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Number-Theoretic Set Theories

PAUL STRAUSS*

In Section 1 we shall describe a system, *WTN*, which is a natural extension of pure number theory, and where all individual variables range over the natural numbers. This system avoids Gödel constructions of undecidable sentences. In Section 2 we prove some elementary theorems in *WTN*. In Section 3 we consider ordinal numbers, and also indicate a proof of the axiom of choice. In Section 4 we consider cardinal numbers, in particular we show that all sets are countable in *WTN*. In Section 5 we consider real numbers. Here we discuss the problem of doing Lebesgue measure theory in *WTN*. In Section 5 we also consider a theorem related to Herbrand's Theorem. In Section 6 we consider related systems (this section can be read right after Section 2).

The system *WTN* was first announced in [7], and has certain similarities with [2], [12], and [13].

1 The system WTN Let *PN* be classical pure number theory, i.e., Peano arithmetic. For definiteness, we consider it formalized as in [4], p. 82, but for simplicity we consider $\sim$ for negation, $\supset$ for implication, and (z) for universal quantification as the only primitive logical connectives (cf. [4], p. 406, Ex. 2). Furthermore, we identify the symbols of the system and strings of symbols with their Gödel numbers according to a customary assignment. We use the logical notation of [5], in particular the dot notation. And we shall use $x_1, x_2, \ldots$ as the individual variables. We sometimes use x, y for x_1, x_2, respectively. We use z, w, u, v, p, q, r, s, t, c, d, with or without subscripts, as meta-variables ranging over the individual variables of our system. We use a and b as terms.

We identify natural numbers with nonnegative integers, although it is of some interest to consider them identified with positive integers instead, particularly in our treatment of real numbers (cf. Section 5), but we shall not do so here.

There is a primitive recursive function ν such that if m is a natural number then $\nu(m)$ is the numeral of m (hence $\nu(m)$ is also a natural number). We sometimes write $\underline{0}, \underline{1}, \ldots, \underline{m}$ for $\nu(0), \nu(1), \ldots, \nu(m)$, respectively.

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If f is a (general) recursive function, then it is numeralwise representable in PN (cf. [4], pp. 200, 295); i.e., there exists a formula $P(u_1, \ldots, u_k, w)$ such that: If $f(m_1, \ldots, m_k) = n$, then $\vdash P(\underline{m}_1, \ldots, \underline{m}_r, \underline{n})$ where $\vdash$ stands for provability in PN; and also, we have $\vdash (E!\,w)P(\underline{m}_1, \ldots, \underline{m}_k, w)$. We suppose that there is an effective procedure for constructing a unique such formula $P(u_1, \ldots, u_k, w)$ when given a recursive f. That is, we assume given a recursive map M such that, if e is the Gödel number of f, then $M(e) = [P(u_1,, \ldots, u_k, w)]$.

Then, for any formula $F(v)$, we write $F(\underline{f}(u_1, \ldots, u_k))$ for $(Ez)\,.\,P(u_1, \ldots, u_k, z)\ F(z)$ where z is free for w in $P(u_1, \ldots, u_k, w)$ and free for v in $F(v)$. When more than one function is represented in a formula, we suppose an effective procedure for eliminating them one at a time (see [4], p. 407).

We adjoin to PN the primitive symbols $T_1, T_2, \ldots$ (or more precisely, $T_1, T_2, \ldots$ are new distinct positive integers of our formal system). If a is a term, then we stipulate that $T_n(a)$ is a formula. We say that the *degree* of a formula A is n, if T_n occurs in A but no T_m occurs for $m > n$. If no T_n occurs in A, we say that its *degree* is 0. We now adjoin the following "truth axioms" or "T-axioms" which are influenced by [10]:

> If A is a sentence (i.e., closed formula) of degree $\leq n$, then $T_{n+1}(\underline{A}) \equiv A$ is an axiom. We call this extended system TN.

The system TN seems to require an ω-rule in order to fully utilize the T-axioms. The author has tried to define such a rule in a manner which is as close to giving a formal system as possible, and which is relatively easy to apply. Other possibilities also present themselves, but the following (viz. rule W, considered in [6]) appears to be the most natural to use. When rule W is added to TN, we get the system WTN which we shall define inductively.

We shall assume that proofs in TN are in tree form and arithmetized in the manner of [4]. We also use the recursive function notation of [4]. If A is a formula of TN, we write:

> $\mathcal{P}\mathfrak{f}_0(A,k)$ for "k is a proof of A in TN, hence $(k)_0 = A$".
>
> $\mathcal{P}\mathfrak{f}_{n+1}(A,k)$ for "$\mathcal{P}\mathfrak{f}_n(A,k)$; or there exists a formula C and natural numbers i and j such that $\mathcal{P}\mathfrak{f}_n(C,i)$ and $\mathcal{P}\mathfrak{f}_n(C \supset A, j)$ and $k = 2^A \cdot 3^i \cdot 5^j$; or there exists a natural number e and formula $B(u)$ such that for every m, $\mathcal{P}\mathfrak{f}_n(B(\underline{m}), \{e\}(m))$ and A is $(u)B(u)$ with $k = 2^A 3^e$".

The last disjunct above with $\{e\}(m)$ in it, we call rule W. We say that A is provable in WTN, if there exists a k and n such that $\mathcal{P}\mathfrak{f}_n(A,k)$. The fact that WTN avoids Gödel constructions of undecidable sentences is due to rule W.

Observe that instead of having TN as our base system, we can eliminate rule 9 of [4], (p. 82) which states: from $C \supset A(u)$ where u is not free in C, infer $C \supset (u)A(u)$. And we can also eliminate the axiom schema of mathematical induction. Both of these postulates can be proved using rule W (cf. [4], p. 406, Ex. 2).

Observe that we can only use rule W a finite number of times in WTN, but this is all we apparently need in our developments. In [6] Shoenfield apparently allows a transfinite number of uses of rule W. Specifically, if a is a Kleene constructive ordinal notation, one can easily define $\mathcal{P}\mathfrak{f}_a(A,k)$ and hence the system W^*TN. If we allow T_a to be a primitive symbol and add the obvious T-axioms, we get the system W^*T^*N.

For $k \geq 1$, there exists a recursive function σ^k such that $\sigma^k(m_1, \ldots, m_k, n)$ is the result of simultaneously substituting m_1 for $x_1, \ldots, m_k$ for x_k in n^*, where n^* is an effectively chosen alphabetic variant of n, so that for every m_i, m_i is free for x_i in n^*.

For $k \geq 1$, write $S^k_{n+1}(a_1, \ldots, a_k, b)$ for $T_{n+1}(\underline{\sigma}^k(\underline{\nu}(a_1), \ldots, \underline{\nu}(a_k), b))$. We usually omit the superscript k, and for $k > 1$, we write a semicolon before b rather than a comma. If $m_1, \ldots, m_k$ are natural numbers and A is a formula of degree $\leq n$, then $S_{n+1}(\underline{m}_1, \ldots, \underline{m}_k; \underline{A})$ states that the ordered k-tuple $\langle m_1, \ldots, m_k \rangle$ satisfies the predicate expressed in our system by A. This is the crucial notion of our theory! Using it, we see that set-theoretic notions can be expressed in our system WTN, which is a natural extension of pure number theory. In particular, $S_{n+1}(x, y)$ is interpreted as $x \in y$.

Define $\theta(m,n) = [(x) \,.\, m \equiv n]$. We write $E_{n+1}(a,b)$ for $T_{n+1}(\underline{\theta}(a,b))$. If A and B are formulas of degree $\leq n$ whose only free variable is x (i.e., x_1), then $E_{n+1}(\underline{A},\underline{B})$ states that the "sets" A and B have the same members. Observe our procedure of choosing subscripts so that the degree of a formula is at least equal to the largest subscript occurring in it.

We use the turnstile, $\vdash$, for provability in WTN.

Using the notion of degree, we are prevented from defining things by means of a vicious circle, or by impredicative definitions (cf. [4], p. 42; [3], p. 37, Def. 2; and [13]). In other words, we are able to assert truths only over things that have been (or could have been) defined previously.

