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A REALIST SEMANTICS FOR COCCHIARELLA'S T*

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0 *Introduction* Russell's paradox has two versions. The first version concerns "the set of all sets which are not members of themselves". The second version concerns "the property of being a property which is not a property of itself": the so-called *Russell property*. This second version of Russell's paradox is called *Russell's paradox of predication.*[1]

Nino Cocchiarella* designed a logistic system, which he christened **T***, whose purpose was to represent the original ontological context behind Russell's paradox of predication [10]. The grammar of **T*** is essentially that of standard second-order logic but goes beyond it by allowing predicate terms to occupy subject positions in the formulas of **T***. Cocchiarella generated the axioms and inference rules of **T*** by explicitly and appropriately generalizing the axioms and inference rules of Church's formulation of standard second-order logic [1] to the extended grammatical context of **T*** and by adding a new axiom schema whose effect is to represent the realist assumption implicit in the ontological background of Russell's paradox of predication that every relation[2] is an individual.

It is a remarkable fact that **T*** is consistent. It even turns out that **T*** is a conservative extension of standard second-order logic. Thus, Russell's "paradox" of predication is not really a paradox after all—at least not in the logistic context of **T***. These discoveries signify the genesis of a radically new, important, and fruitful approach to predication theory.[3]

Be forewarned, however, that **T*** is not without its (apparent) ontological oddities. One particularly interesting example of such an oddity concerns identity. Cocchiarella [10] showed that indiscernibility cannot be construed as identity in **T*** since, in the ontology of **T***, there must be properties which are indiscernible and yet not co-extensive. Going further, Meyer [14] showed that there is no binary relation in the ontology of **T*** satisfying full substitutivity.[4]

Many other nonstandard second- (and even higher-) order theories of predication have grown up around **T***, and we will call these logistic

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systems the *brethren of* T^*. Some of these logistic systems are immune to the ontological oddities of T^*, but others are not.[5]

Cocchiarella devised a formulation of standard second-order logic [8] which is equivalent to that given by Church [1] but whose only inference rule is *modus ponens* and whose axiom set involves neither the notion of proper substitution of a term for a generalized variable of the same type nor the notion of proper substitution of a formula for a generalized predicate variable, i.e., Cocchiarella's formulation of standard second-order logic is *substitution free*.[6] Cocchiarella [3] used this substitution free formulation of standard second-order logic to generate a brother of T^* just as he used Church's formulation of standard second-order logic to create T^*. This brother of T^* we will call $\mathsf{T}_{\mathsf{d}}^{**}$. It is noteworthy that $\mathsf{T}_{\mathsf{d}}^{**}$ is also substitution free. It seems only natural that $\mathsf{T}_{\mathsf{d}}^{**}$ should be equivalent to T^*, but, as Cocchiarella showed, it is not. In fact, there is a problem of more immediate interest concerning $\mathsf{T}_{\mathsf{d}}^{**}$. In [3] it is not shown that the principle of universal instantiation of a predicate term for a generalized predicate variable of the same type, which we will call $(\mathsf{U.I.}_2^*)$, is derivable in $\mathsf{T}_{\mathsf{d}}^{**}$. If $(\mathsf{U.I.}_2^*)$ is not derivable in $\mathsf{T}_{\mathsf{d}}^{**}$, then $\mathsf{T}_{\mathsf{d}}^{**}$ can hardly be understood to capture the meaning of the universal quantifier—much less the original ontological background of Russell's paradox of predication. However, Cocchiarella did show that if $\mathsf{T}_{\mathsf{d}}^{**}$ is supplemented with an especially natural axiom schema, called (A4′), to the effect that every relation is indiscernible from and co-extensive with some relation of the same type, then $(\mathsf{U.I.}_2^*)$ is derivable. The logistic system which results from $\mathsf{T}_{\mathsf{d}}^{**}$ by supplementing it with (A4′) is called T^{**}; we may say that $\mathsf{T}^{**} = \mathsf{T}_{\mathsf{d}}^{**} + (\mathrm{A4}')$. Since it is a trivial matter to verify that $\mathsf{T}_{\mathsf{d}}^{**} + (\mathsf{U.I.}_2^*)$ is equivalent to T^{**}, our feeling of anxiety about the relation of $(\mathsf{U.I.}_2^*)$ to $\mathsf{T}_{\mathsf{d}}^{**}$ may be formulated in this way: Is $\mathsf{T}_{\mathsf{d}}^{**}$ equivalent to T^{**}?

Cocchiarella went on to introduce a so-called *Fregean* model-set-theoretic semantics whose purpose is to reflect the Fregean notion that to say "the property P has the property Q" is to mean that "some individual associated with P has Q". He also introduced a set of formulas, called (Ext*), to the effect that two relations that are co-extensive are indiscernible as well. (Ext*) reflects the Fregean notion grounding the above-mentioned Fregean semantics. Cocchiarella then used his Fregean semantics to show that $\mathsf{T}^* + (\mathrm{Ext}^*)$ is a proper extension of $\mathsf{T}^{**} + (\mathrm{Ext}^*)$ (and therefore of $\mathsf{T}_{\mathsf{d}}^{**} + (\mathrm{Ext}^*)$ which, by the way, he showed to be equivalent to $\mathsf{T}^{**} + (\mathrm{Ext}^*)$) by noting that T^* is an extension of T^{**} and by producing a Fregean frame which is a model of T^{**} but not of T^*. This result implies that T^* is a proper extension of T^{**} (and therefore of $\mathsf{T}_{\mathsf{d}}^{**}$).

Now, this is a very interesting fact, and the question immediately arises as to just why there should be such a difference between T^* and T^{**} since, after all, they appear to have arisen from two equivalent formulations of standard second-order logic in just the same way. Cocchiarella localized the difference between T^* and T^{**} in the following way. Now, whereas Church's formulation of standard second-order logic includes the principle of universal instantiation of a formula for a generalized predicate variable, which we will call $(\mathsf{U.I.}_3)$, Cocchiarella's formulation of standard second-order logic includes a comprehension principle (CP). When each of these formulations of standard second-order logic is generalized to the extended grammatical context of **T***, ($U.I._3$) becomes ($U.I._3^*$), and (CP) becomes (CP*). Cocchiarella showed that the logistic system which results from **T*** by replacing ($U.I._3^*$) with (CP*) is equivalent to **T**** and, furthermore, that the logistic system which results from **T**** by replacing (CP*) with ($U.I._3^*$) is equivalent to **T***. Thus he demonstrated that the difference between **T*** and **T**** lies in the fact that ($U.I._3^*$) is stronger than (CP*). He then noted that a comprehension principle, which we will call (CP**), more general than (CP*) is derivable in **T***. **T**** + (CP**) is called **T*****. Cocchiarella asked the following question: Is **T***** equivalent to **T***? If it is, then we can easily understand how it is that **T**** is weaker than **T*** for in generalizing (CP) to the extended grammatical context of **T*** we have two natural options, viz. (CP*) and (CP**), although, viewed from the context of standard second-order logic, (CP*) is really more natural than (CP**).

The main result of [3] is a completeness theorem for **T**** + (Ext*). As in the usual semantics for standard second-order logic, there is a distinction in Cocchiarella's Fregean semantics between standard, non-standard, and general Fregean frames (general Fregean frames being those Fregean frames which are models of (CP*)), and the completeness theorem for **T**** + (Ext*) is given relative to the general Fregean frames. Cocchiarella noted that it remains an open problem to provide a model-set-theoretic semantics natural to the ontology of **T*** or **T****.[7]

This paper is essentially a response to Cocchiarella [3]. In what follows we introduce a logistic system, **W***, whose grammar is that of **T*** and whose primary syntactical logistic purpose is to capture the meanings of the propositional connectives to be introduced and the universal quantifier. Accordingly, **W*** is weaker than almost all of the brethren of **T***. Then we provide a model-set-theoretic semantics natural to the apparent ontology of **W***, and show that this semantics characterizes **W*** via a strong completeness theorem. Accordingly, we will have provided a strong completeness theorem for every brother of **T*** which is stronger than **W***.

Now, although in **W*** relations are projected grammatically as being entities which, significantly, may serve as the subjects of predication, they are not projected logistically as individuals, for the principle that whatever holds of all individuals also holds of all relations is not derivable in **W***[8] (nor is it intended to be so derivable). Accordingly, the strong completeness theorem for **T***, among the other logistic systems alluded to, which arises from the strong completeness theorem for **W*** is somewhat unfaithful to the intended realist ontology of **T***. We therefore add to **W*** the principle that whatever holds of all individuals holds of all relations, and, using the techniques previously developed for **W***, prove a strong completeness theorem for this supplemented logistic system relative to structures of our semantics which are more natural to the ontology of **T***. This results in a strong completeness theorem for **T*** faithful to its intended realist ontology.

The remainder of this paper is then devoted to investigating various brethren of $\mathbf{T}^*$, the relations between them, and the relation between our semantics and Cocchiarella's Fregean semantics. We produce the logistic system which is characterized by the Fregean semantics, in the process providing it with a strong completeness theorem, and thereby give an alternate route to Cocchiarella's completeness theorems. We produce a realist model of $\mathbf{T}^{***}$ which is not a model of $\mathbf{T}^*$, thereby showing that $\mathbf{T}^{***}$ is *not* equivalent to $\mathbf{T}^*$. We also show that $\mathbf{T}^{***}$ + (Ext*) *is* equivalent to $\mathbf{T}^*$ + (Ext*), thereby showing that $\mathbf{T}^{***}$ is *not* equivalent to $\mathbf{T}^{**}$. Finally, among other things, we show that $\mathbf{T}^{**}_{\mathrm{d}}$ is *not* equivalent to $\mathbf{T}^{**}$, i.e., that it is truly defective.

In spite of the cursory nature of this introduction, this paper has only a minimum of prerequisites. We assume only naive set theory and a basic acquaintance with metalogic. All but the most common notions are rigorously presented, including those of this introduction, and only a few outside results are employed (mostly from [3]).

1 *Grammar* As *logical particles* of the logistic systems which we consider, we include only $\sim$, $\rightarrow$, and Λ: the *negation symbol*, the *material implication symbol*, and the *universal quantifier*, respectively. We assume, of course, that $\sim$, $\rightarrow$, and Λ are distinct.

For each $n \in \omega$,[9] we assume that $V(n)$ is a denumerable set and $C(n)$ is a proper class. Furthermore, for all $n, m \in \omega$, if $n \neq m$, we assume that $V(n)$, $V(m)$, $C(n)$, and $C(m)$ are pairwise disjoint. By an *individual variable*, *individual constant*, or *individual term* we understand an element of $V(0)$, $C(0)$, or $V(0) \cup C(0)$, respectively. For $n \in \omega$, by an *n-place predicate variable*, *n-place predicate constant*, or *n-place predicate term* we understand an element of $V(n + 1)$, $C(n + 1)$, or $V(n + 1) \cup C(n + 1)$, respectively. By a *predicate variable, predicate constant,* or *predicate term* we understand an n-place predicate variable, n-place predicate constant, or n-place predicate term, respectively, for some $n \in \omega$. By a *variable, constant term*, or *term* we understand an individual variable or predicate variable; individual constant or predicate constant; or, individual term or predicate term, respectively. We say that two terms are *of the same type* iff either both are individual terms or, for some $n \in \omega$, both are n-place predicate terms. We use α, β, and γ to refer to individual variables; π, ρ, σ, and τ to refer to predicate variables; μ and ν to refer to variables; and ζ and η to refer to terms.

By a *language* we understand a set of constant terms. If $\mathcal{L}$ is a language, then we say ζ is an *individual $\mathcal{L}$-constant, individual $\mathcal{L}$-term*, etc., iff ζ is an individual constant which is a member of $\mathcal{L}$, ζ is an individual variable or ζ is an individual constant which is a member of $\mathcal{L}$, etc.

Let $\mathcal{L}$ be a language. We say that φ is an *atomic $\mathcal{L}$-formula* iff for some $n \in \omega$, φ is the result of applying an n-place predicate $\mathcal{L}$-term, ζ, to n $\mathcal{L}$-terms, $\eta_0, \ldots, \eta_{n-1}$: $\zeta(\eta_0, \ldots, \eta_{n-1})$; if $n = 0$ we understand this result to be φ itself. We say that φ is an *$\mathcal{L}$-formula* iff φ is a member of the smallest set K such that: (i) every atomic $\mathcal{L}$-formula is an element of K, and (ii) if ψ, $\chi \in K$ and μ is a variable, then $\sim\psi$, $(\psi \rightarrow \chi)$, and $\bigwedge\mu\psi$ are elements of K. We use φ, ψ, and χ to refer to $\mathcal{L}$-formulas; K and Γ to refer to sets of $\mathcal{L}$-formulas; and Δ and Σ to refer to sequences of $\mathcal{L}$-formulas. If φ is an $\mathcal{L}$-formula, then by a *generalization of* φ is understood a formula of the form $\bigwedge\mu_0 \ldots \bigwedge\mu_{n-1}\varphi$, where $n \in \omega$ and $\mu_0, \ldots, \mu_{n-1}$ are variables; if $n = 0$ we understand $\bigwedge\mu_0 \ldots \bigwedge\mu_{n-1}\varphi$ to be φ itself. Where φ, ψ are $\mathcal{L}$-formulas and μ is a variable, we define $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \leftrightarrow \psi)$, and $\bigvee\mu\varphi$ to be $\sim(\varphi \rightarrow \sim\psi)$, $(\sim\varphi \rightarrow \psi)$, $[(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)]$, and $\sim\bigwedge\mu\sim\varphi$, respectively.

Notions of *bondage* and *freedom* remain as usual, but note that an occurrence of a predicate term in some $\mathcal{L}$-formula φ may be bound or free in either *predicate position* or *subject position*. If φ is an $\mathcal{L}$-formula and ζ, η are $\mathcal{L}$-terms, we say that η *may be properly substituted for* ζ *in* φ iff every free occurrence of ζ may be replaced by a free occurrence of η; if this is the case we let $\varphi\begin{bmatrix}\zeta\\ \eta\end{bmatrix}$ be the result of replacing every free occurrence of ζ in φ by η, else we let $\varphi\begin{bmatrix}\zeta\\ \eta\end{bmatrix}$ be φ itself. If φ is an $\mathcal{L}$-formula, $n \in \omega$, $\zeta_0, \ldots, \zeta_{n-1}, \eta_0, \ldots, \eta_{n-1}$ are $\mathcal{L}$-terms, we say that $\eta_0, \ldots, \eta_{n-1}$ *may be simultaneously properly substituted for* $\zeta_0, \ldots, \zeta_{n-1}$ *in* φ iff for each $i \in n$, η_i may be properly substituted for ζ_i in φ; if this is the case we let $\varphi\begin{bmatrix}\zeta_0, \ldots, \zeta_{n-1}\\ \eta_0, \ldots, \eta_{n-1}\end{bmatrix}$ be $\varphi\begin{bmatrix}\zeta_0\\ \alpha_0\end{bmatrix} \ldots \begin{bmatrix}\zeta_{n-1}\\ \alpha_{n-1}\end{bmatrix}\begin{bmatrix}\alpha_0\\ \eta_0\end{bmatrix} \ldots \begin{bmatrix}\alpha_{n-1}\\ \eta_{n-1}\end{bmatrix}$ where $\alpha_0, \ldots, \alpha_{n-1}$ are any n distinct variables which do not occur in φ, else φ itself.

2 *Syntax* We say that $\mathbf{T}$ is a *theory* iff $\mathbf{T}$ is a 2-place sequence $\langle\mathcal{L}, A\rangle$ where $\mathcal{L}$ is a language and A is a set of $\mathcal{L}$-formulas. If $\mathbf{T} = \langle\mathcal{L}, A\rangle$ is a theory we set $\mathcal{L}_{\mathbf{T}} = \mathcal{L}$ and $A_{\mathbf{T}} = A$, and if, in addition, Γ is a set of $\mathcal{L}$-formulas, we set $\mathbf{T} + \Gamma = \langle\mathcal{L}, A \cup \Gamma\rangle$.

Let $\mathbf{T}$ be a theory. If φ is an $\mathcal{L}_{\mathbf{T}}$-formula and Γ is a set of $\mathcal{L}_{\mathbf{T}}$-formulas, we say that Δ is a *derivation of* φ *from* Γ *in* $\mathbf{T}$ iff Δ is a finite sequence $\langle\Delta_0, \ldots, \Delta_{n-1}\rangle$ of $\mathcal{L}_{\mathbf{T}}$-formulas such that $\Delta_{n-1} = \varphi$ and for each $i \in n$ either $\Delta_i \in \Gamma \cup A_{\mathbf{T}}$ or Δ_i follows from preceding sequents in Δ by *modus ponens*, i.e., there are j, $k \in i$ such that $\Delta_k = (\Delta_j \rightarrow \Delta_i)$. If φ is an $\mathcal{L}_{\mathbf{T}}$-formula, we say that Δ is a *proof of* φ *in* $\mathbf{T}$ iff Δ is a derivation of φ from 0 in $\mathbf{T}$.[10] Again, let φ be an $\mathcal{L}_{\mathbf{T}}$-formula and Γ be a set of $\mathcal{L}_{\mathbf{T}}$-formulas. Then we say φ is *derivable from* Γ *in* $\mathbf{T}$, in symbols $\Gamma \vdash_{\mathbf{T}} \varphi$, iff there is a derivation of φ from Γ in $\mathbf{T}$, and we say φ is *provable in* $\mathbf{T}$, in symbols $\vdash_{\mathbf{T}} \varphi$, iff there is a proof of φ in $\mathbf{T}$, i.e., $0 \vdash_{\mathbf{T}} \varphi$.

Let $\mathbf{T}$ and $\mathbf{T}'$ be theories such that $\mathcal{L}_{\mathbf{T}} \subseteq \mathcal{L}_{\mathbf{T}'}$. Then we say $\mathbf{T}$ is a *subsystem of* $\mathbf{T}'$, or $\mathbf{T}'$ is an *extension of* $\mathbf{T}$, iff for every $\mathcal{L}_{\mathbf{T}}$-formula φ, if $\vdash_{\mathbf{T}} \varphi$, then $\vdash_{\mathbf{T}'} \varphi$. If $\mathcal{L}_{\mathbf{T}} = \mathcal{L}_{\mathbf{T}'}$, we say $\mathbf{T}'$ is a *proper extension of* $\mathbf{T}$ iff $\mathbf{T}'$ is an extension of $\mathbf{T}$ and there is an $\mathcal{L}_{\mathbf{T}}$-formula such that $\vdash_{\mathbf{T}'} \varphi$ but $\nvdash_{\mathbf{T}} \varphi$; we say $\mathbf{T}$ and $\mathbf{T}'$ are *equivalent* iff each is a subsystem of the other.

