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First-Order Theories as Many-Sorted Algebras

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Introduction In this paper, by developing a study of first-order logic through many-sorted algebras, we show that every first-order theory is a particular algebra verifying axioms in equational form (see Section 2); therefore we are able to apply Birkhoff's theorems concerning the varieties (see Section 1 and [7] and [8]) and to obtain the Henkin models algebraically, whence the completeness theorem of first-order logic (see Section 3).

The many-sorted (or heterogeneous) algebras, systematized by Birkhoff and Lipson (see [4] and [10]), find a natural application in investigating programming languages: several approaches to the formal definition of semantics for such languages can be developed by means of morphisms between many-sorted algebras (see [2] and [6]).

This paper shows the analogous possibility of algebraizing linguistic features of logic, thus yielding a unique framework for the formal analysis of both programming and logical languages within universal algebra.

The analysis here developed is strongly related to those of [1], [2], [11], and [12]. Moreover, an approach to the topic of present paper is elaborated in the monographs [8] and [9]. Knowledge of these works is not necessary to understand what follows, though it would better enable one to appreciate our results.

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*1 **Preliminaries*** A many-sorted algebra $A = (D, F)$ consists of a family $D = \{(D_i)\}_{i \in I}$ of sets called *domains* of A and a set F of operations such that for all $f \in F$

$$f{:}D_1 \times D_2 \times \ldots \times D_k \to D_0$$

and $D_1, D_2, \ldots D_k$, domains of f, and D_0, codomain of f, are members of D.

Like classic (one-sorted) algebras, two many-sorted algebras (D, F) and (D', F') are said to be *similar*, or of the *same type*, iff: (a) there is a one-to-one correspondence between D and D', (b) there is a one-to-one correspondence between F and F', and (c) corresponding operations have corresponding domains and codomains. The classic notions of *morphism, congruence, subalgebra, quotient algebra,* and *direct product algebra* are extendible to the many-sorted algebras in a natural manner (see [4], [6], and [10]). If a system Σ of notations (many-sorted alphabet) is fixed, then the many-sorted algebra is called a *Σ-algebra*; thus we have *Σ-morphisms, Σ-congruences*, and so on.

Let S be the class of all the many-sorted algebras of a given type. As in the one-sorted case, one constructs the *word algebra* $G(V)$ for S over a family $V = \{V_i\}_{i \in I}$ of *generators*. It is easy to verify that if $f = \{f_i\}_{i \in I}$, where $f_i : V_i \to D_i$ is a family of evaluations of V in the domains of a many-sorted algebra $A \in S$, then there is a unique morphism $\bar{f}{:}G(V) \to A$ which extends f. Furthermore, letting C be a subclass of S, the following (congruence) relation ρ is defined on $G(V)$: $E_1 \rho E_2$ iff $\bar{f}(E_1) = \bar{f}(E_2)$, for any evaluation f and for any $A \in C$. $G(V)/\rho$ is called the *free algebra* over V (generated by V) for C.

In general, it is not always the case that $G(V)/\rho \in C$, but we have the following results due to Birkhoff (see [7] and [8]).

Proposition 1 *If C is closed under the formation of subalgebras and direct product, then $G(V)/\rho \in C$ for any V.*

Proposition 2 *If C is a many-sorted variety (class of algebras defined by axioms in equational form), then C is closed under the formation of subalgebras, epimorphic images, and direct products.*

By the preceding propositions it follows trivially that

Proposition 3 *If C is a variety, then $G(V)/\rho \in C$ for any V.*

*2 **An algebraic metatheory for first-order theories*** A first-order theory is essentially determined by a linguistic structure, i.e., formation rules, and by a deductive structure, i.e., inference rules. In the usual first-order theories the formation rules define a class of formulas by structural induction, starting from individual constants and variables, by means of functors, predicates, and logical operators (connectives and quantifiers). Finally, the deductive structure yields the inductive closure of an initial class of formulas (axioms) by means of inference rules as a class of theorems.