As an elementary illustration, let us consider Russell's paradox. Write A for $\sim S_{n+1}(x,x)$. Then we get $\vdash S_{n+2}(\underline{A},\underline{A}) \equiv T_{n+2}(\underline{\sigma}(\underline{\nu}(\underline{A}), \underline{A}) \equiv T_{n+2}(\underline{\sigma(\underline{A},\underline{A}}) \equiv$ $\sim S_{n+1}(\underline{A},\underline{A})$ which is not contradictory. Observe that the second equivalence above requires us to first establish $\vdash x = y \supset . \, T_{n+1}(x) \equiv T_{n+1}(y)$ which is our first theorem of Section 2 (cf. [4], p. 408).

2 Some elementary theorems

T1 $\quad \vdash x = y \supset . \, T_{n+1}(x) \equiv T_{n+1}(y)$.

Proof: We prove this by two uses of rule W. First, let us choose a particular natural number m, and concentrate on establishing $\vdash x = \underline{m} \supset . \, T_{n+1}(x) \equiv T_{n+1}(\underline{m})$ (which we abbreviate as B_m). Next, we wish to exhibit a recursive function f_m such that for every i, $f_m(i)$ is a proof in TN of $\underline{i} = \underline{m} \supset . \, T_{n+1}(\underline{i}) \equiv T_{n+1}(\underline{m})$ (which we abbreviate as B_{mi}). The proofs of B_{mi} are constructed in essentially two different manners, depending on whether $i = m$ or not. If $i = m$, we first construct a proof of $T_{n+1}(\underline{i}) \equiv T_{n+1}(\underline{m})$, and then of B_{mi} by the propositional calculus. If $i \neq m$, we first construct a proof of $\underline{i} \neq \underline{m}$ using the two Peano axioms $x' = y' \supset x = y$ and $x' \neq \underline{0}$ (cf. [4], p. 82). We then prove B_{mi} using the propositional calculus. Hence we've indicated how to construct f_m. Furthermore, this f_m has been effectively given, i.e., we can construct a Gödel number e_m of f_m. Now employ rule W to obtain $(x)B_m$. Now these e_m's can be effectively enumerated. So, use rule W again to obtain our theorem.

Rule W was used differently in the proof of T1 than in the proofs of theorems to follow. In all other cases, rule W is used in conjunction with the truth axioms.

Let ϕ_n^i be a recursive function that enumerates all formulas A of WTN such that the degree of A is $\leq n$, and for every $j > i$, x_j does not occur free in A. Hence ϕ_n^1 enumerates the sets of degree $\leq n$, and ϕ_n^i enumerates the i-ary relations of degree $\leq n$. We write ${}^i x_j^n$ for $\underline{\phi_n^i(x_j)}$. We also write ${}^i \underline{m}^n$ for $\underline{\phi_n^i(\underline{m})}$ and ${}^i m^n$ for $\phi_n^i(m)$. For $i = 1$, we usually omit the superscript i.

T2.1 $\vdash E_{n+1}(x^n, x^n)$.

T2.2 $\vdash E_{n+1}(x^n, y^n) \supset E_{n+1}(y^n, x^n)$.

T2.3 $\vdash E_{n+1}(x^n, y^n) E_{n+1}(y^n, z^n) \supset E_{n+1}(x^n, z^n)$.

Proof of T2.1: We need to construct a recursive function f such that for every m, $f(m)$ is a proof of $E_{n+1}(\underline{\phi_n(\underline{m})}, \underline{\phi_n(\underline{m})})$. For each m, first construct a proof of $(x) . \phi_n(m) \equiv \phi_n(m)$. Then, using the relevant truth axiom, T1, and the fact that θ and ϕ_n are recursive, we construct the desired f. Finally, use rule W.

The proofs of T2.2 and T2.3, although a little longer, are also easily established.

T3 $\vdash (u_1, \ldots, u_k) . S_{n+1}(u_1, \ldots, u_k; \underline{\phi_n^k(\underline{m})}) \equiv \sigma(u_1, \ldots, u_k; \phi_n^k(m))$.

Proof: $S_{n+1}(\underline{i}_1, \ldots, \underline{i}_k; \underline{\phi_n^k(\underline{m})}) \equiv T_{n+1}(\underline{\sigma}(\underline{\nu}(\underline{i}_1), \ldots, \underline{\nu}(\underline{i}_k); \underline{\phi_n^k(\underline{m})})) \equiv T_{n+1}(\underline{\sigma(\nu(i_1), \ldots, \nu(i_k); \phi_n^k(m))}) \equiv \sigma(\underline{i}_1, \ldots, \underline{i}_k; \phi_n^k(m))$ and use rule W k times.

T4 *If A and B are in the range of ϕ_n, then*

$$\vdash E_{n+1}(\underline{A}, \underline{B}) . \equiv . (x) . S_{n+1}(x, \underline{A}) \equiv S_{n+1}(x, \underline{B}) \ .$$

Proof: Use T3.

The usual Boolean operations on sets and other operations such as the power set are easily defined in WTN. We shall restrict our attention to the intersection of two sets. Let & be the recursive function such that $\&(i,j) = [i \& j]$. Then we have:

T5 $\vdash (u) : S_{n+1}(u, \underline{\&}(v^n, w^n)) \equiv . S_{n+1}(u, v^n) \& S_{n+1}(u, w^n)$.

Proof: Use rule W twice.

For $k \geq 1$, write $Ext_{n+2}^k(a)$ for $(u_1, \ldots, u_k, v_1 \ldots, v_k) : E_{n+1}(u_1^n, v_1^n) . \ldots . E_{n+1}(u_k^n, v_k^n) . S_{n+2}(u_1^n, \ldots, u_k^n; a) . \supset S_{n+2}(v_1^n, \ldots, v_k^n; a)$. For $k = 1$, we omit the superscript k. An a for which $Ext_{n+2}^k(a)$ satisfies the *axiom of extensionality.* In Sections 3, 4, and 5 we have attempted to observe the axiom of extensionality where feasible, hence our theory is essentially an *extensional theory.* But, by ignoring the axiom of extensionality in our developments in WTN, one would obtain an *intensional theory* which might also be of interest. We shall not consider such a theory, however.

3 Ordinal numbers Let A be a formula in the range of ϕ_{n+1}^2, i.e., a binary relation. We often wish to consider the field of A. We now construct a recursive function α such that $\alpha(A)$ is construed to be the set corresponding to the field of A. Hence $\alpha_{n+1}(m) = [(Eu)\sigma(x,u^n;\ m) \vee (Eu)\sigma(u^n,x;\ m)]$. When it is understood by context which α_{n+1} is being used, we usually omit the subscript $n+1$.

We write

$$Con_{n+2}(a) \text{ for } (Ez)a = {}^2z^{n+1} . (u,v) : S_{n+2}(u^n,\underline{\alpha}(a))S_{n+2}(v^n,\underline{\alpha}(a)) . \supset . S_{n+2}(u^n,v^n;\ a) \vee S_{n+2}(v^n,u^n;\ a) \ .$$

In the above, α refers to α_{n+1} since the members of $\underline{\alpha}(a)$ are construed to have degree $\leq n$. $Con_{n+2}(a)$ states that the relation a is connected.

