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DEDUCTIVE COMPLETENESS AND CONDITIONALIZATION IN SYSTEMS OF WEAK IMPLICATION

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I wish to investigate the conditions under which certain systems of implication satisfy deductive completeness of the kind associated with the deduction theorem (in the sense of Curry and Feys [2]). The systems that I investigate have received considerable attention in the last two decades: the implication fragment of Relevance Logic, **R→** (Church's Weak Implication); the implication fragment of strict implication, **S4→**; the implication fragment of Anderson and Belnap's System of Entailment, **E→**; and the system of Ticket Entailment, **T→**. None of these systems satisfy deductive completeness except under certain conditions which may be interpreted as the satisfaction of conditions of relevance (for **R→**), modality (for **S4→**), relevance and modality (for **E→**), and inference ticket/inference distinctions (**T→**). Thus we might say that they are each deductively complete for an extended notion of deductive completeness.

In Section 2, I formulate natural deduction systems **NR→**, **NS4→**, **NE→**, and **NT→** which are deductively equivalent respectively to **R→**, **S4→**, **E→**, and **T→**. Each system involves only two rules, one of which is *modus ponens* and one a form of conditionalization. The conditionalization rule in each case is based on the deduction theorem of the corresponding axiom system. Furthermore, each system is the result of adding a further restriction to only the rule of conditionalization for the previous system. In this form we can see more clearly the relationship between the systems and intuitionistic implication (**H→**); and what relevance, necessity, and ticket entailment amount to. Finally, in Section 3, I show how to formulate **T→** in terms of a restriction on the rule for *modus ponens*, and how adding this restriction to *modus ponens* in **E→**, **R→**, **S4→**, and **H→** affects these systems.

Let **S** be a deductive system with an implication operator, $\supset$, which satisfies the rule *modus ponens* (MP): $A, A \supset B \vdash_{\mathbf{S}} B$. Curry and Feys [2] called such a system deductively complete if, whenever from a premise B, and possibly other premises, we can derive A, then we can derive $B \supset A$ from these other premises alone. It has been shown by Gentzen [3] that

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deductive completeness entails the entire intuitionistic theory of implication (referred to as **H**→). Thus, implications weaker than intuitionistic implication, which for our purposes here we may consider to be any implicative system which does not have $A \supset (B \supset A)$ as a thesis, are not deductively complete. However, we may extend the notion of deductive completeness to systems weaker than **H**→ by noting those conditions under which $B \supset A$ may be derived from a set of premises, given that A is derived from B and those same premises.

1 *Axiom systems and deduction theorem for* **H**→, **S4**→, **R**→, *and* **E**→ The complete implication fragment of classical propositional logic is referred to as **IC**→. After Hacking [4], I call a formula *strict* if it contains at least one occurrence of $\supset$, and *simple* if it does not. I use X and Y to represent sequences of wffs (possibly empty), and A, B, C, to represent well-formed formulas (wffs). If A is strict, I denote this by writing A'. If every member of X is strict, I denote this by X'. If A must be simple, it is denoted by $\check{A}$, and if every member of X is simple, I write $\check{X}$. If neither $'$ nor $\check{}$ is added to X or A, then they may be either strict or simple.

All of the axiom systems in this section have *modus ponens* (MP) as their only inference rule, $A, A \supset B \vdash B$, and so the statement of it is omitted in each case. I use $X' \vdash_{\mathbf{S}} A$ as a metalinguistic operator asserting that A is deducible from the set X in the system **S**. If A follows from the null set $\emptyset$ of premises in **S**, we write $\emptyset \vdash_{\mathbf{S}} A$ or simply $\vdash_{\mathbf{S}} A$. Two systems **S** and **T** will be deductively equivalent if and only if $X \vdash_{\mathbf{S}} A$ if and only if $X \vdash_{\mathbf{T}} A$. If A is a wff then $\mathbf{S}^{A_1, \ldots, A_m}_{v_1, \ldots, v_n} A$ is the result of uniformly substituting the wff A_i for the variable v_i in A (rewriting A if necessary to avoid ambiguities).