Now we express these ideas axiomatically in the theory of many-sorted algebras. Consider the following types of sets:

T, a set of *terms* (let $r, t, t_1, \ldots, t_k$ be variables on T)
F, a set of *formulas* (let $E, E_1, \ldots, E_k$ be variables on F)
B, a set of *Boolean-values,*

and the following types of operations:

predicates, whose elements p are such that $p{:}T^k \to F$ ($k \in N$)
variables, $w{:}T^0 \to T$ (where w is a generic variable)

Since $T^0 = \{h|h{:}\phi \to T\} = \{\phi\}$, a variable yields a particular element which identifies it.

functors,	$f{:}T^k \to T$ (f generic, $k \in N$)
constants,	$c{:}T^0 \to T$ (c generic)
connectives,	$\sim{:}F \to F$, $\vee{:}F \times F \to F$, where $\sim E$ stands for $\sim(E)$ and $E_1 \vee E_2$ for $\vee(E_1, E_2)$
quantifiers,	$\forall_w{:}F \to F$, one for every variable, where $\forall wE$ stands for $\forall_w(E)$
Boolean functions,	$-{:}B \to B$, $+{:}B \times B \to B$, $\cdot{:}B \times B \to B$, where $-b$, $b_1 + b_2$, $b_1 \cdot b_2$ stand for $-(b)$, $+(b_1, b_2)$, $\cdot(b_1, b_2)$, respectively
deduction,	$\delta{:}F \to B$.

The algebras described here are of course relative to a fixed first-order language L determining an algebraic similarity type, therefore our algebraic metatheory is better called L-metatheory. Clearly, given an algebra A of type L, every (first-order) term or formula of L yields a term or a formula of A. Thus, we can extend the usual notions of "free", "bound", "closed", and so on to the algebras of type L in a natural way. Moreover, if G_L is the word algebra of type L over the empty set of generators, then the sets T and F of G_L are practically the first-order terms and formulas of L.

Finally, the crucial difference between this and Example 1 of [2] (see p. 34 and pp. 63-65, Section 4 therein) is our function symbol δ which is new here. See also the similarity type g_4 (of algebras) on pp. 55-58 of [2] in connection with our sorts T and F (cf. also p. 42 of [2]).

Algebraic axioms for first-order theories

$\mathbf{\Delta_0}$ *All the instantiations of the Boolean axioms with the Boolean elements of initial algebra* G_L

$\mathbf{\Delta_1}$ $-\delta E = \delta \sim E$ *for every closed formula* E

$\mathbf{\Delta_2}$ $\delta E_1 + \delta E_2 = \delta(E_1 \vee E_2)$ *for every two closed formulas* E_1, E_2

$\mathbf{\Delta_3}$ $\delta E = \delta \forall wE$ *for every formula* E

$\mathbf{\Delta_4}$ $\delta \forall wE \leqslant \delta E[t/w]$ *for any term* t *free for* w *in* E *(where* $b \leqslant b'$ *stands for* $b + b' = b'$ *and* $E[t/w]$ *is the formula obtained by putting the term* t *in place of all the free occurrences of* w *in* E*).*

If $\Delta = \Delta_0 \cup \ldots \cup \Delta_4$, Γ is a set of enuciates on L, and

$$\theta = \{\delta(E) = 1\}_{E \in \Gamma}$$

then θ is the translation of Γ within the L-metatheory. The intended meaning of the above axioms is the following: When we put $\delta E = 1$ for any $E \in \Gamma$, if B is the Boolean algebra over the subsets of $\{M|M \models E, E \in \Gamma\}$ (where $\models$ is the usual first-order satisfiability), and if we put

$$\delta E = \{M|M \models E', E' \in \Gamma \cup \{E\}\}.$$

then B verifies the axioms $\Delta \cup \theta$.

3 Completeness theorem Here follows the announced proof of the completeness theorem (for classic proofs see [3], [5], and [13]). Let $C(\Delta \cup \theta)$ be the class of all many-sorted algebras verifying the axioms Δ and θ. Of course $C(\Delta \cup \theta)$ is a variety and has free algebras by Proposition 3.

Recall that by Birkhoff's completeness theorem we have

$$\Delta \cup \theta \models b_1 = b_2 \Longleftrightarrow \Delta \cup \theta \vdash_e b_1 = b_2$$

where $\vdash_e$ is the well-known equational calculus (see [7]).