We now define the notion of a well-ordered relation:

$$Wor_{n+2}(a) \text{ for } Con_{n+2}(a) :. (u) :. Ext_{n+2}(u^{n+1}) . (Ev) . S_{n+2}(v^n,u^{n+1})S_{n+2}(v^n,\underline{\alpha}(a)) : \supset : (Ev) : S_{n+2}(v^n,u^{n+1}) S_{n+2}(v^n,\underline{\alpha}(a)) . (w) . S_{n+2}(w^n,u^{n+1})S_{n+2}(w^n,v^n;\ a) \supset E_{n+1}(v^n,w^n) \ .$$

We now define the notion of ordinal similarity (or ordinal equivalence). The definition below may look rather formidable, but really is rather easy to read. The first conjunct states that ${}^2r^{n+1}$ is extensional, the next two state that ${}^2r^{n+1}$ is $1-1$, the next two after that state that ${}^2r^{n+1}$ is onto, and the last two state that ${}^2r^{n+2}$ is order preserving. Now the definition:

$$\begin{aligned} OE_{n+2}(a,b) \text{ for } (Er) :: Ext^2_{n+2}({}^2r^{n+1}) : (t,u,v) . \\ S_{n+2}(t^n,u^n;\ {}^2r^{n+1})S_{n+2}(t^n,v^n;\ {}^2r^{n+1}) \supset E_{n+1}(u^n,v^n) . \\ S_{n+2}(t^n,v^n;\ {}^2r^{n+1})S_{n+2}(u^n,v^n;\ {}^2r^{n+1}) \supset E_{n+1}(t^n,u^n) :. \\ (u) : S_{n+2}(u^n,\underline{\alpha}(a)) . \supset . (Ev) . S_{n+2}(v^n,\underline{\alpha}(b)) \\ S_{n+2}(u^n,v^n;\ {}^2r^{n+1}) :. (u) : S_{n+2}(u^n,\underline{\alpha}(b)) . \supset . (Ev) . \\ S_{n+2}(v^n,\underline{\alpha}(a))S_{n+2}(v^n,u^n;\ {}^2r^{n+1}) :. (u,v) : \\ S_{n+2}(u^n,v^n;\ a) . \supset . (Es,t) . S_{n+2}(s^n,t^n;\ b) \\ S_{n+2}(u^n,s^n;\ {}^2r^{n+1})S_{n+2}(v^n,t^n;\ {}^2r^{n+1}) :. \\ (u,v) : S_{n+2}(u^n,v^n;\ b) . \supset . (Es,t) . S_{n+2}(s^n,t^n;\ a) \\ S_{n+2}(s^n,u^n;\ {}^2r^{n+1})S_{n+2}(t^n,v^n;\ {}^2r^{n+1}) \ . \end{aligned}$$

It is not too difficult to establish that $OE_{n+2}(x,y)$ is an equivalence relation even though the proof is rather long.

We now wish to define the notion of an ordinal number as an equivalence class of $OE_{n+2}(x,y)$ which has a well-ordered representative. So we write

$$Orp_{n+3}(a,b) \text{ for } Wor_{n+2}(b) . (w) . S_{n+3}(w,a) \equiv OE_{n+2}(b,w) \ .$$

Read as "b ordinally represents a".

$$Ord_{n+3}(a) \text{ for } (Ez) . Ext^2_{n+2}({}^2z^{n+1}) . Orp_{n+3}(a,{}^2z^{n+1}) \ .$$

Read as "a is an ordinal number".

Let $\kappa_{n+1}(m,k) = [m . \sigma(y,\underline{k};\ m) . {\sim}E_{n+1}(y,\underline{k})]$. If m is a relation and $k \in \alpha(m)$, then $\kappa_{n+1}(m,k)$ is the initial segment of m determined by k.

There appears to be no difficulty in establishing the basic theorems concerning $Wor_{n+2}(a)$, e.g.,

T6 $\vdash Wor_{n+2}({}^2r^{n+1}) . Ext^2_{n+2}({}^2r^{n+1}) : \supset : (u,v) . S_{n+2}(u^n,v^n;\ {}^2r^{n+1}) S_{n+2}(v^n,u^n;\ {}^2r^{n+1}) \supset E_{n+1}(u^n,v^n)$.

We shall consider one nontrivial and basic theorem. It is the well-known theorem that states: A subset of an initial segment of a well-ordered set is not ordinally equivalent to that set. Translated into *WTN*, we have:

T7 $\vdash Wor_{n+2}({}^2y^{n+1}) \,.\, Ext^2_{n+2}({}^2y^{n+1}) \,.\, S_{n+2}(w^n, \underline{\alpha}({}^2y^{n+1}))\,.$
$(u,v)\,.\, S_{n+2}(u^n, v^n;\ {}^2z^{n+1}) \supset S_{n+2}(u^n, v^n;\ \underline{\kappa}_{n+1}({}^2y^{n+1}, w^n)):$
$\supset : \sim OE_{n+2}({}^2y^{n+1}, {}^2z^{n+1})$.

For the proof of T7 and elsewhere we need the following functions: $\xi(m,n) = m^n = m\ exp\ n$, $\zeta(m,n) = (m)_n$, and $\pi(n) = p_n$ (cf. [4], pp. 222 and 230). T7 is the analogue of Theorem XII. 2.1 of [5], p. 459.

Proof: The crux of the following proof is the variable c. We have, on one hand, an infinite strictly descending sequence $(c)_0$, $(c)_1, \ldots,$ of elements of ${}^2y^{n+1}$, where $(c)_0 = w^n$, or, more precisely, $(c)_0 = \underline{k}^n$ in preparation to rule W. But on the other hand, since ${}^2y^{n+1}$ is well-ordered, this sequence must have a least element, hence a contradiction, which proves our theorem. Observe that f below enumerates this descending sequence. A rigorous proof of T7, however, is not easy, and now follows:

As we've said, we prove T7 by contradiction. Write $F(y, z, w, r)$ for the conjunction of the antecedent of T7 with $OE_{n+2}({}^2y^{n+1}, {}^2z^{n+1})$ except that in the latter factor, "(Er)" is removed. We assume $F(\underline{i}, \underline{j}, \underline{k}, \underline{m})$ (in preparation to four applications of rule W). Write $G(c, v)[k, m, i]$ for $\underline{\zeta}(c, \underline{0}) = \underline{k}^n :. (u):$ $u < v \supset .\, \sigma(\underline{\zeta}(c, u), \underline{\zeta}(c, u');\ {}^2m^{n+1})\sigma(\underline{\zeta}(c, u'), \alpha({}^2i^{n+1}))\,.\,(\underline{Ed})\,.\,\underline{\zeta}(c, u') = d^n$. We now establish a lemma:

L $F(\underline{i}, \underline{j}, \underline{k}, \underline{m}) \vdash (c,v) : G(c,v)[k,m,i] \supset .\, (u)\,.\, u < v \supset .$
$\sigma(\underline{\zeta}(c,u'), \underline{\zeta}(c,u);\ {}^2i^{n+1})\,.\, \sim E_{n+1}(\underline{\zeta}(c,u'), \underline{\zeta}(c,u))$.

L Proof: By induction on u. Assume $F(\underline{i}, \underline{j}, \underline{k}, \underline{m})$, $G(c, v)[k, m, i]$. Basis: And assume $\underline{0} < v$. So $\sigma(\underline{\zeta}(c, \underline{0}), \underline{\zeta}(c, \underline{1});\ {}^2m^{n+1})$. And since $S_{n+2}(\underline{\zeta}(c, \underline{0}),$ $\underline{\alpha}({}^2\underline{i}^{n+1}))$, we get (using rule C of [5] on t, i.e., existential specialization) $S_{n+2}(t^n, \underline{\alpha}({}^2\underline{j}^{n+1}))S_{n+2}(\underline{\zeta}(c, \underline{0}), t^n;\ {}^2\underline{m}^{n+1})$ from the definition of $OE_{n+2}({}^2\underline{i}^{n+1}, {}^2\underline{j}^{n+1})$. So $E_{n+1}(\underline{\zeta}(c, \underline{1}), t^n)$. We now wish to establish $S_{n+2}(t^n,$ $\underline{k}^n;\ {}^2\underline{i}^{n+1})\,.\, \sim E_{n+1}(t^n, \underline{k}^n)$. We do this by rule W on p and later on q (we use p and q to range over natural numbers only for a short while). We have

$$\begin{aligned} &S_{n+2}(\underline{p}^n, \underline{\alpha}({}^2\underline{j}^{n+1}))\,.\equiv.\, T_{n+2}(\sigma(\underline{\nu}(\underline{p}^n), \underline{\alpha}({}^2\underline{j}^{n+1})))\,.\equiv.\\ &\quad (Eu)\sigma(\underline{p}^n, u^n;\ {}^2j^{n+1}) \vee (Eu)\sigma(u^n, \underline{p}^n;\ {}^2j^{n+1})\,.\equiv.\\ &\quad (Eu)S_{n+2}(\underline{p}^n, u^n;\ {}^2\underline{j}^{n+1}) \vee (Eu)S_{n+2}(u^n, \underline{p}^n;\ {}^2\underline{j}^{n+1})\,.\supset.\\ &\quad (Eu)S_{n+2}(\underline{p}^n, u^n;\ \underline{\kappa}_{n+1}({}^2\underline{i}^{n+1}, \underline{k}^n) \vee (Eu)S_{n+2}(u^n, \underline{p}^n;\ \underline{\kappa}_{n+1}({}^2\underline{i}^{n+1}, \underline{k}^n))\ . \end{aligned}$$