H→ is the system of intuitionistic implication which may be axiomatized by:

H→1. $A \supset (B \supset A)$
H→2. $A \supset (B \supset C) \supset . A \supset B \supset . A \supset C$
H→3. $A \supset A$.

Theorem 1 (Deductive Completeness of **H**→) *If* $X, A \vdash_{\mathbf{H}\rightarrow} B$ *then* $\alpha \vdash_{\mathbf{H}\rightarrow} A \supset B$.

S4→ is the implication fragment of **S4** which may be axiomatized by:

S4→1. $A \supset A$
S4→2. $(A \supset . B \supset C) \supset . A \supset B \supset . A \supset C$
S4→3. $A \supset B \supset . C \supset . A \supset B$.

Theorem 2 (Hacking, 1963 [4], Barcan Marcus, 1949) *If* $X, A \vdash_{\mathbf{S4}\rightarrow} B$ *and every member of* X *is strict then* $X \vdash_{\mathbf{S4}\rightarrow} A \supset B$.

The axioms for **R**→ are:

R→1. $A \supset A$
R→2. $A \supset B \supset . C \supset A \supset . C \supset B$
R→3. $(A \supset . B \supset C) \supset . B \supset . A \supset C$
R→4. $(A \supset . A \supset B) \supset . A \supset B$.

The form of the deduction theorem that I give here is due to Norman Martin.

Theorem 3 *If* $X, A \vdash_{\mathbf{R}\rightarrow} B$ *the* $X \vdash_{\mathbf{R}\rightarrow} A \supset B$ *or* $X \vdash_{\mathbf{R}\rightarrow} B$.

Proof: Let $X, A \vdash_{\mathbf{R}\rightarrow} B$. Then there is a derivation sequence $A_1, \ldots, A_n$ of wffs of $R\rightarrow$ such that each A_i is either (i) an element of X, (ii) A, (iii) an axiom, or (iv) follows from earlier steps by MP. By induction on the length of the derivation, we show $X \vdash A \supset A_i$ or $X \vdash A_i$.

Case α: $k = 1$.

Subcase 1: A_k is an element of X. Then $X \vdash A_k$.
Subcase 2: A_k is A. Then $X \vdash A \supset A_k$ by **R**→1.
Subcase 3: A_k is an axiom. Then $X \vdash A_k$.

Case β: Assume true for $k < k_0$; prove for $k = k_0$.

Subcases 1-3: same as Case α, Subcases 1-3.
Subcase 4: There is an $i(1 \leqslant i \leqslant k_0)$ and a $j(1 \leqslant j \leqslant k_0)$ such that A_j is $A_i \supset A_{k_0}$. By the hypothesis of induction, (i) either $X \vdash A \supset A_i$ or $X \vdash A_i$, and (ii) either $X \vdash A \supset (A_i \supset A_{k_0})$ or $X \vdash A_i \supset A_{k_0}$.

4a. $X \vdash A \supset A_i$ and $X \vdash A \supset (A_i \supset A_{k_0})$. But $\vdash_{\mathbf{R}\rightarrow} (A \supset. B \supset C) \supset. A \supset B \supset. A \supset C$, and so by two applications of MP, $X \vdash A \supset A_{k_0}$.
4b. $X \vdash A \supset A_i$ and $X \vdash A_i \supset A_{k_0}$. But $\vdash_{\mathbf{R}\rightarrow} (A \supset A_i) \supset. A_i \supset A_{k_0} \supset. A \supset A_{k_0}$. By two applications of MP, $X \vdash A \supset A_{k_0}$.
4c. $X \vdash A_i$ and $X \vdash A \supset (A_i \supset A_{k_0})$. By **R**→3, $X \vdash A \supset (A_i \supset A_{k_0}) \supset. A_i \supset. A \supset A_{k_0}$. By two applications of MP, $X \vdash A \supset A_{k_0}$.
4d. $X \vdash A_i$ and $X \vdash A_i \supset A_{k_0}$. Then by MP, $X \vdash A_{k_0}$.