Lemma 1 $(\Delta \cup \theta \vdash_e \delta E = \delta E') \Rightarrow \Gamma \vdash E \longleftrightarrow E'$, *where* E, E' *are any closed formulas and* $\vdash$ *is any first-order logical calculus.*

Proof: We state the following fact

(*) $\Delta \cup \theta \vdash_e b_1 = b_2 \Rightarrow \Gamma \vdash \bar{b}_1 \longleftrightarrow \bar{b}_2$

where for any Boolean element b of G_L, $\bar{b}$ is a closed formula obtained as follows: if b is the Boolean term $P(\delta E_1, \ldots, \delta E_k)$ for a polynomial form P in $+$ and $-$, then $\bar{b}$ is $P'(\pi E_1, \ldots, \pi E_k)$, where $\pi E_1, \ldots, \pi E_k$ are the universal closures of $E_1, \ldots, E_k$ respectively, and P' is the corresponding polynomial form of P where $\vee$ and $\sim$ stand for $+$ and $-$, respectively. Further, $\bar{1}$ is $E_0 \vee \sim E_0$ for a fixed enunciate E_0. Of course a formal definition of $\bar{b}$ can easily be given by induction. The proposition (*) implies the enunciate of the lemma because if E is a closed formula then $\overline{\delta E} = E$. Now we prove (*) by induction on the length of the equational proof.

For proofs of length zero we have the following cases:

Case a. $\Delta \cup \theta \vdash_e b = b$
Case b. $\Delta \cup \theta \vdash_e \delta E = 1 \ (E \in \Gamma)$
Case c. $\Delta \cup \theta \vdash_e \delta \sim E = -\delta E$
Case d. $\Delta \cup \theta \vdash_e \delta(E_1 \vee E_2) = \delta E_1 + \delta E_2$
Case e. $\Delta \cup \theta \vdash_e \delta \forall w E = \delta E$
Case f. $\Delta \cup \theta \vdash_e \delta \forall w E + \delta E[t/w] = \delta E[t/w]$
Case g. $\Delta \cup \theta \vdash_e b_1 = b_2$, where $b_1 = b_2 \in \Delta_0$.

Cases a, b, c, d, and e are trivial.

Case f. $\overline{\delta \forall w E + \delta E[t/w]} = \pi \forall w E \vee \pi E[t/w] = \pi E \vee \pi E[t/w]$; but it is clear that $\vdash \pi E \vee \pi E[t/w] \longleftrightarrow \pi E[t/w]$.

Case g. If $b_1 = b_2$ is an instantiation of a Boolean axiom, then $\bar{b}_1 \longleftrightarrow \bar{b}_2$ is an instantiation of the tautology corresponding to the given axiom.

Let us suppose that (*) holds for equational proofs of length n and let $b_1 = b_2$ be obtained with an equational proof of length $n + 1$; then for the last step of the proof we have the following possibilities (expressed in the usual form premisses/conclusion):

i. $$\frac{\Delta \cup \theta \vdash_e b_2 = b_1}{\Delta \cup \theta \vdash_e b_1 = b_2}$$

ii. $$\frac{\Delta \cup \theta \vdash_e b_1 = b_3 \quad \Delta \cup \theta \vdash_e b_3 = b_2}{\Delta \cup \theta \vdash b_1 = b_2}$$

iii. $$\frac{\Delta \cup \theta \vdash_e b'_1 = b'_2 \quad \Delta \cup \theta \vdash_e b''_1 = b''_2}{\Delta \cup \theta \vdash_e b'_1 + b''_1 = b'_2 + b''_2}$$

where $b_1 = b'_1 + b''_1$ and $b_2 = b'_2 + b''_2$

iv. $$\frac{\Delta \cup \theta \vdash_e b'_1 = b'_2}{\Delta \cup \theta \vdash_e -b'_1 = -b'_2}$$

where $b_1 = -b'_1$ and $b_2 = -b'_2$.