Let $\mathcal{L}$ be a language. Then we define θ to be an element of $(A1_{\mathcal{L}}), \ldots, (A6_{\mathcal{L}})$ iff θ is a generalization of an $\mathcal{L}$-formula of the form:

(A1) $\varphi \rightarrow (\psi \rightarrow \varphi)$
(A2) $[\varphi \rightarrow (\psi \rightarrow \chi)] \rightarrow [(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)]$

(A3) $(\sim\varphi \to \sim\psi) \to (\psi \to \varphi)$
(A4) $\Lambda\mu(\varphi \to \psi) \to (\Lambda\mu\varphi \to \Lambda\mu\psi)$
(A5) $\varphi \to \Lambda\mu\varphi$, where μ does not occur free in φ
(A6) $\Lambda\mu\varphi \to \varphi\begin{bmatrix}\mu\\ \zeta\end{bmatrix}$, where ζ is an $\mathscr{L}$-term of the same type as μ, respectively.

We now let $\mathbf{W}_{\mathscr{L}}$ be the theory $\langle\mathscr{L}, A_{\mathscr{L}}\rangle$ where $A_{\mathscr{L}} = (A1_{\mathscr{L}}) \cup \ldots \cup (A6_{\mathscr{L}})$, and from now on we will write $A_{\mathscr{L}}$ for $A_{\mathbf{W}_{\mathscr{L}}}$.

Our primary interest is in $\mathbf{W}_0$, which we will call $\mathbf{W}^*$, and our motivation for discussing non-empty languages in general is that they provide a simple way of proving the following completeness theorems for $\mathbf{W}^*$ as well as some of the other logistic systems we will be considering.

We now prove some useful syntactic lemmas. Where proofs are especially trivial they are omitted.

Lemma 1 *Let* $\mathbf{T}$ *and* $\mathbf{T}'$ *be theories,* Γ *be a set of* $\mathscr{L}_{\mathbf{T}}$*-formulas,* Γ' *be a set of* $\mathscr{L}_{\mathbf{T}'}$*-formulas, and* φ *be an* $\mathscr{L}_{\mathbf{T}}$*-formula. If* $\mathscr{L}_{\mathbf{T}} \subseteq \mathscr{L}_{\mathbf{T}'}$, $\Gamma \cup A_{\mathbf{T}} \subseteq \Gamma' \cup A_{\mathbf{T}'}$, *and* $\Gamma \vdash_{\mathbf{T}} \varphi$, *then* $\Gamma' \vdash_{\mathbf{T}'} \varphi$.

Lemma 2 *Let* $\mathbf{T}$ *be a theory,* Γ *be a set of* $\mathscr{L}_{\mathbf{T}}$*-formulas, and* φ *be an* $\mathscr{L}_{\mathbf{T}}$*-formula. If* $\varphi \in \Gamma \cup A_{\mathbf{T}}$, *then* $\Gamma \vdash_{\mathbf{T}} \varphi$.

Lemma 3 *Let* $\mathbf{T}$ *be a theory,* Γ *be a set of* $\mathscr{L}_{\mathbf{T}}$*-formulas, and* φ *and* ψ *be* $\mathscr{L}_{\mathbf{T}}$*-formulas. If* $\Gamma \vdash_{\mathbf{T}} \varphi$ *and* $\Gamma \vdash_{\mathbf{T}} (\varphi \to \psi)$, *then* $\Gamma \vdash_{\mathbf{T}} \psi$.

We take the notion of *tautology* as given. We state the following lemma for convenience, but in principle succeeding references to it could be routinely, if tediously, omitted.

Lemma 4 *Let* $\mathscr{L}$ *be a language,* Γ *a set of* $\mathscr{L}$*-formulas, and* φ *an* $\mathscr{L}$*-formula. If* φ *is a tautology, then* $\Gamma \vdash_{\mathbf{W}_{\mathscr{L}}} \varphi$.

Proof: See [16] and [12].

Lemma 5 *Let* $\mathscr{L}$ *be a language,* Γ *a set of* $\mathscr{L}$*-formulas, and* φ *and* ψ $\mathscr{L}$*-formulas.* $\Gamma \cup \{\varphi\} \vdash_{\mathbf{W}_{\mathscr{L}}} \psi$ *iff* $\Gamma \vdash_{\mathbf{W}_{\mathscr{L}}} (\varphi \to \psi)$.

Proof: The right-to-left direction follows from Lemmas 1-3. To prove the other direction, assume that $\Gamma \cup \{\varphi\} \vdash_{\mathbf{W}_{\mathscr{L}}} \psi$. Let $\Delta = \langle\Delta_0, \ldots, \Delta_{n-1}\rangle$ be a derivation of ψ from $\Gamma \cup \{\varphi\}$ in $\mathbf{W}_{\mathscr{L}}$. We prove by induction on i that if $i \in n$ then $\Gamma \vdash_{\mathbf{W}_{\mathscr{L}}} (\varphi \to \Delta_i)$. To this end, let $i \in n$ and assume that the hypothesis is true for all $j \in i$. Case 1: $\Delta_i \in \Gamma \cup A_{\mathscr{L}}$. Then $\Gamma \vdash_{\mathbf{W}_{\mathscr{L}}} \Delta_i$ by Lemma 2. But $\Gamma \vdash_{\mathbf{W}_{\mathscr{L}}} [\Delta_i \to (\varphi \to \Delta_i)]$ by (A1) and Lemma 2. Hence, by Lemma 3, $\Gamma \vdash_{\mathbf{W}_{\mathscr{L}}} (\varphi \to \Delta_i)$. Case 2: $\Delta_i = \varphi$. Then $\Gamma \vdash_{\mathbf{W}_{\mathscr{L}}} (\varphi \to \Delta_i)$ by Lemma 4. Case 3: $\Delta_j = (\Delta_k \to \Delta_i)$ where $j, k \in i$. Then by the induction hypothesis we have $\Gamma \vdash_{\mathbf{W}_{\mathscr{L}}} (\varphi \to \Delta_k)$ and $\Gamma \vdash_{\mathbf{W}_{\mathscr{L}}} [\varphi \to (\Delta_k \to \Delta_i)]$. By (A2) and Lemma 2 $\Gamma \vdash_{\mathbf{W}_{\mathscr{L}}} \{[\varphi \to (\Delta_k \to \Delta_i)] \to [(\varphi \to \Delta_k) \to (\varphi \to \Delta_i)]\}$; so by two applications of Lemma 3 we obtain $\Gamma \vdash_{\mathbf{W}_{\mathscr{L}}} (\varphi \to \Delta_i)$. Since $\psi = \Delta_{n-1}$, we are done. QED

Lemma 6 *Let* $\mathscr{L}$ *be a language,* Γ *a set of* $\mathscr{L}$*-formulas, and* φ *an* $\mathscr{L}$*-formula. If* μ *is a variable which does not occur free in any member of* Γ *and* $\Gamma \vdash_{\mathbf{W}_{\mathscr{L}}} \varphi$, *then* $\Gamma \vdash_{\mathbf{W}_{\mathscr{L}}} \Lambda\mu\varphi$.

Proof: Assume that μ does not occur free in any member of Γ and let $\Delta = \langle\Delta_0, \ldots, \Delta_{n-1}\rangle$ be a derivation of φ from Γ in $\mathbf{W}_{\mathcal{L}}$. We show by induction that for each $i \in n$, $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \wedge\mu\Delta_i$. To this end, assume that $i \in n$ and that the hypothesis is true for all $j \in i$. Case 1: $\Delta_i \in \Gamma$. Then μ does not occur free in Δ_i by hypothesis; so $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} (\Delta_i \to \wedge\mu\Delta_i)$ by (A5) and Lemma 2. But $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \Delta_i$ by Lemma 2, too; so $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \wedge\mu\Delta_i$ by Lemma 3. Case 2: $\Delta_i \in A_{\mathcal{L}}$. Then $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \wedge\mu\Delta_i$ by Lemma 2 since every generalization of an element of $A_{\mathcal{L}}$ is again an element of $A_{\mathcal{L}}$. Case 3: $\Delta_j = (\Delta_k \to \Delta_i)$ for some $j, k \in i$. Then by the induction hypothesis we have $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \wedge\mu\Delta_k$ and $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \wedge\mu(\Delta_k \to \Delta_i)$. But $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} [\wedge\mu(\Delta_k \to \Delta_i) \to (\wedge\mu\Delta_k \to \wedge\mu\Delta_i)]$ by (A4); so by two applications of Lemma 3, $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \wedge\mu\Delta_i$. Since $\varphi = \Delta_{n-1}$, we are done. QED

Lemma 7 *Let $\mathcal{L}$ and $\mathcal{L}'$ be languages, Γ a set of $\mathcal{L}$-formulas, and φ an $\mathcal{L}$-formula. If $\mathcal{L} \subseteq \mathcal{L}'$ and $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}'}} \varphi$, then $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \varphi$.*

Proof: Let $\mathcal{L} \subseteq \mathcal{L}'$ and $\Delta = \langle\Delta_0, \ldots, \Delta_{n-1}\rangle$ be a derivation of φ from Γ in $\mathbf{W}_{\mathcal{L}'}$. Let $\zeta_0, \ldots, \zeta_{m-1}$ be those distinct constant terms in $\mathcal{L}' \sim \mathcal{L}$ which occur in the sequents of Δ; let $\mu_0, \ldots, \mu_{m-1}$ be distinct variables that occur in no sequent of Δ and are such that for each $i \in m$, μ_i is of the same type as ζ_i; and, finally, for each $i \in n$, let $\Delta_i' = \Delta_i\begin{bmatrix}\zeta_0, \ldots, \zeta_{m-1}\\ \mu_0, \ldots, \mu_{m-1}\end{bmatrix}$. Then $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \Delta_i'$ for each $i \in n$, and we proceed to prove this fact by induction on i. To this end, assume that $i \in n$ and that the hypothesis is true for all $j \in i$. Case 1: $\Delta_i \in \Gamma$. Then by assumption no constant in $\mathcal{L}' \sim \mathcal{L}$ occurs in Δ_i; so $\Delta_i' = \Delta_i$. Hence $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \Delta_i'$ by Lemma 2. Case 2: $\Delta_i \in A_{\mathcal{L}'}$. Then it is easy to see by inspection that $\Delta_i' \in A_{\mathcal{L}}$; so $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \Delta_i'$ by Lemma 2. Case 3: $\Delta_j = (\Delta_k \to \Delta_i)$ where $j, k \in i$. Then by the induction hypothesis $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \Delta_k'$ and $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} (\Delta_k \to \Delta_i)'$. But $(\Delta_k \to \Delta_i)' = (\Delta_k' \to \Delta_i')$; so $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \Delta_i'$ by Lemma 3. Since $\varphi = \Delta_{n-1} = \Delta_{n-1}'$, we are done. QED

Lemma 8 *Let $\mathcal{L}$ be a language, Γ a set of $\mathcal{L}$-formulas, φ an $\mathcal{L}$-formula, $\zeta \in \mathcal{L}$, and μ a variable. If μ is of the same type as ζ, μ can be properly substituted for ζ in φ, μ, ζ do not occur free in any member of Γ, and $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \varphi$, then $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \wedge\mu\varphi\begin{bmatrix}\zeta\\ \mu\end{bmatrix}$.*

Proof: Let μ be a variable of the same type as ζ which can be properly substituted for ζ in φ and which does not occur free in any member of Γ, and let $\Delta = \langle\Delta_0, \ldots, \Delta_{n-1}\rangle$ be a derivation of φ from Γ in $\mathbf{W}_{\mathcal{L}}$. Now let ν be a variable of the same type as μ which does not occur in any member of Δ, and for each $i \in n$ let $\Delta_i' = \Delta_i\begin{bmatrix}\zeta\\ \nu\end{bmatrix}$. Finally, assume that ν does not occur in any member of Γ. By the method of the proof for Lemma 7 we can prove that for each $i \in n$ $\Gamma' \vdash_{\mathbf{W}_{\mathcal{L}}} \Delta_i'$ where $\Gamma' = \Gamma \cap \{\Delta_0, \ldots, \Delta_{n-1}\}$. Hence $\Gamma' \vdash_{\mathbf{W}_{\mathcal{L}}} \varphi\begin{bmatrix}\zeta\\ \nu\end{bmatrix}$ since $\varphi = \Delta_{n-1}$. Now, since ν does not occur in any member of Γ', we may conclude that $\Gamma' \vdash_{\mathbf{W}_{\mathcal{L}}} \wedge\nu\varphi\begin{bmatrix}\zeta\\ \nu\end{bmatrix}$ by Lemma 6; so $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \wedge\nu\varphi\begin{bmatrix}\zeta\\ \nu\end{bmatrix}$ by Lemma 1. But by (A6) and Lemma 2 $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \left(\wedge\nu\varphi\begin{bmatrix}\zeta\\ \nu\end{bmatrix} \to \varphi\begin{bmatrix}\zeta\\ \mu\end{bmatrix}\right)$; so by Lemma 3 we conclude that $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \varphi\begin{bmatrix}\zeta\\ \mu\end{bmatrix}$. Since μ does not occur free in any member of Γ, $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \Lambda\mu\varphi\begin{bmatrix}\zeta\\ \mu\end{bmatrix}$ by Lemma 6. QED

If $\mathbf{T}$ is a theory and Γ is a set of $\mathcal{L}_{\mathbf{T}}$-formulas, then we say Γ is *consistent in* $\mathbf{T}$ iff there is an $\mathcal{L}_{\mathbf{T}}$-formula φ such that $\Gamma \nvdash_{\mathbf{T}} \varphi$.

Lemma 9 *Let $\mathcal{L}$ be a language and Γ be a set of $\mathcal{L}$-formulas. Then Γ is consistent in $\mathbf{W}_{\mathcal{L}}$ iff there is no $\mathcal{L}$-formula φ such that $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \varphi$ and $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \sim\varphi$.*

Proof: Assume that there is an $\mathcal{L}$-formula, φ, such that $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \varphi$ and $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \sim\varphi$. Let ψ be any $\mathcal{L}$-formula. Then $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} [\varphi \to (\sim\varphi \to \psi)]$ by Lemma 4. By two applications of Lemma 3 we obtain $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \psi$. This shows that Γ is inconsistent in $\mathbf{W}_{\mathcal{L}}$. On the other hand, assume that Γ is inconsistent in $\mathbf{W}_{\mathcal{L}}$. Then the desired result follows trivially. QED

Let $\mathcal{L}$ be a language and Γ be a set of $\mathcal{L}$-formulas. Then we say f is an *$\mathcal{L}$-external constant term realization of* Γ iff f is a 1-1 function whose domain is the set of variables which occur free in some member of Γ, and $f(\mu)$ is, for each μ in the domain of f, a constant term of the same type as μ and not in $\mathcal{L}$. If f is an $\mathcal{L}$-external constant term realization of Γ, then we let $S_f^{\mathcal{L}}$ be the function whose domain is the set of $\mathcal{L}$-formulas and which is such that, for each $\mathcal{L}$-formula φ, $S_f^{\mathcal{L}}(\varphi)$ is the result of replacing each free occurrence in φ of any variable μ in the domain of f by $f(\mu)$. The following lemma makes the observation that free variables in the hypothesis set of a derivation serve as constant terms.

Lemma 10 *Let $\mathcal{L}$ be a language, Γ be a set of $\mathcal{L}$-formulas, and f be an $\mathcal{L}$-external constant term realization of Γ. Then, for each $\mathcal{L}$-formula φ, $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \varphi$ iff $S_f^{\mathcal{L}}[\Gamma] \vdash_{\mathbf{W}_{\mathcal{L} \cup \mathcal{R}(f)}} S_f^{\mathcal{L}}(\varphi)$.*[11] *Accordingly, Γ is consistent in $\mathbf{W}_{\mathcal{L}}$ iff $S_f^{\mathcal{L}}[\Gamma]$ is consistent in $\mathbf{W}_{\mathcal{L} \cup \mathcal{R}(f)}$.*

Proof: Essentially the same as the proof of Lemma 7.

Lemma 11 *Let $\mathcal{L}$ be a language and Γ be a set of $\mathcal{L}$-formulas. Then Γ is consistent in $\mathbf{W}_{\mathcal{L}}$ iff every finite subset of Γ is consistent in $\mathbf{W}_{\mathcal{L}}$.*

Let $\mathbf{T}$ be a theory and Γ be a set of $\mathcal{L}_{\mathbf{T}}$-formulas. We say that Γ is *maximally consistent in* $\mathbf{T}$ iff for every $\mathcal{L}_{\mathbf{T}}$-formula φ, if $\Gamma \cup \{\varphi\}$ is consistent in $\mathbf{T}$, then $\varphi \in \Gamma$.