The entailment fragment of **E** may be axiomatized by:

E→1. $A \supset A$
E→2. $A \supset B \supset. B \supset C \supset. A \supset C$
E→3. $A \supset B \supset. A \supset B \supset C \supset C$
E→4. $(A \supset. B \supset C) \supset. A \supset B \supset. A \supset C$.

When Anderson and Belnap [1] developed **E**→ as a Fitch-style system, the rules for **E**→ were a combination of the rules for **S4**→ and **R**→. Hence, it is no surprise that the deduction theorem for **E**→ combines the restrictions of necessity and relevance in the deduction theorems for **S4**→ and **R**→.

Theorem 4 *If* $X, A \vdash_{\mathbf{E}\rightarrow} B$ *and every member of* X *is strict, then* $X \vdash_{\mathbf{E}\rightarrow} A \supset B$ *or* $X \vdash_{\mathbf{E}\rightarrow} B$.

To prove Theorem 4, I use:

Lemma 1 *If* $X, A \vdash_{\mathbf{E}\rightarrow} B$ *by MP and every member of* X *is strict, then there exists an* A_j *such that* $X, A \vdash_{\mathbf{E}\rightarrow} A_j \supset B$ *and* $X, A \vdash_{\mathbf{E}\rightarrow} A_j$ *and* A_j *is either strict or is* A.

Proof: As usual, we denote that X is strict by writing X'. If $X, A \vdash B$ then there is a derivation sequence $A_1, \ldots, A_n$ such that each A_i is (i) an axiom, (ii) a member of X, (iii) A, or (iv) by MP from previous steps. By proof by induction on the number K of applications of MP in the derivation, we prove that if $X', A \vdash A_i$ by MP, then there is some A_j such that $X', A \vdash A_j$ and $X', A \vdash A_j \supset A_i$ and A_j is strict or is A.

Case α: $K = 1$. $X', A \vdash A_i$ is by MP then for some A_j we have $X', A \vdash A_j \supset A_i$ and $X', A \vdash A_j$. Since A_j is not by MP, then A_j is A, A_j is in X', or A_j is an axiom. If A_j is in X' or A_j is an axiom, then A_j is strict. Hence A_j is strict or is A.

Case β: Assume true for $K < K_0$, prove for $K = K_0$. If $X', A \vdash A_i$ by MP, then for some A_j, $X', A \vdash A_j \supset A_i$ and $X', A \vdash A_j$. If $X', A \vdash A_j$ then either (i) A_j is A, (ii) A_j is in X', (iii) A_j is an axiom, or (iv) A_j is by MP. If A_j is in X' or A_j is an axiom, then A_j is strict. If A_j is by MP, then by the hypothesis of induction, there exists an A_m such that $X', A \vdash A_m$; $X', A \vdash A_m \supset A_j$; and A_m is strict or is A. But if $X', A \vdash A_m \supset A_j$ and $X', A \vdash A_j \supset A_i$, then since $\vdash_{\mathbf{E}\rightarrow} (A_m \supset A_j) \supset ((A_j \supset A_i) \supset (A_m \supset A_i))$, and MP twice, we have $X', A \vdash A_m \supset A_i$. Thus there exists a formula, C, such that $X', A \vdash C$ and $X', A \vdash C \supset A_i$ and C is either strict or is A.

We may now prove Theorem 4. If $X, A \vdash B$ then we have a derivation sequence $A_1, \ldots, A_n$ such that each A_i is (i) in axiom (ii) a member of X, (iii) A, or (iv) from previous steps by MP. By proof by induction on the length k of the derivation, we prove $X \vdash A_i$ or $X \vdash A \supset A_i$.

Case α: $k = 1$.

Subcase 1: A_i is an axiom. Then $X \vdash A_i$.

Subcase 2: A_i is in X. Then $X \vdash A_i$.

Subcase 3: A_i is A. Then $X \vdash A \supset A_i$ by $\mathbf{E}\rightarrow 1$.

Case β: Assume true for $k < k_0$, prove for $k = k_0$.

Subcases 1-3: same as Case α, Subcases 1-3.

