)(y)(y \in x \equiv \sim(y \in z))$ as a case of the abstraction schema in naive set theory) and they therefore have to be included in the general criterion. The question is whether or not such a formula should be classified as a reflexive contradiction if its matrix entails an inequality between two free variables or between a free variable and a bound variable. In the example given, the matrix entails $x \neq z$ and we have to decide whether or not it is a reflexive contradiction on the basis of this.

The fact that the matrix entails $x \neq z$ tells us that $(\exists x)(y)(y \in x \equiv \sim(y \in z))$ is a reflexive contradiction if x and z are instantiated with the same value. But such an instantiation is not permissible since if we move to $(y)(y \in x_0 \equiv \sim(y \in z))$, x_0 constant, we cannot now take x_0 as a value of z

since this amounts to generalizing on the free variable in a formula which has arisen by existential instantiation. Such a move would amount to treating the initial formula as if it were $(\exists x)(z)(y)(y \in x \equiv \sim(y \in z))$–which certainly is a reflexive contradiction in terms of C_2–but this is not derivable from the initial formula. If, on the other hand, we make the permissible generalization on z to get $(z)(\exists x)(y)(y \in x \equiv \sim(y \in z))$, we do not have a reflexive contradiction since no permissible instantiation presupposes $x = z$. Here the proviso on (i′) fails to be satisfied.

In general, if QA is such that A entails an inequality between a free variable and an e-variable, this is not incompatible with permissible instantiations. For if the e-variable is instantiated first, the value chosen cannot subsequently be used to instantiate the free variable since such a move would amount to generalizing on the free variable in a formula which has arisen by existential instantiation. On the other hand, if the free variable is instantiated first, the ordinary conditions on existential instantiation prohibit the use of the same value to instantiate the e-variable.

If we think of free variables as if they were u-variables bound by quantifiers which precede Q, then the conclusion that an implied inequality between a free variable and an e-variable is not incompatible with permissible instantiations follows immediately since the proviso on (i′) is never satisfied. By the same token, an implied inequality between a u-variable and a free variable or between two free variables is incompatible with permissible instantiation cases. So, for example, $(x)(y)(y \in x \equiv \sim(y \in z))$ and $(y)(y \in x \equiv \sim(y \in z))$ are reflexive contradictions since in either case we are entitled to take the same instantiation value for both x and z.

This is not to say, however, that in testing whether $QA(z)$ is a reflexive contradiction, where z is free, we should replace it by the surrogate formula $(z)QA(z)$. This would defeat the purpose of providing a criterion for formulas in pnf, since it involves trading in logical equivalence for mutual derivability, but it would also yield an incorrect condition in the case of an implied inequality between a free variable and a constant. For suppose A entails $z \neq c$, where z is free and c is a constant in QA. If z were a u-variable this inequality would be enough to determine QA as a reflexive contradiction; but where z is a free variable, this is so only in case c has not arisen in QA by previous existential instantiation.

We can now state the general criterion in terms of pnfs as follows:

C_3 Let QA be a formula in pnf such that $x_1, \ldots, x_n$ are e-variables, $y_1, \ldots, y_m$ are u-variables, $z_1, \ldots, z_k$ are free variables, and $c_1, \ldots, c_r$ are individual constants. As in C_2, QA may contain predicate parameters or constants, but no schematic letters.

Standard Case. Let D be a formula consisting of any single inequality or any disjunction of two or more inequalities in the set:

$\{(x_i \neq y_j)$, all i, j, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, such that $(\exists x_i)$ precedes (y_j) in Q;
$(y_i \neq y_j)$, $i \neq j$ and $1 \leqslant i,j \leqslant m$;
$(c_i \neq y_j)$, all i, j, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant m$;
$(z_i \neq y_j)$, all i, j, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$;

$(z_i \neq z_j)$, all i, j, $1 \leqslant i, j \leqslant m$;
$(z_i \neq c_j)$, all i, j, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant r$, provided c_j has not arisen in QA as the result of existential instantiation}.

Then QA is a reflexive contradiction, and consequently $\sim QA$ is a thesis, if $A \supset D$ is a thesis for any such D.

Limiting Case. Let D' be a single inequality of the form $a \neq a$, where a is any variable or constant. Then QA is a limiting case of a reflexive contradiction if $A \supset D'$ is a thesis for any such D'.

As before, since $A \supset D$ and $A \supset D'$ are quantifier-free formulas, we can apply a decision procedure to determine whether or not any such formula is a thesis for all the possible inequality conditions D or D' which can be formed from the specified set of inequalities. The final condition concerning inequalities between free variables and constants makes for a slight complication, but this can be met by introducing a tagging device to mark constants which arise by existential instantiation. Hence we can give a recipe for determining whether or not an arbitrary formula B satisfies C_3 by putting it into pnf QA and applying the procedure to all possible cases of $A \supset D$ and $A \supset D'$. If, in terms of this test, QA is a reflexive contradiction, then by the logical equivalence between QA and B, so is B. It should be noticed, however, that since no provision is made for descriptive terms in QA, it is necessary, before applying the test, to eliminate any definite description occurring in B by means of its contextual definition before putting B into pnf. For heuristic purposes descriptive terms can be treated as constants to test the application of C_3 (see the discussion of *Aut* in Section 1.3 below), but a strict application requires their elimination. The reason for this is explained in Section 4.4.

1.3 The proposal I now wish to put forward is that, for the standard cases, $\vdash A \supset D$ is a necessary and sufficient condition for an arbitrary formula in pnf QA to be a reflexive contradiction (correspondingly $\vdash A \supset D'$ for the nonstandard cases). This claim cannot be proved formally since we do not have an alternative and independently justified formal criterion for a formula to be a reflexive contradiction which can be proved equivalent to $\vdash A \supset D$. The position is, rather, that $\vdash A \supset D$ is being offered as a formal expression of the intuitive concept of a reflexive contradiction.

What can be proved is that if $\vdash A \supset D$ then $\vdash \sim QA$. The proof is essentially a generalization of the argument following the two-variable condition C in Part I, Section 2.4, taking into account the additional cases, and a nonformal outline of it has already been given in the previous section in the preamble leading up to C_3. But this is not a proof of sufficiency in the above sense, though it confirms the claim indirectly. Thus, the claim of intuitive sufficiency is: (1) if $\vdash A \supset D$ then QA is a reflexive contradiction; and we should also expect: (2) if QA is a reflexive contradiction, then $\vdash \sim QA$; hence $\vdash A \supset D$ then $\vdash \sim QA$. A direct proof of the latter, however, bypasses the intuitive claim.

Similar remarks can be made about necessity. Here the intuitive claim is: (1′) if QA is a reflexive contradiction then $\vdash A \supset D$. But we should not expect to prove the formal analogue: if $\vdash \sim QA$ then $\vdash A \supset D$ (or the missing premiss: (2′) if $\vdash \sim QA$ then QA is a reflexive contradiction), since this amounts to identifying every contradiction as a reflexive contradiction.

Direct proofs of intuitive sufficiency and necessity are not possible, therefore, but a refutation of the claim is possible in terms of counterexamples. And there are some simple cases which might seem to show that the claim is false, at least in respect of necessity. In fact, however, they fail to do so and instead illustrate different points.

Consider a formula $(y)A(y)$ where y is the only variable in A and no constants occur in A. Suppose that $\vdash A \supset (y \neq y)$ is not a thesis. Since this is the only relevant condition in terms of C_3, $(y)A(y)$ is not classified as a reflexive contradiction by C_3. Now suppose that $\vdash A \supset (y \neq c)$, where c is a constant not occurring in A; then $\vdash \sim(y)A(y)$. Moreover, this thesis arises directly from an incompatibility between the inequality $y \neq c$ implied by the matrix and the presupposed equality $y = c$ in a permissible instantiation case. Further, no other instantiations yield a formal contradiction, and indeed we may suppose that such instantiations are satisfied in some model. So, it may seem, $(y)A(y)$ under these assumptions is a standard case of a reflexive contradiction which is not covered by C_3. Hence $\vdash A \supset D$, for D defined in C_3, is not a necessary condition.

This fails as a counterexample, however, simply because there is no reflexivity involved. The case is no different from that which would arise if someone were to say "Everyone is happy" and then discover a person Tom who is unhappy. The assertion is thereby falsified, but it is not self-falsified. Consider, by contrast, the quite different situation which would arise if someone were to say "Everyone is happy" while being aware of the fact that he himself is unhappy. Here the reflexivity is in the context but it could be represented as part of the utterance since in effect what is being said is "Everyone is happy (including me), but I am unhappy". Once this is done, however, C_3 is satisfied. Reflexivity is characterized in C_3 by the requirement that all the variables and constants involved in the implied inequalities should also appear in the formula.

What is true about the proposed counterexample is that the other concomitant intuitive conditions which are satisfied by standard cases of reflexive contradictions (namely, an incompatibility between an implied inequality and a permissible instantiation, and a presumption that all other instantiations are true) are satisfied by it. Since it lacks the essential reflexivity, however, what the example illustrates is that these concomitant conditions are not themselves sufficient to characterize reflexive contradictions. Instead they characterize a wider class which includes reflexive contradictions and some near-relatives. These additional cases would need to be taken into account in a more comprehensive analysis of contradiction in terms of identity.

A different kind of case which might seem to count against necessity is the following version of Grelling's paradox (cf. Part I, Section 1.1(d)).

We suppose that there is an adjective Aut such that, for any adjective y, y has the feature described by Aut ($d(y,Aut)$) iff y has the feature described by y ($d(y,y)$); i.e., we adopt the meaning postulate

(i) $(y)(d(y,Aut) \equiv d(y,y))$.

Now introduce *Het* by

(ii) $(y)(d(y,Het) \equiv {\sim}d(y,Aut))$.

From (i) we get

(iii) $(y)({\sim}d(y,Aut) \equiv {\sim}d(y,y))$

and this together with (ii) gives the Grelling sentence

(iv) $(y)(d(y,Het) \equiv {\sim}d(y,y))$.

If we now apply C_3 in a heuristic fashion by treating *Aut* and *Het* as constants, then it is easily seen that neither (i) nor (ii) satisfies it. The matrix of (i) does not entail $y \neq Aut$, $y \neq y$, or $Aut \neq Aut$, the only relevant cases; and similarly the matrix of (ii) does not entail either $y \neq Het$ or $y \neq Aut$ or their disjunction, nor does it entail $a \neq a$ where a is a variable or constant in the formula. The matrix of (ii) does entail $Het \neq Aut$, but this is irrelevant since an implied inequality between two different constants is not incompatible with any permissible instantiation. The position then is that neither (i) nor (ii) is a reflexive contradiction in terms of C_3, yet together they imply (iv), which we know is a reflexive contradiction and which is in fact identified as such by C_3 since the matrix entails $y \neq Het$. Hence, if it is thought that either (i) or (ii) should be classified as a reflexive contradiction because they jointly imply a reflexive contradiction, then, since neither satisfies C_3, the condition $\vdash A \supset D$ (or $\vdash A \supset D'$) is not necessary.