Lemma 12 *Let $\mathcal{L}$ be a language and Γ be a set of $\mathcal{L}$-formulas. If Γ is maximally consistent in $\mathbf{W}_{\mathcal{L}}$, then, for every $\mathcal{L}$-formula φ, $\varphi \in \Gamma$ iff $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \varphi$.*

Proof: Assume that Γ is maximally consistent in $\mathbf{W}_{\mathcal{L}}$. Let φ be any $\mathcal{L}$-formula. If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \varphi$ by Lemma 2. Assume, on the other hand, that $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \varphi$ but $\varphi \notin \Gamma$. Then, since Γ is maximally consistent in $\mathbf{W}_{\mathcal{L}}$, $\Gamma \cup \{\varphi\}$ is inconsistent in $\mathbf{W}_{\mathcal{L}}$. Hence, $\Gamma \cup \{\varphi\} \vdash_{\mathbf{W}_{\mathcal{L}}} \sim\varphi$; wherefore, by Lemma 5, $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} (\varphi \to \sim\varphi)$. It follows by Lemma 3 that $\Gamma \vdash_{\mathbf{W}_{\mathcal{L}}} \sim\varphi$. We therefore conclude by Lemma 9 that Γ is inconsistent in $\mathbf{W}_{\mathcal{L}}$, but this contradicts our assumption that Γ is maximally consistent in $\mathbf{W}_{\mathcal{L}}$. Accordingly, we have shown the right-to-left direction. QED

Lemma 13 *Let $\mathcal{L}$ be a language and Γ be a set of $\mathcal{L}$-formulas. If Γ is consistent in $\mathbf{W}_{\mathcal{L}}$, then Γ is a subset of some set of $\mathcal{L}$-formulas which is maximally consistent in $\mathbf{W}_{\mathcal{L}}$.*

Proof: Assume that Γ is consistent in $\mathbf{W}_{\mathcal{L}}$. Let $\Sigma = \langle \Sigma_\beta \rangle_{\beta \in \alpha}$ be an enumeration of the $\mathcal{L}$-formulas for some ordinal α. We define by transfinite induction on $\beta \leqslant \alpha \dotplus 1$[12]: $\Gamma'_0 = \Gamma$; $\Gamma'_{\beta \dotplus 1} = \Gamma'_\beta \cup \{\Sigma_\beta\}$ if $\Gamma'_\beta \cup \{\Sigma_\beta\}$ is consistent in $\mathbf{W}_{\mathcal{L}}$, else Γ'_β; and, for β a limit ordinal, $\Gamma'_\beta = \bigcup_{\gamma \in \beta} \Gamma'_\gamma$. Let $\mathrm{K} = \Gamma'_\alpha$. It is easy to show by transfinite induction on β that Γ'_β is consistent in $\mathbf{W}_{\mathcal{L}}$ for each $\beta \leqslant \alpha$, and, in particular, therefore, K is consistent in $\mathbf{W}_{\mathcal{L}}$. We next show that K is maximally consistent in $\mathbf{W}_{\mathcal{L}}$ as well. To this end, assume that φ is an $\mathcal{L}$-formula with the property that $\mathrm{K} \cup \{\varphi\}$ is consistent in $\mathbf{W}_{\mathcal{L}}$. Now, $\varphi = \Sigma_\beta$ for some $\beta \in \alpha$; so $\Gamma'_\beta \cup \{\Sigma_\beta\}$ is consistent in $\mathbf{W}_{\mathcal{L}}$, whence $\varphi \in \Gamma'_{\beta \dotplus 1} \subseteq \mathrm{K}$. This proves our assertion. Since $\Gamma \subseteq \mathrm{K}$, we are done. QED

Lemma 14 *Let $\mathcal{L}$ be a language, α be an ordinal, $\langle \Sigma_\beta \rangle_{\beta \in \alpha}$ be a sequence of $\mathcal{L}$-formulas, $\langle \mu_\beta \rangle_{\beta \in \alpha}$ be a sequence of variables, $\langle \zeta_\beta \rangle_{\beta \in \alpha}$ be a sequence of constant terms, and Γ be a set of $\mathcal{L}$-formulas. If $\zeta_\beta \neq \zeta_\gamma$ for all $\beta \neq \gamma$ in α such that μ_β and μ_γ occur free in Σ_β and Σ_γ, respectively, $\zeta_\beta \notin \mathcal{L}$ and is of the same type as μ_β for each $\beta \in \alpha$ such that μ_β occurs free in Σ_β, Γ is consistent in $\mathbf{W}_{\mathcal{L}}$, and no member of $\{\mu_\beta : \beta \in \alpha\}$ occurs free in any element of Γ, then $\Gamma \cup \left\{\left(\sim\Lambda\mu_\beta\Sigma_\beta \rightarrow \sim\Sigma_\beta\begin{bmatrix}\mu_\beta\\ \zeta_\beta\end{bmatrix}\right) : \beta \in \alpha\right\}$ is consistent in $\mathbf{W}_{\mathcal{L}'}$, where $\mathcal{L}' = \mathcal{L} \cup \{\zeta_\beta : \beta \in \alpha$ and μ_β occurs free in $\Sigma_\beta\}$.*

Proof: Assume the hypotheses. For convenience, let $\Delta_\beta = \left(\sim\Lambda\mu_\beta\Sigma_\beta \rightarrow \sim\Sigma_\beta\begin{bmatrix}\mu_\beta\\ \zeta_\beta\end{bmatrix}\right)$ for each $\beta \in \alpha$. We define by transfinite induction on β: $\mathrm{K}_0 = \Gamma$, $\mathrm{K}_{\beta \dotplus 1} = \mathrm{K}_\beta \cup \{\Delta_\beta\}$ for $\beta \in \alpha$, and $\mathrm{K}_\beta = \bigcup_{\gamma \in \beta} \mathrm{K}_\gamma$ for $\beta \leqslant \alpha$ a limit ordinal. Note that ζ_β does not occur in Δ_β if μ_β does not occur free in Σ_β; so every member of K_α is an $\mathcal{L}'$-formula. We now show by transfinite induction on β that for each $\beta \leqslant \alpha$, K_β is consistent in $\mathbf{W}_{\mathcal{L}'}$. That K_0 is consistent in $\mathbf{W}_{\mathcal{L}'}$ follows from Lemmas 7 and 9. Now let $\beta \in \alpha$ and assume that K_β is consistent in $\mathbf{W}_{\mathcal{L}'}$ but $\mathrm{K}_{\beta \dotplus 1} = \mathrm{K}_\beta \cup \{\Delta_\beta\}$ is not. Then $\mathrm{K}_\beta \cup \{\Delta_\beta\} \vdash_{\overline{\mathbf{W}_{\mathcal{L}'}}} \sim\Delta_\beta$; so by Lemma 5 $\mathrm{K}_\beta \vdash_{\overline{\mathbf{W}_{\mathcal{L}'}}} (\Delta_\beta \rightarrow \sim\Delta_\beta)$. Since $[(\Delta_\beta \rightarrow \sim\Delta_\beta) \rightarrow \sim\Delta_\beta]$ is a tautology, it follows from Lemmas 4 and 3 that $\mathrm{K}_\beta \vdash_{\overline{\mathbf{W}_{\mathcal{L}'}}} \sim\Delta_\beta$, i.e., $\mathrm{K}_\beta \vdash_{\overline{\mathbf{W}_{\mathcal{L}'}}} \sim\left(\sim\Lambda\mu_\beta\Sigma_\beta \rightarrow \sim\Sigma_\beta\begin{bmatrix}\mu_\beta\\ \zeta_\beta\end{bmatrix}\right)$. Again, since $[\sim(\varphi \rightarrow \sim\psi) \rightarrow \varphi]$ and $[\sim(\varphi \rightarrow \sim\psi) \rightarrow \psi]$ are tautologies for any $\mathcal{L}'$-formulas φ and ψ, it follows from Lemmas 4 and 3 that $\mathrm{K}_\beta \vdash_{\overline{\mathbf{W}_{\mathcal{L}'}}} \sim\Lambda\mu_\beta\Sigma_\beta$ and $\mathrm{K}_\beta \vdash_{\overline{\mathbf{W}_{\mathcal{L}'}}} \Sigma_\beta\begin{bmatrix}\mu_\beta\\ \zeta_\beta\end{bmatrix}$. Our hypotheses guarantee that μ_β occurs free in no element of K_β and that ζ_β occurs in no element of K_β. Hence, by Lemma 8, $\mathrm{K}_\beta \vdash_{\overline{\mathbf{W}_{\mathcal{L}'}}} \Lambda\mu_\beta\Sigma_\beta$. But this implies by Lemma 9 that K_β is inconsistent in $\mathbf{W}_{\mathcal{L}'}$, which is contrary to assumption. Hence, if $\beta \in \alpha$ and K_β is consistent in $\mathbf{W}_{\mathcal{L}'}$, then so is $\mathrm{K}_{\beta \dotplus 1}$. Next let $\beta \leqslant \alpha$ be a limit ordinal and assume that for each $\gamma \in \beta$ K_γ is consistent in $\mathbf{W}_{\mathcal{L}'}$. Then so is $\mathrm{K}_\beta = \bigcup_{\gamma \in \beta} \mathrm{K}_\gamma$ by Lemma 11. Consequently, our assertion is proved, and, in particular, we have shown that K_α is consistent in $\mathbf{W}_{\mathcal{L}'}$. But $\mathrm{K}_\alpha = \Gamma \cup \{\Delta_\beta : \beta \in \alpha\}$. QED

3 *Semantics* Let $\mathcal{L}$ be a language. We say that $\mathfrak{A}$ is an *$\mathcal{L}$-structure* iff $\mathfrak{A}$ is a 4-place sequence $\langle \mathcal{D}, \langle D_n\rangle_{n\in\omega}, H, I\rangle$ such that: (i) for all $n \in \omega$, $0 \neq D_n \subseteq \mathcal{D}$; (ii) $H \subseteq \mathcal{D} \times \bigcup_{n\in\omega} ({}^n\mathcal{D})^{13}$; and (iii) I is a function whose domain is the set of $\mathcal{L}$-terms and which is such that for every individual $\mathcal{L}$-term, ζ, $I(\zeta) \in D_0$ and for every $n \in \omega$ and n-place predicate $\mathcal{L}$-term, η, $I(\eta) \in D_{n+1}$. Now let $\mathfrak{A} = \langle \mathcal{D}, \langle D_n\rangle_{n\in\omega}, H, I\rangle$ be an $\mathcal{L}$-structure. If ζ is an $\mathcal{L}$-term, then we define *the category of* ζ *in* $\mathfrak{A}$ to be D_0 if ζ is an individual $\mathcal{L}$-term and D_{n+1} if ζ is an n-place predicate $\mathcal{L}$-term. If $d \in \mathcal{D}$ and ζ is an $\mathcal{L}$-term, then we set $\mathfrak{A}\begin{bmatrix}\zeta\\ d\end{bmatrix} = \left\langle \mathcal{D}, \langle D_n\rangle_{n\in\omega}, H, I\begin{pmatrix}\zeta\\ d\end{pmatrix}\right\rangle$, where $I\begin{pmatrix}\zeta\\ d\end{pmatrix} = (I \sim \{(\zeta, I(\zeta))\}) \cup \{(\zeta, d)\}$, i.e., $I\begin{pmatrix}\zeta\\ d\end{pmatrix}$ is the function which agrees with I everywhere except, possibly, at ζ where it takes the value d. If $\mathcal{L}' \subseteq \mathcal{L}$, then by $\mathfrak{A}|\mathcal{L}'$ we will understand $\langle \mathcal{D}, \langle D_n\rangle_{n\in\omega}, H, I|\mathcal{L}'\rangle$, where $I|\mathcal{L}'$ is the function whose domain is the set of $\mathcal{L}'$-terms and which agrees with I on those terms.

Let $\mathcal{L}$ be a language, $\mathfrak{A} = \langle \mathcal{D}, \langle D_n\rangle_{n\in\omega}, H, I\rangle$ be an $\mathcal{L}$-structure, and φ be an $\mathcal{L}$-formula. We define φ to be *true in* $\mathfrak{A}$ by induction on $\mathcal{L}$-formulas as follows:

1. if $\varphi = \zeta(\eta_0, \ldots, \eta_{n-1})$ is an atomic $\mathcal{L}$-formula, then φ is true in $\mathfrak{A}$ iff $I(\zeta)H\langle I(\eta_0), \ldots, I(\eta_{n-1})\rangle$, i.e., iff $(I(\zeta), \langle I(\eta_0), \ldots, I(\eta_{n-1})\rangle) \in H$
2. if $\varphi = \sim\psi$, then φ is true in $\mathfrak{A}$ iff ψ is not true in $\mathfrak{A}$
3. if $\varphi = (\psi \rightarrow \chi)$, then φ is true in $\mathfrak{A}$ iff either ψ is not true in $\mathfrak{A}$ or χ is true in $\mathfrak{A}$
4. if $\varphi = \Lambda\mu\psi$, then φ is true in $\mathfrak{A}$ iff for every d in the category of μ in $\mathfrak{A}$, ψ is true in $\mathfrak{A}\begin{bmatrix}\mu\\ d\end{bmatrix}$.

We write $\models_{\mathfrak{A}} \varphi$ for φ is true in $\mathfrak{A}$. We say that $\mathfrak{A}$ is a *model of* φ iff $\models_{\mathfrak{A}} \varphi$. Let Γ be a set of $\mathcal{L}$-formulas. We say that $\mathfrak{A}$ is a *model of* Γ, in symbols $\models_{\mathfrak{A}}\Gamma$, iff $\mathfrak{A}$ is a model of every member of Γ. Now let $\aleph$ be a class (which may be proper) of $\mathcal{L}$-structures. We say that φ is an *$\aleph$-valid consequence of* Γ, in symbols $\Gamma \models_{\aleph} \varphi$, iff every member of $\aleph$ which is a model of Γ is also a model of φ. We also say that φ is *$\aleph$-valid*, in symbols $\models_{\aleph} \varphi$, iff φ is an $\aleph$-valid consequence of 0, i.e., iff $0 \models_{\aleph} \varphi$.

Let $\mathcal{L}$ be a language and $\mathfrak{A} = \langle \mathcal{D}, \langle D_n\rangle_{n\in\omega}, H, I\rangle$ be an $\mathcal{L}$-structure. Then we will say that $\mathfrak{A}$ is *strict* iff (1) $D_n \cap D_m = 0$ for all $n, m \in \omega$ such that $n \neq m$, (2) $\mathcal{D} = \bigcup_{n\in\omega} D_n$, and (3) $H \subseteq \bigcup_{n\in\omega} (D_{n+1} \times {}^n\mathcal{D})$. By the *cardinality* of $\mathfrak{A}$ we will understand the cardinality of $\mathcal{D}$. For convenience we let $\aleph_V^{\mathcal{L}}$ be the class of $\mathcal{L}$-structures and $\aleph_S^{\mathcal{L}}$ be the class of strict $\mathcal{L}$-structures.

We now state a few useful semantical lemmas. In most cases the results are so obvious that they need no proof.

Lemma 15 *Let $\mathcal{L}$ be a language,* Γ *and* K *be sets of $\mathcal{L}$-formulas,* $\aleph$ *and* $\beth$ *be classes of $\mathcal{L}$-structures, and φ be an $\mathcal{L}$-formula. If* $\Gamma \subseteq \mathrm{K}$, $\beth \subseteq \aleph$, *and* $\Gamma \models_{\aleph} \varphi$, *then* $\mathrm{K} \models_{\beth} \varphi$.

Lemma 16 *Let $\mathcal{L}$ be a language, Γ be a set of $\mathcal{L}$-formulas, $\aleph$ be a class of $\mathcal{L}$-structures, and φ be an $\mathcal{L}$-formula. If $\varphi \in \Gamma$, then $\Gamma \models_{\aleph} \varphi$.*

Lemma 17 *Let $\mathcal{L}$ be a language, $\mathfrak{A}$ be an $\mathcal{L}$-structure, φ be an $\mathcal{L}$-formula, and μ be a variable. If μ does not occur free in φ, then, for every d in the category of μ in $\mathfrak{A}$, $\models_{\mathfrak{A}} \varphi$ iff $\models_{\mathfrak{A}\left[\begin{smallmatrix}\mu\\ d\end{smallmatrix}\right]} \varphi$.*

Proof: A trivial induction on $\mathcal{L}$-formulas.

Lemma 18 *Let $\mathcal{L}$ be a language, $\mathfrak{A} = \langle \mathcal{D}, \langle D_n \rangle_{n \in \omega}, H, I \rangle$ be an $\mathcal{L}$-structure, φ be an $\mathcal{L}$-formula, and ζ and η be $\mathcal{L}$-terms. If $I(\eta)$ is in the category of ζ in $\mathfrak{A}$, and η can be properly substituted for ζ in φ, then $\models_{\mathfrak{A}} \varphi\begin{bmatrix}\zeta\\ \eta\end{bmatrix}$ iff $\models_{\mathfrak{A}\left[\begin{smallmatrix}\zeta\\ I(\eta)\end{smallmatrix}\right]} \varphi$.*

Proof: Let $\mathcal{L}$ be a language and ζ and η be $\mathcal{L}$-terms. We prove by induction on $\mathcal{L}$-formulas φ that, for every $\mathcal{L}$-structure $\mathfrak{A} = \langle \mathcal{D}, \langle D_n \rangle_{n \in \omega}, H, I \rangle$, if $I(\eta)$ is in the category of ζ in $\mathfrak{A}$, and η can be properly substituted for ζ in φ, then $\models_{\mathfrak{A}} \varphi\begin{bmatrix}\zeta\\ \eta\end{bmatrix}$ iff $\models_{\mathfrak{A}\left[\begin{smallmatrix}\zeta\\ I(\eta)\end{smallmatrix}\right]} \varphi$. The only case of this induction worth considering is when φ is of the form $\Lambda\mu\psi$. If $\mu = \zeta$, then $\varphi\begin{bmatrix}\zeta\\ \eta\end{bmatrix} = \varphi$, and the desired result is seen to follow from Lemma 17. If $\mu = \eta$, then, since η can be properly substituted for ζ in φ, it follows that ζ does not occur free in φ; so $\varphi\begin{bmatrix}\zeta\\ \eta\end{bmatrix} = \varphi$, and the desired result again follows by Lemma 17. Finally, if $\mu \neq \zeta$ and $\mu \neq \eta$, then the conclusion we want is drawn from induction hypothesis and the last clause in the definition of truth in an $\mathcal{L}$-structure.

QED

Lemma 19 *Let $\mathcal{L}$ be a language, $\aleph$ be a class of $\mathcal{L}$-structures, and φ be an $\mathcal{L}$-formula. If $\varphi \in A_{\mathcal{L}}$, then $\models_{\aleph} \varphi$.*

Proof: Assume the hypothesis. If φ falls under (A1)-(A4), then the result is trivial. If φ falls under (A5), then Lemma 17 guarantees the result. Finally, if φ falls under (A6), then the desired result follows from Lemma 17 in conjunction with Lemma 18. QED

Lemma 20 *Let $\mathcal{L}$ be a language, Γ be a set of $\mathcal{L}$-formulas, $\aleph$ be a class of $\mathcal{L}$-structures, and φ and ψ be $\mathcal{L}$-formulas. If $\Gamma \models_{\mathfrak{A}} \varphi$ and $\Gamma \models_{\mathfrak{A}} (\varphi \rightarrow \psi)$, then $\Gamma \models_{\mathfrak{A}} \psi$.*

Lemma 21 *Let $\mathcal{L}$ be a language, Γ be a set of $\mathcal{L}$-formulas, $\aleph$ be a class of $\mathcal{L}$-structures, and φ be an $\mathcal{L}$-formula. If $\Gamma \vdash_{W_{\mathcal{L}}} \varphi$, then $\Gamma \models_{\aleph} \varphi$.*

Proof: By Lemmas 16, 19, 15, and 20.