Each disjunct gives the desired result, so let us consider just the first one. So assume $S_{n+2}(\underline{p}^n, \underline{q}^n;\ \underline{\kappa}_{n+1}({}^2\underline{i}^{n+1}, \underline{k}^n))$ (in preparation for rule W on $\underline{q}$). So $T_{n+2}(\underline{\sigma}(\underline{\nu}(\underline{p}^n), \underline{\nu}(\underline{q}^n);\ \underline{\kappa}_{n+1}({}^2\underline{i}^{n+1}, \underline{k}^n)))$. So $\sigma(\underline{p}^n, \underline{q}^n;\ \kappa_{n+1}({}^2i^{n+1}, k^n))$. So $\sigma(\underline{p}^n, \underline{q}^n;\ {}^2i^{n+1})\,.\, \sigma(\underline{q}^n, \underline{k}^n;\ {}^2i^{n+1})\,.\, \sim E_{n+1}(\underline{q}^n, \underline{k}^n)$. Since $Wor_{n+2}({}^2\underline{i}^{n+1})$, we get by transitivity $\sigma(\underline{p}^n, \underline{k}^n;\ {}^2\underline{i}^{n+1})$. Now if $E_{n+1}(\underline{p}^n, \underline{k}^n)$, then $\sigma(\underline{q}^n, \underline{p}^n;$ ${}^2i^{n+1})$. Hence $E_{n+1}(\underline{q}^n, \underline{p}^n)$ by antisymmetry, so $E_{n+1}(\underline{q}^n, \underline{k}^n)$ which contradicts. Hence $\sim E_{n+1}(\underline{p}^n, \underline{q}^n)$. So we've established $S_{n+2}(\underline{p}^n, \underline{\alpha}({}^2\underline{j}^{n+1})) \supset .$ $S_{n+2}(\underline{p}^n, \underline{k}^n;\ {}^2\underline{i}^{n+1}) \sim E_{n+1}(\underline{p}^n, \underline{k}^n)$. The other disjunct gives the same result. Rule W and the fact that $S_{n+2}(t^n, \underline{\alpha}({}^2\underline{j}^{n+1}))$ give $S_{n+2}(t^n, \underline{k}^n;\ {}^2\underline{i}^{n+1})$

$\sim E_{n+1}(t^n, \underline{k}^n)$. Since $Ext^2_{n+2}({}^2\underline{i}^{n+1})\,.\,E_{n+1}(\underline{\zeta}(c, \underline{0}), \underline{k}^n)$, we get $S_{n+2}(\underline{\zeta}(c, \underline{1}), \underline{\zeta}(c, \underline{0}); {}^2\underline{i}^{n+1})\,.\sim E_{n+1}(\underline{\zeta}(c, \underline{1}), \underline{\zeta}(c, \underline{0}))$.

Now for the induction hypothesis: And assume $u' < v$. Induction hypothesis gives $S_{n+2}(\underline{\zeta}(c, u'), \underline{\zeta}(c, u); {}^2\underline{i}^{n+1})$. $\sim E_{n+1}(\underline{\zeta}(c, u'), \underline{\zeta}(c, u))$. So (Es, t). $S_{n+2}(s^n, t^n; {}^2\underline{j}^{n+1})\ S_{n+2}(\underline{\zeta}(c, u'), s^n; {}^2\underline{m}^{n+1})\ S_{n+2}(\underline{\zeta}(c, u), t^n; {}^2\underline{m}^{n+1})$. But $S_{n+2}(\underline{\zeta}(c, u'), \underline{\zeta}(c, u''); {}^2\underline{m}^{n+1})\ S_{n+2}(\underline{\zeta}(c, u), \underline{\zeta}(c, u'); {}^2\underline{m}^{n+1})$. So $E_{n+1}(\underline{\zeta}(c, u''), s^n)E_{n+1}(\underline{\zeta}(c, u'), t^n)$. So $S_{n+2}(\underline{\zeta}(c, u''), \underline{\zeta}(c, u'); {}^2\underline{i}^{n+1})$. Now suppose $E_{n+1}(\underline{\zeta}(c, u'', \underline{\zeta}(c, u'))$. Since $Ext^2_{n+2}({}^2\underline{m}^{n+1})$, we get $S_{n+2}(\underline{\zeta}(c, u), \underline{\zeta}(c, u''); {}^2\underline{m}^{n+1})$. Hence $E_{n+1}(\underline{\zeta}(c, u'), \underline{\zeta}(c, u))$ which contradicts. Hence $\sim E_{n+1}(\underline{\zeta}(c, u''), \underline{\zeta}(c, u'))$. This proves L.

Now let

$$f(k,m,i) = [(Ec,v) :.\ G(c,v)[k,m,i] : (u)\,.\,\underline{\zeta}(c,v'+u) = \underline{0} : E_{n+1}(\underline{\zeta}(c,v),x)]\,.$$

We wish to use $\underline{f}(\underline{k}, \underline{m}, \underline{i})$ as a u^{n+1} in $Wor_{n+2}({}^2\underline{i}^{n+1})$. First observe that $Ext_{n+2}(\underline{f}(\underline{k}, \underline{m}, \underline{i}))$. This follows easily from $\vdash G(c, v)[k, m, i] \supset .\,(Ed)\,.\,\underline{\zeta}(c, v) = d^n$ which is easily proved by induction on v.

Since $\vdash G(\underline{\xi}(\underline{2}, \underline{k}^n), \underline{0})[k, m, i]$, we get $\vdash \sigma(\underline{k}^n, f(k, m, i))$, hence $\vdash S_{n+2}(\underline{k}^n, \underline{f}(\underline{k}, \underline{m}, \underline{i}))$. Also $S_{n+2}(\underline{k}^n, \underline{\alpha}({}^2\underline{i}^{n+1}))$ from $F(\underline{i}, \underline{j}, \underline{k}, \underline{m})$. Hence $(Et) : S_{n+2}(t^n, \underline{f}(\underline{k}, \underline{m}, \underline{i}))\,.\,S_{n+2}(t^n, \underline{\alpha}({}^2\underline{i}^{n+1}))\,.\,(s)\,.\,S_{n+2}(s^n, \underline{f}(\underline{k}, \underline{m}, \underline{i}))\ S_{n+2}(s^n, t^n; {}^2\underline{i}^{n+1}) \supset E_{n+1}(t^n, s^n)$. Remove "$(Et)$" above, by rule C. Since $\vdash S_{n+2}(t^n, \underline{f}(\underline{k}, \underline{m}, \underline{i})) \equiv \sigma(t^n, f(k, m, i))$, we get $\sigma(t^n, f(k, m, i))$. Also, $S_{n+2}(t^n, \underline{\alpha}({}^2\underline{i}^{n+1}))$ gives $(Eq)\,.\,S_{n+2}(q^n, \underline{\alpha}({}^2\underline{j}^{n+1}))\,.\,S_{n+2}(t^n, q^n; {}^2\underline{m}^{n+1})$. Now use rule C on q. Write b for $c\,.\,\underline{\xi}(\underline{\pi}(v'), q^n)$. So we get $G(b, v')[k, m, i]$. This follows from $\vdash (u) : u < v' \supset \underline{\zeta}(c, u) = \underline{\zeta}(b, u)$ and $\vdash (c, v) : (u)\,.\,\underline{\zeta}(c, v' + u) = \underline{0}\,.\supset.\,(q)\,.\,\underline{\zeta}(b, v') = q^n$. These two statements can be proved in PN by formalizing Gödel's β-function (cf. [4], p. 244, Remark 1) or, from the numeralwise representability of ζ, ξ, π, we simply use rule W.