But this example would count against the necessity of $\vdash A \supset D$ only if there were independent grounds for claiming that one or other of the meaning postulates (i) and (ii) is a reflexive contradiction. The fact that jointly they imply a reflexive contradiction is not of itself sufficient for making such a claim. What is justified in terms of the example is the weaker claim that the conjunction of (i) and (ii) should be classified as a reflexive contradiction. However this does not count against C_3 since in fact the conjunction is so identified. When (i) and (ii) are conjoined and the quantifier is brought to the front, the matrix entails $y \neq Het$. But there are no grounds for saying that because the conjunction of two formulas is a reflexive contradiction, so one or other or both of the components must be—any more than there are grounds for saying that because $p \;\&\; {\sim}p$ is a contradiction, so either p, ${\sim}p$, or both must be. To justify saying that either (i) or (ii) is a reflexive contradiction it would be necessary to show that a reflexive instantiation case of one or the other is a formal contradiction. But this requirement is not satisfied. Instantiating the universal quantifier in (i) with *Aut* is innocuous; and similarly, no contradiction arises if the quantifier in (ii) is instantiated with either *Aut* or *Het*.

However, the example does illustrate an important point, namely that C_3 stands as a test for determining whether or not a *particular* formula is a reflexive contradiction. In terms of it, (i) and (ii) are not; but C_3 is satisfied by the conjunctive formula (i) & (ii) and by the implied (iv), and it is these which *are* the reflexive contradictions, however, they arise, whether as consequences of (i) and (ii) or by direct stipulation. Thus C_3 is a test for individual formulas and as such guarantees nothing about their consequences. On the other hand, their consequences are themselves testable, so there is no weakness here. But the fact that reflexive contradictions can arise as consequences of formulas which are not themselves reflexive contradictions does raise some problems, especially in set theory. In set theory, too, the question of necessity is different. These points are taken up in Section 5.

2 A general criterion for Curry paradoxes It is now a simple matter to obtain a general criterion for Curry paradoxes in a form which illustrates their family resemblance to reflexive paradoxes.

2.1 Suppose for the moment we have a formula QA such that $\vdash A \supset (D \vee q)$ where D is as in C_3 and q is a sentential variable which does not occur in A. This case raises no particular problem since if we can show $\vdash A \supset (D \vee q)$, we can substitute any contradiction for q and hence show $\vdash A \supset D$; so C_3 is satisfied and QA is identified as a reflexive contradiction. Suppose now, however, that q does occur in A. The substitution of a contradiction for q in $\vdash A \supset (D \vee q)$ will not now lead to $\vdash A \supset D$, but instead to $\vdash A' \supset D$ where A' is a substitution instance of A in which all occurrences of q are replaced by the substituted contradiction. Hence, QA' is identified as a reflexive contradiction but QA is not. This is similar to the *Aut* case, and again it may not seem to be a problem since all such substitution instances, considered simply as particular formulas rather than as consequences of QA, are classified as reflexive contradictions by C_3. There is, however, an important difference between such cases and the *Aut* example due to the nature of the disjunction $D \vee q$.

To see this, consider the general two-variable case of a formula $(\exists x)(y)A(y,x)$ such that $\vdash A \supset. (y \neq x) \vee q$. As we have seen, the interesting case arises when q is a wf part of A, but the initial argument is independent of this. Thus, since $\vdash A \supset. (y \neq x) \vee q$, we have $\vdash A(x,x) \supset. (x \neq x) \vee q$; hence $\vdash A(x,x) \supset q$. But we also have $\vdash (y)A(y,x) \supset A(x,x)$, hence $\vdash (y)A(y,x) \supset q$. Consequently, $\vdash (\exists x)(y)A(y,x) \supset q$. Now if A does not contain q then as we have seen it is identifiable from the beginning as a reflexive contradiction; i.e., we have $\vdash \sim(\exists x)(y)A(y,x)$, and the conclusion we have arrived at is immediately derivable from $\sim p \supset (p \supset q)$. But if A does contain q, the conclusion $\vdash (\exists x)(y)A(y,x) \supset q$ still stands. Hence even though $(\exists x)(y)A(y,x)$ is not identifiable as a reflexive contradiction, its assumption leads immediately to absolute inconsistency. In this case, however, $(\exists x)(y)A(y,x)$ is certainly a paradoxical formula even though it is not a simple reflexive contradiction.

The limiting cases as defined in C_3 and extended to include $\vdash A \supset (D' \vee q)$ are now no longer trivial but instead lead to the same result. Thus we can pass immediately from $\vdash A(y,x) \supset. (a \neq a) \vee q$ to $\vdash A(y,x) \supset q$; hence $\vdash (\exists x)(y)A(y,x) \supset q$.

Similar considerations in respect of both standard and limiting cases apply, however many variables or constants occur in A. Hence we have in general,

C_4 Where QA, D and D' are as in C_3 and B is a sentential variable which occurs in A, QA is a Curry formula if either $A \supset (D \vee B)$ or $A \supset (D' \vee B)$ are theses.

This, too, provides us with a recursive test.

It should be noticed that no argument has been given to show that if QA is a Curry formula then $\vdash \sim QA$; and in general this will not be so. On the other hand if $\vdash \sim QA$ then $\vdash QA \supset B$. Hence, trivially, every reflexive contradiction is also a Curry formula.

In case B is a nonvalid sentential wff rather than a single variable, weaker kinds of Curry formulas arise in some instances, though in other cases QA is a straightforward Curry formula. For example, let $A(y,x)$ entail $y \neq x \vee (p \mathbin{\&} q)$; then $A(x,x)$ entails $p \mathbin{\&} q$, hence q, in which case $(\exists x)(y)A(y,x)$ is a Curry formula. On the other hand, if $A(y,x)$ entails $(y \neq x) \vee (p \vee q)$, then we have $\vdash (\exists x)(y)A(y,x) \supset (p \vee q)$, which is not reducible to a Curry formula by, say, substituting q for p, without changing A. Such formulas are, however, as equally damaging in their consequences as the simple Curry formulas. Similar considerations apply if B is any nonvalid wff which occurs as a wf part of A. Hence,

C_5 Where QA, D and D' are as in C_3 and B is a nonvalid sentential wff all of whose variables occur in A, or a nonvalid wff which is a wf part of A, then QA is a weak Curry formula if either $\vdash A \supset (D \vee B)$ or $\vdash A \supset (D' \vee B)$.

We therefore have the position that C_5 includes C_4 as a special case (where B is a single variable), C_4 includes C_3 (where B does not occur in A), and C_3 includes C_2 (where Q is regimented), which in turn includes C.

2.2 To illustrate the application of C_4 consider the formula,

$$(\exists x)(y)(f(y,x) \equiv (f(y,y) \supset q)).$$

A special case of this, namely $(\exists x)(y)(y \in x \equiv (y \in y \supset q))$, which arises as an instance of the naive abstraction schema, is shown to lead to a Curry paradox in Meyer, Routley, and Dunn [4]. It is identified as a Curry sentence by C_4 as follows:

$$\begin{aligned}
&(y = x) \supset (f(y,x) \equiv f(y,y))\\
&(y = x) \mathbin{\&} \sim q \mathbin{.\supset.} (f(y,x) \equiv f(y,y)) \mathbin{\&} \sim q\\
&\qquad\qquad\qquad . (f(y,x) \equiv .f(y,y) \mathbin{\&} \sim q)\\
&\qquad\qquad\qquad . (f(y,x) \equiv \sim(f(y,y) \supset q)\\
&\qquad\qquad\qquad . \sim(f(y,x) \equiv (f(y,y) \supset q));\\
&\text{i.e., } (f(y,x) \equiv (f(y,y) \supset q)) \supset (y \neq x \vee q).
\end{aligned}$$

This condition tells us that if we postulate any case of $(\exists x)(y)(f(y,x) \equiv (f(y,y) \supset q))$, with a predicate constant for f, we shall obtain a contradiction if we instantiate x and y with the same value and then substitute a contradiction for q. And this is so even though the formula itself is not contradictory; i.e., we do not have $\vdash \sim (\exists x)(y)(f(y,x) \equiv (f(y,y) \supset q)$; essentially, however, the moves which are made yield an instance which is a reflexive contradiction. Neither substitution step need be made, however, because we know from $\vdash (f(y,x) \equiv (f(y,y) \supset q)) \supset (y \neq x \vee q)$ and C_4 that we can pass straight to absolute inconsistency since $(\exists x)(y)(f(y,x) \equiv (f(y,y) \supset q)) \supset q$ is provable, by the argument preceding C_4.

3 The general criterion for a resolution The intuitive criterion R_1 given in Part I, Section 3.1, is vague in two respects. First because it contains a reference to C_1 in which the actual inequalities involved in a reflexive contradiction are not specified; secondly, because the actual modifications which have to be made to remove the value given to one variable from the range of a quantifier over another variable are not specified. The first point can now be met by formulating the criterion in terms of C_2–C_5. The second vagueness will remain, however, simply because there are alternative techniques which can be applied to achieve an appropriate quantificational restriction. At the general level, therefore, no precise uniform specification is possible. But given a restriction to a particular kind of technique, a precise recipe can be given. One such technique is examined in the following Sections 4 and 5.

There are variants of R_1 corresponding to each of the conditions C_2–C_5 but they do not differ in essentials. We can therefore formulate a general condition R_i in terms of C_i, $2 \leqslant i \leqslant 5$.

Let QA, D, and D' be as in C_i.

Consider first the case in which D is a single inequality between an e-variable and a u-variable, say $x_1 \neq y_1$ where $(\exists x_1)$ precedes (y_1) in Q. The contradiction involved in the reflexive case will arise in case QA is instantiated in such a way as to presuppose $x_1 = y_1$, i.e., by taking the same instantiation value for $(\exists x_1)$ and (y_1). To achieve this it is necessary to instantiate the existential quantifier first. To avoid the contradiction therefore, some technique has to be found such that the value chosen to instantiate the existential quantifier is no longer available as a permissible instantiation value for the universal quantifier. Thus, the value given to the e-variable must be removed from the range of the universal quantifier.