Lemma 22 *Let $\mathcal{L}$ and $\mathcal{L}'$ be languages. If $\mathcal{L} \subseteq \mathcal{L}'$, φ is an $\mathcal{L}$-formula, Γ is a set of $\mathcal{L}$-formulas, and $\aleph$ is a class of $\mathcal{L}'$-structures, then $\Gamma \models_{\aleph} \varphi$ iff $\Gamma \models_{\aleph'} \varphi$, where $\aleph' = \{\mathfrak{A} | \mathcal{L} : \mathfrak{A} \in \aleph\}$.*

Proof: A trivial induction on $\mathcal{L}$-formulas.

4 *Completeness*

Theorem 23 *Let $\mathcal{L}$ be a language and Γ be a set of $\mathcal{L}$-formulas. If Γ is consistent in $\mathbf{W}_{\mathcal{L}}$, then there is a strict $\mathcal{L}$-structure (whose cardinality is $\omega + |\mathcal{L}|$) which is a model of Γ.*

Proof: Assume that Γ is consistent in $\mathbf{W}_{\mathcal{L}}$. Let f be an $\mathcal{L}$-external constant realization of Γ, $\mathcal{L}' = \mathcal{L} \cup \mathcal{R}(f)$, and $\Gamma' = S_f^{\mathcal{L}}[\Gamma]$. By construction no variable occurs free in any member of Γ', i.e., every member of Γ' is closed. Furthermore, by Lemma 10, Γ' is consistent in $\mathbf{W}_{\mathcal{L}'}$.

Let $\Sigma = \langle \Sigma_\beta \rangle_{\beta \in \alpha}$ be, for some ordinal α, an enumeration of all the $\mathcal{L}'$-formulas which contain at most one free variable. For each $\beta \in \alpha$, let μ_β be the variable which occurs free in Σ_β if such a variable exists, else let μ_β be any variable. Next let $\langle \zeta_\beta \rangle_{\beta \in \alpha}$ be a pairwise distinct enumeration of constant terms which are not members of $\mathcal{L}'$ and are such that, for each $\beta \in \alpha$, ζ_β is of the same type as μ_β. Finally, let $\mathcal{L}'' = \mathcal{L}' \cup \{\zeta_\beta\colon \beta \in \alpha\}$. It follows from Lemma 14 that $\Gamma' \cup \left\{\left(\sim\Lambda\mu_\beta\Sigma_\beta \rightarrow \sim\Sigma_\beta\begin{bmatrix}\mu_\beta\\ \zeta_\beta\end{bmatrix}\right)\colon \beta \in \alpha\right\}$ is consistent in $\mathbf{W}_{\mathcal{L}''}$. It then follows from Lemma 13 that $\Gamma' \cup \left\{\left(\sim\Lambda\mu_\beta\Sigma_\beta \rightarrow \sim\Sigma_\beta\begin{bmatrix}\mu_\beta\\ \zeta_\beta\end{bmatrix}\right)\colon \beta \in \alpha\right\}$ is a subset of some set K of $\mathcal{L}''$-formulas which is maximally consistent in $\mathbf{W}_{\mathcal{L}''}$.

We use K to construct an $\mathcal{L}''$-structure. Let $\mathcal{D} = \mathcal{L}''$; $D_n = C(n) \cap \mathcal{L}''$ for each $n \in \omega$; $H = \{(\zeta, \langle \eta_0, \ldots, \eta_{n-1}\rangle)\colon n \in \omega,\ \zeta, \eta_0, \ldots, \eta_{n-1} \in \mathcal{L}'', \text{ and } \zeta(\eta_0, \ldots, \eta_{n-1}) \in \mathrm{K}\}$; and I be any function which coincides with the identity function on the constant $\mathcal{L}''$-terms and is such that $I(\mu)$ is a constant $\mathcal{L}''$-term of the same type as μ for every variable μ. Then $\mathfrak{A} = \langle \mathcal{D}, \langle D_n\rangle_{n \in \omega}, H, I\rangle$ is a strict $\mathcal{L}''$-structure. Furthermore, if we chose the sequents of Σ to be pairwise distinct, then the cardinality of $\mathfrak{A}$ is just $\omega + |\mathcal{L}|$.

We now prove by induction on the number of occurrences of logical particles in φ that for every closed $\mathcal{L}''$-formula φ, $\varphi \in \mathrm{K}$ iff $\models_{\mathfrak{A}} \varphi$. Case 1: The number of occurrences of logical particles in φ is 0. Then φ is atomic and the desired result follows from our definition of H. Case 2: Assume that the number of occurrences of logical particles in φ is $n + 1$ and that the result holds for every closed $\mathcal{L}''$-formula in which the number of occurrences of logical particles is n. If φ is of the form $\sim\psi$ or $(\psi \rightarrow \chi)$, then the desired result again follows easily. Hence, assume that φ is of the form $\Lambda\mu\psi$. For the left-to-right direction, assume that $\Lambda\mu\psi \in \mathrm{K}$. Then, for every constant $\mathcal{L}''$-term ζ of the same type as μ, $\psi\begin{bmatrix}\mu\\ \zeta\end{bmatrix} \in \mathrm{K}$ by (A6), Lemmas 2, 3, and 12; so $\models_{\mathfrak{A}} \psi\begin{bmatrix}\mu\\ \zeta\end{bmatrix}$ by the induction hypothesis, whence $\models_{\mathfrak{A}\begin{bmatrix}\mu\\ \zeta\end{bmatrix}} \psi$ by Lemma 17. Since the category of μ in $\mathfrak{A}$ is just the set of constant $\mathcal{L}''$-terms which are of the same type as μ, this implies that $\models_{\mathfrak{A}} \Lambda\mu\psi$, i.e., that $\models_{\mathfrak{A}} \varphi$. For the right-to-left direction assume that $\models_{\mathfrak{A}} \Lambda\mu\psi$ but $\Lambda\mu\psi \notin \mathrm{K}$. Since $\Lambda\mu\psi$ is closed, ψ contains at most one free variable; so there is some $\beta \in \alpha$ such that $\psi = \Sigma_\beta$. Now, if ψ itself is closed, then by (A6), and Lemmas 2, 3, 6, and 12 we find $\Lambda\mu_\beta\psi \in \mathrm{K}$, and by Lemma 17 we find $\models_{\mathfrak{A}} \Lambda\mu_\beta\psi$. In any case, we may assume that $\mu = \mu_\beta$. Now, since we have assumed that

$\Lambda\mu\psi \notin K$, it follows that $\sim\Lambda\mu_\beta\Sigma_\beta \in K$. But, by construction, $\left(\sim\Lambda\mu_\beta\Sigma_\beta \rightarrow \sim\Sigma_\beta\begin{bmatrix}\mu_\beta \\ \zeta_\beta\end{bmatrix}\right) \in K$; so $K \vdash_{\mathbf{W}_\mathcal{L}} \sim\Sigma_\beta\begin{bmatrix}\mu_\beta \\ \zeta_\beta\end{bmatrix}$ by Lemmas 2 and 3. Accordingly, $\sim\Sigma_\beta\begin{bmatrix}\mu_\beta \\ \zeta_\beta\end{bmatrix} \in K$ by Lemma 12. On the other hand, $\models_{\mathfrak{A}} \Lambda\mu_\beta\Sigma_\beta$; so $\models_{\mathfrak{A}\begin{bmatrix}\mu_\beta \\ \zeta_\beta\end{bmatrix}} \Sigma_\beta$, whence $\models_{\mathfrak{A}} \Sigma_\beta\begin{bmatrix}\mu_\beta \\ \zeta_\beta\end{bmatrix}$ by Lemma 18. Thus, by the induction hypothesis, $\Sigma_\beta\begin{bmatrix}\mu_\beta \\ \zeta_\beta\end{bmatrix} \in K$. But then K is inconsistent in $\mathbf{W}_\mathcal{L}$—a contradiction. Hence, the right-to-left direction is proved, and we have established our assertion.

Since $\Gamma' \subseteq K$ and every member of Γ' is closed, it follows that $\models_{\mathfrak{A}} \Gamma'$. Since I is essentially arbitrary on the variables, we may require that $f \subseteq I$, and, when we do so require, we find that $\models_{\mathfrak{A}} \Gamma$ by Lemma 18. Hence, by Lemma 22, $\models_{\mathfrak{A}|\mathcal{L}} \Gamma$. QED

The proof of Theorem 23 has been adapted for our purposes from the proof of the corresponding theorem for first-order logic in Mendelson [13], pp. 65-67.

Let $\mathcal{L}$ be a language and $\mathfrak{A} = \langle \mathcal{D}, \langle D_n \rangle_{n \in \omega}, H, I \rangle$ be an $\mathcal{L}$-structure. If $\langle \mathfrak{m}_n \rangle_{n \in \omega}$ is a sequence of cardinals, then we say that $\mathfrak{A}$ has *cardinality structure* $\langle \mathfrak{m}_n \rangle_{n \in \omega}$ iff for each $n \in \omega$, $|D_n| = \mathfrak{m}_n$. By adjusting the cardinality of repetitions of $\mathcal{L}$-formulas in Σ of the proof of Theorem 23, we can prove:

Theorem 24 *Let $\mathcal{L}$ be a language, Γ be a set of $\mathcal{L}$-formulas, and $\mathfrak{m} = \langle \mathfrak{m}_n \rangle_{n \in \omega}$ be a sequence of cardinals. If $\mathfrak{m}_n \geq \omega + |\mathcal{L}|$ for each $n \in \omega$, and Γ is consistent in $\mathbf{W}_\mathcal{L}$, then there is a strict $\mathcal{L}$-structure which is a model of Γ and whose cardinality structure is $\mathfrak{m}$.*

Theorem 23 provides the means for proving the following two completeness theorems, the latter of which is a corollary of the former.

Theorem 25 *Let $\mathcal{L}$ be a language, Γ be a set of $\mathcal{L}$-formulas, and φ be an $\mathcal{L}$-formula. $\Gamma \vdash_{\mathbf{W}_\mathcal{L}} \varphi$ iff $\Gamma \models_{\aleph_S^\mathcal{L}} \varphi$.*

Proof: In light of Lemma 21, we need only prove the right-to-left direction. To this end, assume that $\Gamma \models_{\aleph_S^\mathcal{L}} \varphi$ but $\Gamma \nvdash_{\mathbf{W}_\mathcal{L}} \varphi$. Then $\Gamma \cup \{\sim\varphi\}$ is consistent in $\mathbf{W}_\mathcal{L}$. For, otherwise, $\Gamma \cup \{\sim\varphi\} \vdash_{\mathbf{W}_\mathcal{L}} \varphi$; so $\Gamma \vdash_{\mathbf{W}_\mathcal{L}} (\sim\varphi \rightarrow \varphi)$ by Lemma 5, whence $\Gamma \vdash_{\mathbf{W}_\mathcal{L}} \varphi$ by Lemmas 4 and 3—a contradiction. Accordingly, Theorem 23 guarentees that $\Gamma \cup \{\sim\varphi\}$ has a model which is a strict $\mathcal{L}$-structure. But such an $\mathcal{L}$-structure cannot also be a model of φ. QED

Theorem 26 *Let $\mathcal{L}$ be a language, Γ be a set of $\mathcal{L}$-formulas, and φ be an $\mathcal{L}$-formula. Then $\Gamma \vdash_{\mathbf{W}_\mathcal{L}} \varphi$ iff $\Gamma \models_{\aleph^\mathcal{L}} \varphi$.*

As mentioned in Section 1, the preceding strong completeness theorems yield corresponding strong completeness theorems for any theory **T** at least as strong as $\mathbf{W}_{\mathcal{L}_\mathbf{T}}$. If **T** is a theory, we say that $\mathfrak{A}$ is a **T**-structure iff $\mathfrak{A}$ is an $\mathcal{L}_\mathbf{T}$-structure which is a model of $A_\mathbf{T}$. We let $\aleph_{(S)\mathbf{T}}$ be the class of (strict) **T**-structures.

Theorem 27 *Let* $\mathbf{T}$ *be a theory,* Γ *be a set of* $\mathcal{L}_{\mathbf{T}}$*-formulas, and* φ *be an* $\mathcal{L}_{|\mathbf{T}}$*-formula. If* $\mathbf{T}$ *is an extension of* $\mathbf{W}_{\mathcal{L}_{\mathbf{T}}}$*, then* $\Gamma \vdash_{\mathbf{T}} \varphi$ *iff* $\Gamma \models_{\aleph_{(S)\mathbf{T}}} \varphi$.

Proof: By Theorems 25 and 26, $\Gamma \vdash_{\mathbf{T}} \varphi$ iff $\Gamma \cup A_{\mathbf{T}} \vdash_{\mathbf{W}_{\mathcal{L}|\mathbf{T}}} \varphi$ iff $\Gamma \cup A_{\mathbf{T}} \models^{\mathcal{L}_{\mathbf{T}}}_{\aleph_{(S)(V)}} \varphi$ iff $\Gamma \models_{\aleph_{(S)\mathbf{T}}} \varphi$. QED

5 *Results and Applications* In this section we are concerned only with that part of our theory corresponding to the language 0, viz. with $\mathbf{W}^*$ and its extensions. Henceforth, when reference to a language is omitted, we assume reference to 0, e.g., *formula, structure,* etc., refer to 0-formula, 0-structure, etc. Furthermore, unless otherwise indicated, *theory* refers to theories whose associated languages are 0.

5.1 *A Substitution Free Axiom Set for* $\mathbf{W}^*$ In this section we will formulate a substitution free axiom set for $\mathbf{W}^*$. See [8] for the general importance of substitution free axiom sets, particularly as regards modal contexts. We will see how the substitution free axiom set for $\mathbf{W}^*$ becomes simplified and expanded as we pass to extensions of $\mathbf{W}^*$ in later sections.

We say that t is a *type* iff $t \in \omega$. If α is an individual variable, we say that *the type of* α is 0. And if $n \in \omega$ and π is an n-place predicate variable, we say that *the type of* π is $n + 1$.

Let $n \in \omega$ and $t = \langle t_0, \ldots, t_{n-1}\rangle$ be an n-place sequence of types. If $i \in n + 1$, μ is a variable, and φ is a formula, then we say that μ has a *t, i-occurrence* in φ iff an atomic formula of the form $\pi(\nu_0, \ldots, \nu_{i-1}, \mu, \nu_i, \ldots, \nu_{n-1})$ occurs in φ where, for each $j \in n$, the type of $\nu_j = t_j$. (We understand $\pi(\nu_0, \ldots, \nu_{i-1}, \mu, \nu_i, \ldots, \nu_{n-1})$ to be: $\pi(\mu, \nu_0, \ldots, \nu_{n-1})$ if $i = 0$; $\pi(\nu_0, \ldots, \nu_{n-1}, \mu)$ if $i = n$; and $\pi(\mu)$ if $n = 0$.) If π is an n-place predicate variable and φ is a formula, we say that π has a *t-occurrence* in φ iff an atomic formula of the form $\pi(\mu_0, \ldots, \mu_{n-1})$ occurs in φ where, for each $i \in n$, the type of $\mu_i = t_i$.

If $i \in n+1$ and μ, μ' are variables, we say that φ is a *t, i-indiscernibility formula for* μ, μ' iff φ is a formula of the form

$$\Lambda\nu_0 \ldots \Lambda\nu_{n-1}\Lambda\pi(\pi(\nu_0, \ldots, \nu_{i-1}, \mu, \nu_i, \ldots, \nu_{n-1}) \leftrightarrow \pi(\nu_0, \ldots, \nu_{i-1}, \mu', \nu_i, \ldots, \nu_{n-1}))\ ,$$

where μ has a t, i-occurrence in φ; μ, μ' are each distinct from π, $\nu_0, \ldots, \nu_{n-1}$; and $\pi, \nu_0, \ldots, \nu_{n-1}$ are pairwise distinct. If μ, μ' are variables, we say that φ is a *general indiscernibility formula for* μ, μ' iff there are an $n \in \omega$, n-place sequence of types t, and $i \in n + 1$ such that φ is a t, i-indiscernibility formula for μ, μ'. If π, σ are n-place predicate variables, we say that φ is a *t-coextensivity formula for* π, σ iff φ is a formula of the form $\Lambda\mu_0 \ldots \Lambda\mu_{n-1}(\pi(\mu_0, \ldots, \mu_{n-1}) \leftrightarrow \sigma(\mu_0, \ldots, \mu_{n-1}))$, where π, σ are each distinct from $\mu_0, \ldots, \mu_{n-1}$; $\mu_0, \ldots, \mu_{n-1}$ are pairwise distinct; and, for each $i \in n$, the type of $\mu_i = t_i$. If π, σ are n-place predicate variables, we say that φ is a *general coextensivity formula for* π, σ iff there is an n-place sequence of types t such that φ is a t-coextensivity formula for π, σ.

We now define three new sets of formulas. We say that θ is an element of (A7), (A8), or (A9) iff θ is a generalization of any formula of the form:

(A7) $\vee\beta(\varphi_0 \wedge \ldots \wedge \varphi_{n-1})$, where β is an individual variable, $n \in \omega$, and there is an individual variable α such that (i) $\alpha \neq \beta$, and (ii) for each $i \in n$, φ_i is a general indiscernibility formula for α, β

(A8) $\vee\sigma(\varphi_0 \wedge \ldots \wedge \varphi_{n-1} \wedge \psi_0 \wedge \ldots \wedge \psi_{m-1})$, where σ is a predicate variable, $n, m \in \omega$, and there is a predicate variable π of the same type as σ such that (i) $\pi \neq \sigma$, (ii) for each $i \in n$, φ_i is a general indiscernibility formula for π, σ, and (iii) for each $i \in m$, ψ_i is a general coextensivity formula for π, σ

(A9) $\varphi \rightarrow (\psi \leftrightarrow \chi)$, where there are an $n \in \omega$, n-place sequence of types t, $i \in n + 1$, and distinct variables μ, μ' such that (i) φ is a t, i-indiscernibility formula for μ, μ', (ii) $\psi = \pi(\nu_0, \ldots, \nu_{i-1}, \mu, \nu_i, \ldots, \nu_{n-1})$ is an atomic formula in which μ' has a t, i-occurrence, and (iii) $\chi = \pi(\nu_0, \ldots, \nu_{i-1}, \mu', \nu_i, \ldots, \nu_{n-1})$.

Now let $\mathbf{W}^*_{sf}$ be the theory whose axiom set is the union of (A1)-(A5) and (A7)-(A9). We will prove that $\mathbf{W}^*_{sf}$ is equivalent to $\mathbf{W}^*$. Since it is clear that $\mathbf{W}^*_{sf}$ is a subsystem of $\mathbf{W}^*$, it will suffice to prove that every instance of (A6) is a theorem of $\mathbf{W}^*_{sf}$. Note that Lemmas 4, 5, and 6 hold for $\mathbf{W}^*_{sf}$.