Now L gives $S_{n+2}(\underline{\zeta}(b, v'), \underline{\zeta}(b, v); {}^2\underline{i}^{n+1})\ \sim E_{n+1}(\underline{\zeta}(b, v'), \underline{\zeta}(b, v))$. But $E_{n+1}(\underline{\zeta}(b, v), t^n)\,.\,\underline{\zeta}(b, v') = q^n$, by definition of f, last conjunct. So $S_{n+2}(q^n, t^n; {}^2\underline{i}^{n+1})$, so $E_{n+1}(q^n, t^n)$ which contradicts. Hence we've established T7 for $\underline{i}$, $\underline{j}$, $\underline{k}$, $\underline{m}$ in place of y, z, w, r respectively. Now use rule W four times.

The axiom of choice and the well-ordering of the universe appear to be provable in *WTN*, as we indicate below.

Write

$$VO_{n+1} \text{ for } (Eu) : E_{n+1}(u^n, x)\,.\,(v)\,.\,E_{n+1}(v^n, y) \supset u \le v\ .$$
$$V^i_n \text{ for } (Eu)\,.\,{}^iu^n = x\ .$$

For $i = 1$, we usually omit the superscript.

Observe, in passing, that if V is $x = x$, we get $S_{n+1}(\underline{V}, \underline{V})$, so the axiom of foundation can be refuted in *WTN*.

By means of VO_{n+1}, we may well order V_n. We also use VO_{n+1} for a proof of the axiom of choice, which we now state:

T8 $\vdash (x) : S_{n+3}(x^{n+1}, z^{n+2})\,.\supset.\,Ext_{n+2}(x^{n+1})\,.\,(Ey)\,.\,S_{n+2}(y^n, x^{n+1})\,.:\supset:.\,(Er) :.\,(u, v, w)\,.\,S_{n+3}(u, v^n; {}^2r^{n+2})\ S_{n+3}(u, w^n; {}^2r^{n+2}) \supset E_{n+1}(v^n, w^n) : (x) : S_{n+3}(x^{n+1}, z^{n+2}) \supset .\ (Ey)\,.\,S_{n+3}(x^{n+1}, y^n; {}^2r^{n+2})\ S_{n+2}(y^n, x^{n+1})$.

To prove this, we write R for $S_{n+2}(y, x) . (w) . S_{n+2}(w^n, x) \supset VO_{n+1}(y, w^n)$. Then R is the desired "choice function".

4 Cardinal numbers We leave it to the reader to define the notion of cardinal equivalence, $CE_{n+2}(a,b)$. It is similar to the definition of $OE_{n+2}(a,b)$ but shorter.

It is not too difficult to establish that $CE_{n+2}(x,y)$ is an equivalence relation.

We now wish to define a cardinal number as an equivalence class of $CE_{n+2}(x,y)$. So we write

$$Crp_{n+3}(a,b) \text{ for } (w) . S_{n+3}(w,a) \equiv CE_{n+2}(b,w) \ .$$

Read as "b cardinally represents a".

$$Card_{n+3}(a) \text{ for } (Ez) . Ext_{n+2}(z^{n+1}) . Crp_{n+3}(a, z^{n+1}) \ .$$

Read as "a is a cardinal number".

We now define the countable cardinals in such a way that they are extensional:

$$\text{Let } \psi(k) = [x' = \underline{k}],\ \gamma_{n+1}(0) = \psi(0),\ \gamma_{n+1}(k+1) = [\gamma_{n+1}(k) \vee E_{n+1}(x, \underline{\psi}(\underline{k})],\ \text{and } \bar{\gamma}_{n+2}(k) = CE_{n+2}(x, \underline{\gamma}_{n+1}(\underline{k})).$$

We see that $\gamma_{n+1}(k)$ is an extensional set with k distinct members, and that $\vdash Card_{n+3}(\underline{\gamma}_{n+2}(\underline{k}))$. We easily get such theorems as $\vdash x \neq y \supset {\sim}E_{n+1}(\underline{\psi}(x), \underline{\psi}(y))$, and $\vdash x \neq y \supset {\sim}CE_{n+2}(\underline{\gamma}_{n+1}(x), \underline{\gamma}_{n+1}(y))$. We write

$$N_{n+1} \text{ for } (Ez)E_{n+1}(x, \underline{\psi}(z))$$

and

$$\bar{\omega}_{n+2} \text{ for } CE_{n+2}(x, \underline{N_{n+1}}) \ .$$

We easily get $\vdash Card_{n+3}(\underline{\bar{\omega}_{n+2}})$.

We now prove that *all* sets are countable:

T9 $\vdash (x) : (Ez)CE_{n+2}(x^{n+1}, \underline{\gamma}_{n+1}(z)) \vee CE_{n+2}(x^{n+1}, \underline{N_{n+1}})$.

Proof: We consider two cases.

Case 1. $(Ew)(u) : S_{n+2}(u^n, x^{n+1}) \supset . (Ev) . E_{n+1}(u^n, v^n) . v < w . S_{n+2}(v^n, x^{n+1})$. We abbreviate this formula to $(Ew)A(w)$. Note that x also occurs free in $A(w)$.

We shall prove:

(1) $(w, x) : A(w) \supset (Ez)CE_{n+2}(x^n, \underline{\gamma}_{n+1}(z))$ by induction on w. Basis: $w = \underline{0}$. Then $A(\underline{0})$ yields that x^{n+1} is empty, hence $CE_{n+2}(x^{n+1}, \underline{\gamma}_{n+1}(\underline{0}))$.

Now assume induction hypothesis and $A(w')$. If $A(w)$, then there is no trouble, so assume ${\sim}A(w)$, i.e.,

(2) $(Eu) :. S_{n+2}(u^n, x^{n+1}) : (v) : E_{n+1}(u^n, v^n) . S_{n+2}(v^n, x^{n+1}) . \supset w \leq v.$

There is a least such u that satisfies (2). By rule C, take p for that u. We get $w \leq p$. But $A(w')$ yields $p < w'$. Hence $p = w$. Let $g(j, k) = [j \,.\, {\sim}E_{n+1}(x, \underline{k})]$. Assume $S_{n+2}(u^n, \underline{g}(x^{n+1}, w^n))$. Then $(Ev) \,.\, E_{n+1}(u^n, v^n) \,.\, v < w.\ S_{n+2}(v^n, \underline{g}(x^{n+1}, w^n))$ since $A(w')$, $v < w'$, and ${\sim}E_{n+1}(v^n, w^n)$, where we use (1) with $\underline{g}(x^{n+1}, w^n)$ in place of x^{n+1}. So, by induction hypothesis, we get $(Ez)\overline{C}E_{n+2}(\underline{g}(x^{n+1}, w^n), \underline{\gamma}_{n+1}(z))$. Use rule C on z and r where ${}^2r^{n+1}$ is the equivalence relation involved. Let $h(i, j, k) = [i \vee . E_{n+1}(x, \underline{j})\, E_{n+1}(y, \underline{k})]$. Then $CE_{n+2}(x^{n+1}, \underline{\gamma}_{n+1}(z'))$ where the equivalence relation involved is $\underline{h}({}^2r^{n+1}, w^n, \underline{\psi}(z))$. Hence Case 1 yields the first disjunct of T9.

