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SEMI-MONOTONE SERIES OF ORDINALS

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The task of calculating the sum of a series of ordinals is greatly facilitated by the absorption law, and with regard to the use of this law, increasing series of ordinals provide the example par excellence: if you can only keep going long enough, you can forget about some of the early terms. In this note* we define a generalization of the increasing series, and by means of this generalization we obtain some conditions under which a series of ordinals has minimal sum.

Let $s = (s_\xi)_{\xi<\alpha}$ be a sequence of ordinals: we denote by "$\mathrm{o}(s)$" the ordinal α—the "order-type" of the sequence—and by "$\Sigma(s)$" the sum of the associated series. A second sequence $t = (t_\zeta)_{\zeta<\mathrm{o}(t)}$ is called a "permutation" of s if $\mathrm{o}(t) = \mathrm{o}(s)$ and there is a bijection $f: \mathrm{o}(s) \to \mathrm{o}(t)$ such that $s_\xi = t_{f(\xi)}$ for all $\xi < \mathrm{o}(s)$. In some of our previous papers ([1], [2], [3]) we have been interested in the set $S(s) = \{\Sigma(t); t \text{ is a permutation of } s\}$, and in [3] we had occasion to introduce the concept of a semi-monotone sequence. A sequence $s = (s_\xi)_{\xi<\alpha}$ of ordinals is called "semi-monotone" if for each pair (μ, ν) of ordinals $\mu, \nu < \alpha$ there is a third ordinal $\psi < \alpha$ such that $\nu \leq \psi$ and $s_\mu \leq s_\psi$. We showed in [3] that if s is a semi-monotone sequence of positive ordinals with $\mathrm{o}(s)$ a regular initial ordinal, then for any permutation t of s we have t semi-monotone and $\Sigma(t) = \Sigma(s)$. Since we shall be mainly concerned with series of ordinals, we make the blanket assumption that unless the contrary is either obvious or explicitly stated, all sequences referred to henceforth will have only positive terms.

Let r, s be any two sequences. We define $r \dot{\cup} s$ to be the sequence t such that $\mathrm{o}(t) = \mathrm{o}(r) + \mathrm{o}(s)$, $t_\xi = r_\xi$ for $\xi < 0(r)$, and $t_{\mathrm{o}(r)+\zeta} = s_\zeta$ for $\zeta < \mathrm{o}(s)$. If $s = r \dot{\cup} t \dot{\cup} u$ for some sequences r, t, u, then we call t a "segment" of s, and say that t is initial (final) if $\mathrm{o}(r) = 0(\mathrm{o}(u) = 0)$. For any sequence s with $\mathrm{o}(s) \neq 0$, we put $\sup(s) = \sup\{s_\xi; \xi < 0(s)\}$. Finally, we say that s is non-empty if $\mathrm{o}(s) \neq 0$.

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Result 1 *Let s be any nonempty sequence. Then s is semi-monotone if and only if* $\sup(r) = \sup(s)$ *for every nonempty final segment r of s.*

Proof: Suppose that s is semi-monotone and let $r = (s_\xi)_{\zeta<\xi<o(s)}$ be any nonempty final segment of s. If $\zeta = 0$ then of course $r = s$ and so $\sup(r) = \sup(s)$. Hence we may assume that $\zeta > 0$, and we take any ordinal τ with $0 \leqslant \tau < \zeta$. By semi-monotonicity there exists $\theta < o(s)$ such that $\zeta \leqslant \theta$ and $s_\tau \leqslant s_\theta$. It follows that $\sup(r) \geqslant \sup(t)$, where t is the unique sequence defined by $s = t \dot\cup r$. Since it is clear that $\sup(s) = \max\{\sup(t), \sup(r)\}$, we see that $\sup(r) = \sup(s)$.

Now suppose that $\sup(r) = \sup(s)$ for every nonempty final segment r of s, and take any pair (μ, ν) of ordinals $\mu, \nu < o(s)$. Now if $o(s) = \beta + 1$ for some β, then we must have $s_\beta = \sup(s)$, and it follows at once that s is semi-monotone. Hence we may assume that $o(s)$ is an infinite limit ordinal, from which it follows that $\zeta = \max\{\mu, \nu\} + 1 < o(s)$. Put $r = (s_\xi)_{\zeta\leqslant\xi<o(s)}$; then r is a nonempty final segment of s and so $\sup(r) = \sup(s)$. Since $\mu < \zeta$, it follows that there exists $\theta < o(s)$ such that $\nu \leqslant \theta$ and $s_\mu \leqslant s_\theta$, as desired.