Similar considerations hold in respect of inequalities between free variables and u-variables, and between constants and u-variables. That is, in each case, the initial value chosen must be eliminated from the range of the relevant universal quantifier. In the different case in which there is an implied inequality between two u-variables, the order of instantiation is unimportant. In this case too, of course, every instantiation value of the first quantifier has to be removed from the range of the second—effectively, that is, the two quantifiers must be restricted to different sorts. Similarly, in the case of inequalities between two free variables, the removal of permissible instantiation cases which are incompatible with these inequalities effectively restricts the variables to different sorts.

The general condition which is required, therefore, in the case in which only one inequality occurs in D (neglecting Curry formulas for the moment) is:

(i) (a) Let D be of the form $a \neq b$ where b is a u-variable. Then, if a is a constant, or if a is a free variable, or if a is an e-variable where the quantifier over a precedes the quantifier over b in Q, or if a is a u-variable, the contradiction is removed if the value chosen to instantiate a (or the value a itself if a is a constant) is removed from the range of the quantifier over b.

(b) Let D be of the form $a \neq b$ where b is a free variable and a is either a free variable or a constant which has not arisen by existential instantiation in QA. Then the contradiction is removed if the value chosen to instantiate a (or the value a itself where a is a constant) is removed from the range of b.

It is not necessary to specify separately that in the case of an inequality between two u-variables or two free variables all possible instantiations on one variable have to be excluded from the range of the second, i.e., that the variables have to be restricted to different sorts, since this is implied by (i) and the ordinary conditions of instantiation. It is worth noting, however, that this implication holds only in such cases. Hence a general restriction of all variables involved in inequalities to different sorts represents overkill (e.g., type theory).

Consider now the case in which D is a disjunction of two or more inequalities, say D is $(x_1 \neq y_1) \vee (x_2 \neq y_2)$ where x_1, x_2 are e-variables and y_1, y_2 are u-variables. Here the contradiction involved in the reflexive case depends on two successive instantiations, one of which presupposes $x_1 = y_1$ while the other presupposes $x_2 = y_2$. Thus we have to take a common instantiation value for $(\exists x_1)$ and (y_1) and a different common value for $(\exists x_2)$ and (y_2)–the value has to be different since the value chosen to instantiate $(\exists x_1)$ cannot subsequently be used to instantiate $(\exists x_2)$. In the special case in which the same variable occurs in both disjuncts, e.g., D is $(x_1 \neq y_1) \vee (x_2 \neq y_1)$, the first instantiation involves taking a common value for $(\exists x_1)$ and (y_1) while the second involves taking a different common value for $(\exists x_2)$ and (y_1) (cf. the cyclic paradoxes, Part I, Section 2.5). In either case, however, since both instantiations are required to derive the contradiction, it is only necessary to block one of them in order to block the contradiction. And in general, whenever D contains two or more distinct disjuncts, only one need be considered and it is unimportant which one. A unique resolution can be given, however, by utilizing a lexicographical ordering and choosing the first inequality in D. Hence this case is already covered in (i) above.

Similarly, (i) also covers the case of Curry formulas (ordinary and weak) where the matrix A entails $D \vee B$ (not $D' \vee B$, see below). The reason for this is that the derivation of the contradiction, as well as the argument to show that if A entails $D \vee B$ then QA entails B (see the discussion preceding C_4), depend on an initial instantiation which is incompatible with an inequality in D. Hence these moves are blocked if restrictions are imposed which prevent such instantiations. We can therefore neglect the disjunct B and include Curry formulas of this type under (i).

These last cases in which D contains two or more disjuncts are quite different from those cases which arise if A entails two or more different disjunctions. For example, suppose we have $A \supset D_1$ and $A \supset D_2$. In view of the above, we can restrict attention to the case in which D_1 and D_2 are each single inequalities distinct from each other (though they may have a shared variable). A double condition of this kind will tell us that two independent contradictions are derivable from QA. Hence each has to be blocked separately by applying (i) twice. By contrast, if A entails a disjunction which contains two or more independent disjuncts, only one contradiction can be derived by making successive dependent instantiations. Hence only one application of (i) is necessary here.

Consider now cases for which $\vdash A \supset D'$, where D' is of the form $a \neq a$. We know that all instantiation cases of QA are formal contradictions. For if $A \supset (a \neq a)$, we also have $A \supset (b \neq b)$, etc., for each variable and constant in A. It would therefore be perverse to express a resolution of these cases along the lines suggested for standard cases, e.g., by saying that the contradiction will be removed if the value used to instantiate the quantifier over a (b, etc.) is removed from the range of the quantifier over a (b, etc.), for that is simply another way of saying that no value can consistently be given to any variable. Such contradictions can in fact be removed in terms of particular techniques (see Section 4 below) but in a trivial and uninteresting way. We shall not therefore include them in the general form of resolution.

In the case of Curry formulas which are such that $\vdash A \supset (D' \vee B), D'$ can be eliminated and we have $\vdash A \supset B$. Nontrivial resolutions can be given for these cases, though not in terms of quantificational restrictions. These too, therefore, will be excluded from the general form and considered separately within the framework of a particular technique.

The general condition is then as follows:

R$_\mathbf{i}$ *Standard Case*. Let QA and D be as in $\mathrm{C_i}$ where $\vdash A \supset D$, and let $a \neq b$ be the first disjunct in D under a lexicographical ordering.

(a) If b is a u-variable and either
 (i) a is a constant
or (ii) a is a free variable
or (iii) a is a u-variable
or (iv) a is an e-variable where the quantifier over a precedes the quantifier over b in Q;
(b) If b is a free variable and either
 (i) a is a free variable
or (ii) a is a constant which has not arisen in QA by existential instantiation;

then the contradiction is removed if the value chosen to instantiate a (or the value a itself if a is a constant) is removed from the range of b.

Repeated Standard Case. Let $\vdash A \supset D_1$, $\vdash A \supset D_2, \ldots, \vdash A \supset D_n$, where none of $D_1, \ldots, D_n$ contain a common disjunct (though they may have shared variables). Then the contradictions arising from QA are removed if the standard case is applied separately in respect of each distinct D_i.

Modified Standard Case. Where $D_1, \ldots, D_n$ each have at least one disjunct in common, only one such disjunction need be considered provided the standard case is applied in respect of the common disjunct (or any one common disjunct if there is more than one) whether or not it is first in lexicographical ordering in any D_i.

4 Resolution by redefinition I now want to look at one particular application of $\mathrm{R_i}$ where the value chosen to instantiate one variable is removed from the range of a second by means of an explicit formal restriction in a quantifier or the matrix. This seems to be the most natural and obvious way of applying

R_i. Given classical quantification theory in which restricted quantifiers have to be introduced by means of antecedent conditions, we can confine ourselves to modifications in the matrix. That is, if QA is a reflexive contradiction or a Curry formula, this application consists in constructing formulas of the form $Q(S \supset A)$ where S is the appropriate restriction. Any formula which entails $Q(S \supset A)$, e.g., $Q(A \;\&\; S)$, will also be regarded as a resolution by explicit restriction.

Any technique which consists in replacing QA by QA^* where A^* is a modified form of A will be called a resolution by redefinition. The reason for calling this kind of technique a resolution by redefinition is obvious in terms of the simple cases. Thus, let QA be $(\exists x)(y)(c(y,x) \equiv {\sim}c(y,y))$ where c is the predicate constant 'is catalogued in'. Given the usual context the assumption is that the existential quantifier is satisfied uniquely and what is being presented, therefore, is an implicit definition of a unique individual C (the catalogue of all catalogues which do not list themselves) where C is $(\iota x)(y)(c(y,x) \equiv {\sim}c(y,y))$ (Part I, Section 1.1). Hence if the contradiction is removed by replacing QA by, say, $(\exists x)(y)(S(y) \supset (c(y,x) \equiv {\sim}c(y,y))$, where $S(y)$ is some condition on the universal quantifier, a different individual C^* is being implicitly defined by $(\iota x)(y)(S(y) \supset (c(y,x) \equiv {\sim}c(y,y))$. That is, C is being redefined.

This bears on a question raised at the beginning, namely what exactly does a resolution amount to since the removal of reflexive contradictions cannot change paradoxical items such as the catalogue, the barber (B), the Russell class (R), etc., from inconsistent concepts to consistent concepts. Since we have a formal proof that C, B, R, etc., do not exist, there can be no technique which will enable us to define *these* items in such a way that it is no longer inconsistent to postulate their existence. If, for example, C is defined by $(\iota x)(y)(c(y,x) \equiv {\sim}c(y,y))$, then it does not exist and it *is* inconsistent to postulate its existence; and if it is not so defined then it is not C. Hence if redefinition is the aim, the most we can hope for is a definition of a different catalogue C^* which, so far as entries are concerned, is as close to C as it is possible to be without running into inconsistency. That is, C's counterpart C^* must list as many as possible of those catalogues which C was intended to list while satisfying the condition that it is not inconsistent to affirm the existence of C^*. To require that it should not be inconsistent to affirm the existence of C^*, however, is not the same as saying that we should be committed by the resolution to affirming its existence. This distinction relates to the earlier discussion of plausibility (Part I, Section 2.2).

Thus it was suggested in Part I that, in the case of the simple paradoxes at least, since the postulational definitions of C, B, R, etc. have the same formal structure, any resolution should apply equally to all the paradoxes. Hence if redefinition is to provide a uniform resolution, there must be a general technique for defining counterparts P^* of paradoxical items P (defined by $(\iota x)(y)(f_0(y,x) \equiv {\sim}f_0(y,y))$ for a given predicate constant f_0) such that $E!P^*$ is not inconsistent. But there is a difference between a definition of P^* such that $E!P^*$ is not inconsistent and a definition such that $E!P^*$ is true. Given that $E!P^*$ is not inconsistent, we may perhaps wish to affirm it; but that is a separate matter which has to be separately motivated. For if there is a general technique for constructing consistent definitions of counterparts, we shall be able to define, say, L^*, a counterpart of L (the ultimate left-most object) such that $E!L^*$ is consistent; but there could be no motivation for affirming $E!L^*$. The existence of a neoultimate left-most object can be ruled out as false by meaning even if the assumption of its existence is no longer formally inconsistent by virtue of the reflexive case. Hence, as was noted earlier, it should not be expected that whatever technique is employed to remove the formal inconsistency should result in a modified sentence which is true. This would be to conflate the problem of paradoxicality with that of plausibility.

If we now add to the condition that $E!P^*$ be not inconsistent the further requirement that P^* be as close to P as it is possible to be without being inconsistent, then we are effectively requiring a minimal resolution, i.e., a resolution which eliminates just the contradictory instantiation cases of the original QA and no others. That is, the permissible instantiation cases should form a maximally consistent set when P is replaced by P^*.