Subcase 4: If A_i is by MP, then for some $j < i$, $X, A \vdash A_j$ and $X, A \vdash A_j \supset A_i$. By Lemma 1, we may guarantee that this is true for some A_j such that A_j is strict or A_j is A. By the hypothesis of induction, either $X \vdash A_j$ or $X \vdash A \supset A_j$, and either $X \vdash A_j \supset A_i$ or $X \vdash A \supset (A_j \supset A_i)$.

4a. $X \vdash A_j$ and $X \vdash A_j \supset A_i$. Thus by MP, $X \vdash A_i$.

4b. $X \vdash A_j$ and $X \vdash A \supset (A_j \supset A_i)$. If A_j is strict, $X \vdash A \supset (A_j \supset A_i) \supset. A_j \supset. A \supset A_i$. By two applications of MP, $X \vdash A \supset A_i$. If A_j is A, then $X \vdash A$ and $X \vdash A \supset (A \supset A_i)$ and by MP twice $X \vdash A_i$.

4c. $X \vdash A \supset A_j$ and $X \vdash A_j \supset A_i$. By $\mathbf{E}\rightarrow 2$, $X \vdash A \supset A_j \supset. A_j \supset A_i \supset. A \supset A_i$, and by MP twice, $X \vdash A \supset A_i$.

4d. $X \vdash A \supset A_j$ and $X \vdash A \supset (A_j \supset A_i)$. By $\mathbf{E}\rightarrow 4$, $X \vdash A \supset (A_j \supset A_i) \supset. A \supset A_j \supset. A \supset A_i$. By MP twice, $X \vdash A \supset A_i$.

2 *Natural deduction systems* We may regard $X \vdash_{\mathbf{S}} A$ as a metalinguistic operator asserting that A is deducible from the set X in **S**. In the natural deduction systems to follow we will keep the structural rules for $\vdash$ fixed. The structural rules are:

1. If A is a member of X, then $X \vdash A$.
2. If $Y \vdash B_i$ for each B_i in $X = \{B_1, \ldots, B_n\}$, and $X \vdash A$, then $Y \vdash A$.
3. If $Y \subseteq X$ and $Y \vdash A$, then $X \vdash A$.

If A follows from the null set $\emptyset$ of premises in **S**, we write $\emptyset \vdash_{\mathbf{S}} A$, or simply $\vdash_{\mathbf{S}} A$. If A does not follow from X in **S**, we express this by $X \nvdash_{\mathbf{S}} A$. Besides the structural rules for $\vdash$, the systems will be constructed using the following rules:

1. $$\frac{X \vdash A \qquad X \vdash A \supset B}{X \vdash B}$$ *Modus Ponens* (MP)

2. $$\frac{X, A \vdash B}{X \vdash A \supset B}$$ Conditionalization (C)

3. $$\frac{X', A \vdash B}{X' \vdash A \supset B}$$ Necessary Conditionalization (NC)

4. $$\frac{X, A \vdash B \qquad X \nvdash_{\mathbf{IC}\to} B}{\mathbf{S}^{A_1, \ldots, A_n}_{v_1, \ldots, v_n}(X) \vdash \mathbf{S}^{A_1, \ldots, A_n}_{v_1, \ldots, v_n}(A \supset B)}$$ Relevant Conditionalization (RC)

5. $$\frac{X', A \vdash B \qquad X' \nvdash_{\mathbf{IC}\to} B}{\mathbf{S}^{A_1, \ldots, A_n}_{v_1, \ldots, v_n}(X') \vdash \mathbf{S}^{A_1, \ldots, A_n}_{v_1, \ldots, v_n}(A \supset B)}$$ Necessary Relevant Conditionalization (NRC)

The rules RC and NRC are quite unlike the usual rules given in natural deduction systems. The point is to restrict conditionalization to cases where the premise A is "really" needed to deduce B. If B already follows from X, we may say that A is irrelevant to B, and hence that B does not follow relevantly from A given the premises X. However, we want to allow for cases where, although B does follow from X; $X, A \vdash B$ is an instance of some case $Y, C \vdash D$, where $Y \nvdash D$. Hence we allow that $X \vdash A \supset B$ may also be permitted if it is a substitution instance of some $\vdash$ statement for which RC (or NRC) is admissable. It should also be noted that unlike the deduction rule of each system, the conditionalization rule, so stated, requires that B not follow from X in **IC**→ (**IC**→ is the implication fragment of the classical propositional calculus). This is simply to guarantee that it is decidable whether or not B follows from X.