Lemma 28 *Let* $[\varphi \rightarrow (\psi \rightarrow \chi)] \in$ (A9). *Then* $\vdash_{\mathbf{W}^*_{sf}} [\varphi \rightarrow (\chi \leftrightarrow \psi)]$.

Proof: Since $[\varphi \rightarrow (\psi \rightarrow \chi)] \in$ (A9), φ is of the form

$$\wedge\nu_0 \ldots \wedge\nu_{n-1}\wedge\pi(\pi(\nu_0, \ldots, \nu_{i-1}, \mu, \nu_i, \ldots, \nu_{n-1}) \leftrightarrow \pi(\nu_0, \ldots, \nu_{i-1}, \mu', \nu_i, \ldots, \nu_{n-1})),$$

where $\mu, \mu', \pi, \nu_0, \ldots, \nu_{n-1}$ are pairwise distinct. By Lemma 2 and (A9),

$$\vdash_{\mathbf{W}^*_{sf}} [\varphi \rightarrow (\pi(\nu_0, \ldots, \nu_{i-1}, \mu', \nu_i, \ldots, \nu_{n-1}) \leftrightarrow \pi(\nu_0, \ldots, \nu_{i-1}, \mu, \nu_i, \ldots, \nu_{n-1}))].$$

By several applications of Lemma 6, (A4), (A5), and Lemmas 4 and 3, we obtain $\vdash_{\mathbf{W}^*_{sf}} (\varphi \rightarrow \varphi')$, where φ' is obtained from φ by interchanging μ and μ'. Now, $[\varphi' \rightarrow (\chi \rightarrow \psi)] \in$ (A9). It follows from Lemmas 2, 4, and 3 that $\vdash_{\mathbf{W}^*_{sf}} [\varphi \rightarrow (\chi \leftrightarrow \psi)]$. QED

Lemma 29 *Let* $n \in \omega$, t *be an* n*-place sequence of types,* $i \in n + 1$, μ *and* μ' *be variables, and* ψ *and* χ *be formulas. If* φ *is a* t, i*-indiscernibility formula for* μ, μ', *and* χ *results from* ψ *by replacing a free* t, i*-occurrence of* μ *by a free occurrence of* μ', *then* $\vdash_{\mathbf{W}^*_{sf}} [\varphi \rightarrow (\psi \leftrightarrow \chi)]$.

Proof: We prove this by induction on formulas with respect to ψ. If $\mu = \mu'$, then there is nothing to prove in any case; so assume that $\mu \neq \mu'$. Case 1: ψ is atomic. Then the desired result follows by (A9). Case 2: ψ is of the form $\sim\psi'$ or $(\psi' \rightarrow \psi'')$. Then the desired result follows by *tautologous transformations*, i.e., Lemmas 4 and 3, and induction hypothesis. Case 3: ψ is of the form $\wedge\nu\psi'$. Then χ may be written as $\wedge\nu\chi'$, and as the induction hypothesis we assume that $\vdash_{\mathbf{W}^*_{sf}} [\varphi \rightarrow (\psi' \leftrightarrow \chi')]$. By tautologous transformations we therefore obtain $\vdash_{\mathbf{W}^*_{sf}} [\varphi \rightarrow (\psi' \rightarrow \chi')]$, whence, by

generalization, i.e., Lemma 6, $\vdash_{\overline{W^*_{sf}}} \wedge\nu[\varphi \rightarrow (\psi' \rightarrow \chi')]$. Applying (A4), and Lemmas 2 and 3, we get $\vdash_{\overline{W^*_{sf}}} [\wedge\nu\varphi \rightarrow (\psi \rightarrow \chi)]$. Since our assumption requires that $\nu \neq \mu$, μ', $(\varphi \rightarrow \wedge\nu\varphi) \in$ (A5). It follows by Lemma 2 and tautologous transformations that $\vdash_{\overline{W^*_{sf}}} [\varphi \rightarrow (\psi \rightarrow \chi)]$. Similarly, $\vdash_{\overline{W^*_{sf}}} [\varphi \rightarrow (\chi \rightarrow \psi)]$. Hence, again by tautologous transformations, $\vdash_{\overline{W^*_{sf}}} [\varphi \rightarrow (\psi \leftrightarrow \chi)]$.
QED

Lemma 30 *Let μ and ν be variables and ψ and χ be formulas. If χ is obtained from ψ by replacing some (or none or all) free subject position occurrences of μ in ψ by ν, then there are an $n \in \omega$ and general indiscernibility formulas $\varphi_0, \ldots, \varphi_{n-1}$ for μ, ν such that* $\vdash_{\overline{W^*_{sf}}} [\varphi_0 \wedge \ldots \wedge \varphi_{n-1} \rightarrow (\psi \leftrightarrow \chi)]$.

Proof: If $\mu = \nu$, then the result is trivial. Otherwise, merely apply Lemma 29 and tautologous transformations repeatedly. QED

Lemma 31 *Let μ, ν be variables and φ be a formula. If μ and ν are of the same type, and μ does not occur free in predicate position in φ, then* $\vdash_{\overline{W^*_{sf}}} \left(\wedge\mu\varphi \rightarrow \varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix}\right)$.

Proof: First assume that $\mu \neq \nu$ and that ν can be properly substituted for μ in φ. By Lemma 30 there are an $n \in \omega$ and general indiscernibility formulas $\psi_0, \ldots, \psi_{n-1}$ for μ, ν such that $\vdash_{\overline{W^*_{sf}}} \left[\psi_0 \wedge \ldots \wedge \psi_{n-1} \rightarrow \left(\varphi \leftrightarrow \varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix}\right)\right]$. Let $\Psi = \psi_0 \wedge \ldots \wedge \psi_{n-1}$. It follows by tautologous transformations that $\vdash_{\overline{W^*_{sf}}} \left[\varphi \rightarrow \left(\Psi \rightarrow \varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix}\right)\right]$ and, therefore, that $\vdash_{\overline{W^*_{sf}}} \left[\varphi \rightarrow \left(\sim\varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix} \rightarrow \sim\Psi\right)\right]$. Accordingly, by generalization, (A4), and tautologous transformations, we obtain $\vdash_{\overline{W^*_{sf}}} \left[\wedge\mu\varphi \rightarrow \left(\wedge\mu\sim\varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix} \rightarrow \wedge\mu\sim\Psi\right)\right]$, whence $\vdash_{\overline{W^*_{sf}}} \left[\wedge\mu\sim\varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix} \rightarrow \left(\wedge\mu\varphi \rightarrow \wedge\mu\sim\Psi\right)\right]$. Since μ does not occur free in $\sim\varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix}$, it follows that $\left(\sim\varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix} \rightarrow \wedge\mu\sim\varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix}\right) \in$ (A5); so, by Lemmas 2, 4, and 3, $\vdash_{\overline{W^*_{sf}}} \left[\sim\varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix} \rightarrow \left(\wedge\mu\varphi \rightarrow \wedge\mu\sim\Psi\right)\right]$. Consequently, by more tautologous transformations, $\vdash_{\overline{W^*_{sf}}} \left[\vee\mu\Psi \rightarrow \left(\wedge\mu\varphi \rightarrow \varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix}\right)\right]$. But $\vee\mu\Psi \in$ (A7) $\cup$ (A8). Therefore $\vdash_{\overline{W^*_{sf}}} \left(\wedge\mu\varphi \rightarrow \varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix}\right)$ by Lemmas 2 and 3. If $\mu = \nu$, or ν cannot be properly substituted for μ in φ, then $\varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix} = \varphi$. In this case, let ν' be a variable distinct from and of the same type as μ which does not occur in φ. Then $\vdash_{\overline{W^*_{sf}}} \left(\wedge\mu\varphi \rightarrow \varphi\begin{bmatrix}\mu\\ \nu'\end{bmatrix}\right)$ by what we have just proved. It follows from generalization, (A4), and Lemma 3 that $\vdash_{\overline{W^*_{sf}}} \left(\wedge\nu'\wedge\mu\varphi \rightarrow \wedge\nu'\varphi\begin{bmatrix}\mu\\ \nu'\end{bmatrix}\right)$. But $(\wedge\mu\varphi \rightarrow \wedge\nu'\wedge\mu\varphi) \in$ (A5), and we have already shown that $\vdash_{\overline{W^*_{sf}}} \left(\wedge\nu'\varphi\begin{bmatrix}\mu\\ \nu'\end{bmatrix} \rightarrow \varphi\right)$. Accordingly, by Lemmas 2, 4, and 3, $\vdash_{\overline{W^*_{sf}}} (\wedge\mu\varphi \rightarrow \varphi)$.
QED

Lemma 32 *Let $n \in \omega$, t be an n-place sequence of types, π and σ be n-place predicate variables, and ψ and χ be formulas. If φ is a t-coextensivity formula for π, σ, and χ results from ψ by replacing a free t-occurrence of π in ψ by a free occurrence of σ, then* $\vdash_{\mathbf{W}^*_{sf}} [\varphi \to (\psi \leftrightarrow \chi)]$.

Proof: Just like that of Lemma 29, except for Case 1: ψ is atomic. Let $\mu_0, \ldots, \mu_{n-1}$ be distinct variables which do not occur in $[\varphi \to (\psi \leftrightarrow \chi)]$ and are such that for each $i \in n$, the type of μ_i is t_i. Then by Lemma 31 $\vdash_{\mathbf{W}^*_{sf}} [\varphi \to (\pi(\mu_0, \ldots, \mu_{n-1}) \leftrightarrow \sigma(\mu_0, \ldots, \mu_{n-1})]$. Repeatedly applying generalization, (A4), and Lemmas 2, 4, and 3, and (A5), we obtain $\vdash_{\mathbf{W}^*_{sf}} [\varphi \to \Lambda\mu_0 \ldots \Lambda\mu_{n-1}(\pi(\mu_0, \ldots, \mu_{n-1}) \leftrightarrow \sigma(\mu_0, \ldots, \mu_{n-1}))]$. But $\vdash_{\mathbf{W}^*_{sf}} [\Lambda\mu_0 \ldots \Lambda\mu_{n-1}(\pi(\mu_0, \ldots, \mu_{n-1}) \leftrightarrow \sigma(\mu_0, \ldots, \mu_{n-1})) \to (\psi \leftrightarrow \chi)]$ by Lemma 31. Hence, by Lemmas 4 and 3, $\vdash_{\mathbf{W}^*_{sf}} [\varphi \to (\psi \leftrightarrow \chi)]$. QED

By repeatedly applying Lemma 32 and then using Lemma 30, we can now prove:

Lemma 33 *Let π and σ be predicate variables and ψ and χ be formulas. If π and σ are of the same type, and χ is obtained from ψ by replacing some (or none or all) free occurrences of π in ψ by free occurrences of σ, then there are n, $m \in \omega$, general indiscernibility formulas $\varphi_0, \ldots, \varphi_{n-1}$ for π, σ, and general coextensivity formulas $\varphi_0', \ldots, \varphi_{n-1}'$ for π, σ such that* $\vdash_{\mathbf{W}^*_{sf}} [\varphi_0 \wedge \ldots \wedge \varphi_{n-1} \wedge \varphi_0' \wedge \ldots \wedge \varphi_{n-1}' \to (\psi \leftrightarrow \chi)]$.

If we use Lemma 33 instead of Lemma 30 in the proof of Lemma 31 we thereby obtain a proof of:

Lemma 34 *Let π and σ be predicate variables and φ be a formula. If π and σ are of the same type, then* $\vdash_{\mathbf{W}^*_{sf}} \left(\Lambda\pi\varphi \to \varphi\begin{bmatrix}\pi\\ \sigma\end{bmatrix}\right)$.

Lemma 31 (for generalized individual variables) and Lemma 34 together imply the following theorem, which in turn shows that $\mathbf{W}^*_{sf}$ is, indeed, equivalent to $\mathbf{W}^*$.

Theorem 35 *Every instance of* (A6) *is a theorem of* $\mathbf{W}^*_{sf}$.

5.2 *Relations as Individuals* Although in $\mathbf{W}^*$ relations are projected grammatically as being entities which may serve as subjects of predication, they fail to be fully projected logistically as individuals. As a formal representative of a realist ontology, such as the intended ontology of $\mathbf{T}^*$, then, $\mathbf{W}^*$ is too weak. We will remedy this situation by considering, instead of $\mathbf{W}^*$, the theory $\mathbf{W}^{**}$ which is defined to be $\mathbf{W}^*$ + (A10), where (A10) is the set of all formulas θ such that θ is a generalization of some formula of the form:

(A10) $\left(\Lambda\alpha\varphi \to \varphi\begin{bmatrix}\alpha\\ \mu\end{bmatrix}\right)$, where α is an individual variable, and μ is any variable.

Now, we already have a completeness theorem for $\mathbf{W}^{**}$, viz. Theorem 27. But as a semantical reflection of the intended realist ontology for $\mathbf{W}^{**}$, Theorem 27 is quite unnatural because of its inclusion of, say, strict structures wherein the category of the individual variables is disjoint from the categories of the predicate variables. The semantics which we now describe, however, is faithful to this realist ontology.

We say that $\mathfrak{A}$ is an *Aristotelian structure*, or, for brevity, *A-structure*, iff $\mathfrak{A} = \langle \mathscr{D}, \langle D_n \rangle_{n\in\omega}, H, I\rangle$ is a structure and $\bigcup_{n\in\omega} D_{n+1} \subseteq D_0$. If $\mathfrak{A} = \langle \mathscr{D}, \langle D_n \rangle_{n\in\omega}, H, I\rangle$ is an A-structure, then we say that $\mathfrak{A}$ is *semi-strict* iff (i) $\mathscr{D} = D_0$, (ii) for all n, $m \in \omega$, if $n \neq m$, then $D_{n+1} \cap D_{m+1} = 0$, and (iii) $H \subseteq \bigcup_{n\in\omega} (D_{n+1} \times {}^n D_0)$. We define $\aleph_A$ to be the class of A-structures, $\aleph_{SA}$ to be the class of semi-strict A-structures, and, if $\mathbf{T}$ is a theory, $\aleph_{(S)A\mathbf{T}}$ to be $\aleph_{\mathbf{T}} \cap \aleph_{(S)A}$.

Theorem 36 *Let Γ be a set of formulas. If Γ is consistent in $\mathbf{W}^{**}$, then there is a semi-strict A-structure which is a model of Γ.*

Proof: We can directly utilize the proof of Theorem 23. Take this proof, replace Γ by $\Gamma \cup$ (A10) at the outset, and, when it comes time to construct the model, set D_0 equal to $\mathscr{D}$ instead of $C(0) \cap \mathscr{L}''$. That the A-structure so defined is a model of Γ follows in essentially the same way as before. QED

We can now derive the following two theorems in the usual manner.

Theorem 37 *Let φ be a formula and Γ be a set of formulas. Then $\Gamma \vdash_{\mathbf{W}^{**}} \varphi$ iff $\Gamma \models_{\aleph_{(S)A}} \varphi$.*

Theorem 38 *Let Γ be a set of formulas, $\mathbf{T}$ be a theory, and φ be a formula. If $\mathbf{T}$ is an extension of $\mathbf{W}^{**}$, then $\Gamma \vdash_{\mathbf{T}} \varphi$ iff $\Gamma \models_{\aleph_{(S)A\mathbf{T}}} \varphi$.*

We can also provide $\mathbf{W}^{**}$ with a substitution free axiom set. If $n \in \omega$, $i \in n+1$, and μ, ν are variables, we say that φ is an *n, i-indiscernibility formula* for μ, ν iff φ is a t, i-indiscernibility formula for μ, ν where t is the n-place sequence each of whose sequents is 0. We say that φ is a *special indiscernibility formula* for μ, ν iff there is an $n \in \omega$ and an $i \in n$ such that φ is an n, i-indiscernibility formula for μ, ν. (A11), (A12), or (A13) is defined as the set of all θ such that θ is any generalization of any formula of the form:

(A11) $\bigvee \beta(\varphi_0 \wedge \ldots \wedge \varphi_{n-1})$, where β is an individual variable, $n \in \omega$, and there is a variable μ distinct from β such that for each $i \in n$, φ_i is a special indiscernibility formula for μ, β

(A12) $\bigvee \sigma[\varphi_0 \wedge \ldots \wedge \varphi_{n-1} \wedge \bigwedge \alpha_0 \ldots \bigwedge \alpha_{m-1}(\pi(\alpha_0, \ldots, \alpha_{m-1}) \leftrightarrow \sigma(\alpha_0, \ldots, \alpha_{m-1}))]$, where n, $m \in \omega$, and π, σ are distinct m-place predicate variables such that (i) for each $i \in n$, φ_i is a special indiscernibility formula for π, σ, and (ii) $\alpha_0, \ldots, \alpha_{m-1}$ are pairwise distinct individual variables

(A13) $\varphi \rightarrow (\psi \leftrightarrow \chi)$, where there are an $n \in \omega$, $i \in n+1$, and distinct variables μ, μ' such that (i) φ is an n, i-indiscernibility formula for μ, μ', (ii) $\psi = \pi(\nu_0, \ldots, \nu_{i-1}, \mu, \nu_i, \ldots, \nu_{n-1})$ is an atomic formula, and (iii) $\chi = \pi(\nu_0, \ldots, \nu_{i-1}, \mu', \nu_i, \ldots, \nu_{n-1})$.

We define $\mathbf{W}^{**}_{sf}$ to be the theory whose axiom set is the union of

(A1)-(A5) and (A11)-(A13). It is a simple matter to adjust the proofs of Lemmas 28-34 to show that $\mathbf{W}^{**}_{sf}$ is equivalent to $\mathbf{W}^{**}$.

5.3 *Reducing the Indiscernibilities* Let $n \in \omega$ and t be an n-place sequence of types. If π, σ are n-place predicate variables, $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables, and φ is a t-coextensivity formula for π, σ, then $\vdash_{\mathbf{W}^{**}} [\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \sigma(\alpha_0, \ldots, \alpha_{n-1})) \rightarrow \varphi]$ by (A10), generalization, (A4), and (A5). Similarly, if $i \in n + 1$, μ, ν are variables, φ is an n, i-indiscernibility formula for μ, ν, and ψ is a t, i-indiscernibility formula for μ, ν, then $\vdash_{\mathbf{W}^{**}} (\varphi \rightarrow \psi)$. Thus, in $\mathbf{W}^{**}$, t, i-indiscernibility is reduced to n, i-indiscernibility.

