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JUNCTIONS

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1 *Propositional Logic*

1.1 *Definitions* Consider the following array of symbols:

$$\begin{array}{c|c|c} r & s & \hat{p} \\ & \hat{q} & r \\ \hline \hat{s} & \multicolumn{2}{l}{s \quad p} \\ \hat{p} & \multicolumn{2}{l}{} \end{array} \quad .$$

Such an expression will be called a *junction*. The vertical and horizontal bars are to be interpreted as conjunction and disjunction signs, respectively, and the capped letters as negations. Thus, a junction is simply a formula of the propositional calculus written in terms of horizontal conjunctions, vertical disjunctions, and literals (single letters or their negations). The *junctives* (connectives) indicate grouping unambiguously without parentheses. Innermost junctives are omitted. The positions of the literals are called *junctures*. Junctions within junctions are called *subjunctions*.

1.2 *Normal forms and truth conditions* So understood, the familiar conjunctive and disjunctive normal forms of the junction given above are easily shown to be as follows:

$$r\hat{s}\hat{p} \wedge rs \wedge rp \wedge s\hat{q}\hat{s}\hat{p} \wedge s\hat{q}s \wedge s\hat{q}p \wedge \hat{p}r\hat{s}\hat{p} \wedge \hat{p}rs \wedge \hat{p}rp$$
$$rs\hat{p} \vee rsr \vee r\hat{q}\hat{p} \vee r\hat{q}r \vee \hat{s}sp \vee \hat{p}sp \quad .$$

These forms can be written down at once by direct inspection, clearly, but need not be, since the needed conjuncts and disjuncts coincide with immediately visible downward and forward paths through the junction. We will refer to this coincidence (suitably generalized) of normal form terms with junctive paths as the *normal form theorem* for junctions. From this theorem we obtain at once the following basic laws:

(1) Given a complete and consistent truth-assignment A for the literals of a junction J, J is true for A if and only if the literals true in A are consistent with the literals on some forward path of J.

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(2) A junction is valid (inconsistent) if and only if it contains a pair of contradictory literals on each downward (forward) path.

1.3 *Proof by inspection* In testing for validity or inconsistency it is not necessary to inspect separately all downward or forward paths. Instead, we inspect the junction for occurrences of contradictory literals. The discovery of a pair of such occurrences eliminates without further inspection and at a single blow all those paths containing the contradictory pair. Proceeding thus by elimination, even complicated junctions quickly fall apart when they are valid or inconsistent. And when they are neither, we show invalidity or consistency by finding a single path free of contradiction.

1.4 *Reduction to junctive form* To obtain a junction from a formula written in standard notation without biconditionals, drive negations inward by DeMorgan's Laws, eliminating conditionals along the way by means of the equivalence of $A \supset B$ with $-A \vee B$ and of $-(A \supset B)$ with $A \wedge -B$. The desired junction can be written down at once from direct inspection of the original formula, as the following illustrates:

$$_{-}(\hat{r} \vee {}_{-}((s \vee \hat{q}) \supset p\hat{r})) \supset -(sp \vee {}_{-}(s \supset \hat{p}))$$

$$\begin{array}{c} \begin{array}{c|c|c} r & s & \hat{p} \\ & \hat{q} & r \end{array} \\ \hline \begin{array}{c|cc} \hat{s} & s & p \\ \hat{p} & & \end{array} \end{array} .$$

The three negation signs added just beneath the original formula during the process of writing down the junction make it easy to keep track of what remains to be done. Note how perspicuous the junction seems, relative to the original formula.

1.5 *Junctions with biconditionals* The dual of a junction is its reflection on the major diagonal (a line of slope -1 passing through the upper left-hand corner of the junction). So the negation of a junction is obtained by negating its literals and interchanging horizontal and vertical junctives. This can be done by first writing in the junctives and then filling in the negated literals, or the negation can be written down directly, proceeding naturally from left to right and top to bottom. An equivalent negative form can also be obtained by a simple 90° rotation, again negating literals. Given these means of writing negations, the biconditional of a pair of junctions A and B can be written in any of three ways:

$$\begin{array}{c} \begin{array}{c|c} A & B \end{array} \\ \hline \begin{array}{c|c} -B & -A \end{array} \end{array} \qquad \begin{array}{c|c} \begin{array}{c} A \\ \hline -B \end{array} & \begin{array}{c} B \\ \hline -A \end{array} \end{array} \qquad \begin{array}{c|c} A & B \\ \hline -B & -A \end{array} .$$