We now consider one general form of explicit restriction and then return to the simple cases to illustrate its application.

4.1 If the value chosen to instantiate a quantifier over a is to be removed from the range of a quantifier over b, the simplest form of antecedent restriction is obtained by taking the antecedent to be $a \neq b$, i.e., by taking just that inequality which creates the need for a restriction in the first place. This, of course, is Frege's resolution; but to meet the conditions for a general resolution it has to be more complex than the form in which Frege gave it.

Consider first the case where QA is such that A entails a single inequality $a \neq b$ which satisfies the conditions of the standard case in $\mathrm{R_i}$. Let A' be the contradictory instantiation case obtained from A by taking the same value for a and b, say a_0, and arbitrary permissible instantiations for the other variables. A' is contradictory since $A' \supset (a_0 \neq a_0)$. No other instantiations are contradictory, however (though they may be false by meaning). Consider now $Q(a \neq b. \supset A)$. By making the same instantiations which led to the original contradiction, we obtain $a_0 \neq a_0. \supset A'$. This is trivially valid and no other instantiations are contradictory (though they need not be valid). For let A'' be any other instantiation case of QA obtained by taking say a_0 for a, b_0 for b, and arbitrary permissible values for the other variables. Then since A'' is not contradictory, neither is $a_0 \neq b_0. \supset A''$. We thus have a minimal resolution for this case since just exactly that instantiation which gives rise to the original reflexive contradiction has been blocked.

In case $A \supset D$, where D is a disjunction of inequalities, the first of which is, say, $a \neq b$, the antecedent restriction $a \neq b$ is again sufficient to block the contradiction (cf. $\mathrm{R_i}$). Here, the instantiation case A' obtained by taking a_0 for both a and b and arbitrary permissible values for the other variables is not now a contradiction but instead an essential premiss in a chain argument which leads to a contradiction. Hence since the corresponding instantiation case for $Q(a \neq b. \supset A)$ is $a_0 \neq a_0. \supset A'$, A' can no longer be detached and used as a premiss. As before, other instantiation cases remain acceptable even though some of them also stand as essential premisses in the chain argument. To eliminate all such premisses QA has to be replaced by $Q(D \supset A)$, but this takes us beyond a minimal resolution. The similar case in which A entails $D_1, \ldots, D_n$, where each D_i contains a common disjunct, is met by taking just that disjunct as an antecedent restriction.

In case A entails $D_1, \ldots, D_n$ such that none of the D_i contain a common disjunct, it is necessary to select one disjunct from each (which can be specified as the first in each in lexicographical order) and to take their conjunction as the antecedent restriction. For here, contradictions arise independently in respect of each D_i.

With this type of resolution, it is possible to cover the nonstandard cases in a similar manner. Thus, let $A \supset (a \neq a)$. Then we have a trivial resolution if we replace QA by $Q(a \neq a. \supset A)$. Less trivial, however, is the nonstandard case of a Curry formula. For if $A \supset .(a \neq a) \vee B$, where B is a sentential wff or a wf part of A, we have $A \supset B$. Here a minimal resolution is obtained if we replace QA by $Q(B \supset A)$. This does not restrict the quantification and it is therefore a limiting case.

Thus the general form of a Fregean resolution is as follows:

FR$_i$ *Standard Case*. Let QA and D be as in C_i where $\vdash A \supset D$, and let $a \neq b$ be the first disjunct in D in lexicographical order. Then the contradiction is removed if QA is replaced by $Q(a \neq b. \supset A)$.

Repeated Standard Case. Let $\vdash A \supset D_1$, $\vdash A \supset D_2, \ldots, \vdash A \supset D_n$, where none of $D_1, D_2, \ldots, D_n$ contain a common disjunct. Let $a_1 \neq b_1$, $a_2 \neq b_2$, $\ldots, a_n \neq b_n$ be respectively the first terms in lexicographical order in D_1, $D_2, \ldots, D_n$. Then the contradictions are removed if QA is replaced by $Q((a_1 \neq b_1) \;\&\; (a_2 \neq b_2) \;\&\; \ldots \;\&\; (a_n \neq b_n). \supset A)$.

Modified Standard Case. Where $D_1, \ldots, D_n$ each have at least one disjunct in common, select one such common disjunct, say $a \neq b$. Then the contradiction is removed if QA is replaced by $Q(a \neq b. \supset A)$.

Nonstandard Cases

(i) Let $A \supset D'$, where D' is a single inequality of the form $a \neq a$. Then the contradiction is removed if QA is replaced by $Q(a \neq a. \supset A)$.

(ii) Let $A \supset (D' \vee B)$, where B is a sentential wff all of whose variables occur in A or a wff which is a wf part of A. Then the contradiction is removed if QA is replaced by $Q(B \supset A)$. (Ordinary (and weak) Curry formulas which are such that $A \supset .D \vee B$ are included in the standard case above—see the discussion preceding R_i).

Where QA is a reflexive contradiction or a Curry formula and S is a restrictive antecedent condition determined by FR_i, the modified formula $Q(S \supset A)$ will be called a Fregean counterpart of QA. The application of C_i followed by FR_i therefore provides us with a recursive technique for constructing Fregean counterparts.

4.2 To construct a Fregean counterpart B of a given formula A is not thereby to construct an acceptable formula. Whether or not B is acceptable (in the sense that it can be affirmed) depends on whether or not the original reflexive contradiction A is plausible. We therefore have to provide a general condition for plausibility to distinguish the cases. However, plausibility is an appropriate concept only in the case of formulas which contain predicate constants and no predicate parameters, no free individual variables and no sentential variables. Consider, for example, the formula A_0, $(\exists x)(y)(f(y,x) \equiv {\sim}f(y,y))$, which satisfies C_2. Applying FR_2 we construct the Fregean counterpart B_0, $(\exists x)(y)(x \neq y . \supset .f(y,x) \equiv {\sim}f(y,y))$. The condition for plausibility (Part I, Section 2.2) requires us to put meaning postulates on f and to determine whether this set of postulates $M(f)$ is consistent with A_0 independently of the reflexive case. But this is not possible where f is a parameter. On the other hand, since f is standing in for arbitrary predicate constants, to ask whether A_0 is plausibie is effectively to ask whether $\{M(f), A_0\}$ is consistent, when the reflexive case is excluded, for all possible substitutions of predicate constants for f. This, however, is the same as asking whether B_0 is an acceptable formula for all possible substitutions on f, since B_0 excludes just the reflexive case. But this will be so iff B_0 is a thesis of quantification theory. Hence we may say that A_0 is plausible just in case its Fregean counterpart B_0 is a thesis. Notice, however, that although A_0 is implausible in terms of this criterion, special cases of it which contain a predicate constant in place of the parameter are nevertheless plausible, for example, $(\exists x)(y)(c(y,x) \equiv {\sim}c(y,y))$, since we should expect $M(c)$ to be consistent with it if the reflexive case is excluded. On the other hand, other cases such as $(\exists x)(y)(r(y,x) \equiv {\sim}r(y,y))$, where '$r$' is 'to the right of', are implausible.

Similar considerations apply to formulas which contain free variables and sentential variables. In the case of formulas which contain both predicate constants and predicate parameters, the question of plausibility is essentially the question of whether the counterpart formula is a thesis if the predicate constants are replaced by parameters.

This extension in the concept of plausibility is required only for the sake of completeness in stating the general criterion. The interesting cases are those which contain no predicate parameters, free variables, or sentential variables. The general criterion is as follows:

PFR$_i$ Let A satisfy C_i and let B be a Fregean counterpart of A.

Standard Case. A contains predicate constants $f_0, \ldots, f_n$, no predicate parameters, no free variables, and no sentential variables. Then A is plausible and B is acceptable iff $\{M(f_0, \ldots, f_n), A\}$ is consistent when the reflexive instantiation case(s) of A is (are) excluded; otherwise A is implausible and B is unacceptable.

Nonstandard Case. A contains at least one predicate parameter, or one free variable, or one sentential variable. Then A is plausible and B is acceptable iff $\vdash B'$, where B' is obtained from B by replacing such distinct predicate constants as occur in B by distinct predicate parameters not already occurring in B; otherwise A is implausible and B is unacceptable.

The actual application of PFR$_i$ will of course be limited since in many cases appropriate meaning postulates cannot be given. Hence the determination of plausibility is not characterized by a recursive procedure.

4.3 There are a number of differences between FR_i and Frege's own resolution which we now examine.

Note first that FR_i depends on a prior application of C_i, and C_i is a test which applies only to individual formulas. Moreover, the expression to be tested must be a genuine formula (or sentence), i.e., it must not contain schematic letters. Hence FR_i does not sanction the introduction of modified schemata.

The reason for this is easy to see. We know, for example, that any formula of the form $(\exists x)(y)(A(y,x) \equiv {\sim}A(y,y))$ will be a reflexive contradiction since the matrix will entail $y \neq x$ whatever formula A is. But suppose we now apply FR_i directly to this schema to obtain $(\exists x)(y)(y \neq x. \supset A(y,x) \equiv {\sim}A(y,y))$. Here we have achieved nothing. We cannot say, for example, that any formula of this form is not a reflexive contradiction. For we can choose a case of A, say A', which contains variables (free or bound) which are different from x and y and is such that the matrix $(y \neq x. \supset .A'(y,x) \equiv {\sim}A'(y,y))$ entails another inequality which satisfies the conditions of C_i. Hence the antecedent restriction applied to the original schema is insufficient to block all reflexive contradictions which can arise from it for suitable choices of A. By contrast, where we have a particular formula in this form, say $(\exists x)(y)(f(y,x) \equiv {\sim}f(y,y))$, then the antecedent restriction $y \neq x$ is sufficient to block the contradiction. For since f is a parameter standing in for two-place predicate constants only, and is not a schematic letter, "extra" variables cannot be introduced as they can in the case of the schema. Moreover, other cases of the original schema which do contain "extra" variables are themselves dealt with as individual formulas in terms of C_i and adequate antecedent restrictions can be imposed if their matrices entail other inequalities.

This is the first way in which FR_i differs from Frege's resolution. Frege's proposal was of course limited to a resolution of Russell's paradox, and in that respect it is less comprehensive than FR_i, but in another respect it is more comprehensive since in effect it took the form of imposing an antecedent restriction on a schema for this limited case.