It will be convenient to go further and reduce n, i-indiscernibility to 0, 0-indiscernibility. If μ, ν are variables, let us say that φ is a *simple indiscernibility formula for* μ, ν iff φ is a 0, 0-indiscernibility formula for μ, ν, i.e., φ is of the form $\bigwedge\pi(\pi(\mu) \leftrightarrow \pi(\nu))$, where π is distinct from μ and ν. Now let (A14) be the set of all formulas θ such that θ is any generalization of any formula of the form:

(A14) $\bigwedge\pi(\pi(\mu) \leftrightarrow \pi(\nu)) \rightarrow \varphi$, where $\bigwedge\pi(\pi(\mu) \leftrightarrow \pi(\nu))$ is a simple indiscernibility formula for μ, ν, and φ is a special indiscernibility formula for μ, ν.

One syntactical advantage of studying the theory $\mathbf{W}^{**}$ + (A14) is that it can be given a particularly simple substitution free axiom set. To this end, let (A15), (A16), or (A17) be defined as the set of all formulas θ such that θ is any generalization of some formula of the form:

(A15) $\bigvee\alpha\bigwedge\pi(\pi(\mu) \leftrightarrow \pi(\alpha))$, where α is an individual variable, μ is any variable distinct from α, and π is distinct from μ and α

(A16) $\bigvee\sigma[\bigwedge\tau(\tau(\pi) \leftrightarrow \tau(\sigma)) \wedge \bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \sigma(\alpha_0, \ldots, \alpha_{n-1}))]$, where $n \in \omega$, π and σ are distinct n-place predicate variables, τ is distinct from π and σ, and $\alpha_0, \ldots, \alpha_{n-1}$ are pairwise distinct individual variables

(A17) $\bigwedge\pi(\pi(\mu) \leftrightarrow \pi(\nu)) \rightarrow (\varphi \leftrightarrow \psi)$, where μ, ν, and π are pairwise distinct, ψ is an atomic formula, and ψ is obtained from φ by replacing an occurrence of μ in subject position by an occurrence of ν.

M^{**} is defined to be the theory whose axiom set is the union of (A1)-(A5) and (A15)-(A17). It is not difficult to show that M^{**} is equivalent to $\mathbf{W}^{**}$ + (A14). To show that $\mathbf{W}^{**}$ is a subsystem of M^{**}, just adjust the proofs of Lemmas 28-34. Every instance of (A14) can be derived in M^{**} by generalizing on (A17) and then applying (A4) and (A5). Then $\mathbf{W}^{**}$ + (A14) is shown to be a subsystem of M^{**}. Now, (A15) $\subseteq$ (A11), (A16) $\subseteq$ (A12), and every instance of (A17) can be derived in $\mathbf{W}^{**}$ + (A14) from (A14) and (A13). Thus, M^{**} is shown to be a subsystem of $\mathbf{W}^{**}$ + (A14). Accordingly, M^{**} is, indeed, equivalent to $\mathbf{W}^{**}$ + (A14).

We can even go further. If μ, ν are variables, we will say that φ is an *indiscernibility formula for* μ, ν iff φ is of the form $\bigwedge\pi(\pi(\mu) \rightarrow \pi(\nu))$ where π is distinct from μ and ν. If μ, ν are variables, then we will write $\mu \equiv \nu$ for any indiscernibility formula for μ, ν. Now let (A18) be the set of all formulas θ such that θ is any generalization of any formula of the form:

(A18) $\mu \equiv \nu \rightarrow \bigwedge\pi(\pi(\mu) \leftrightarrow \pi(\nu))$, where μ, ν are distinct variables, and π is distinct from μ, ν.

Then simple indiscernibility is reduced to indiscernibility in M^{**} + (A18), and no further reduction is possible. As might be expected, M^{**} + (A18) can be given an even simpler substitution free axiom set than M^{**}. To this end, we define (A19), (A20), or (A21) to be the set of all formulas θ such that θ is any generalization of any formula of the form:

(A19) $\bigvee\alpha\mu \equiv \alpha$, where α is an individual variable distinct from the variable μ

(A20) $\bigvee\sigma[\pi \equiv \sigma \wedge \bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \sigma(\alpha_0, \ldots, \alpha_{n-1}))]$, where $n \in \omega$, π, σ are distinct n-place predicate variables, and $\alpha_0, \ldots, \alpha_{n-1}$ are pairwise distinct individual variables

(A21) $\mu \equiv \nu \rightarrow (\varphi \rightarrow \psi)$, where μ, ν are distinct variables, ψ is an atomic formula, and φ is obtained from ψ by replacing an occurrence of μ in subject position by an occurrence of ν.

The theory whose axiom set is the union of (A1)-(A5) and (A19)-(A21) is Cocchiarella's M^*, introduced in [9].[14] It is a simple matter to verify that M^* is equivalent to M^{**} + (A18).

5.4 *Comprehension and Instantiation of Formulas for Generalized Predicate Variables* Cocchiarella's formulation of standard second-order logic [8] contains a comprehension principle, and Church's formulation of standard second-order logic [1] contains a principle for the substitution of formulas for generalized predicate variables. When generalizing standard second-order logic to the extended grammatical-logistic context of $\mathbf{W}^{**}$, therefore, it is of great interest to investigate the addition of generalizations of such principles to $\mathbf{W}^{**}$. One of the most interesting facts about $\mathbf{W}^{**}$ is that it affords a great variety of such generalizations, not all of which are equivalent. This is quite in contrast to the logical context of standard second-order logic where there is only one natural comprehension principle or principle for the substitution of formulas for generalized predicate variables.

We say that φ is a *comprehension formula* iff φ is a formula of the form $\bigvee\pi\bigwedge\mu_0 \ldots \bigwedge\mu_{n-1}(\pi(\mu_0, \ldots, \mu_{n-1}) \leftrightarrow \psi)$, where $n \in \omega$, and $\pi, \mu_0, \ldots, \mu_{n-1}$ are pairwise distinct. We also say that φ is a *standard second-order formula* iff φ is a formula in which no predicate variable occurs in subject position.

We begin our study of comprehension principles by noting the axiomatic simplifications which result from adding two especially simple comprehension principles to $\mathbf{W}^{**}$. Let (CPA) be the set of all θ such that θ is any generalization of any comprehension formula of the form:

(CPA) $\bigvee\pi\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \varphi)$, where φ is an atomic standard second-order formula in which π does not occur, and $\alpha_0, \ldots, \alpha_{n-1}$ are individual variables.

Then $\mathbf{W}^{**}$ + (CPA) is equivalent to M^{**} + (CPA). That $\mathbf{W}^{**}$ + (CPA) is a subsystem of M^{**} + (CPA) follows from the previously mentioned fact that **W**** is a subsystem of M^{**}. To show that M^{**} + (CPA) is a subsystem of **W**** + (CPA), and, hence, that they are equivalent, it will suffice to show that every instance of (A14) is provable in **W**** + (CPA) since M^{**} is itself equivalent to **W**** + (A14). To this end, let $n \in \omega$, π be a 1-place predicate variable, α, β, $\gamma_0, \ldots, \gamma_{n-1}$ be pairwise distinct individual variables, $i \in n + 1$, and σ be an $(n + 1)$-place predicate variable distinct from π. Let $\varphi = \sigma(\gamma_0, \ldots, \gamma_{i-1}, \alpha, \gamma_i, \ldots, \gamma_{n-1})$ and $\psi = \sigma(\gamma_0, \ldots, \gamma_{i-1}, \beta, \gamma_i, \ldots, \gamma_{n-1})$. By (A6) and the usual transformations, $\vdash_{\mathbf{W^{**}}} \{\Lambda\pi(\pi(\alpha) \leftrightarrow \pi(\beta)) \rightarrow [\Lambda\alpha(\pi(\alpha) \leftrightarrow \varphi) \rightarrow (\varphi \leftrightarrow \psi)]\}$, whence $\vdash_{\mathbf{W^{**}}} \{\Lambda\pi(\pi(\alpha) \leftrightarrow \pi(\beta)) \rightarrow [\sim(\varphi \leftrightarrow \psi) \rightarrow \sim\Lambda\alpha(\pi(\alpha) \leftrightarrow \varphi)]\}$ by tautologous transformations. It follows from generalization, (A4), and (A5) that $\vdash_{\mathbf{W^{**}}} \{\Lambda\pi(\pi(\alpha) \leftrightarrow \pi(\beta)) \rightarrow [\sim(\varphi \leftrightarrow \psi) \rightarrow \Lambda\pi\sim\Lambda\alpha(\pi(\alpha) \leftrightarrow \varphi)]\}$ and, hence, by tautologous transformations, that $\vdash_{\mathbf{W^{**}}} \{V\pi\Lambda\alpha(\pi(\alpha) \leftrightarrow \varphi) \rightarrow [\Lambda\pi(\pi(\alpha) \leftrightarrow \pi(\beta)) \rightarrow (\varphi \leftrightarrow \psi)]\}$. Accordingly, $\vdash_{\mathbf{W^{**}}+(\mathrm{CPA})} [\Lambda\pi(\pi(\alpha) \leftrightarrow \pi(\beta)) \rightarrow (\varphi \leftrightarrow \psi)]$. From this result we can easily derive every instance of (A14) using generalization and (A4) and (A5). Thus, **W**** + (CPA) is equivalent to M^{**} + (CPA).

Let (CPN) be the set of all θ such that θ is any generalization of any comprehension formula of the form:

(CPN) $V\pi\Lambda\alpha_0 \ldots \Lambda\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \varphi)$, where $\alpha_0, \ldots, \alpha_{n-1}$ are individual variables, φ is either an atomic standard second-order formula or the negation of an atomic standard second-order formula, and π does not occur in φ.

Then **W**** + (CPN) is equivalent to M^* + (CPN). Since **W**** + (CPN) is obviously equivalent to M^{**} + (CPN), and M^* is equivalent to M^{**} + (A18), it suffices to prove that every instance of (A18) is a theorem of M^{**} + (CPN). But this follows in essentially the same way as in the preceding paragraph.

Consequently, if Γ is a set of formulas and every element of (CPN) is provable in **W**** + Γ, then **W**** + Γ is equivalent to M^* + Γ with its simpler axiom set.

Now, Cocchiarella's formulation of standard second-order logic contains only one inference rule, *modus ponens*, and its axioms are just the standard second-order instances of (A1)-(A5), (A19), (A21), and (CP), where (CP) is the set of all θ such that θ is any generalization of any comprehension formula of the form:

(CP) $V\pi\Lambda\alpha_0 \ldots \Lambda\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \varphi)$, where $\alpha_0, \ldots, \alpha_{n-1}$ are individual variables, φ is a standard second-order formula, and π does not occur free (in predicate position) in φ.[15]

The most straightforward generalization of this formulation of standard second-order logic to the extended grammatical context of non-standard second-order logic is just the theory whose set of axioms is the union of (A1)-(A5), (A19), (A22), where (A22) is the set of all θ such that θ is any generalization of any formula of the form:

(A22) $\vee\sigma\pi \equiv \sigma$, where σ is a predicate variable distinct from and of the same type as π,

(A21), and (CP*), where (CP*) is the set of all θ such that θ is any generalization of any comprehension formula of the form:

(CP*) $\vee\pi\wedge\alpha_0 \ldots \wedge\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \varphi)$, where $\alpha_0, \ldots, \alpha_{n-1}$ are individual variables, and π does not occur free in φ;

we call this theory $\mathbf{T}_d^{**}$. This is the theory that Cocchiarella initially introduced in [3].[16]

By all rights $\mathbf{T}_d^{**}$ should be equivalent to M^* + (CP*), which we will call $\mathbf{T}^{**}$, and the only difference between these two theories is that whereas $\mathbf{T}_d^{**}$ contains (A22) in its axiom set, $\mathbf{T}^{**}$ contains the "slightly" stronger set of axioms (A20), which, after [3], we will also call (A4′). As a matter of fact, $\mathbf{T}_d^{**}$ also has included in its axiom set, subsumed under (CP*), all formulas of the form $\vee\sigma\wedge\alpha_0 \ldots \wedge\alpha_{n-1}(\sigma(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \pi(\alpha_0, \ldots, \alpha_{n-1}))$, where $n \in \omega$, $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables, and σ is distinct from π; and from this, generalization, (A4), and (A5) we can prove in $\mathbf{T}_d^{**}$ every formula of the same form but with π and σ interchanged in the biconditional. Cocchiarella shows in [3] that every instance of (A10) is provable in $\mathbf{T}_d^{**}$, that every instance of (A6) in which μ does not have a free occurrence in predicate position in φ is provable in $\mathbf{T}_d^{**}$, and that every instance of (A6) in which μ does not have a free occurrence in subject position in φ is provable in $\mathbf{T}_d^{**}$. Clearly, then, $\mathbf{T}^{**}$ is equivalent to $\mathbf{T}_d^{**}$ + (A4′) which, in turn, is equivalent to $\mathbf{T}_d^{**}$ + (U.I.$_2^*$), where (U.I.$_2^*$) is the set of instances of (A6) in which μ is a predicate variable.

In spite of all this, $\mathbf{T}_d^{**}$ is not equivalent to $\mathbf{T}^{**}$; and we now prove this. Let $D_0 = \omega$, and, for each $n \in \omega$, let $D_{n+1} = \mathcal{P}({}^nD_0)$. Let $O = \{\langle i\rangle: i \in \omega \text{ and } i \text{ is odd}\}$ and $E = \{\langle i\rangle: i \in \omega \text{ and } i \text{ is even}\}$. Let $f: \bigcup_{n\in\omega} D_n \to \omega$ such that: (1) for all $i \in \omega$, $f(i) = i$, (2) $f(O) = 0$, and (3) $f[D_2 \sim \{O\}] \subseteq \omega \sim \{0\}$. Let π^* and σ^* be distinct 1-place predicate variables. Finally, let $c: \bigcup_{n\in\omega} V(n+1) \to \omega$ such that (1) $c(\sigma^*) = 0$, and (2) $c(\pi^*) = 1$.

If α is an individual variable, we will say that the category of α is D_0; if $n \in \omega$ and π is an n-place predicate variable, we will say that the category of π is D_{n+1}. We will say that $\mathfrak{A}$ is an assignment iff $\mathfrak{A}: \bigcup_{n\in\omega} V(n) \to \bigcup_{n\in\omega} D_n$ such that for every variable μ, $\mathfrak{A}(\mu)$ is a member of the category of μ. If $\mathfrak{A}$ is an assignment, π is a predicate variable, and μ is any variable, we will define $F_{\mathfrak{A}}(\pi, \mu)$ to be $f(\mathfrak{A}(\mu)) + c(\pi)$. If φ is a formula, we define φ to be true with regard to any assignment $\mathfrak{A}$, in symbols $\models_{\mathfrak{A}} \varphi$, by induction on formulas φ as follows:

1. if $\varphi = \pi(\mu_0, \ldots, \mu_{n-1})$ is atomic, then $\models_{\mathfrak{A}} \varphi$ iff $\langle F_{\mathfrak{A}}(\pi, \mu_0), \ldots, F_{\mathfrak{A}}(\pi, \mu_{n-1})\rangle \in \mathfrak{A}(\pi)$
2. if $\varphi = \sim\psi$, then $\models_{\mathfrak{A}} \varphi$ iff $\not\models_{\mathfrak{A}} \psi$
3. if $\varphi = (\psi \to \chi)$, then $\models_{\mathfrak{A}} \varphi$ iff $\not\models_{\mathfrak{A}} \psi$ or $\models_{\mathfrak{A}} \chi$
4. if $\varphi = \wedge\mu\psi$, then $\models_{\mathfrak{A}} \varphi$ iff for each d in the category of μ, $\models_{\mathfrak{A}\binom{\mu}{d}} \psi$.

If φ is a formula, we will say that φ is valid, in symbols $\models \varphi$, iff φ is true with regard to every assignment.

It is an easy exercise to verify that every axiom of $\mathbf{T}_{\mathrm{d}}^{**}$ is valid. From this it follows that every theorem of $\mathbf{T}_{\mathrm{d}}^{**}$ is valid. But it is not the case that every instance of (A4′) is valid; in particular, $\not\models \bigvee\sigma^*[\bigwedge\tau(\tau(\pi^*) \leftrightarrow \tau(\sigma^*)) \wedge \bigwedge\alpha(\pi^*(\alpha) \leftrightarrow \sigma^*(\alpha))]$, where τ is distinct from π^* and σ^*, and α is an individual variable. To prove this, let $\mathfrak{A}$ be any assignment with the property that $\mathfrak{A}(\pi^*) = O$ and assume that $\models_{\mathfrak{A}} \bigvee\sigma^*[\bigwedge\tau(\tau(\pi^*) \leftrightarrow \tau(\sigma^*)) \wedge \bigwedge\alpha(\pi^*(\alpha) \leftrightarrow \sigma^*(\alpha))]$. Then there is a $P \in D_2$ such that (i) for every $Q \in D_2$, $\langle F_{\mathfrak{A}\binom{\sigma^*}{P}\binom{\tau}{Q}}(\tau, \pi^*)\rangle \in \mathfrak{A}\binom{\sigma^*}{P}\binom{\tau}{Q}(\tau)$ iff $\langle F_{\mathfrak{A}\binom{\sigma^*}{P}\binom{\tau}{Q}}(\tau, \sigma^*)\rangle \in \mathfrak{A}\binom{\sigma^*}{P}\binom{\tau}{Q}(\tau)$, and (ii) for every $d \in D_0$, $\langle F_{\mathfrak{A}\binom{\sigma^*}{P}\binom{\alpha}{d}}(\pi^*, \alpha)\rangle \in \mathfrak{A}\binom{\sigma^*}{P}\binom{\alpha}{d}(\pi^*)$ iff $\langle F_{\mathfrak{A}\binom{\sigma^*}{P}\binom{\alpha}{d}}(\sigma^*, \alpha)\rangle \in \mathfrak{A}\binom{\sigma^*}{P}\binom{\alpha}{d}(\sigma^*)$. On the one hand, (i) implies that for every $Q \in D_2$, $\langle 0 + c(\tau)\rangle \in Q$ iff $\langle f(P) + c(\tau)\rangle \in Q$. Choosing Q to be $\{\ulcorner c(\tau)\urcorner\}$ shows that $f(P)$ must be 0. But f was constructed so that only O is assigned the value 0. Hence, (i) implies that $P = O$. On the other hand, (ii) implies that for all $d \in D_0$, $\langle d + 1\rangle \in O$ iff $\langle d + 0\rangle \in P$. This requires that $P = E$. Thus, (i) and (ii) are contradictory; so $\not\models_{\mathfrak{A}} \bigvee\sigma^*[\bigwedge\tau(\tau(\pi^*) \leftrightarrow \tau(\sigma^*)) \wedge \bigwedge\alpha(\pi^*(\alpha) \leftrightarrow \sigma^*(\alpha))]$, whence $\not\models \bigvee\sigma^*[\bigwedge\tau(\tau(\pi^*) \leftrightarrow \tau(\sigma^*)) \wedge \bigwedge\alpha(\pi^*(\alpha) \leftrightarrow \sigma^*(\alpha))]$. Accordingly, $\mathbf{T}^{**}$ is a proper extension of $\mathbf{T}_d^{**}$.