Case 2. The negation of Case 1, i.e.,

(3) $(w)(Eu) :. S_{n+2}(u^n, x^{n+1}) : (v) : E_{n+1}(u^n, v^n) \,.\, S_{n+2}(v^n, x^{n+1}) \,.\, \supset w \leq v.$

This proof is fashioned after the one for T7. In this case c gives a 1-1 correspondence between x^{n+1} and $\underline{N}_{n+1}$. This correspondence has the form

$$c = 2^{2^{a_0 3^0}} . 3^{2^{a_1 3^1}} \ldots p_k^{2^{a_k . 3^k}}$$

where the a_i belong to x^{n+1}. We have $a_0 < a_1 < \ldots < a_k$. This 1-1 correspondence is given explicitly by f below. Now for the formal proof: In preparation for rule W, replace x by $\underline{m}$. For the ${}^2r^{n+1}$ in the definition of $CE_{n+2}(\underline{m}^{n+1}, \underline{N}_{n+1})$, we take:

$$\begin{aligned} f(m) = [\,&(Ec, v) ::: \sigma(\underline{\phi}_n(\underline{\zeta}(\underline{\zeta}(c, \underline{0}), \underline{0})), m^{n+1}) \,.\, (u) \,.\, u < \underline{\zeta}(\underline{\zeta}(c, \underline{0}), \underline{0}) \supset \\ &{\sim}\sigma(u^n, m^{n+1}) : \underline{\zeta}(\underline{\zeta}(c, \underline{0}), \underline{1}) = \underline{\psi}(\underline{0}) ::. (u) :: u < v \supset :. \\ &\underline{\zeta}(\underline{\zeta}(c, u'), \underline{1}) = \underline{\psi}(u') : (t) : t < \underline{\zeta}(\underline{\zeta}(c, u'), \underline{0}) \,.\, \sigma(t^n, m^{n+1}) \,.\supset . \\ &(Es) \,.\, s < u' \,.\, E_{n+1}(\underline{\phi}_n(\underline{\zeta}(\underline{\zeta}(c, s), \underline{0})), t^n) :. (s) : s < u' \supset \\ &{\sim}E_{n+1}(\underline{\phi}_n(\underline{\zeta}(\underline{\zeta}(c, s), \underline{0})), \underline{\phi}_n(\underline{\zeta}(\underline{\zeta}(c, u'), \underline{0}))) :. \sigma(\underline{\phi}_n(\underline{\zeta}(\underline{\zeta}(c, u'), \underline{0})), m^{n+1}) \\ &::. E_{n+1}(x, \underline{\phi}_n(\underline{\zeta}(\underline{\zeta}(c, v), \underline{0}))) \,.\, E_{n+1}(y, \underline{\zeta}(\underline{\zeta}(c, v), \underline{1}))]. \end{aligned}$$

Observe that f has degree $n + 1$.

We first show that f is $1 - 1$:

We obtain $\vdash Ext^2_{n+2}(\underline{f}(\underline{m}))$ since $\underline{\zeta}(\underline{\zeta}(c, v), \underline{1}) = \underline{\psi}(v)$. Now assume $S_{n+2}(p_1^n, q_1^n; \underline{f}(\underline{m}))$ and $S_{n+2}(p_2^n, q_2^n; \underline{f}(\underline{m}))$. Use rule C on $f(m)$ with x, y first replaced by p_1^n, q_1^n, and then by p_2^n, q_2^n obtaining c_1, v_1, c_2, v_2. We now prove

(4) $v_1 \leq v_2 \supset : (u) : u \leq v_1 \supset . \underline{\zeta}(\underline{\zeta}(c_1, u), \underline{0}) = \underline{\zeta}(\underline{\zeta}(c_2, u), \underline{0}) \,.\, \underline{\zeta}(\underline{\zeta}(c_1, u), \underline{1}) = \underline{\zeta}(\underline{\zeta}(c_2, u), \underline{1})$ by induction on u, assuming $v_1 \leq v_2$.

Basis: $u = \underline{0}$ is easy. Now assume induction hypothesis and $u' \leq v_1$. Assume $\underline{\zeta}(\underline{\zeta}(c_1, u'), \underline{0}) < \underline{\zeta}(\underline{\zeta}(c_2, u'), \underline{0})$. So $(Es) \,.\, s < u' \,.\, E_{n+1}(\underline{\phi}_n(\underline{\zeta}(\underline{\zeta}(c_2, s), \underline{0}), \underline{\phi}_n(\underline{\zeta}(\underline{\zeta}(c_1, u'), \underline{0})))$. But, by induction hypothesis $\underline{\zeta}(\underline{\zeta}(c_2, s), \underline{0}) = \underline{\zeta}(\underline{\zeta}(c_1, s), \underline{0})$, hence $E_{n+1}(\underline{\phi}_n(\underline{\zeta}(\underline{\zeta}(c_1, s), \underline{0})), \underline{\phi}_n(\underline{\zeta}(\underline{\zeta}(c, u'), \underline{0})))$ which contradicts. We obtain a similar contradiction if $\underline{\zeta}(\underline{\zeta}(c_2, u'), \underline{0}) < \underline{\zeta}(\underline{\zeta}(c_1, u'), \underline{0})$. Hence $\underline{\zeta}(\underline{\zeta}(c_1, u'), \underline{0}) = \underline{\zeta}(\underline{\zeta}(c_2, u'), \underline{0})$. Also we see that $\underline{\zeta}(\underline{\zeta}(c_1, u'), \underline{1}) = \underline{\zeta}(\underline{\zeta}(c_2, u'), \underline{1}) = \underline{\psi}(u')$. So (4) is proved. Hence we get $E_{n+1}(p_1^n, p_2^n) \equiv E_{n+1}(q_1^n, q_2^n)$. And so $f(m)$ is $1 - 1$.

We prove $(p) : S_{n+2}(p^n, \underline{m}^{n+1}) \supset . (Eq) \,.\, S_{n+2}(q^n, \underline{N}_{n+1}) \,.\, S_{n+2}(p^n, q^n; \underline{f}(\underline{m}))$ by straightforward strong induction on p, extending the c, v as was done in the proof of T7.

Finally, we wish to prove:

(5) $(p):S_{n+2}(p^n, \underline{N}_{n+1}) \supset . (Eq) . S_{n+2}(q^n, \underline{m}^{n+1}) . S_{n+2}(q^n, p^n; \underline{f}(\underline{m}))$.

We also prove this by strong induction on p. Basis: $p = \underline{0}$. Then apply (3) with $w = \underline{0}$, and take the least such u of (3) for the q of (5). Now assume induction hypothesis, and assume $S_{n+2}(p'^n, \underline{N}_{n+1})$. Apply (3) again with $w = p'$, hence obtaining a new u of (3) for the q of (5). To obtain this result, we must extend the c, v in the definition of $f(m)$ from satisfying (5) for numbers $\leq p$, to satisfying (5) for p'.

From T9, we easily obtain:

T10 $\vdash Card(x) \supset . (Ez)E_{n+1}(x, \bar{\underline{\gamma}}_{n+2}(z)) \vee E_{n+3}(x, \bar{\underline{\omega}}_{n+2})$.

We state the following theorem without proof, it follows from T9:

Theorem 1 *If f is a recursive function, A is in the range of ϕ_{n+1}, $\vdash \sigma(\underline{f}(x), A)$, and $\vdash x \neq y \supset \sim E_{n+1}(f(x), f(y))$, then $\vdash CE_{n+2}(\underline{A}, \underline{N}_{n+1})$.*

Using Theorem 1, we obtain

T11 $\vdash CE_{n+2}(\underline{N}_{n+1}, \underline{V}_n)$.

Furthermore, we get

T12 $\vdash CE_{n+3}(\underline{V}_n, \underline{V}_{n+1})$.

Proof: Obtain $CE_{n+3}(\underline{V}_n, \underline{N}_{n+2})$ and $CE_{n+3}(\underline{N}_{n+2}, \underline{V}_{n+1})$. Then use transitivity of CE_{n+3}.

However, we can prove that there are sets of degree $= n + 1$ which are not equivalent to any sets of degree $\leq n$, i.e.,

T13 $\vdash (Eu)(v) \sim E_{n+2}(u^{n+1}, v^n)$.

Proof: Take u^{n+1} to be $\underline{\sim S_{n+1}(x, x^n)}$.

Also we obtain that V_n is properly included in V_{n+1}, i.e.,

T14 $\vdash (u) . S_1(u, \underline{V}_n) \supset S_1(u, \underline{V}_{n+1}) : (Eu) . S_1(u, V_{n+1}) . \sim S_1(u, \underline{V}_n)$.

Proof: Observe that $\vdash (x) . \underline{V}_n \supset \underline{V}_{n+1}$ because ϕ_n and ϕ_{n+1} are primitive recursive; hence we obtain this by formalizing Gödel's β-function (cf. [4], p. 244, Remark 1) or, since ϕ_n and ϕ_{n+1} are numeralwise representable, simply use rule W. The second conjunct follows from T13.