As is well known, Frege's resolution was initially a proposal to modify the identity condition on classes, $(\hat{x}(\phi x) = \hat{z}(\psi z)) \equiv (y)(\phi y \equiv \psi y)$, by restricting the quantifier in such a way that y does not take $\hat{x}(\phi x)$ as a value in ϕy or $\hat{z}(\psi z)$ as a value in ψy. However, the resolution is usually expressed by a formal restriction on the naive abstraction scheme, $(y)(y \in \hat{x}(\phi x) \equiv \phi y)$; in particular by reformulating it as $(y)(y \neq \hat{x}(\phi x). \supset .y \in \hat{x}(\phi x) \equiv \phi y)$ or in the stronger form $(y)(y \in \hat{x}(\phi x) \equiv .\phi y \,\&\, y \neq \hat{x}(\phi x))$. We need only consider the weaker form since consequences of it are consequences of the stronger. Given the Frege-Russell substitution conventions on the letters ϕ, ψ, etc., the letters cannot simply be regarded as parameters which are standing in for one-place predicate constants. On the contrary, an expression such as 'ϕy' stands in for arbitrary wffs which contain y free. Thus they are schematic letters and the identity principle and the abstraction principle are rightly regarded as schemata. Hence, in view of what was said above, we should expect that for suitable choices of ϕ the matrix of the modified abstraction schema will entail other inequalities which satisfy C_i and which are not part of the antecedent restriction. That is, we should expect to be able to generate other reflexive contradictions from the modified schema. This relates to the demonstration by Quine [5] and Geach [1] that the modified abstraction schema is inconsistent.

The second way in which FR_i differs from Frege's resolution is connected to the first. For since the modification is made to a schema, particular cases of the original schema which are not reflexive contradictions, and which therefore require no modification, nevertheless now appear as modified. For since the schema is modified, so are all instances of it. Thus we do not have a minimal resolution. By contrast, FR_i permits only the modification of just those instances which need to be modified; all other instances remain unaffected.

So far as these two points are concerned, therefore, the important differences between FR_i and the application of Frege's own resolution in the particular area of set theory, is that the former provides us with a case-by-case resolution, where the modification can differ from case to case, while the latter provides us with a uniform modification on all cases. The uniform modification is inadequate, since it fails to eliminate all reflexive contradictions; and at the same time it is too comprehensive since it results in the modification of formulas which are not contradictory. However, these considerations are mainly of importance when examining what needs to be done to eliminate reflexive contradictions in naive set theory. In other contexts, the question of modifying schemata does not arise. We shall therefore postpone further discussion of these points to Section 5. There is, however, a further difference between FR_i and Frege's own resolution which has a wider application. It arises from the treatment of descriptive terms in C_i.

4.4 In terms of C_i definite descriptions have to be eliminated before the test for a reflexive contradiction is applied and hence before FR_i is applied. There is of course a strong temptation to treat them as constants, but we now show that if C_i contained a condition for descriptive terms analogous to the condition on constants, then FR_i would fail to block the contradiction even in the simple cases.

As before, let C, the catalogue of all catalogues which do not list themselves, be $(\imath x)(y)(c(y,x) \equiv {\sim}c(y,y))$, and consider the standard contradictory assumption,

$$\boldsymbol{\alpha}^0 \qquad (y)(c(y,C) \equiv {\sim}c(y,y)).$$

Treating C as a constant for the purposes of applying C_i and FR_i we obtain the Fregean counterpart,

$$\boldsymbol{\alpha}^1 \qquad (y)(y \neq C. \supset c(y,C) \equiv {\sim}c(y,y)),$$

since the matrix of α^0 entails $y \neq C$. Given the *PM*-theory of descriptions, however, α^1 is inconsistent. This is immediate if C has a primary occurrence in α^1 since α^1 is then of the form $\psi((\imath x)\phi x)$ which entails $E!(\imath x)\phi x$ (*PM* *14.21); i.e., α^1 entails $E!C$, but we already know that ${\sim}E!C$ is a thesis (Part I, Section 1.1(a)). If C has a secondary occurrence, the argument is more complex, but there is a simpler argument which applies independently of whether C has a primary or secondary occurrence. The argument has the further advantage of showing that a conditional instantiation rule for descriptions, $(y)Ay \supset (E!(\imath x)Bx \supset A(\imath x)Bx)$ (cf. *PM* *14.18), does not provide adequate protection.

Conditional instantiation does block the obvious route. Thus we cannot employ ordinary instantiation to move from α^1 to $(C \neq C. \supset (c(C,C) \equiv \sim c(C,C))$, hence $C = C$, i.e., $E!C$ (*PM* *14.28). Instead we have $E!C \supset (C \neq C. \supset (c(C,C) \equiv \sim c(C,C))$, which is innocuous. However, from α^1 we have $(y)(y \neq C) \supset (y)(c(y,C) \equiv \sim c(y,y))$. But $(y)(y \neq C)$ is equivalent to $\sim E!C$ (*PM* *14.204), and we have $\sim E!C$ as a thesis. Hence, $(y)(c(y,C) \equiv \sim c(y,y))$; but this is α^0.[1]

Similar considerations apply in respect of all the usual paradoxical items which give rise to the simple paradoxes. That is, α^1-type modifications of α^0-type assumptions are inconsistent with *PM* description-theory. In effect, such modifications provide no resolution at all since they simply lead back to the original paradoxical assumption. In particular, this is true of the Russell class R, $(\imath x)(y)(y \in x \equiv \sim(y \in y))$. That is,

$$(y)(y \neq R. \supset (y \in R \equiv \sim(y \in y)))$$

fails as a resolution. But now consider Frege's own resolution of the class paradox. As a special case of the modified abstraction schema we have,

F $\quad (y)(y \neq \hat{x}(\sim(x \in x)). \supset (y \in \hat{x}(\sim(x \in x)) \equiv \sim(y \in y))$,

where $\hat{x}(\sim(x \in x))$ is the Russell class. That is, given our use of descriptive terms instead of class abstracts, it is in exactly the unacceptable form above. Hence even in this particular case of the modified abstraction schema, FR_i provides no analogue of Frege's own resolution.

This is not to say that Frege's own resolution is inconsistent in the particular case of the Russell paradox, as well as being inconsistent in other cases. For one thing, there is no reason to require class abstracts to satisfy the conditions of *PM* description-theory. Thus we need not be committed to $(\alpha = \alpha) \equiv E!\alpha$ where α is an abstract; in which case we can allow the instantiation move on F to the conclusion $\hat{x}(\sim(x \in x)) = \hat{x}(\sim(x \in x))$. Alternatively, there is no reason to be committed to a *PM* theory of descriptions; in which case we can take abstracts to be descriptions and block the above argument elsewhere. Frege's own theory of descriptions would be satisfactory in this respect since an improper description, which denotes no ordinary element of the domain, denotes the null element. So, for example, we have $(\exists y)(y = C)$; and this being so the move from $(y)(y \neq C) \supset (y)(c(y,C) \equiv \sim c(y,y))$ back to α^0 is stopped, though conditional instantiation is still required to block the earlier moves (cf. Scott [6]).

The main point of these remarks is simply to emphasize the significant differences between the application of FR_i and Frege's own resolution, even though they are similarly motivated and have superficial similarities. Both C_i and FR_i could be simplified either by dispensing with descriptions altogether or by adopting a different theory which allows for them to be treated as constants. But this would be to draw a veil over some important issues. For one of the main components of the paradoxes is the fact that we can construct descriptions which are not satisfied by any existing items; in such cases their behaviour is different from that of constants and it is a main virtue of *PM*-theory that it characterizes this difference.

4.5 The above difficulties do not arise in the case of generalized forms of the paradoxes. For example, if we begin with the assumption $(\exists x)(y)(c(y,x) \equiv \sim c(y,y))$ in place of $(y)(c(y,C) \equiv \sim c(y,y))$, then FRi supplies the counterpart $(\exists x)(y)(y \neq x. \supset (c(y,x) \equiv \sim c(y,y)))$. If we now instantiate to obtain $(y)(y \neq x_0. \supset (c(y,x_0) \equiv \sim c(y,y))$, then although we have a formula which has the structure of α^1, it cannot be α^1, i.e., x_0 cannot be C, for the conditions on instantiation require x_0 to be a constant, i.e., a term which designates an element of the domain, or imposing uniqueness, a description which designates an element of the domain (which C does not). Assuming the latter, then x_0 is $(\iota x)(y)(y \neq x. \supset (c(y,x) \equiv \sim c(y,y))$, i.e., it is that catalogue which, excluding itself from consideration as an eligible candidate, lists all and only those catalogues which do not list themselves. That is, x_0 is C's counterpart C^* which satisfies the condition $(y)(y \neq C^*. \supset (c(y,C^*) \equiv \sim c(y,y))$. Thus C^* is not an "ordinary" catalogue. For the antecedent condition imposes the requirement that C^* itself be *excluded from consideration as a possible candidate for listing or nonlisting* in C^*. That is, C^* cannot be considered for listing in C^* and it cannot be considered for exclusion from C^*. Hence it is a catalogue of a different kind from those catalogues which are eligible for consideration. It is not that C fails to be an "ordinary" catalogue, for C is not a catalogue at all. The temptation to affirm the existence of C, and the temptation to think that we can make a start on compiling it, derives from the fact that we *can* compile C^*. And the reason for this is simply that the description C^* has the antecedent restriction built into it, whereas C does not. The conditions on C^* require us to exclude it from consideration, but the conditions on C do not. And although α^1, $(y)(y \neq C. \supset (c(y,C) \equiv \sim c(y,y)))$, looks as if it is imposing such a condition on C, in fact it is not because C remains as the inconsistent $(\iota x)(y)(c(y,x) \equiv \sim c(y,y))$. Thus, α^1 does nothing to change the conditions on C; in particular, it does not redefine it. That is why it fails as a resolution. But what α^1 is trying to express is consistent, namely

$$\boldsymbol{\alpha^2} \qquad (y)(y \neq C^*. \supset (c(y,C^*) \equiv \sim c(y,y))),$$

where C^* is $(\iota x)(y)(y \neq x. \supset (c(y,x) \equiv \sim c(y,y)))$.

Since C^* is defined in this way, α^2 has the form $\phi((\iota x)\phi x)$ (unlike α^1 which has the form $\psi((\iota x)\phi x)$, given that $(\iota x)\phi x$ has a primary occurrence); but this is equivalent to $E!(\iota x)\phi x$ (*PM* *14.22), i.e., α^2 is equivalent to $E!C^*$. Hence to affirm α^2 just is to affirm $E!C^*$. Nothing of what has been said so far, however, commits us to this affirmation. The above discussion shows only that α^2 is consistent, i.e., that it offers a genuine resolution as opposed to the spurious resolution offered by α^1. Given that the original paradox is plausible, then we can affirm α^2 since $E!C^*$ is precisely what we want to affirm. However, since all α^2-type resolutions are equivalent to $E!P^*$, where P^* is the counterpart of the original paradoxical item P, they cannot all be affirmed. In particular, they cannot be affirmed if the original paradox is implausible.