(Derivations by means of substitutions are not considered because in our case we have axiom schemata rather than axioms with variables.) But it is trivial to verify that each translation of the above rules (where any b is replaced by $\bar{b}$ and $+, -, =$ are replaced by $\vee, \sim, \longleftrightarrow$, respectively) holds in the calculus $\vdash$; furthermore, by induction hypothesis, the translations of the premisses are derived in the calculus $\vdash$. Thus in all the cases we have $\Delta \cup \theta \vdash \bar{b}_1 \longleftrightarrow \bar{b}_2$.

Lemma 2 *If Γ is consistent according to formal first-order deducibility, then the axioms $\Delta \cup \theta$ are consistent according to the equational calculus.*

Proof: If by $\Delta \cup \theta \vdash_e 1 = 0$, then for the previous lemma we also have $\Gamma \vdash \bar{0} \longleftrightarrow \bar{1}$, whence the thesis.

Lemma 3 *Let C be a class of similar many-sorted algebras. If a free algebra of generators V is in C, then it is the quotient algebra $G(V)/\rho$ where $G(V)$ is the word algebra of generators V, and ρ is the least one within the congruences R for which $G(V)/R \in C$. ($R_1 \leqslant R_2$ iff $xR_1y \Rightarrow xR_2y$.)*

Proof: See [7] and [8].

Lemma 4 *If Γ is a consistent set of first-order enunciates, then in the free algebra $G(V)/\rho$ of $C(\Delta \cup \theta)$, we have $1 \neq 0$.*

Proof: By Lemma 2, $\Delta \cup \theta \not\vdash_e 1 = 0$, therefore $1 \neq 0$ holds in $G(V)/\rho$ by Lemma 3.

Lemma 5 *If Γ is a consistent set of first-order enunciates, then in $C(\Delta \cup \theta)$ there exists an algebra A where $B = \{0, 1\}$.*

Proof: In the initial algebra $A_0 = G_L/\rho$ of variety $C(\Delta \cup \theta)$, consider a congruence ε determined by a Boolean morphism from the Boolean algebra B of A_0 in $\{0, 1\}$, i.e., an ultrafilter in B. Clearly $A = A_0/\varepsilon$ has only the two truth values and, being a quotient algebra, is an epimorphic image of A_0. But $C(\Delta \cup \theta)$ is a variety, so by Proposition 2, it is closed under formation of epimorphic images, therefore $A_0/\varepsilon \in C(\Delta \cup \theta)$.

We note that the previous lemma plays the role of a Lindenbaum completion.

Completeness Theorem *If Γ is a consistent set of enunciates according to first-order formal deducibility, then it has a model.*

Proof: First we recall that if Γ is consistent, then the expansion Γ' obtained by adding to Γ the Henkin axioms

$$\sim\forall wE \rightarrow \sim E[c_E/w]$$

for any formula E having only one free variable w is also consistent (see [13], Lemma 4.2.3, p. 46). Now, using the algebra $A = (T, F, B, \ldots)$ obtained by applying Lemma 5 to Γ', we define a model $M = (D, c^M \ldots, f^M \ldots, p^M \ldots)$ of Γ as follows: let $D = \{t \in T \mid$ no variable w is in $t\}$ (the elements of T are strings; in fact, T is also domain of a word algebra); $c^M, \ldots$ are the constants of A; $f^M, \ldots$ are the functors of A restricted to D; and $p^M, \ldots$ are defined by

(*) $\quad p^M(t_1, \ldots, t_k) = 1$ iff $\delta(p(t_1, \ldots, t_k)) = 1$ in A.

(In the algebra A, $\delta(E) = 1$ or $\delta(E) = 0$ for every atomic enunciate E.)

Now we show that

(**) $\quad \delta(E) = 1 \Rightarrow M \models E$

whence

$$E \in \Gamma \Rightarrow M \models E$$

thus concluding the proof.

(**) is established by induction on the complexity of the formulas. (For brevity, let w be the sequence $w_1, \ldots, w_n$ of all free variables of E and let r be a sequence $r_1, \ldots, r_n$ on D (i.e., $r \in D^n$) and let $E[r/w]$ be the formula $((E[r_1/w_1]) \ldots [r_n/w_n])$.)