5 ***Real numbers*** As the reader has come to realize by now, formal developments in *WTN* are rather cumbersome. In this section, I spare the reader the details of a development of real numbers, which is more unwieldy than the developments of Sections 3 and 4. With this in mind, let us proceed.

A real number is to be a set of positive integers of the form $m = 2^{m_1} . 3^{m_2} . 5^{m_3}$ where $m_3 \neq 0$ and m is to be associated with the rational number $\frac{m_1 - m_2}{m_3}$ such that under this association, the set is the lower half of a Dedekind cut. Write $\langle z_1, z_2, z_3 \rangle$ for $\underline{\xi}(\underline{2}, z_1) \cdot \underline{\xi}(\underline{3}, z_2) \cdot \underline{\xi}(\underline{5}, z_3)$.

In order to see the complexities involved we give a definition of the real numbers of degree $\leq n$.

$$\begin{aligned}&Real_{n+1}(a) \text{ for } (u) ::. S_{n+1}(u,a) . \supset :. (Eu_1, u_2, u_3) :\\ &\quad u = \langle u_1, u_2, u_3\rangle . u_3 \neq \underline{0} . (Ev_1, v_2, v_3) . S_{n+1}(\langle v_1, v_2, v_3\rangle, a) . v_3 \neq \underline{0} .\\ &\quad u_1 v_3 + u_3 v_2 < u_3 v_1 + u_2 v_3 : (v_1, v_2, v_3) : v_3 \neq \underline{0} .\\ &\quad v_1 u_3 + v_3 u_2 \leq v_3 u_1 + v_2 u_3 . \supset S_{n+1}(\langle v_1, v_2, v_3\rangle, a) ::.\\ &\quad (Eu, v_1, v_2, v_3) : S_{n+1}(u,a) . v_3 \neq \underline{0} . \sim S_{n+1}(\langle v_1, v_2, v_3\rangle, a) .\end{aligned}$$

Read "a is a real number of degree $\leq n$".

We then define the basic operations on reals, such as addition, $add(a,b)$; subtraction, $sub(a,b)$; and multiplication, $mult(a,b)$ – observing that this can be done in such a way that, if a and b are reals of degree $\leq n$, then so are $add(a,b)$, $sub(a,b)$, and $mult(a,b)$. It is easy to define the predicates $<$ and $\leq$ for reals.

One then obtains that if A is a nonempty set of degree $=n+1$ all of whose members are real numbers of degree $\leq n$, which are bounded above by a real number of degree $\leq n$, then the set has a least upper bound (l.u.b.) which is a real of degree $\leq n+1$. Indeed, the l.u.b. is the union of the bounded set of reals. This result apparently cannot be sharpened, i.e., the l.u.b. may not have degree $\leq n$.

By using Theorem 1 of Section 4, one obtains

T15 $\quad \vdash CE_{n+2}(\underline{N_{n+1}}, \underline{Real_{n+1}(x)})$.

One should have no difficulty in defining $a = \sum_{i=0}^{\infty} b_i$. That is, we write $Sum_{n+2}(a,b)$ for "the sum of the sequence b (hence b is a binary relation) of reals of degree $\leq n$ converges to the real number a of degree $\leq n$". Hence one may define $Meas_{n+3}(a,b)$ for "a is a set of degree $\leq n+1$ whose members are reals of degree $\leq n$, and b is a real of degree $\leq n+1$ which is the Lebesgue outer measure of a".

However, it will turn out that the Lebesgue outer measure of the unit interval [0, 1] will equal 0. This will not conflict with the classical result that the measure equals 1, since the Heine-Borel theorem cannot be proved in *WTN*. The crucial point is that a set of reals each of degree $\leq n$ may have a limit point of degree $=n+1$. Moreover, the $1-1$ function that maps the unit interval [0, 1] onto N_{n+1} has *the same degree as [0, 1] itself* (cf. T9 above). So unlike the situation in [12], p. 251, this $1-1$ function cannot be distinguished from more "legitimate" $1-1$ functions. Hence the theory of Lebesgue measure cannot be developed in *WTN* in the classical manner. Indeed, if there were such a countably additive measure, then *WTN* would be inconsistent.

The first thought the author had to resolve this problem was to deal with a density measure δ, such that if b is a set dense in the interval (c,d), then $\delta(b) = d - c$ provided $c \leq d$. But then, both the rationals and irrationals in the unit interval would have density measure 1, which is no good.

The referee has suggested a possible remedy, viz., using Bishop's definition of measure (cf. [1], pp. 155–159).

We now discuss another approach. We first observe that probability and statistics are based on measure theory. Let us consider the probability of choosing a particular real number r in the unit interval. This probability $P(\{r\}) = 0$ in the standard approach. That is, if we accept the premise that a probability of 0 states impossibility, then it is impossible that any particular number would be picked. This seems to indicate that $P(\{r\})$ should be >0. This observation seems to require that we develop measure theory a la A. Robinson's "non-standard analysis".

The following theorem perhaps should be called a "thesis" or "quasi-theorem", especially since it requires further elaboration. Suppose that a person who is primarily a mathematician wishes to see whether certain sentences are provable in *WTN* or not, without going through the formidable details. He may desire to formalize a sentence of *WTN* very carefully, and then agree that it is "obviously true" hence probably provable. Observe that all the formal theorems we considered in this article from T1 to T15 are "obviously true", and we provided, in many cases, a rigorous proof of them. We wish to connect the notion of "obviously true" with "provable in *WTN*" as best as we can.

Theorem 2 *If A is a sentence of WTN, such that by reading its content carefully it becomes "obviously true" then* $\vdash A$.

Proof: Write "*OT*" for "obviously true". What follows may be regarded as a definition of *OT*.

Let B be the sentence obtained from A by advancing all unbounded quantifiers to the front and contracting them (cf. [4], p. 285). Next, obtain C from B by replacing the existential quantifiers by the numeralwise representable recursive functions and constant now to be defined. We take an example. Suppose B has the form $(Eu)(v)(Ew)(z)(Et)D(u, v, w, z, t)$, where $D(u, v, w, z, t)$ has no unbounded quantifiers, then C is $D(\underline{m}, v, \underline{f}(v), z, \underline{g}, (v, z))$. We write $C(v, z)$ for C.

Observe that A is OT implies B is OT implies C is OT, and $\vdash C$ implies $\vdash B$ implies $\vdash A$. So all we have to show is that C is OT implies $\vdash C$.

For C to be OT seems to imply that the functions $\underline{f}$ and $\underline{g}$ can be explicitly constructed. Observe that only bounded quantifiers occur in C. Hence, in a sense, C is "primitive recursive" except of course some T_n's may occur in it. So numeralwise representability should be established for C. That is, if C^{-1} is the informal predicate that C represents, then for all i, j if $C^{-1}(i, j)$ is true, then $\vdash C(\underline{i}, \underline{j})$. So use rule W two times and we get $\vdash (v, z)C(v, z)$ hence $\vdash A$.

Observe that we used such functions f in the long proofs of T7 and T9 (above). Also, observe that we must replace all occurrences such as $\sigma(u, k^n)$ by $S_{n+1}(u, \underline{k}^n)$ in C before using rule W (cf. T3, Section 2, above).

Theorem 2 is obviously related to Herbrand's Theorem.

An informal development of analysis, along the lines indicated above, should first be undertaken (cf. [13]).

6 Related systems Let $T'N$ be the extension of TN obtained by adjoining to TN the "T'-axioms": If m is not a sentence of degree $\leq n$, then $\sim T_{n+1}(\underline{m})$ is an axiom. We now prove:

Theorem 3

(1) *TN is consistent relative to PN, and*

(2) *$T'N$ is consistent relative to TN. Hence*
(3) *$T'N$ is consistent relative to PN.*

Proof: (1) follows because in any proof only finitely many T-axioms $T_n(\underline{A}) \equiv A$ are used. If they are for the formulas $A_1, \ldots, A_k$, then consider interpreting $T_n(x)$ as $(x = \underline{A}_1 . A_1) \vee \ldots \vee (x = \underline{A}_k . A_k)$. The details of making this precise are left to the reader. (This proof is due to the referee.)