As we have noted, the criterion which has been given for plausibility is often inapplicable since it is difficult to provide appropriate meaning postulates. But in the case of the simple paradoxes at least, we now have an alternative since the question of whether or not the original paradox is plausible is the question of whether or not the corresponding α^2-type formula is acceptable. Hence if we can provide a model which satisfies the appropriate α^2-condition, then we have shown that the paradox is plausible.

In the case of the catalogue, there is an obvious model for the elementary form of order theory which α^2 imposes. Let there be two kinds of catalogues each of which lists books and only books: those in book form (book-catalogues) and those in card-index form (index-catalogues). Let some of the book-catalogues list themselves as books in the library and let some fail to list themselves. If we now propose to compile a catalogue C^* of all those book-catalogues which fail to list themselves, α^2 imposes the requirement that C^* be an index-catalogue since it then fails to be an eligible candidate for consideration as a listable or nonlistable item in C^* because it is not itself a book.

In general, a resolution of the simple paradoxes in terms of $\mathrm{FR_i}$ imposes the requirement that the domain over which the quantifiers range be partitioned into two mutually exclusive and nonempty subdomains, in effect into two categories or kinds of individuals, though the quantifiers range unrestrictedly over the whole domain in the sense that any individual in either of the subdomains can be taken as an instantiation value for any quantifier. Thus we have a many-sorted theory without many-sorted quantifiers.

4.6 These conditions on the domain can be made explicit in terms of other variants of a resolution by redefinition, in particular by variants which impose an antecedent predicate-restriction. Thus, let B_0 be the predicate '. . . is a book catalogue' and consider $(\exists x)(y)(B_0 y \supset (c(y,x) \equiv {\sim}c(y,y)))$. Instantiating, we have as an analogue of α^2,

$$\alpha^3 \qquad (y)(B_0 y \supset (c(y,x_0) \equiv {\sim}c(y,y))),$$

i.e., provided y is a book catalogue, it is listed in x_0 iff it is not listed in itself. Given uniqueness, x_0 is that counterpart of C, C^{**}, defined by $(\imath x)(y)(B_0 y \supset (c(y,x) \equiv {\sim}c(y,y)))$. Instantiating again we have ${\sim}B_0(x_0)$; hence ${\sim}(y)B_0 y$. Thus if α^3 is affirmed, we are committed to the conclusion that not everything in the domain is a book catalogue, and in particular, that x_0 is not. If the domain were restricted to book catalogues, we should be forced to ${\sim}\alpha^3$, i.e., ${\sim}E!x_0$.

Since to affirm α^3 is to characterize the original paradox as plausible, it follows that plausibility depends on there being a suitable or "natural" partitioning of the domain. If the domain is homogeneous, α^3 cannot be affirmed. This relates to a problem raised in Part I, Section 1.2, where it was noted that a simple antecedent restriction to catalogues is inconsistent with restricting the domain to catalogues. We can restrict the domain to catalogues iff the antecedent restriction is expressed in terms of a subset of catalogues. In general, in the case of the simple paradoxes, resolutions by redefinition which take the form $(\exists x)(y)(Sy \supset A(y,x))$, where $(\exists x)(y)A(y,x)$ is the original paradoxical formula, always impose the requirement that the domain admits of a suitable partitioning into the two nonempty subsets $\{y: Sy\}$

and $\{y\colon \sim Sy\}$. If this requirement cannot be met, the original paradox is implausible.

That there are α^3-type resolutions of the other familiar paradoxes is well known. For the Russell paradox, we can adopt the familiar course of taking the domain to be all classes, partitioned into sets and proper (ultimate) classes, and then take the restricting predicate to be '. . . is a set'. Then R^{**}, the counterpart of R defined by $(\imath x)(y)(Set\ y \supset .(y \in x \equiv \sim y \in y))$, is a nonset. Note, however, that as in the case of the catalogue, this kind of resolution will not survive a restriction of the domain to sets, for we then simply restore the original form of the paradox, now expressed in terms of R^{**} instead of R, and conclude $\sim E!R^{**}$.

In some cases there is more than one choice of partitioning. Thus, for the barber we can restrict the domain to villagers (as in the original paradox) but partition it into those who are clean-shaven and those who are not and take the restricting antecedent predicate to be '. . . is clean-shaven'. Then B^{**}, the counterpart of B, shaves all and only those clean-shaven villagers who do not shave themselves, while he himself turns out to be bearded. Alternatively, we can extend the domain of the original paradox to include nonvillagers and take the restricting predicate to be '. . . is a villager', in which case B^{**} is a visiting barber.

What is obvious about such resolutions, however, is that the paradox is avoided by changing the conditions in terms of which it was originally expressed. That is, we set a new scene which differs essentially from the old and for that reason alone is nonparadoxical. The old problem is not solved, it is simply replaced by a new situation which presents no problem. This is really what is meant by saying that nothing can change the fact that the paradoxical items P postulated by the simple paradoxes do not exist and that any closely resembling nonparadoxical situation invokes counterparts which cannot, in any circumstances, be P.

In general, then, modified α-instances like α^3 provide us with a case-by-case resolution of the simple paradoxes but they impose the following conditions:

(a) they introduce a counterpart of the original paradoxical item
(b) they are domain-relative in the sense that they fail as resolutions if the domain is restricted in an appropriate way
(c) they require the domain to be partitioned into a minimum of two nonempty subsets such that the counterpart is an element of one subset and the items to which it is related are elements of the other. That is, where x_0 is the counterpart in question and f_0 is the predicate constant in a particular case, the minimum requirement is that $x_0 \notin \{y\colon f_0(y,x_0)\}$ and $\{z\colon z \notin \{y\colon f_0(y,x_0)\}\} \neq \phi$.

Given $x_0 \notin \{y\colon f_0(y,x_0)\}$ and $E!x_0$, the second condition $\{z\colon z \notin \{y\colon f_0(y,x_0)\}\} \neq \phi$ is satisfied. And given $\sim(\exists z)f_0(x_0,z)$, then $x_0 \notin \{y\colon f_0(y,x_0)\}$ is satisfied; for if x_0 does not stand in the relation f_0 to any individual, then it does not stand in the relation f_0 to itself. Hence if we can guarantee $\sim(\exists z)f_0(x_0,z)$ and $E!x_0$ we shall satisfy the requirements of an α^3-type resolution of the simple paradoxes. But both conditions are satisfied if we affirm,

$$\alpha^4 \qquad (y)((\exists z)f_0(y,z) \supset (f_0(y,x_0) \equiv {\sim}f(y,y)),$$

where x_0 is $(\imath x)(y)((\exists z)f_0(y,z) \supset (f_0(y,x) \equiv {\sim}f_0(y,y)))$. For then α^4 just is $E!x_0$, and the reflexive instantiation case yields ${\sim}(\exists z)f_0(x_0,z)$. Alternatively we can take α^4 in the stronger form $(y)(f_0(y,x_0) \equiv .{\sim}f_0(y,y)\ \&\ (\exists z)f_0(y,z))$ where x_0 is now $(\imath x)(y)(f_0(y,x) \equiv .{\sim}f_0(y,y)\ \&\ (\exists z)f_0(y,z))$. Hence a Quinean-type condition provides a general form of predicate-restriction resolution for the simple paradoxes. The plausibility of the original paradox then turns on the question of whether α^4 can be affirmed; and that is the question of whether or not the domain admits of a plausible (or "natural") partitioning into the two nonempty subsets $\{y: (\exists z)f_0(y,z)\}$ and $\{y: {\sim}(\exists z)f_0(y,z)\}$ such that x_0 is an element of the latter.

Writing α^4 as,

$$(y)(f_0^*y \supset (f_0(y,x_0) \equiv {\sim}f_0(y,y)),$$

where f_0^*y is $(\exists z)f_0(y,z)$, the α^3-type resolutions arise as special cases. Thus c^*y is plausibly interpreted as 'y is a book catalogue' since $(\exists z)c(y,z)$ says that y is listed in some catalogue (book or index), but given the earlier assumption that card indexes are not themselves listed in any catalogue (book or index), then y is a book catalogue. Similarly, s^* is the predicate '. . . is clean-shaven' since $(\exists z)s(y,z)$ says that someone (maybe y himself, or maybe the barber) shaves y. Again, ϵ^*y says that y is a membership-eligible entity (set). Where f_0 is the predicate-constant r (to the right of), the reflexive case yields ${\sim}(\exists z)r(x_0,z)$, i.e., x_0 is to the right of no item, so x_0 remains the ultimate left-most object and this case of α^4 cannot be affirmed. Here r^* is the predicate '. . . is right-hand eligible'.

Although α^4 provides us with a generalized form of a predicate-restricting resolution which can be recursively applied where only one distinct predicate constant occurs in the original formula, it is not easily generalizable to more complex cases which contain more than one predicate constant. The problem of finding a suitable restricting condition $S(y)$ remains open for such cases. Hence considered as a general technique, predicate-restriction is less comprehensive than $\mathrm{FR_i}$. The application of $\mathrm{FR_i}$ is independent of the predicate constants occurring in the original formula and it is uniformly applicable to all cases. Since the paradoxes are themselves independent of the particular predicate constants involved, this represents an advantage. A further advantage is that $\mathrm{FR_i}$ invokes the minimum antecedent condition required to block the reflexive contradiction in each case. It thus treats paradox resolution as a purely formal enterprise stemming from a purely formal requirement. Hence it requires no further rationale. By contrast, predicate-restriction builds in a rationale by providing a "plausible" predicate to effect the required partitioning of the domain. Thus it moves us into the realm of solutions, as distinct from resolutions.

5 The class paradoxes and naive set theory I now want to consider the possible application of $\mathrm{FR_i}$ to remove the paradoxes in naive set theory.

5.1 The position we have is that given standard quantification theory with *PM* description-theory, and the predicate constant ϵ, we can prove that certain classes do not exist if descriptions are utilized as class abstracts. More particularly, let **Q** be a system of quantification theory with identity which contains no predicate parameters and only one primitive predicate constant (apart from identity), namely ϵ, and let the definite descriptor $\imath$ be introduced by the usual *PM* contextual definition. Then in **Q** we can establish $\sim E!R$ as well as other similar theses which determine the nonexistence of classes (e.g., such as those postulated by cyclic-paradox formulas). We know that **Q** is consistent. If, however, we add to **Q** an abstraction schema of the form,

A $\quad E!(\imath x)(y)(y \epsilon x \equiv Ay)$

then we get an immediate inconsistency by taking Ay to be $\sim(y \epsilon y)$ since we then have $E!R$. And similarly in the case of other reflexive-paradoxical classes, a nonexistence theorem of **Q** will be inconsistent with an instance of A or with some consequence of one or more instances of A. Thus A generates existence theorems which are reflexive contradictions; and since they are reflexive contradictions, their negations are already theses of **Q**. These are not the only inconsistencies which arise since there are instances of A which are Curry formulas, and the negations of Curry formulas are not existing theses of **Q**. Here the simple inconsistency arises indirectly by way of absolute inconsistency.