(CP*) is clearly the most natural generalization of (CP) when one is thinking along the lines of definitional extensions of, say, $\mathbf{W}^{**}$, but there is another way of looking at the restriction on free occurrences of the existentially quantified predicate variable in the formula being comprehended in instances of (CP), and that is seeing it as applying only to free occurrences in *predicate* position. We let (CP**) be the set of all θ such that θ is any generalization of any comprehension formula of the form:

(CP**) $\bigvee\pi\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \varphi)$, where $\alpha_0, \ldots, \alpha_{n-1}$ are individual variables, and π does not occur free in φ in predicate position.

Then $\mathbf{T}^{***}$ is defined to be $M^* +$ (CP**). Since (CP*) $\subseteq$ (CP**), it is clear that $\mathbf{T}^{***}$ is an extension of $\mathbf{T}^{**}$. We will show later that $\mathbf{T}^{***}$ is actually a proper extension of $\mathbf{T}^{**}$.

We now turn to the substitution of formulas for generalized predicate variables. Let $n \in \omega$, π be an n-place predicate variable, $\alpha_0, \ldots, \alpha_{n-1}$ be distinct individual variables, and ψ and φ be formulas. We say that ψ *can be properly substituted for* π *with regard to* $\alpha_0, \ldots, \alpha_{n-1}$ *in* φ iff

1. π does not occur free in predicate position in φ within a subformula of φ of the form $\bigwedge\mu\chi$ where μ is a variable distinct from $\alpha_0, \ldots, \alpha_{n-1}$ which occurs free in ψ; and

2. for all variables $\mu_0, \ldots, \mu_{n-1}$, if $\pi(\mu_0, \ldots, \mu_{n-1})$ occurs in φ in such a way that the (predicate position) occurrence of π is a free occurrence, then, for each $i \in n$, there is no subformula of ψ of the form $\bigwedge\mu_i\chi$ in which α_i has a free occurrence.

If ψ can be properly substituted for π with regard to $\alpha_0, \ldots, \alpha_{n-1}$ in φ, then we let $\check{\mathbf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\psi}\varphi\Big|$ be the result of replacing each subformula $\pi(\mu_0, \ldots, \mu_{n-1})$ of φ in which π occurs free in predicate position in φ by $\psi\begin{bmatrix}\alpha_0, \ldots, \alpha_{n-1}\\ \mu_0, \ldots, \mu_{n-1}\end{bmatrix}$; else we let $\check{\mathbf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\psi}\varphi\Big|$ be φ itself.

Let (U.I.$^*_3$*) be the set of all θ such that θ is any generalization of any formula of the form:

(U.I.$^*_3$*) $\Lambda\pi\varphi \rightarrow \check{\mathbf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\psi}\varphi\Big|$, where $n \in \omega$, π is an n-place predicate variable, $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables, ψ and φ are formulas, and π does not occur free in subject position in φ.

Cocchiarella has noted [3] that M^* + (U.I.$^*_3$*) is equivalent to **T****. So the correspondence between comprehension and substitution of formulas for generalized predicate variables can be quite close in the context of nonstandard second-order logic, much as it is for standard second-order logic.

The restriction in (U.I.$^*_3$*) that π not occur free in subject position is really quite natural since in general we cannot, of course, substitute a formula for a predicate variable in *subject* position, and in any other principle of substitution for a generalized predicate variable we would expect to have to replace *every* free occurrence of the generalized variable by a free occurrence of that to which it is being instantiated in the generalized formula. It is an interesting fact that we need not make this restriction when we generalize the standard second-order principle of substitution of a formula for a generalized predicate variable to the context of **W****. We let (U.I.$^*_3$) be the set of all θ such that θ is any generalization of any formula of the form:

(U.I.$^*_3$) $\Lambda\pi\varphi \rightarrow \check{\mathbf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\psi}\varphi\Big|$, where $n \in \omega$, π is an n-place predicate variable, $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables, and ψ and φ are formulas.

We will define **T*** to be M^* + (U.I.$^*_3$). **T*** was the first nonstandard second-order logistic system to be created, introduced by Cocchiarella in [10] to analyze Russell's paradox of predication. In [10] Cocchiarella shows that **T*** is not only consistent, but is even a conservative extension of standard second-order logic.

It is of interest to delineate in just what way **T*** is related to **T****. Since (U.I.$^*_3$*) $\subseteq$ (U.I.$^*_3$), **T*** is at least an extension of **T****. Cocchiarella has shown [3] that **T*** is equivalent to the theory which results from **T**** by adding a particularly simple form of (U.I.$^*_3$) to its axiom set, viz. (U.I.$^*_4$), where (U.I.$^*_4$) is the set of all θ such that θ is any generalization of any formula of the form:

(U.I.$^*_4$) $\bigwedge\pi\varphi \rightarrow \check{\mathbf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\sigma(\alpha_0, \ldots, \alpha_{n-1})}\varphi\Big|$, where $n\in\omega$, π is an n-place predicate variable, and $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables.

Actually, it is shown in [3] that **T*** is even equivalent to $\mathbf{T}^{**}_{\mathbf{d}}$ + (U.I.$^*_4$).

We now give an even simpler formulation of **T*** relative to **T****. Let (I^{**}) be the set of all θ such that θ is any generalization of any formula of the form:

(I^{**}) $\bigvee\rho[\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \rho(\alpha_0, \ldots, \alpha_{n-1})) \wedge \sigma \equiv \rho]$, where $n\in\omega$, $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables, and π, σ, and ρ are distinct.

Then **T*** is equivalent to **T**** + (I^{**}). We first show that every instance of (I^{**}) is provable in **T***. To this end, let $n\in\omega, \alpha_0, \ldots, \alpha_{n-1}$ be distinct individual variables, and π, σ, and ρ be distinct n-place predicate variables. For convenience, set $\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \rho(\alpha_0, \ldots, \alpha_{n-1})) = \varphi$, and let ψ be the result of replacing ρ in φ with π. By (U.I.$^*_4$) we have $\vdash_{\mathbf{T}^*} [\bigwedge\rho\sim(\varphi\wedge\sigma\equiv\rho) \rightarrow \sim(\psi\wedge\sigma\equiv\rho)]$. From this it follows that $\vdash_{\mathbf{T}^*}[\bigwedge\rho\sim(\varphi\wedge\sigma\equiv\rho) \rightarrow \sim\sigma\equiv\rho]$. Then, by generalization, (A4), and (A5), we have $\vdash_{\mathbf{T}^*}[\bigwedge\rho\sim(\varphi\wedge\sigma\equiv\rho) \rightarrow \bigwedge\rho\sim\sigma\equiv\rho]$. Hence, $\vdash_{\mathbf{T}^*}[\bigwedge\rho\sim(\varphi\wedge\sigma\equiv\rho) \rightarrow \sim\sigma\equiv\sigma)]$; so $\vdash_{\mathbf{T}^*}\bigvee\rho(\varphi\wedge\sigma\equiv\rho)$. Accordingly, by generalization, every instance of (I^{**}) is provable in **T***, whence we have shown that **T**** + (I^{**}) is a subsystem of **T***. It is an easy exercise, using (I^{**}) and appropriate forms of Lemmas 30 and 33, to show that every instance of (U.I.$^*_4$) is provable in **T**** + (I^{**}). Hence, **T*** is a subsystem of **T**** + (I^{**}). We have, therefore, proved that **T*** is, indeed, equivalent to **T**** + (I^{**}), and this is a substitution free formulation of **T*** as well.

Cocchiarella also notes in [3] that **T***** is a subsystem of **T***. The proof of this is quite simple. Let n, π, $\alpha_0, \ldots, \alpha_{n-1}$, and φ be as in (CP**). Then

$$\vdash_{\mathbf{T}^*}[\bigwedge\pi\sim\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \varphi) \rightarrow \sim\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}(\varphi\leftrightarrow\varphi)].$$

From this (CP**) follows quite readily by tautologous transformations and generalization. It is of immediate interest to know whether **T***** is actually equivalent to **T*** since they arise from **T**** in analogous ways. It turns out that **T*** is *proper* extension of **T*****.

To prove this, we will construct a semi-strict A-structure which is a model of the axioms of **T***** but not of **T***. To this end, let **T**, F, **T**$'$, $d_0, d_1, \ldots$ be any sequence of pairwise distinct objects. Let $\mathscr{D} = D_0 = \{\mathbf{T}, F, \mathbf{T}', d_0, \ldots\}$. Let $A = \{\mathbf{T}\}$ and $B = \{F, \mathbf{T}', d_0, \ldots\}$. For each $n\in\omega$, let $S_n = \mathscr{P}(^n\{A, B\})$. Let $\langle D_{n+1}\rangle_{n\in\omega}$ form a disjoint partition of D_0 such that $D_1 = \{\mathbf{T}, F, \mathbf{T}'\}$ and, for each $n\in\omega$, if $n \neq 0$, then the cardinality of D_{n+1} is equal to the cardinality of S_n. Then, for each $n\neq 0$, let f_n be a bijection from D_{n+1} onto S_n. Now let $H_0 = \{(\mathbf{T}, \langle\,\rangle), (\mathbf{T}', \langle\,\rangle)\}$, and, for each $n\in\omega$, if $n\neq 0$, let $H_n = \{(a, \langle b_0, \ldots, b_{n-1}\rangle): a\in D_{n+1}$ and there is a $\langle C_0, \ldots, C_{n-1}\rangle\in f_n(a)$ such that $b_0\in C_0, \ldots,$ and $b_{n-1}\in C_{n-1}\}$. Finally, let $H = \bigcup_{n\in\omega} H_n$, and let I be such that $\mathfrak{A} = \langle\mathscr{D}, \langle D_n\rangle_{n\in\omega}, H, I\rangle$ is a structure.

Now, it is clear that $\mathfrak{A}$ is a semi-strict A-structure. A little reflection shows that the equivalence classes of the elements of D_0 under indiscernibility are just A and B and that since, for each $n \in \omega$, all and only "representations" of subsets of the set of n-place sequences of A and B are included amongst the extensions of the elements of D_{n+1}, $\mathfrak{A}$ is a model of T^{**}. Furthermore, since (I^{**}) holds in $\mathfrak{A}$ for each $n \neq 0$, it is clear that (CP$'^{**}$) does, too. But (I^{**}) does not hold for $n = 0$; so $\mathfrak{A}$ is not a model of T^{*}. It only remains to show that (CP**) holds in for $n = 0$. To this end, let π be a 0-place predicate variable and φ be any formula in which π does not occur free in predicate position. We need to show that there is a $P \in D_1$ such that $\models_{\mathfrak{A}\left[\frac{\pi}{P}\right]} \pi$ iff $\models_{\mathfrak{A}\left[\frac{\pi}{P}\right]} \varphi$, i.e., $PH\langle\,\rangle$ iff $\models_{\mathfrak{A}\left[\frac{\pi}{P}\right]} \varphi$. If $\not\models_{\mathfrak{A}\left[\frac{\pi}{F}\right]} \varphi$, then $P = F$ satisfies the condition. Otherwise, $P = \mathsf{T}'$ satisfies the condition since F and T' are in the same equivalence class under indiscernibility. Thus, $\mathfrak{A}$ is a model of T^{***}, and we have proved that T^{*} is a proper extension of T^{***}.

For the purpose of constructing structures which are models of T^{***}, it would generally be useful to have a simpler characterization of T^{***} relative to T^{**}, since, after all, it is fairly easy to construct structures which are models of T^{**}. It is not difficult to show that T^{***} is equivalent to T^{**} + (A23), where (A23) is the set of all θ such that θ is any generalization of any formula of the form:

(A23) $\bigvee\pi\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \sigma(\pi, \alpha_0, \ldots, \alpha_{n-1}))$, where $n \in \omega$, $\alpha_0, \ldots, \alpha_{n-1}$ are pairwise distinct individual variables, and π and σ are distinct,

and this characterization serves our purpose to some extent.

In [9] Cocchiarella has explored an entirely different approach to extending (CP). If φ is a formula, we say that φ is *stratified* iff there is a function f whose domain is the set of variables which occur in φ, whose range is included in ω, and which is such that for every atomic formula $\pi(\mu_0, \ldots, \mu_{n-1})$ which occurs in φ, $f(\pi) = \max(f(\mu_0), \ldots, f(\mu_{n-1})) + 1$. We let $S\mathsf{T}^{*} = M^{*}$ + (SCP$'^{*}$), where (SCP*) is the set of all θ such that θ is any generalization of any comprehension formula of the form:

(SCP$'^{*}$) $\bigvee\pi\bigwedge\mu_0 \ldots \bigwedge\mu_{n-1}(\pi(\mu_0, \ldots, \mu_{n-1}) \leftrightarrow \varphi)$, where φ is a stratified formula in which π does not occur free.

We will only take time to remark that although indiscernibility cannot satisfy full substitutivity in T^{**} (see, e.g., [10]), it does satisfy full substitutivity in $S\mathsf{T}^{*}$ (see [9]). Thus, if $S\mathsf{T}^{*}$ is consistent, both T^{**} and $S\mathsf{T}^{*}$ are proper extensions of M^{*} + (CP$'$).

5.5 *The Fregean Semantics* We now describe the Fregean semantics which Cocchiarella introduced in [3]. Our main results are two: (1) we will produce the minimal theory characterized by the Fregean semantics, and (2) we will show that T^{*} + (Ext*) is equivalent to T^{***} + (Ext*), where (Ext*) is defined below.

We will say that $\mathfrak{A}$ is a *Fregean structure*, *F-structure* for brevity, iff $\mathfrak{A}$ is a 4-place sequence $\langle \mathcal{D}, \langle D_n\rangle_{n\in\omega}, f, I\rangle$ such that (i) $\mathcal{D} \neq 0$, (ii) for each

$n \in \omega$, $0 \neq D_n \subseteq \mathcal{P}({}^n\mathscr{D})$, (iii) f is a function whose domain is $\mathscr{D} \cup \bigcup_{n\in\omega} D_n$, whose range is included in $\mathscr{D}$, and which is such that for all $d \in \mathscr{D}$, $f(d) = d$,[17] and (iv) I is a function whose domain is the set of variables and which is such that for each individual variable α, $I(\alpha) \in \mathscr{D}$, and, for each $n \in \omega$ and n-place predicate variable π, $I(\pi) \in D_n$. Let $\mathfrak{A} = \langle \mathscr{D}, \langle D_n\rangle_{n\in\omega}, f, I\rangle$ be an F-structure. If α is an individual variable, we say that *the category of* α *in* $\mathfrak{A}$ is $\mathscr{D}$; if $n \in \omega$ and π is an n-place predicate variable, we say that *the category of* π *in* $\mathfrak{A}$ is D_n. If φ is a formula, we define φ to be *true in* any F-structure $\mathfrak{A} = \langle \mathscr{D}, \langle D_n\rangle_{n\in\omega}, f, I\rangle$, in symbols $\models_{\mathfrak{A}} \varphi$, by induction on formulas φ as follows:

1. if $\varphi = \pi(\mu_0, \ldots, \mu_{n-1})$ is atomic, then $\models_{\mathfrak{A}} \varphi$ iff $\langle f(I(\mu_0)), \ldots, f(I(\mu_{n-1}))\rangle \in I(\pi)$
2. if $\varphi = \sim\psi$, then $\models_{\mathfrak{A}} \varphi$ iff $\not\models_{\mathfrak{A}} \psi$
3. if $\varphi = (\psi \rightarrow \chi)$, then $\models_{\mathfrak{A}} \varphi$ iff $\not\models_{\mathfrak{A}} \psi$ or $\models_{\mathfrak{A}} \chi$
4. if $\varphi = \bigwedge\mu\psi$, then $\models_{\mathfrak{A}} \varphi$ iff for all x in the category of μ in $\mathfrak{A}$, $\models_{\mathfrak{A}\left[\begin{smallmatrix}\mu\\ x\end{smallmatrix}\right]} \psi$,

where, of course, $\mathfrak{A}\begin{bmatrix}\mu\\ x\end{bmatrix}$ is taken to be $\left\langle \mathscr{D}, \langle D_n\rangle_{n\in\omega}, f, I\begin{pmatrix}\mu\\ x\end{pmatrix}\right\rangle$. The notions of model, validity, etc., are understood as usual. We let $\aleph_F$ be the class of F-structures, and if **T** is a theory, we define $\aleph_{F\mathbf{T}}$ to be the class of F-structures which are models of the axioms of **T**.

The main obstacle to giving a model-set-theoretic semantics for **T*** and its brethren is interpreting formulas such as $\pi(\pi)$. In the usual kind of semantics, π would denote a set and predication would be interpreted as set membership. Thus, $\pi(\pi)$ would always be construed as false. But it is easily seen that there is, in the ontology of **T****, for example, a property which holds of everything—and, therefore, of itself. Our realist semantics gets around this problem by not necessarily mirroring predication as set-membership. The Fregean semantics gets around this problem by regarding subject position occurrences of predicate variables as denoting not the relations that they denote when in predicate position but, rather, associated individuals. In this way the Fregean semantics can continue to mirror predication as set-membership.

It is clear that every axiom of **W**** is valid in the Fregean semantics. We define (Ext**) to be the set of all θ such that θ is any generalization of any formula of the form:

(Ext**) $\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \sigma(\alpha_0, \ldots, \alpha_{n-1})) \rightarrow \varphi$, where $n \in \omega$, $\alpha_0, \ldots, \alpha_{n-1}$ are pairwise distinct individual variables, and φ is a special indiscernibility formula for π, σ.

It is also clear that every instance of (Ext**) is valid in the Fregean semantics. We define **W***F** to be **W**** + (Ext**). It turns out that **W***F** is characterized by the full Fregean semantics.

Theorem 39 *Let* Γ *be a set of formulas. If* Γ *is consistent in* **W***F**, *then there is an* F*-structure which is a model of* Γ.