Initial step:

$$\begin{aligned} \delta p(t_1 \ldots t_k) = 1 &\Rightarrow \delta \forall w p(t_1 \ldots t_k) = 1, \text{ by } \Delta \\ &\Rightarrow \delta p(t_1 \ldots t_k)[r/w] = 1 \text{ for every } r \in D^n, \text{ by } \Delta \\ &\Rightarrow M \models p(t_1 \ldots t_k). \end{aligned}$$

Let us suppose that (**) holds for formulas E, E_1, E_2 of a given complexity; we now prove (**) for

Case 1. $\sim E$
Case 2. $E_1 \vee E_2$
Case 3. $\forall w_0 E$.

In Case 1 we have four possibilities:

a. $E = p(t_1 \ldots t_k)$
b. $E = \sim E'$
c. $E = E' \vee E''$
d. $E = \forall w_0 E'$.

We show that (**) holds in each of them:

a. $$\begin{aligned} \delta \sim p(t_1 \ldots t_k) = 1 &\Rightarrow \delta \sim p(t_1 \ldots t_k)\,[r/w] = 1 \text{ for every } r \in D^n, \text{ by } \Delta \\ &\Rightarrow \delta p(t_1 \ldots t_k)\,[r/w] = 0 \text{ for every } r \in D^n, \text{ by } \Delta \\ &\Rightarrow M \not\models p(t_1 \ldots t_k)\,[r/w] \text{ for every } r \in D^n, \text{ by } (*) \\ &\Rightarrow M \models \sim p(t_1 \ldots t_k) \end{aligned}$$

b. $$\begin{aligned} \delta \sim\sim E' = 1 &\Rightarrow \delta \sim\sim E'[r/w] = 1 \text{ for every } r \in D^n, \text{ by } \Delta \\ &\Rightarrow \delta E'[r/w] = 1 \text{ for every } r \in D^n, \text{ by } \Delta \end{aligned}$$

$\Rightarrow M \models E'[r/w]$ for every $r \in D^n$, by induction hypothesis
$\Rightarrow M \not\models \sim E'[r/w]$ for every $r \in D^n$
$\Rightarrow M \models \sim\sim E'[r/w]$ for every $r \in D^n$
$\Rightarrow M \models \forall w \sim\sim E'$
$\Rightarrow M \models \sim\sim E'$

c. $\delta \sim (E' \vee E'') = 1 \Rightarrow \delta \sim (E' \vee E'')\,[r/w]) = 1$ for every $r \in D^n$, by Δ
$\Rightarrow \delta \sim (E'[r/w] \vee E''[r/w]) = 1$ for every $r \in D^n$, by Δ
$\Rightarrow -(\delta\, E'[r/w] + \delta E''[r/w]) = 1$ for every $r \in D^n$, by Δ
$\Rightarrow \delta \sim E'[r/w] = 1$ and $\delta \sim E''[r/w] = 1$ by Δ
$\Rightarrow M \models \sim E'[r/w]$ and $M \models \sim E''[r/w]$, by induction hypothesis
$\Rightarrow M \models \sim(E'[r/w] \vee E''[r/w])$, by induction hypothesis
$\Rightarrow M \models (\sim E' \vee E'')\,[r/w]$, by induction hypothesis
$\Rightarrow M \models \forall w \sim (E' \vee E'')$
$\Rightarrow M \models \sim(E' \vee E'')$

d. $\delta \sim \forall w_0 E' = 1 \Rightarrow \delta(\sim E'[r/w])\,[c'/w_0] = 1$ for every $r \in D^n$, by Δ and Henkin axiom, where c' is the Henkin constant of E'
$\Rightarrow M \models (\sim E'[r/w])\,[c'/w_0]$ by induction hypothesis
$\Rightarrow M \models \sim \forall w_0 E'$.