Let **NH**→ be the system with rules MP and C; **NS4**→ the system with rules NC and MP; **NR**→ the system with rules MP and RC; and **NE**→ with rules MP and NRC. We prove:

Theorem 5 *The following relations hold between these systems:*

a. $X \vdash_{\mathbf{H}\to} A$ *iff* $X \vdash_{\mathbf{NH}\to} A$
b. $X \vdash_{\mathbf{S4}\to} A$ *iff* $X \vdash_{\mathbf{NS4}\to} A$
c. $X \vdash_{\mathbf{R}\to} A$ *iff* $X \vdash_{\mathbf{NR}\to} A$
d. $X \vdash_{\mathbf{E}\to} A$ *iff* $X \vdash_{\mathbf{NE}\to} A$.

Proof: The reader may verify that each of the axioms of **H**→ (**S4**→, **R**→, **E**→) is provable in **NH**→ (**NS4**→, **NR**→, **NE**→); and since each system has MP as a rule, they are deductively contained in the systems. Furthermore, the deduction theorem for each of **H**→, **S4**→, **R**→, **E**→, shows that they contain the conditionalization rule of the corresponding **N**-system as a derived rule. Thus anything provable in the natural deduction system is provable in the corresponding axiom system.

It is well known that **H**→ deductively contains **R**→ and **S4**→, and that each of these systems deductively contains **E**→ (but not vice versa). The natural deduction formulations given above effectively display the relationship between implication in each of the systems, and the relationship between implication and provability within each of them. That is, each system is constructed by adding one further restriction to the conditionalization rule. Another system that may be treated in this manner is Ticket Entailment, **T**→. The system **T**→ may be axiomatized by MP and

T→1. $A \supset A$
T→2. $A \supset B \supset . B \supset C \supset . A \supset C$
T→3. $A \supset B \supset . C \supset A \supset . C \supset B$
T→4. $A \supset (B \supset C) \supset . A \supset B \supset . A \supset C$
T→5. $A \supset B \supset . (A \supset . B \supset C) \supset . A \supset C$.

The Fitch-style natural deduction system equivalent to **T**→ involves a restriction on the rule MP, but otherwise has the same rules as the system equivalent to **E**→. Hence it seems plausible to suppose that **T**→ will have the same deduction theorem as **E**→. However, the proof of the deduction theorem for **E**→ makes use of restricted permutation which is not provable in **T**→. The problem is that, although the motivation for **T**→ involved a natural deduction system with weak MP, the axiom system for **T**→ makes use of ordinary MP. Thus the restriction on MP must appear as part of the deduction theorem. The following lemma is used and, as usual, I use X' to denote that every member of X is strict.

Lemma 2 *If* X', $A' \vdash_{\mathbf{T}\to} B$ *by MP, then there exists an* A_j *such that* X', $A' \vdash_{\mathbf{T}\to} A_j$ *and* X', $A \vdash_{\mathbf{T}\to} A_j \supset B$ *and* A_j *is strict or is* A.

Proof: The proof is similar to the proof of Lemma 1 and is therefore omitted.

Theorem 6 *If* X, $A' \vdash_{\mathbf{T}\to} B$ *and every member of* X *is strict, and* B *results from MP only if the antecedent of the conditional was simple, then* $X \vdash_{\mathbf{T}\to} A \supset B$ *or* $X \vdash_{\mathbf{T}\to} B$.

Proof: If X', $A \vdash_{T\rightarrow} B$ then we have a derivation sequence $A_1, \ldots, A_n$ such that each A_i is (i) an element of X', (ii) A, (iii) an axiom, or (iv) from earlier steps by MP. Proof will be by induction on the length k of the derivation that if X', $A \vdash A_i$ results from MP only if the antecedent of the conditional was simple, then $X \vdash A \supset A_i$ or $X \vdash A_i$.