The third way is simplest, and it can substantially reduce the number of paths to be considered. It amounts to extending the notion of a junction and its paths in an obvious manner. In writing a junction by direct inspection from a formula containing biconditionals, simply write

$$\begin{array}{c|c} A & B \\ \hline & \end{array} \qquad \left(\begin{array}{c|c} A & \\ \hline B & \end{array}\right)$$

for each biconditional (negated biconditional), go on to the rest of the junction, and plan on filling the blanks later.

1.6 *Rules of simplification* Junctions and subjunctions can often be simplified by means of the following rules:

(1) Treat any valid or inconsistent subjunction of a junction as true or false and resolve, i.e., eliminate a true conjunct or a false disjunct, treat a conjunction with a false conjunct as false and a disjunction with a true disjunct as true, and eliminate all but one corner of a biconditional in which a true or false corner appears.

(2) If some subjunction S is a conjunct (disjunct) of a junction J, treat all other occurrences of S or $-S$ in J as true or false (false or true), respectively, and resolve.

Simplification by either of these rules proceeds by inspection and by erasure. The junction given above, for example, resolves to the following:

$$\begin{array}{c|l} r & s \\ & \hat{q} \end{array} \; .$$

1.7 *Infinite junctions* We consider briefly now the notion of an *infinite junction*, i.e., an imagined junctive array for an infinite conjunction or disjunction of ordinary finite junctions. Infinite junctions are an easy and perspicuous avenue to the following familiar and basic law of propositional logic which we will refer to as the *infinite junction law*:

> An infinite conjunction (disjunction) is consistent (invalid) if every finite initial string of conjuncts (disjuncts) is.

Given any infinite junction J satisfying the hypothesis of this law, we have only to define a non-contradictory path P through J, by induction, as follows:

> Given the first n literals of P, with $n \geqslant 0$, choose the $(n + 1)$th literal in such a way that the resulting subpath of P is infinitely continuable, i.e., lies on finite initial paths of arbitrary length which are free of contradictory literals.

By hypothesis there must exist non-contradictory initial paths of arbitrary finite length in J. This guarantees a first literal, after which a next literal is always guaranteed by the hypothesis of the induction.

2 *Predicate Logic*

2.1 *Prenex junctive form* Consider now an example in predicate logic made familiar to logicians by DeMorgan:

$$-(\forall x(Fx \supset Gx) \supset -\forall y(\exists u(Fu \wedge Hyu) \supset -\exists v(Gv \wedge Hyv)))$$

$$\begin{array}{ll|l|ll|l|l|l|c|c}
x\bar{y}\bar{u}v & \hat{F}x & Fu\ Hyu & \hat{G}v & . & Fb\ Hab & \hat{G}b & \hat{F}b & .\ . & . \\
.abb & Gx & & \hat{H}yv & . & & \hat{H}ab & Gb & & . \\
b\ldots & & & & & & & & &
\end{array}$$

Here the expression

$$\begin{array}{ll|l|l}
x\bar{y}\bar{u}v & \hat{F}x & Fu\ Hyu & \hat{G}v \\
 & Gx & & \hat{H}yv
\end{array}$$

is simply the original formula written in prenex junctive form, with existential variables distinguished from universal variables by overbars. The expression

$$\begin{array}{l}
.abb \\
b\ldots
\end{array}$$

is, in effect, our proof of the inconsistency of this quantified formula. It is to be interpreted as a claim that when the junction is instantiated twice, first with the variables of the first row and then with those of the second, the conjunction of the resulting junctions will be inconsistent. This claim is easily checked by inspecting the junction on the right. Dots indicate positions which we might fill in various ways largely indifferent to the proof.

2.2 *Inconsistency proofs* In order to justify this example and generalize from it, what is needed, clearly, is an appropriate restriction on the variables of instantiation in the proof. We will call a rectangular array of variables written beneath the prefix of a formula F written in prenex junctive form a *proof* of F. The variables of a proof P appearing beneath a given quantifier will be called its *instances*. A proof is to be regarded not only as a rectangular array but also as a discursive sequence to be read in the usual way, from left to right and top to bottom. The sequence of all the variables on any row of P up to and including an existential instance will be called an *existential string*. Then the desired restriction is as follows:

> The topmost occurrence of each existential string in P must terminate with a variable which has no previous occurrences in P and no free occurrences in F, i.e., a fresh existential instance is needed for each new existential string.