In Cases 2 and 3 we have analogous deductions:

2. $\delta(E_1 \vee E_2) = 1 \Rightarrow \delta(E_1[r/w] \vee E_2[r/w]) = 1$ for every $r \in D^n$, by Δ
$\Rightarrow \delta E_1[r/w] + \delta E_2[r/w] = 1$ for every $r \in D^n$, by Δ
$\Rightarrow \delta E_1[r/w] = 1$ or $\delta E_2[r/w] = 1$ for every $r \in D^n$, by Lemma 5
$\Rightarrow M \models E_1[r/w]$ or $M \models E_2[r/w]$ for every $r \in D^n$, by induction hypothesis
$\Rightarrow M \models E_1[r/w] \vee E_2[r/w]$ for every $r \in D^n$
$\Rightarrow M \models \forall w(E_1 \vee E_2)$
$\Rightarrow M \models E_1 \vee E_2$

3. $\delta \forall w_0 E = 1 \Rightarrow \delta(E[r/w])\,[r_0/w_0] = 1$ for every $r \in D^n$, $r_0 \in D$, by Δ
$\Rightarrow M \models (E[r/w])\,[r_0/w_0]$, by induction hypothesis
$\Rightarrow M \models \forall w_0 E$.

The given algebraic construction of the Henkin models suggests naturally a first-order logical calculus $\vdash^L$ in equational form defined by

$$\Gamma \vdash^L E \Longleftrightarrow \Delta \cup \theta \cup H \vdash_e \delta E = 1$$

where

i. E is a first-order enunciate in the language L of Γ
ii. θ and Δ are as in Section 2, but in the language L_H obtained by adding to L the Henkin constants
iii. $H = \{\delta E \mid E'$ is a Henkin axiom for $\Gamma\}$.

Thus, we have the following

Corollary $\quad \Gamma \vdash^L E \Longleftrightarrow \Gamma \vdash E$

for any Hilbert-type first-order logical calculus $\vdash$.

Proof: The implication $\Rightarrow$ follows by the proof of previous Lemma 1. For the converse, it is convenient to consider the logical calculus $\vdash$ having as inference rules modus ponens and generalization and as axioms all tautologies and the schemata

$$E_1 \equiv \forall x A \to A[t/x] \qquad \text{(where } t \text{ is any term free for } x \text{ in } A\text{)}$$
$$E_2 \equiv \forall x(A \to B) \to (A \to \forall x B) \quad \text{(where all occurrences of } x \text{ in } A \text{ are bound)}$$

$\Gamma \models_e E_1$ because $\delta E_1 = 1$ is really Axiom Δ_4, and $\Gamma \models_e E_2$ because for any constant c by Δ_4 it follows that

$$\delta(\forall x(A \to B) \to (A \to \forall x B)) \geqslant \delta((A \to B[c/x]) \to (A \to B[c/x])),$$

by Δ_0 we have

$$\delta((A \to B[c/x]) \to (A \to B[c/x])) = 1,$$

therefore $\delta E_2 = 1$.

Further, the following deductions (where some obvious steps are omitted) show that modus ponens and generalization are derived rules in our calculus $\vdash'$. (For brevity we suppose that A and B have only one free variable w.)

1.
$$\begin{array}{lll}
\delta(A \to B) = 1 & & \\ \hline
\delta\pi(A \to B) = 1 & \delta A = 1 & \\ \hline
\delta(A \to B)[c_B/w] = 1 & \delta\pi A = 1 & c_B \text{ is the Henkin constant} \\ \hline
\delta({\sim}A' \vee B') = 1 & \delta A[c_B/w] = 1 & \text{associated with } B \\ \hline
-\delta A' + \delta B' = 1 & \delta A' = 1 & A' \equiv A[c_B/w] \\ \hline
(-\delta A') \cdot 1 + \delta B' = 1 & & B' \equiv B[c_B/w] \\ \hline
(-\delta A') \cdot \delta A' + \delta B' = 1 & & \\ \hline
0 + \delta B' = 1 & & \\ \hline
\delta B' = 1 & & \\ \hline
\delta{\sim}B' = 0 \quad \delta{\sim}\pi B \leqslant \delta{\sim}B' & & \text{(Henkin Axiom)} \\ \hline
\delta{\sim}\pi B = 0 & & \\ \hline
\delta B = 1 & &
\end{array}$$

2.
$$\begin{array}{ll}
\delta A = 1 \quad \delta A = \delta\forall w A & \text{(Axiom } \Delta_3\text{)} \\ \hline
\delta\forall w A = 1 &
\end{array}$$

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