Result 2 *Let s be any nonempty sequence. There exists a finite family $\{s^i\}_{i<n}$ of semi-monotone sequences s^i such that* $\sup(s^i) \geqslant \sup(s^{i+1})$ *for $i < n - 1$ and $s = s^0 \dot\cup s^1 \dot\cup \dots \dot\cup s^{n-1}$.*

Proof: Since this result is trivial if $o(s) = 1$, we may proceed by induction on the order-type of a sequence. Thus let s be given, and consider the set $C = \{\sup(r);\ r \text{ a nonempty final segment of } s\}$. C is a nonempty set of ordinals, and we take $\zeta < o(s)$ such that $\sup((s_\xi)_{\zeta\leqslant\xi<o(s)}) = \min C$ with ζ minimal in this respect. Put $r = (s_\xi)_{\zeta\leqslant\xi<o(s)}$ and $t = (s_\xi)_{\xi<\zeta}$. Now r is certainly nonempty, and if u is any nonempty final segment of r then because u is also a nonempty final segment of s we have $\sup(r) = \min C \leqslant \sup(u) \leqslant \sup(r)$, and so $\sup(u) = \sup(r)$.

Thus r is semi-monotone, and so if $o(t) = 0$, then we are through. Hence we may assume $o(t) \neq 0$, and because $o(t) < o(s)$ the induction hypothesis tells us that $t = t^0 \dot\cup t^1 \dot\cup \dots \dot\cup t^{m-1}$ for some finite family $\{t^j\}_{j<m}$ of semi-monotone sequences t^j with $\sup(t^j) \geqslant \sup(t^{j+1})$ for all $j < m - 1$. It follows that the full result will be established once we show that $\sup(t^{m-1}) \geqslant \sup(r)$.

Suppose that this is not the case and put $u = t^{m-1} \dot\cup r$. Then we would have $u = (s_\xi)_{\psi\leqslant\xi<o(s)}$ for some ordinal $\psi < \zeta$: but $\sup(u) = \max\{\sup(t^{m-1}), \sup(r)\} = \sup(r) = \min C$, and we have contradicted our choice of ζ.

The inequalities given in the preceding result cannot of course be made strict; one has only to consider the sequence $(s_k)_{k<\omega}$ with $s_0 = \omega$ and $s_k = k$ for $k > 0$ in order to see this.

Result 3 *Let s be any nonempty semi-monotone sequence with $o(s)$ a limit ordinal. Then $\Sigma(s) = \sup(s)\alpha$ for some ordinal α with $1 \leqslant \alpha \leqslant o(s)$.*

Proof: If this is not the case, then $\Sigma(s) = \sup(s)\alpha + \delta$ for some ordinals α, δ with $0 < \delta < \sup(s)$. Now $o(s)$ is a limit ordinal, and so there must be a nonempty final segment r of s with $\Sigma(r) \leqslant \delta$. But s is semi-monotone, and so $\sup(r) = \sup(s)$. Since $\Sigma(r) \geqslant \sup(r)$, this yields a contradiction, whence $\Sigma(s) = \sup(s)\alpha$ for some ordinal α. The bounds for α are obvious.

We now wish to address ourselves to the following problem. It is clear that for any nonempty sequence s at all, we have $\Sigma(s) \geqslant \sup(s)$: under what conditions do we have equality?

Result 4 *Let s be any nonempty sequence with $\Sigma(s) = \sup(s)$. Then s is semi-monotone; furthermore, if $o(s)$ is a limit ordinal, then $\sup(s)$ is a prime component.*

Proof: Let r be any nonempty final segment of s, let t be such that $s = t \dot{\cup} r$, and suppose that $\sup(t) = \sup(s)$. Then we certainly have $\Sigma(s) \geqslant \sup(t) + \Sigma(r)$, whence $\Sigma(s) > \sup(s)$. Thus $\sup(t) < \sup(s)$. But $\sup(s) = \max\{\sup(t), \sup(r)\}$, and so $\sup(s) = \sup(r)$. Therefore s is semi-monotone.

Now suppose that $o(s)$ is a limit ordinal and take any $\alpha < \sup(s)$. We must have $\alpha \leqslant \Sigma(t)$ for some proper initial segment t of s, and we let r be such that $s = t \dot{\cup} r$. Since $o(t) < o(s)$, r is nonempty, whence $\sup(r) = \sup(s)$. Thus $\alpha + \sup(s) \leqslant \Sigma(t) + \sup(r) \leqslant \Sigma(s) = \sup(s)$, and so $\sup(s)$ is a prime component.

In the next result we denote by "$\mathrm{cf}(\alpha)$" the cofinality of a nonzero limit ordinal α, and recall that an initial ordinal κ is said to be regular if $\mathrm{cf}(\kappa) = \kappa$.

Result 5 *Let s be any sequence with $o(s)$ a regular initial ordinal. Then:*

(1) $\Sigma(s) = \sup(s)$ *if and only if*

(a) $\sup(s)$ *is a prime component;*
(b) $\mathrm{cf}(\sup(s)) = o(s)$*;*
(c) $s_\xi < \sup(s)$ *for all* $\xi < o(s)$*:*

(2) *If s is semi-monotone, then* $\Sigma(s) = \sup(s)$ *or* $\Sigma(s) = \sup(s)o(s)$.