In general, however, since **Q** is consistent we know that whatever inconsistencies arise when A is taken as an additional postulate are either instances of A or consequences which rest essentially on A in the sense that one or more instances of A occur essentially in their derivation. To the extent that such inconsistencies arise from the generation of reflexive contradictions and Curry formulas, therefore, the problem of removing them is solved if,

(1) we can determine which formulas are reflexive contradictions or Curry formulas;
(2) we can modify these formulas in such a way that the contradiction is removed.

Now we know that we can solve each of these problems in general. C_i provides us with a recursive procedure for determining whether an arbitrary wff B of **Q** is a reflexive contradiction or a Curry formula; and FR_i provides us with a recursive procedure for constructing noncontradictory counterparts of B. But the addition of A to **Q** does not extend the set of wff. Hence if we apply C_i as a filter and FR_i as a fix, it seems we can remove such inconsistency as arises from the generation of reflexive contradictions and Curry formulas.

Call the naive theory determined by **Q** together with A, **N**. Alternatively, we can take **N** to be determined by **Q** and the axiom schema,

$\mathbf{A_1}$ $\quad (\exists x)(y)(y \epsilon x \equiv Ay)$

together with Extensionality, and then recover A as a theorem schema. A then guarantees that the satisfiers of A_1 are unique and we can use $s_0, \ldots, s_n$ to abbreviate definite descriptions which are instantiation values of the *e*-quantifier in A_1 and are such that for each s_i, $E!s_i$. This simplifies some of the discussion if we assume that all the reflexive contradictions and Curry formulas which arise from A also arise from A_1 and hence that the addition of extensionality contributes no further formulas of this kind. (There is no problem in the other direction, i.e., if we take A as the axiom schema, since all consequences of A_1 are consequences of A.) Given this, the position is that if the only inconsistencies which arise in **N** are due to the generation of reflexive contradictions and Curry formulas from A_1, then there seems to be, at least in principle, a technique which can be applied to the formulas of **N** in such a way as to yield a modified system **N*** which is consistent. Put otherwise, a proof that **N*** is consistent would show two things: first that the only inconsistencies in **N** are due to the generation of reflexive and Curry contradictions from A_1; secondly, that C_i is a necessary and sufficient condition for detecting such formulas and FR_i is a sufficient condition for constructing noncontradictory counterparts.

Notice, however, that the obvious step of simply applying C_i and FR_i to instances of A_1 is inadequate. This would certainly enable us to filter out and modify axioms which are reflexive or Curry contradictions. For example, C_3 would filter out $(\exists x)(y)(y \epsilon x \equiv \sim(y \epsilon y))$ and FR_3 would supply $(\exists x)(y)(y \neq x. \supset .y \epsilon x \equiv \sim(y \epsilon y))$ in its place. But we should not in this way capture those contradictions which arise from two or more instances of A_1, neither of which in itself is a reflexive or Curry contradiction. A simple example is provided by an analogue of the contradiction arising from *Aut* (Section 1.3). Thus, take as an instance of A_1: (i) $(\exists x)(y)(y \epsilon x \equiv y \epsilon y)$, and instantiate to obtain: (ii) $(y)(y \epsilon s_0 \equiv y \epsilon y)$; s_0 is the class of all self-membered classes, K. Now take as a second instance of A_1: (iii) $(\exists x)(y)(y \epsilon x \equiv \sim(y \epsilon s_0))$. Instantiating we have: (iv) $(y)(y \epsilon s_1 \equiv \sim(y \epsilon s_0))$. From (ii) and (iv), then: (v) $(y)(y \epsilon s_1 \equiv \sim(y \epsilon y))$. Thus s_1 is R, introduced as the complement of K. This will be called the *K-R* argument in what follows.

As in the derivation of the contradiction from *Aut*, there is a heuristic argument which treats s_0 and s_1 as constants to show that the two instances of A_1, (i) and (iii), are not reflexive contradictions, yet (v) is. Hence even if (v) is rejected as an axiom (or strictly, as an instantiation case of an axiom), it will nevertheless still arise as a thesis from axioms which are not filtered out by C_i.[2] To avoid this, we therefore need to apply C_i as a filter on every thesis, rather than as a filter on just the set of axioms.

The following remarks are merely suggestions about what kind of system we might expect to get in this way, together with some indication of the problems involved in setting it up.

5.2 The system which is envisaged, **N***, differs markedly from those which result from usual ways of applying a resolution by redefinition in set theory for, in general, this kind of resolution is expressed by modifying the schema A_1, or A, in one way or another. This, however, is not open to us (Section 4.3). The filter C_i can be applied only to formulas. But this has two advantages: first, only those formulas which need to be modified get modified; and second, the modification is minimal in each case.

These differences are clear if we consider what seems to be the nearest relative to **N***, namely the system proposed by Hintikka ([2] and [3]).

Hintikka's proposal consists in modifying A_1 in such a way that every bound variable is restricted by an inequality condition wherever it occurs and every free variable is restricted by an antecedent inequality condition. Hence, every axiom is modified by inequality conditions which involve all the variables in it. By contrast, the application of C_i and FR_i does not result in the modification of all axioms, and even where modification is required it need not (and frequently does not) involve all the variables in the formula. The only variables which are involved are those which occur in relevant inequality conditions entailed by the matrix.

In effect, then, the proposal to develop $\mathbf{N}^*$ by employing C_i and FR_i to filter out and modify certain theses of $\mathbf{N}$ aims at a Hintikka-type of modification but applied in a case-by-case way to just those formulas which are either reflexive contradictions or Curry formulas and otherwise imposes no restrictions on the construction of classes.

5.3 The main problem in setting up $\mathbf{N}^*$ is that we have to devise some way of screening every thesis of $\mathbf{N}$ which rests on one or more instances of A_1. This could be done in terms of a recursive enumeration of $\mathbf{N}$-theses where the appropriate C_i and FR_i are applied to each formula as it comes up; but this is hardly a practical solution. Instead, what might seem to be required is something along the following lines.

Call formulas which fail to satisfy C_i, i.e., those which get past the filter, starred formulas (they can be marked with a star). Secondly, if A is a formula which does satisfy C_i, replace it by its counterpart A_c as determined by FR_i and star A_c. Now adopt the following two rules:

N*1 If $\vdash_{\mathbf{N}} A$ and A^*, then $\vdash_{\mathbf{N}^*} A^*$
N*2 If $\vdash_{\mathbf{N}} A$ but not A^*, then $\vdash_{\mathbf{N}^*} A_c^*$.

In terms of this, the theses of $\mathbf{N}^*$ are the starred theses of $\mathbf{N}$ together with the counterparts of unstarred theses of $\mathbf{N}$. However, there are some obvious difficulties.

For one thing, we know that we have $\vdash_{\mathbf{N}} A$ and $\vdash_{\mathbf{N}} \sim A$ for arbitrary A. We have no guarantee, however, that C_i will filter out all such contradictions. C_i is designed only to filter out reflexive contradictions and Curry formulas and it is at least theoretically possible that for some A both A and $\sim A$ will be starred, in which case we shall have $\vdash_{\mathbf{N}^*} A^*$ and $\vdash_{\mathbf{N}^*} \sim A^*$ for some A. More generally, we have $\vdash_{\mathbf{N}} B$, for arbitrary B. But not every formula satisfies C_i. Hence, we shall have $\vdash_{\mathbf{N}^*} B^*$ for some B^* such that B^* would not be a thesis of $\mathbf{N}$ were it not for the inconsistencies in $\mathbf{N}$, in which case it is an unacceptable thesis of $\mathbf{N}^*$.

These points suggest that what we should be looking at are derivations in $\mathbf{N}$ which rest on A_1 rather than at arbitrary theses of $\mathbf{N}$, for we want to block proofs which lead to inconsistency by way of reflexive and Curry contradictions. That is, our interest is in theses of $\mathbf{N}$ which are "good", in the sense that they do not rest on contradictions and are not themselves contradictory.

We can restrict attention to minimal derivations which rest on A_1, i.e., which are of the form $A_1, \ldots, A_n \vdash_{\mathbf{N}} B$ where: (i) none of the A_i is eliminable, and (ii) at least one A_i is such that it is an instance of A_1 or contains a wf part which is an instance of A_1.

Suppose, now, that each A_i gets past the filter. Even so, we cannot affirm $A_1, \ldots, A_n \vdash_{\mathbf{N}^*} B$ since, as we have seen, reflexive contradictions can arise as the last line of a proof sequence such that none of the preceding members of the sequence is a reflexive contradiction. Hence B has to be screened also. Thus, in place of $\mathbf{N}^*1$ and $\mathbf{N}^*2$ we require:

N*1′ If $A_1, \ldots, A_n \vdash_{\mathbf{N}} B$ and each of $A_1, \ldots, A_n$ and B are starred, then $A_1^*, \ldots, A_n^* \vdash_{\mathbf{N}^*} B^*$

N*2′ If $A_1, \ldots, A_n \vdash_{\mathbf{N}} B$ and $A_1, \ldots, A_n$ are starred but B is not, then $A_1^*, \ldots, A_n^* \vdash_{\mathbf{N}^*} B_c^*$.

It is immediate from $\mathbf{N}^*2'$ that the rules of **N** are not closed under starring. A simple example is provided by the following proof in **N** which utilizes starred premisses taken from the *K-R* argument:

(ii)* $(y)(y \in s_0 \equiv y \in y)$
(iv)* $(y)(y \in s_1 \equiv \sim y \in s_0))$,

hence,

(vi) $(y)(y \in s_0 \equiv y \in y)\ \&\ (y)(y \in s_1 \equiv \sim(y \in s_0))$.

However, (vi) is not starred since the matrix of its pnf, $(y)(z)(y \in s_0 \equiv y \in y.\ \&\ .z \in s_1 \equiv \sim(z \in s_0))$, entails $y \neq s_1 \vee z \neq s_1$. Hence, adjunction is not closed under starring and $\mathbf{N}^*$ is therefore a nonclassical system. The **Q**-part of $\mathbf{N}^*$ remains classical since all derivations in **Q** satisfy $\mathbf{N}^*1'$, i.e., the rules are closed under starring for derivations in **Q** and all theses of **Q** get an automatic star.