Proofs conforming to this restriction will be called *well-formed*. The conjunction of the junctions obtained by replacing the variables of the junction of F by successive rows of P will be called the *instantiation* of F by P. Finally, we say that P is *valid* for F if P is well-formed and the instantiation of F by P is inconsistent. We will also say that an incompleted proof P is well-formed if we can fill all empty universal positions in P with a single fresh variable and then go through P in order, filling empty existential positions compatibly with our restriction. Then a well-formed incomplete proof will be called valid if its incomplete instantiation is inconsistent regardless of how the missing variables are filled in.

2.3 *Consistency* Our restriction is a natural one. If we consider the meanings of the quantifiers of an existential string, we see that an existential instance depends on the instances preceding it in the string and should be kept distinct from each of these, as well as from all free variables of the given formula. When this is done for an entire row of instances, the junction for these instances is a legitimate instantiation of the given quantified junction. By this we mean that if the quantified junction is consistent, its instantiation by this row must also be consistent, as we see at once by imagining the quantifiers instantiated one at a time from the outside. Similarly, if I is the instantiation of F by a valid proof P, then F must be inconsistent; otherwise we could go through the entire proof in order, instantiating one variable at a time with our restriction in mind, to obtain a consistent interpretation of I. These observations suffice to show the consistency of our method for establishing inconsistency by means of valid proofs.

2.4 *Duality* If we reflect a prenex junctive formula and its forward reading inconsistency proof on the major diagonal, we obtain a downward reading validity proof of its dual. Thus the validity of the DeMorgan example could have been established directly, instead of through the inconsistency of its negation, by means of the following dualized proof:

$$\begin{array}{l} \;x.b \\ |ya\,. \\ |ub\,. \\ \;vb\,.\quad . \end{array}$$

So we now agree that quantifier prefixes and their proofs may on occasion be written vertically, just above their junctions, with vertical prefix bars indicating universal quantifiers, and vertical proofs establishing validity rather than inconsistency. The expressions $|x$, $|y$, . . . and $\overline{x}$, $\overline{y}$, . . . for universal and existential quantifiers agree nicely with our expressions for conjunctions and disjunctions, and will be taken as official quantifier notation. The omission of vertical bars in horizontal quantifier prefixes and of horizontal bars in vertical prefixes is to be regarded as convenient unofficial abbreviation. Given these conventions, the downward proof technique for validity requires no further account or justification beyond a simple appeal to its obvious duality with the forward technique for inconsistency.

2.5 *Reduction to prenex form* Reduction of a quantified formula free of biconditionals to prenex junctive form proceeds quite as easily as when no quantifiers are present. We first require that all quantified variables be distinct, supplying prime symbols, if necessary, to ensure this. Then as negations are driven inward, quantifiers are picked up and taken into the prefix in the given order of their occurrence, writing $|x$ for $\forall x$ or $-\exists x$ and $\overline{x}$ for $\exists x$ or $-\forall x$. Otherwise, quantifiers are simply ignored in constructing the appropriate junction. The process is aided as before by judicious use of extra negation signs, two of which can be observed in the DeMorgan example above. In practice, it is more convenient to violate the given order of the quantifiers and pick up existentials before universals (or the reverse in downward proofs) while moving inward. Applying this device to the DeMorgan example, we easily obtain the following prefix and simplified proof:

$$\begin{array}{c} \bar{y}\bar{u}xv \\ abbb \end{array} .$$

Finally, if q_1 and q_2 are quantifier prefixes of a biconditional formula $q_1A_1 \equiv q_2A_2$, we write the following:

$$q_1q_2q_1^*q_2^* \qquad \begin{array}{c|c} A_1 & A_2 \\ \hline A_2^* & A_1^* \end{array} .$$