Proof: (1) Assume that $\Sigma(s) = \sup(s)$: then (a) follows from the preceding result and (b) follows from $\sup(s) = \lim_{\zeta < o(s)}\Sigma((s_\xi)_{\xi<\zeta})$ and $\mathrm{cf}(o(s)) = o(s)$. Finally, if $s_\xi = \sup(s)$ for some $\xi < o(s)$, then because $o(s)$ is a limit ordinal we would have $\Sigma(s) > \sup(s)$.

Now assume that $\Sigma(s) \neq \sup(s)$; then of course $\Sigma(s) > \sup(s)$, and so we may define t to be the unique initial segment of s such that $\Sigma(t) \geqslant \sup(s)$ and $\Sigma(u) < \sup(s)$ for every proper initial segment u of t. Suppose that conditions (a), (c) hold; then it follows easily that $o(t)$ is a limit ordinal, from which we conclude that $\Sigma(t) = \sup(s)$. But then we have $\mathrm{cf}(\sup(s)) = \mathrm{cf}(o(t)) \leqslant o(t) < o(s)$, and so (b) fails.

(2) From Result 3 we have $\Sigma(s) = \sup(s)\alpha$ for some ordinal α with $1 \leqslant \alpha \leqslant o(s)$. Now we have to have $\mathrm{cf}(\Sigma(s)) = o(s)$, and so if α is limit, this leads to $\mathrm{cf}(\alpha) = o(s)$, whence $\alpha = o(s)$. Thus it suffices to show that if $\Sigma(s) = \sup(s)(\beta + 1)$ for some β, then $\beta = 0$.

Suppose then that $\Sigma(s) = \sup(s)(\beta + 1)$ for some $\beta \geqslant 0$. It follows that s has a nonempty final segment r with $\Sigma(r) = \sup(s)$. Now of course $o(r) = o(s)$

and $\sup(r) = \sup(s)$. Hence $\Sigma(r) = \sup(r)$, and from (1) we conclude that (a), (b), (c) hold with respect to r, whence (a), (b), (c) hold with respect to s (that conditions (a), (b) hold with respect to s is clear, whilst (c) from the fact that r is a nonempty final segment of a semi-monotone sequence). Thus by (1), $\Sigma(s) = \sup(s)$, i.e., $\beta = 0$.

Unfortunately, it seems that nothing much can be done with the other extreme, the case in which $o(s)$ is a successor ordinal.

Result 6 *Let s be any sequence such that $o(s) = \beta + 1$ for some ordinal β. Then $\Sigma(s) = \sup(s)$ if and only if $s_\beta = \sup(s) \geqslant \Sigma((s_\xi)_{\xi<\beta})\omega$.*

Proof: We already know that if $\Sigma(s) = \sup(s)$, then s is semi-monotone, from which it follows that $s_\beta = \sup(s)$. The rest is obvious.

There remains the case in which $o(s)$ is a nonregular limit ordinal. Putting $c(s) = cf(o(s))$, we see that $o(s) = c(s)\alpha$ for some ordinal α with either $\alpha = \beta + 1$ for some β or else α is limit and $cf(\alpha) = c(s)$.

Result 7 *Let s be any nonempty sequence with $o(s)$ a nonregular limit ordinal and $o(s) = c(s)(\beta + 1)$ for some β. Then $\Sigma(s) = \sup(s)$ if and only if $s = t \dot{\cup} r$ for some t, r with $o(r) = c(s)$ and $\Sigma(r) = \sup(s) \geqslant \Sigma(t)\omega$.*

Proof: If s has such a decomposition, then the result is clear. On the other hand, if $\Sigma(s) = \sup(s)$, then s is semi-monotone and we can take any final segment r of s with $o(r) = c(s)$ and let t be such that $s = t \dot{\cup} r$. Then $\Sigma(r) = \sup(r) = \sup(s)$ by semi-monotonicity and Result 5 (1), and the rest follows.

We are finally faced with the case in which $o(s) = c(s)\alpha$ for some limit ordinal α, and here again we are at this stage unable to give a particularly satisfactory criterion.

Result 8 *Let s be any nonempty sequence such that $o(s)$ is a nonregular limit ordinal and $o(s) = c(s)\alpha$ for some limit ordinal α. Then $\Sigma(s) = \sup(s)$ if and only if the following hold:*

(1) *$\sup(s)$ is a prime component;*
(2) *$cf(\sup(s)) = c(s)$;*
(3) *$\Sigma(s^\xi) < \sup(s)$ for all $\xi < c(s)$, where $\{s^\xi\}_{\xi<c(s)}$ is some family of sequences such that $s = \dot{\bigcup}\{s^\xi;\ \xi < c(s)\}$.*

Proof: We know that $cf(\alpha) = c(s)$, and so we can express s as $\dot{\bigcup}\{s^\xi;\ \xi < c(s)\}$, where each s^ξ is a segment of s. But now $(\Sigma(s^\xi))_{\xi<c(s)}$ is a sequence whose order-type is a regular initial ordinal, and we can apply Result 5 (1).

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