Proof: Assume that Γ is consistent in **W***F**. Then $\Gamma \cup$ (Ext**) is consistent in $\mathbf{W}^{**}$. Accordingly, by Theorem 36 there is a semi-strict A-structure $\langle \mathcal{D}, \langle D_n \rangle_{n\in\omega}, H, I\rangle$ which is a model of $\Gamma \cup (\mathrm{Ext}^{**})$. Let K be the set of all functions J such that the domain of J is the set of variables, and, for each variable μ, $J(\mu)$ is a member of the category of μ in $\langle \mathcal{D}, \langle D_n\rangle_{n\in\omega}, H, I\rangle$. For each $J \in K$, let $\mathfrak{A}(J)$ be $\langle \mathcal{D}, \langle D_n\rangle_{n\in\omega}, H, J\rangle$.

We next define a set of equivalence relations. For each $n \in \omega$ and $i \in n+1$ let $R_{n,i}$ be the set of all ordered pairs (a, b) such that $a, b \in D_0$ and for all $d_0, \ldots, d_{n-1} \in D_0$ and $P \in D_{n+1}$, $PH\langle d_0, \ldots, d_{i-1}, a, d_i, \ldots, d_{n-1}\rangle$ iff $PH\langle d_0, \ldots, d_{i-1}, b, d_i, \ldots, d_{n-1}\rangle$. It is clear that $R_{n,i}$ is an equivalence relation for each $n \in \omega$ and $i \in n+1$; for each such n and i and $a \in D_0$, let $[a]_{n,i}$ be the equivalence class of a under $R_{n,i}$. Let F be the function whose domain is D_0 and which is such that for all $a \in D_0$, $F(a) = \langle [a]_{n,i}\rangle_{n,i}$, where n, i ranges over all $n \in \omega$ and $i \in n+1$. Set $\mathcal{E} = F[D_0]$. Let G be the function whose domain is $\bigcup_{n\in\omega} D_{n+1}$ and which is such that for each $n \in \omega$ and $P \in D_{n+1}$, $G(P) = \{\langle F(a_0), \ldots, F(a_{n-1})\rangle : PH\langle a_0, \ldots, a_{n-1}\rangle\}$. G is well-defined because $\langle \mathcal{D}, \langle D_n\rangle_{n\in\omega}, H, I\rangle$ is semi-strict. For each $n \in \omega$, set $E_n = G[D_{n+1}]$. Let f be the function whose domain is $\mathcal{E} \cup \bigcup_{n\in\omega} E_n$ and which is such that (i) for each $x \in \mathcal{E}$, $f(x) = x$, and (ii) for each $n \in \omega$ and $X \in E_n$, if $X = G(P)$, then $f(X) = F(P)$. f is well-defined because $\mathfrak{A}(I)$ is a model of (Ext^{**}) and because if $n \in \omega$, $P, Q \in D_{n+1}$, and $G(P) = G(Q)$, then for all $d_0, \ldots, d_{n-1} \in D_0$, $PH\langle d_0, \ldots, d_{n-1}\rangle$ iff $QH\langle d_0, \ldots, d_{n-1}\rangle$, for, assume that $PH\langle d_0, \ldots, d_{n-1}\rangle$. Then $\langle F(d_0), \ldots, F(d_{n-1})\rangle \in G(P)$. But $G(P) = G(Q)$ by assumption; so $\langle F(d_0), \ldots, F(d_{n-1})\rangle \in G(Q)$. Hence there are $\langle e_0, \ldots, e_{n-1}\rangle \in D_0$ such that $QH\langle e_0, \ldots, e_{n-1}\rangle$ and $\langle F(e_0), \ldots, F(e_{n-1})\rangle = \langle F(d_0), \ldots, F(d_{n-1})\rangle$. But if $F(a) = F(b)$, then for each $n \in \omega$ and $i \in n+1$, $R_{n,i}(a, b)$. It follows from this fact that $QH\langle d_0, \ldots, d_{n-1}\rangle$. The converse direction follows similarly. Accordingly, f is, indeed, well-defined.

For each $J \in K$, let J' be the function whose domain is the set of variables and which is such that: (i) for each individual variable α, $J'(\alpha) = F(J(\alpha))$, and (ii) for each $n \in \omega$ and n-place predicate variable π, $J'(\pi) = G(J(\pi))$. For each $J \in K$, let $\mathfrak{B}(J')$ be the F-structure $\langle \mathcal{E}, \langle E_n\rangle_{n\in\omega}, f, J'\rangle$.

We now prove by induction on formulas with regard to φ that for all $J \in K$, $\models_{\mathfrak{A}(J)} \varphi$ iff $\models_{\mathfrak{B}(J')} \varphi$. Case 1: $\varphi = \pi(\mu_0, \ldots, \mu_{n-1})$. Assume that $\models_{\mathfrak{A}(J)} \pi(\mu_0, \ldots, \mu_{n-1})$. Then $J(\pi)H\langle J(\mu_0), \ldots, J(\mu_{n-1})\rangle$; so $\langle F(J(\mu_0)), \ldots, F(J(\mu_{n-1}))\rangle \in G(J(\pi))$. Now, if ν is any individual variable, then $f(J'(\nu)) = f(F(J(\nu))) = F(J(\nu))$; and if ν is any predicate variable, then $f(J'(\nu)) = f(G(J(\nu))) = F(J(\nu))$. In either case, $f(J'(\nu)) = F(J(\nu))$. Accordingly, $\langle f(J'(\mu_0)), \ldots, f(J'(\mu_{n-1}))\rangle \in G(J(\pi)) = J'(\pi)$; so $\models_{\mathfrak{B}(J')} \pi(\mu_0, \ldots, \mu_{n-1})$. The converse direction follows similarly, noting, as we did when proving that f is well-defined, that if $\langle F(d_0), \ldots, F(d_{n-1})\rangle \in G(P)$, then $PH\langle d_0, \ldots, d_{n-1}\rangle$. Case 2: φ is of the form $\sim\psi$ or $(\psi \rightarrow \chi)$. Then the desired result trivially follows from the induction hypothesis. Case 3: φ is of the form $\Lambda\mu\psi$. Then the result again follows trivially from the induction hypothesis keeping in mind that the induction hypothesis applies to all $J \in K$ and that $\{J' : J \in K\}$ contains all functions L such that $\mathfrak{B}(L)$ is an F-structure.

It follows, in particular, that for all formulas φ, $\models_{\mathfrak{A}(I)} \varphi$ iff $\models_{\mathfrak{B}(I')}$. Hence, $\mathfrak{B}(I')$ is a model of Γ. QED

From Theorem 39 we can derive the following two theorems in the usual manner.

Theorem 40 *Let Γ be a set of formulas and φ be a formula. Then* $\Gamma \vdash_{\mathbf{W}F^*} \varphi$ *iff* $\models_{\aleph_F} \varphi$.

Theorem 41 *Let* $\mathbf{T}$ *be a theory,* Γ *be a set of formulas, and* φ *be a formula. If* $\mathbf{T}$ *is an extension of* $\mathbf{W}F^*$, *then* $\Gamma \vdash_{\mathbf{T}} \varphi$ *iff* $\Gamma \models_{\aleph_{F\mathbf{T}}} \varphi$.

It is of immediate interest to examine the theories which result from supplementing extensions of $\mathbf{W}^{**}$ with (Ext**). The first thing to notice is that M^* + (Ext**) is equivalent to M^* + (Ext*), where (Ext*) is the set of all θ such that θ is any generalization of any instance of (Ext**) of the form:

(Ext*) $\wedge \alpha_0 \ldots \wedge \alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \sigma(\alpha_0, \ldots, \alpha_{n-1})) \rightarrow \pi \equiv \sigma$.

Furthermore, it is clear that M^* + (Ext*) + (I^{**}) is equivalent to M^* + (Ext*) + (I^*), where (I^*) is the set of all θ such that θ is any generalization of any formula of the form:

(I^*) $\pi \equiv \sigma$, where π and σ are any two distinct predicate variables of the same type.

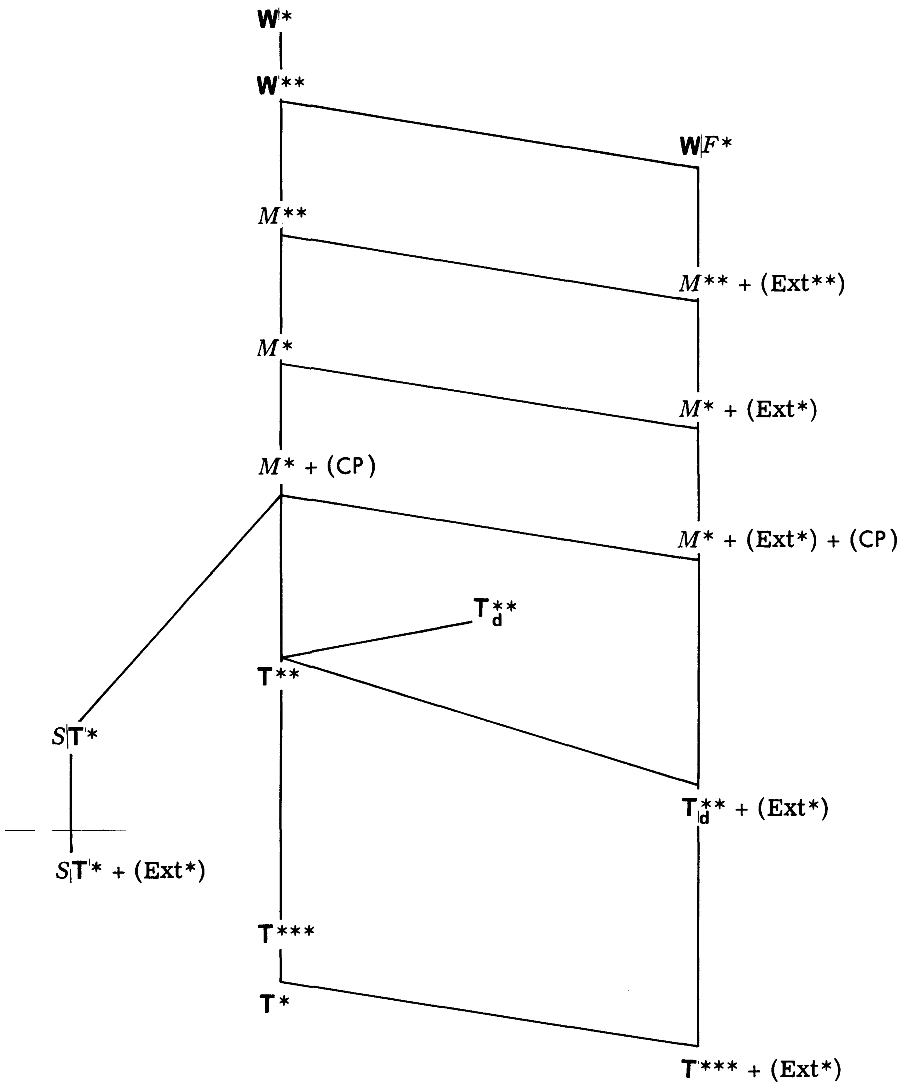
Since $\mathbf{T}^*$ is equivalent to $\mathbf{T}^{**}$ + (I^{**}), it follows that $\mathbf{T}^*$ + (Ext*) is equivalent to $\mathbf{T}^{**}$ + (Ext*) + (I^*), as Cocchiarella has proved in [3]. From this it is easy to see that $\mathbf{T}^*$ + (Ext*) is a proper extension of $\mathbf{T}^{**}$ + (Ext*). This result also appears in [3].

Interestingly enough, it turns out that $\mathbf{T}^{***}$ + (Ext*) is equivalent to $\mathbf{T}^*$ + (Ext*). To see this, let $\mathfrak{A} = \langle \mathcal{D}, \langle D_n \rangle_{n \in \omega}, f, I \rangle$ be any F-structure which is a model of $\mathbf{T}^{***}$ + (Ext*). Let n be any element of ω. Let R be the set of all ordered pairs (a, b) such that $a, b \in \mathcal{D}$ and for all $P \in D_1$, if $a \in P$, then $b \in P$. $R(a, b)$ means simply that a and b are indiscernible. (CP**) guarantees that $0 \in D_n$. Now let P be any element of $\mathcal{D}_n$. By (CP**) there is a $Q \in D_n$ such that for all $a_0, \ldots, a_{n-1} \in \mathcal{D}$, $\langle a_0, \ldots, a_{n-1} \rangle \in Q$ iff $R(f(Q), f(0))$ and $\langle a_0, \ldots, a_{n-1} \rangle \in P$. If $Q = 0$, then $R(f(Q), f(0))$ by (Ext*); and if $Q \neq 0$, then there are $a_0, \ldots, a_{n-1} \in \mathcal{D}$ such that $\langle a_0, \ldots, a_{n-1} \rangle \in Q$, whence $R(f(Q), f(0))$ and $\langle a_0, \ldots, a_{n-1} \rangle \in P$. In either case, $R(f(Q), f(0))$. Hence, for all $a_0, \ldots, a_{n-1} \in \mathcal{D}$, $\langle a_0, \ldots, a_{n-1} \rangle \in Q$ iff $\langle a_0, \ldots, a_{n-1} \rangle \in P$. Therefore $R(f(Q), f(P))$ by (Ext*); so $R(f(P), f(0))$. Since this holds for arbitrary $P \in D_n$, it follows that the elements of D_n are indiscernible from each other. Since n is arbitrary and $\mathfrak{A}$ is an arbitrary element of $\aleph_{F, \mathbf{T}^{***} + (\mathrm{Ext}^*)}$, it follows from Theorem 41 that every instance of (I^*) is provable in $\mathbf{T}^{***}$ + (Ext*). Therefore, $\mathbf{T}^{***}$ + (Ext*) is equivalent to $\mathbf{T}^*$ + (Ext*). This result also shows that $\mathbf{T}^{***}$ is not equivalent to $\mathbf{T}^{**}$.

As $S\mathbf{T}^*$ is closely related to the theory of simple types, lacking only its grammatical "peculiarities", monadic $S\mathbf{T}^*$ + (Ext*) is intimately related to Jensen's *NFU*, i.e., New Foundations with urelements (see [11]). Likewise, if we add to monadic $S\mathbf{T}^*$ + (Ext*) the assumption that every individual is a property, we obtain a system intimately related to Quine's *NF*, i.e., New Foundations [15]. Cocchiarella has discussed these theories at length in [9].

6 *Remarks* It should be noted that the semantics which we have developed here can be applied to a great variety of systems not mentioned. For example, first-order predicate logic without identity, standard second-order logic, monadic **W***. Another byproduct of this paper is a substitution free axiom set for first-order predicate logic without identity, which can be easily abstracted from the substitution free axiom set which we have given for **W***.

The various theories discussed in this paper may be summarized as follows, where a connecting line segment indicates proper extension:



NOTES

1. See [10] for this analysis of Russell's paradox.

2. Unless otherwise stated or unless in set-theoretic contexts, 'relation' refers to relations, properties, and propositions.

3. For technical results and philosophical discussion concerning **T*** and the logistic systems which have grown up around it, see Meyer [14] and Cocchiarella [2]–[10]. The conceptual order of dependence of these papers is probably best analyzed as: [8], [10], [14], [3], [5], [2], [9], [4], [6], [7].

4. For further discussion of these oddities see [10], [14], [3], [5], [9], [7].

5. See [2]–[10].

6. For the significance of substitution free axiom sets see [8].

7. The preceding four paragraphs are not a wholly faithful summary of Cocchiarella [3]. They are only meant to pick out those results of [3] which are relevant to motivating this paper. For example, we have somewhat altered the terminology of [3].

8. My thanks to Professor Cocchiarella for pointing this out to me. See [2] for a thorough discussion of this distinction.

9. ω is understood to be the first limit ordinal, and it is understood to be the set of natural numbers in the usual way so that for all $n, m \in \omega$, $n < m$ iff $n \in m$.

10. By the convention of Note 9, 0 is the empty set.

11. If F is a function and A is a subset of the domain of F, we define $F[A]$ to be the set of all $F(x)$ such that $x \in A$; and $\mathcal{R}(F)$ is understood to be the range of F.

12. We take $\alpha \dot{-} 1 = \alpha$ for α a limit ordinal, else we take it to be the number of which it is the successor.

13. If A and B are sets, ${}^{A}B$ is defined to be the set of all functions whose domains are A and whose ranges are included in B. Thus, ${}^{n}\mathcal{D}$ is the set of all n-place sequences whose sequents are in $\mathcal{D}$.

14. Actually, in Cocchiarella's M^* (A1)–(A3) are replaced by the set of generalizations of tautologies.

15. Actually, Cocchiarella's (CP) requires that $\alpha_0, \ldots, \alpha_{n-1}$ be included among the free variables of φ, but our (CP) is readily derivable from Cocchiarella's, as he himself has noted in, e.g., [3], by appending to φ any tautologous formula whose free variables are just $\alpha_0, \ldots, \alpha_{n-1}$.

16. The remark of Note 15 applies here, too.

17. In what follows it will be necessary to distinguish the empty subsets of the ${}^{n}\mathcal{D}$. One way to do this formally is to regard f not as a function on $\mathcal{D} \cup \bigcup_{n \in \omega} D_n$, but as a sequence of functions $\langle f_n \rangle_{n \in \omega}$ such that for each $n \in \omega$, the domain of f_{n+1} is D_n and the range of f_{n+1} is included in $\mathcal{D}$ and such that f_0 is the identity function on $\mathcal{D}$. Then instead of applying the usual f we apply the appropriate f_n. The reason we need to distinguish the various empty subsets of the ${}^{n}\mathcal{D}$ is that *only* 0 may occur common to the ${}^{n}\mathcal{D}$, and it is clear from the following that we do not wish to identify any n-place relations with any m-place relations where $n \neq m$. If, however, we do not distinguish the various null relations, we will have to add to (Ext**) and (Ext*) wherever they occur the set of axioms (Z^*), where (Z^*) is the set of all θ such that θ is any generalization of any formula of the form:

$(Z^*)\quad [\sim V\alpha_0 \dots V\alpha_{n-1}\pi(\alpha_0, \dots, \alpha_{n-1}) \wedge \sim V\beta_0 \dots V\beta_{m-1}\sigma(\beta_0, \dots, \beta_{m-1})] \rightarrow \phi$,

where $n, m \in \omega$, $\alpha_0, \dots, \alpha_{n-1}$ and $\beta_0, \dots, \beta_{m-1}$ are groups of pairwise distinct individual variables, π, σ are distinct, and ϕ is a special indiscernibility formula for π, σ.

NOTE: (Added in proof, December 18, 1979). Lemma 28 on page 15 should be deleted.

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