This is equivalent to

$$\begin{array}{c|c} q_1A_1 & q_2A_2 \\ \hline q_2^*A_2^* & q_1^*A_1^* \end{array} ,$$

where q^* is obtained from q by putting primes on the variables in q and interchanging universal and existential quantifiers, and A^* is obtained from the negation of A by putting primes on all the variables in A which belong to q. For this treatment of biconditionals to work properly, our original formula must be free of any primed occurrences of the variables in q_1 and q_2. And we can often rearrange the variables in the quantifier prefix $q_1q_2q_1^*q_2^*$ to better advantage, moving existentials farther to the front, subject only to the avoidance of passing an existential through a universal in whose scope it lies. Thus, the variables to be put in the quantifier prefix are first obtained by simple inspection of the biconditional (negated biconditional), which is then written as

$$\begin{array}{c|c} A_1 & A_2 \\ \hline & \end{array} \qquad \left(\begin{array}{c|c} A_1 & \\ \hline A_2 & \end{array}\right) ,$$

the blanks to be filled later by negating A_1 and A_2 and supplying primes to the variables as required.

2.6 *Completeness* For any prenex junctive formula F containing n universal variables in its prefix, consider an infinite array A constructed beneath the prefix of F as follows:

(1) For some infinite list of potential instantial variables (including all free variables of F), first enumerate all the different strings of variables of length n in some arbitrary fashion and write these strings beneath the universal quantifiers of F.

(2) Next fill in existential positions consecutively, each with the first variable which is compatible with the restriction on existential instances.

A proof for F will be called *normal* if it is an initial string of rows in A.

Now suppose that no normal proof of F is valid. Then, by the infinite junction law, the infinite conjunction of instantiations of F by the rows of A is consistent. So this infinite conjunction is true for some interpretation in a universe of objects designated by the variables. Finally, we see by (1) and (2) that F must be true in this interpretation. For, given arbitrary instances of the initial string of universal quantifiers in F, there exist uniquely determined instances of the subsequent string of existentials such that, . . ., for any instances of the final string of universals (some of these strings may be empty), the instantiation of F by the resulting row of instances is a conjunct of the infinite conjunction. So the assumption that no normal proof of F is valid entails that F is consistent. This shows that the method of establishing inconsistency by normal proof is complete.

2.7 *Sets of formulas* Consider another version of the DeMorgan example:

$$\forall x\,(Fx \supset Gx), \quad \exists x\,(Fx \wedge Hax), \quad \therefore \exists x\,(Gx \wedge Hax)$$

$$\begin{array}{llll} x\bar{x}x & x\ \hat{F}x & \bar{x}\ Fx\ Hax & x\ \hat{G}x \\ .b. & b\ Gx & b & b\ \hat{H}ax \\ b.. & & & \\ ..b & & & \end{array}$$

This illustrates our proposed technique for establishing the inconsistency of a finite set of formulas (or a conjunction). We first reduce each formula separately to prenex form. Quantifier variables in different formulas need not be distinct. We then write a quantifier prefix for the entire set by stringing the separate prefixes together in order, regarding the variables of different formulas as distinct variables. Next we construct a well-formed proof for the set which combines separate sub-proofs constructed for the individual formulas. Except for unfilled positions indicated by dots, the rows in the proof are the same as the rows in the sub-proofs. We construct the proof by writing a row of instances under one of the separate prefixes and the same row (together with dots) under the combined prefix, then writing another row under the same or a different sub-prefix and also under the combined prefix, and so on. The proof will be valid if the conjunction of instantiations by the sub-proofs is inconsistent, as we see from the structure of the instantiation by the proof. Briefly, then, the new technique consists in prenexing formulas of the set separately and instantiating by rows of instances of the individual prefixes until an inconsistency is reached. Completeness of the technique is easily seen from the fact that for any valid proof

$$\begin{array}{c} r_{11} r_{12} \ldots r_{1n} \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ r_{m1} r_{m2} \ldots r_{mn} \end{array}$$

for a set (conjunction) of formulas, we can always find a valid proof of the form

$$\begin{array}{l}
r_{11}\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\ r_{12}\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\ r_{1n} \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
r_{m1}\ \cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\ r_{m2}\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\ r_{mn} \quad ,
\end{array}$$

where r_{ij} is a row of instantial variables for the quantifier prefix of the jth formula of the set. Note, in conclusion, that our method is readily applied to disjunctions as well as conjunctions, by duality, and to infinite as well as finite sets, an infinite conjunction (disjunction) being inconsistent (valid) if and only if some finite initial string of the conjuncts (disjuncts) is.

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