The application of $\mathbf{N}^*2'$ to the sequence (ii)*, (iv)* $\vdash_{\mathbf{N}}$ (vi), requires us to replace (vi) by its counterpart to obtain a proof sequence in $\mathbf{N}^*$. Notice, however, that the heuristic arguments which we have employed so far, which treat s_0 and s_1 as if they were constants, fail to supply a counterpart of (vi). In particular, although the matrix of the pnf of (vi) entails $y \neq s_1 \vee z \neq s_1$ we cannot apply $\mathrm{FR_i}$ directly to produce $(y)(z)(y \neq s_1. \supset (y \in s_0 \equiv y \in y.\ \&\ .z \in s_1 \equiv \sim(z \in s_0)))$. This is not a genuine counterpart since the antecedent restriction contains a description. If we were to adopt it, the reflexive instantiation case which arises by taking $y = s_1$ and $z = s_1$ would yield $s_1 = s_1$; but s_1 is R, so we have $E!R$ which contradicts the existing thesis $\sim E!R$. This reaffirms the discussion in Section 4.4 and emphasizes that a correct application of $\mathrm{C_i}$ and $\mathrm{FR_i}$ requires the prior elimination of descriptions. Thus the abbreviations s_0 and s_1 in (vi) have to be replaced by $(\iota x)(y)(y \in x \equiv y \in y)$ and $(\iota z)(w)(w \in z \equiv \sim(w \in (\iota x)(y)(y \in x \equiv y \in y)))$, respectively; these have then to be eliminated before constructing the pnf of (vi) and applying $\mathrm{C_i}$. The genuine counterpart of (vi) will then have an antecedent restriction in which no descriptions occur.

Now consider the relevance of $\mathbf{N}^*2'$ to the *K-R* argument. If we just take the argument as it stands, then we have: (1)*, (ii)*, (iii)*, (iv)* $\vdash_{\mathbf{N}}$ (v), where (v) is not starred. Hence it has to be replaced by its counterpart before we get an $\mathbf{N}^*$ sequence. However, in view of the discussion above, it is clear that the argument gets blocked at an earlier stage. For although (vi) does not occur in the *K-R* argument as it is presented, it would do so in an expanded form of the argument since adjunction on (ii) and (iv) is implicit in the move to (v). From this point of view, (vi) is a step on the way to (v). But as we have seen, (vi) is not a consequence of (ii) and (iv) in **N***; hence neither is (v). In an expanded form of the argument, therefore, we should expect that the counterpart of (v) arises directly from the counterpart of (ii) & (iv).

This general approach of using C_i and FR_i to screen derivations in **N** has obvious advantages over the earlier proposal to screen theses, but it is still not without difficulties. These arise because: (i) it is limited to derivations which contain instances of A_1 as premisses or as the wf part of premisses, and (ii) it can only work efficiently if all steps in the derivation are explicit. Thus we could ensure an adequate application of C_i and FR_i in this way only if all derivations in **N** which rest directly or indirectly on A_1 were expanded to complete proofs. By this are meant proofs which run right back to the axioms on which they rest and which employ no derived rules.

This suggests an alternative approach which consists in setting up **N*** independently of **N**. Thus, let **N*** be a set of postulates for **Q** together with A_1, Extensionality, and the following primitive rules:

$\mathbf{C_i}$	Rules for starring
$\mathbf{FR_i}$	Rules for constructing counterparts of unstarred formulas
MP*1	If $\vdash A^*$ and $\vdash(A \supset B)^*$ and B^*, then $\vdash B^*$
MP*2	If $\vdash A^*$ and $\vdash(A \supset B)^*$ and not B^*, then $\vdash B_c^*$
Gen*	If $\vdash A^*$, then $\vdash((a)A)^*$.

Gen* is standard since starring is not affected by universal closure. Thus if A is in pnf so is $(a)A$. Moreover A and $(a)A$ have the same matrix; hence the same inequalities are entailed. Let a occur as a free variable in A (other cases are trivial). If A is starred, its matrix does not entail any inequalities of the form $a \neq b$ where b is a u-variable, a free variable, or a constant in A. Hence the matrix of $(a)A$ does not entail any inequalities of the form $a \neq b$ where a is now a u-variable and b is as above. But these are the only relevant inequalities since no e-variables satisfy the condition that the quantifiers in which they occur precede the quantifier over a in $(a)A$. A similar argument establishes that if A is not starred, neither is $(a)A$. If existential instantiation is introduced as a primitive rule, the usual restraints have to be put on the application of Gen*.

The starring conditions on the primitive rules will carry through to all derived rules, and in consequence these secondary rules will usually have complex constraints on them. Consider, for example, the usual derivation of adjunction:

	(1)	$\vdash A \supset (B \supset .A \,\&\, B)$
Hyp	(2)	$\vdash A$
	(3)	$\vdash(B \supset .A \,\&\, B)$
Hyp	(4)	$\vdash B$
	(5)	$\vdash A \,\&\, B$

In **N***, however, we first require that any instance of (1) be a starred formula. For even though (1) is a theorem schema of **Q**, this does not guarantee that every instance of (1) is a theorem of **N***; and indeed we should expect in some cases that if one or more of the schematic letters is replaced by an instance of A_1, the formula so obtained will not be starred. Second, we require A^*; third, that any instance of (3) be starred; fourth that B be starred; and finally that A & B be starred. Hence adjunction gets through only in the cumbersome form:

Adj*1 If $\vdash A^*$ and $\vdash B^*$ then $\vdash (A \;\&\; B)^*$ provided:
(i) $(A \supset (B \supset .A \;\&\; B))^*$
(ii) $(B \supset .A \;\&\; B)^*$
(iii) $(A \;\&\; B)^*$

Adj*2 If $\vdash A^*$ and $\vdash B^*$ then $\vdash (A \;\&\; B)^*_c$ provided:
(i) $(A \supset (B \supset .A \;\&\; B))^*$
(ii) $(B \supset .A \;\&\; B)^*$
(iii) $(A \;\&\; B)$ is not starred.

Evidently derived rules have little practical value in **N***.

If **N*** is set up in this way, however, the primitive rules guarantee that only starred formulas get through as theses, i.e., that no thesis will be a reflexive contradiction or a Curry formula. We therefore expect **N*** to be consistent if these formulas are the only source of inconsistency in **N**.

5.4 **N*** is absolutely consistent, trivially, since certain formulas fail to get a star and cannot therefore be theses. However, since **N*** is nonclassical, there is no argument from absolute consistency to simple consistency, even though $A \;\&\; {\sim}A. \supset B$ is a theorem-schema of **Q**. What we are assured of, then, is that if **N*** is simply inconsistent, it is paraconsistent since the inconsistency does not spread to absolute inconsistency.

It is easy to see that it is theoretically possible for **N*** to be simply inconsistent yet absolutely consistent. For suppose we have $\vdash{\sim}(\exists x)(\exists y)A_0(x,y)$ and $\vdash(\exists x)(\exists y)A_0(x,y)$, for some $A_0(x,y)$ containing just the two variables x and y. The pnf of their conjunction is $(\exists x)(\exists y)(z)(w)(A_0(x,y) \;\&\; {\sim}A_0(z,w))$; but the matrix of this entails $x \neq z \vee y \neq w$, hence it is not a starred formula. Consequently Adj*1 fails to be satisfied and we cannot affirm $\vdash(\exists x)(\exists y)A_0(x,y) \;\&\; {\sim}(\exists x)(\exists y)A_0(x,y)$. So even if $A_0 \;\&\; {\sim}A_0. \supset B_0$ is a starred thesis where B_0 is an arbitrary wff, we cannot apply MP*1 to conclude $\vdash B_0^*$.

One minor point of interest so far as consistency is concerned is that the usual proviso on A_1: $(\exists x)(y)(y \in x \equiv Ay)$, namely that x not be free in Ay, seems not to be required in **N***, at least in simple cases. For suppose we choose ${\sim}(y \in x)$ for Ay. Then the instance $(\exists x)(y)(y \in x \equiv {\sim}(y \in x))$ is a limiting case of a reflexive contradiction since its matrix entails $x \neq x$. Hence it is eliminated by the filter. On the other hand, the innocuous instance $(\exists x)(y)(y \in x \equiv y \in x)$ which is eliminated by the usual proviso is not eliminated by the filter.

Note, finally, that the question of plausibility coincides with the question of consistency in **N***. For if the only two primitive predicate constants which can occur in a formula are $\in$ and $=$, then the meaning postulates $M(\in,=)$ are just the axioms on identity and membership. Consequently, the question of whether a given reflexive contradiction is plausible always reduces to the question of whether it is consistent with these axioms independently of the reflexive case. But that is just the question of whether **N*** is consistent.

5.5 It is not claimed for **N*** that it has any practical advantages over more conventional theories which apply a resolution by redefinition by means of a modification on the abstraction schema. Rather, **N*** is offered as a theoretical system which lies between **N** and conventional theories and which represents a minimally modified variant of **N** if the sole concern is to eliminate inconsistency. If **N*** is consistent, then those of its theses which are not counterpart formulas constitute the maximally consistent set of **N**-theses. When counterpart theses are added, we have the maximally consistent set of minimally modified formulas of **N**. What is interesting, however, is that although **N*** lies between the classical (though inconsistent) **N** and classical (presumed consistent) conventional theories, it is itself nonclassical. What we can conclude, then, if **N*** is consistent, is that no classical conventional theory can stand as a minimally modified **N**. All such theories, that is to say, will always contain more restrictions on the construction of sets than are necessary to avoid inconsistency.

NOTES

1. It might be thought that if we insist on conditional instantiation, then the usual contradiction from α^0 is blocked since we cannot move to $c(C,C) \equiv \sim c(C,C)$; instead we have $E!C \supset (c(C,C) \equiv \sim c(C,C))$, hence $\sim E!C$. So it might seem that there is no paradox in the first place and that the contradiction arises from a mishandling of descriptive terms. However, given that there is at least one catalogue, y_0, which does not list itself (and certainly there is no paradox if not), we have $\sim c(y_0,y_0)$. But by ordinary instantiation on α^0, since y_0 is a constant, we have $c(y_0,C) \equiv \sim c(y_0,y_0)$; hence $c(y_0,C)$. Here C has a primary occurrence, so by *PM** 14.21 we also have $E!C$.

2. To show this in detail it is necessary to eliminate the description from (iii), which in full is

$$(\exists x)(y)(y \in x \equiv \sim(y \in (\iota z)(w)(w \in z \equiv w \in w))),$$

before C_i is applied.

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