To prove (2), simply replace every occurrence of a T'-axiom in a proof by $\underline{m} = \underline{m}$. The proof remains the same otherwise, since $\sim T_n(\underline{m})$ does not interplay with the rest of the proof.

Shoenfield in [6] showed that W^*PN is complete. Hence it may well be that $W^*T'N$ is complete. Indeed if we let rule W' be the unrestricted ω-rule (i.e., if for every m, $A(\underline{m})$, then infer $(u)A(u)$), then we easily see that $W'T'N$ is complete, using a similar argument to the proof that $W'PN$ is complete.

Let rule W'' be like rule W except that the fact that, for every m, $\{e\}(m)$ is a proof of $A(\underline{m})$ must be provable in PN. We easily get a proof predicate, $\mathcal{P}\mathfrak{f}''_n(A, k)$ for $W''TN$, along the lines indicated by the definition $\mathcal{P}\mathfrak{f}_n(A, k)$ given in Section 1. We see that $\mathcal{P}\mathfrak{f}''_n(A, k)$ is primitive recursive since all natural number variables in the definition are bounded by k. Hence $W''PN$, $W''TN$, and $W''T'N$ are all incomplete since they are formal systems.

We now describe the systems SN and WSN which are natural extensions of PN (cf. [9]). We adjoin as primitive symbols $\in_1, \in_2, \ldots$, and stipulate that if a and b are terms, then $a \in_n b$ is a formula. We say that the *$\in$-degree* of a formula A is n, if $\in_n$ occurs in A, but no $\in_m$ occurs for $m > n$. If no $\in_n$ occurs in A, we say that its *$\in$-degree* is 0. Next we adjoin the following axioms to our system. Let x be the first variable of our system, and let A be a formula of $\in$-degree $\leq n$ whose only free variable is x, then $(x) . x \in_{n+1} \underline{A} \equiv A$ is an axiom. This is system SN. We obtain WSN by adjoining rule W as usual. We obviously have

Theorem 4 *WSN is interpretable in WTN.*

Proof: Use T3 of Section 2.

We shall now consider systems of ramified analysis, which are not number-theoretic systems (in the sense of the first sentence of this paper) but occur in much literature (cf. particularly [2], especially pp. 7 and 21).

All these systems are extensions of PN, where we add the predicate $\in$ and big variables $X^n, Y^n, Z^n, \ldots$ for $n \geq 0$. We call n the R-degree (for "ramified degree") of the variable X^n, etc. If a is a term of PN, then $a \in X^n$ is a formula of R-degree n. Next, we say that the R-degree of a formula A is n, if n is the maximum of all $k + 1$ such that a variable X^k occurs bound in A and of all m such that a variable Y^m occurs free in A. If no big variables occur in A, we say that A has R-degree -1.

Consider the following comprehension axiom schema, which is used in all our ramified systems:

(CS) $(EY^n)(x) . x \in Y^n \equiv A$ where A is a formula that may or may not contain big variables, but Y^n does not occur free in A.

We get the system R_n if the R-degree of A is $\leq n$. According to Feferman,

R_0 is the system Weyl would be most happy with. The union of the R_n we call R (for "ramified analysis").

We adjoin rule W to R, obtaining WR. Following Feferman, we may also consider transfinite systems W^*R^*, etc. (cf. Section 1).

Theorem 5 *WTN is interpretable in WR.*

Proof: This proof is due to Feferman (private communication). We make use of Wang's development of a truth definition in R_{n+1} for R_n (cf. [11]). $T_1(x)$ is interpreted as "x is the Gödel number of a true arithmetic sentence", which is given in R_0. Using this interpretation, we can associate with each sentence of WTN of degree ≤ 1 its interpretation in R_0. Then $T_2(x)$ is interpreted as "x is the Gödel number of a sentence of degree ≤ 1, whose interpretation is true". This can be given in R_1, etc.

Theorem 6 *WR is interpretable in WSN.*

Proof: Let ϕ_n enumerate the formulas of WSN of $\in$-degree $\leq n$, whose only free variable is x. We have $\vdash (x) . x \in_{n+1} \underline{\phi_n(m)} \equiv \phi_n(m)$ (using the analogue of T1). This proof is complicated by the fact that for WR, free variables of the same R-degree as the set to be defined by the comprehension axiom schema can appear on the right of the equivalence. So suppose that $\phi_n(m)$ above has a subformula, $\phi_n(j)$, which is of $\in$-degree $= n$. Let $\phi^*_{n+1}(m)$ be the formula obtained from $\phi_n(m)$ by replacing "$\phi_n(j)$" by "$u \in_{n+1} \underline{\phi_n(j)}$", then we would get $\vdash (x) . x \in_{n+1} \underline{\phi_n(m)} \equiv \phi^*_{n+1}(m)$. Next, we would get $\vdash (Ey)(x) . x \in_{n+1} y^n \equiv \phi^*_{n+1}(m)$. Finally, by rule W on j, we would get $\vdash (Ey)(x) . x \in_{n+1} y^n \equiv \phi^{**}_{n+1}(m))$ where $\phi^{**}_{n+1}(m)$ is obtained from $\phi^*_{n+1}(m)$ by replacing "$u \in_{n+1} \underline{\phi_n(j)})$" by "$u \in_{n+1} v^n$" where v does not occur in $\phi^*_{n+1}(m)$. This shows that WR is interpretable in WSN.

Hence we have by Theorems 4, 5, and 6:

Theorem 7 *WTN, WR, and WSN are mutually interpretable.*

We close by considering another system AS, called *arithmetical set theory*, which is a number-theoretic set theory and is PN itself! That is, consider the following abbreviations in PN, which enumerate the arithmetical predicates. If a and b are terms, write

$a \in 3^b$ for $\{b\}(a) = 0$,
$a \in 2 \cdot 3^b$ for $(Ex)\{b\}(a,x) = 0$,
$a \in 2^2 \cdot 3^b$ for $(x)\{b\}(a,x) = 0$,
$a \in 2^3 \cdot 3^b$ for $(x)(Ey)\{b\}(a,x,y) = 0$, etc.

In general, we have $a \in 2^n \cdot 3^b$ where the numeral n has a status similar to that of degree, say in WTN. We call n the *form* and b the *base* of the set $2^n \cdot 3^b$. This system was first announced in [8].

Observe that we have dispensed with the distinction between $\underline{f}$, $\underline{n}$, . . . and f, n, . . . since no confusion can result.

Let ψ_k be a primitive recursive function which enumerates the Gödel numbers of the k-ary primitive recursive functions. We can do this in such a way that ψ_k is monotone increasing. Observe that there exists a primitive recursive function, η, such that $\eta(n) = k$. We write ${}^nb$ for $2^n \cdot 3^{\psi_{\eta(n)}(b)}$. We call b the *index* of the set ${}^nb$.

We spare the reader a formal development of the theorems of AS. As a matter of fact, some serious difficulties arise. However, we give the reader one simple example:

Let e be a Gödel number of the representing function of the predicate $x = x$. Then we can prove in $AS \vdash (x) x \in 3^e$, i.e., $\vdash (x)\{e\}(x) = 0$. This can be established by formalizing Gödel's β-function (cf. [4], p. 244, Remark 1). By universal specialization, we get $\vdash 3^e \in 3^e$. Hence the axiom of foundation can be refuted in AS.

If we consider $\in$ a primitive symbol, we obtain the system AS'. That is, we have as axioms, $(x, y) . x \in 3^y \equiv \{y\}(x) = 0$, etc. Then we can prove in $AS' \vdash (Ey)(x) . x \in y$. So, in this manner, as we develop some basic theorems in AS' or WAS', we could begin to dispense with the cumbersome need to keep track of degree or form in our developments.

Finally, if *PNI* is the intuitionistic system of [4] (pp. 82, 101), we get the systems *WTNI*, *AS'I*, etc., where the definition of "form" is complicated somewhat. These systems may be of interest to intuitionists.

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600 West 169th Street, Apartment 65
New York, New York 10032