Case α: $k = 1$.

Subcase 1: A_i is a member of X'. Then $X' \vdash A_i$.

Subcase 2: A_i is A. Then $X' \vdash A \supset A_i$ by **T**→1.

Subcase 3: A_i is an axiom. Then $X' \vdash A_i$.

Case β: Assume true for $k < k_0$, prove for $k = k_0$.

Subcases 1-3: same as Case α, Subcases 1-3.

Subcase 4: X', $A \vdash A_{k_0}$ is by MP. Then for some previous steps we have $X', A \vdash A_j$ and $X', A \vdash A_j \supset A_{k_0}$. By Lemma 2, we can guarantee that A_j is strict or A_j is A. But by hypothesis, A_j is simple. Hence, A_j is A. Thus we have X', $A \vdash A$ and X', $A \vdash A \supset A_{k_0}$. By the hypothesis of induction: (i) either $X' \vdash A$ or $X' \vdash A \supset A$ and (ii) either $X' \vdash A \supset A_{k_0}$ or $X' \vdash A \supset (A \supset A_{k_0})$.

4a. $X' \vdash A$ and $X' \vdash A \supset A_{k_0}$. Then by MP, $X' \vdash A_{k_0}$.

4b. $X' \vdash A$ and $X' \vdash A \supset (A \supset A_{k_0})$. By using MP twice we arrive at $X' \vdash A_{k_0}$.

4c. $X' \vdash A \supset A$ and $X' \vdash A \supset A_{k_0}$. Then by **T**→2, and MP, $X' \vdash A \supset A_{k_0}$.

4d. $X' \vdash A \supset A$ and $X' \vdash A \supset (A \supset A_{k_0})$. By **T**→5, $X' \vdash A \supset A \supset. (A \supset (A \supset A_{k_0}) \supset. A \supset A_{k_0}$. Thus by MP, $X' \vdash A \supset A_{k_0}$.

If B results from an application of MP wherein the antecedent of $A \supset B$ was simple, we shall denote this by $'B$. Let **DT**→ be the sequent calculus with the rules MP and

$$\frac{\begin{array}{c} X', A \vdash B \\ X' \nvdash_{\mathbf{IC}\rightarrow} B \end{array}}{\mathsf{S}^{A_1,\ldots,A_n}_{v_1,\ldots,v_n}(X') \vdash \mathsf{S}^{A_1,\ldots,A_n}_{v_1,\ldots,v_n}(A \supset B) \text{ only if } 'B} \quad \text{(SNRC)} \quad .$$

Lemma 3 $\quad X \vdash_{T\rightarrow} A$ *iff* $X \vdash_{\mathbf{DT}\rightarrow} A$.

Thus **DT**→ results from **NE**→ by adding one further restriction to the conditionalization rule for **NE**→.

Since part of the restriction on the conditionalization rule for **DT**→ involves prior applications of MP, it would be more natural to have the restriction on MP and use the conditionalization rule of **NE**→. Let **NT**→ be the sequent calculus with rules NRC and

$$\frac{\begin{array}{c} X \vdash A \\ X \vdash A \supset B \end{array}}{X \vdash B \text{ if } A \text{ is simple or } X \text{ is empty.}} \quad \text{(SMP)}.$$

Lemma 3 $X \vdash_{\mathbf{T}\rightarrow} A$ *iff* $X \vdash_{\mathbf{NT}\rightarrow} A$.

The motivation behind the original construction of $\mathbf{T}\rightarrow$ was that necessary truths, in general, were only used in proofs as "inference tickets" ([1], pp. 41-49). That is, we may use them in proofs to deduce formulas from our hypothesis, but they should not actually become part of the theorem. SMP achieves this by only allowing entailments, e.g., $A \supset B$, to be used as the inference ticket from A to B. If A were allowed to be strict in SMP, then A itself might be an inference ticket. In that case, B would be a derived inference ticket that became part of the theorem we were deriving. We also allow for unrestricted MP in the case where X is empty. In that case A and $A \supset B$ would both be inference tickets from which we were deriving the inference ticket B. Thus we allow for the proof of derived inference rules from other inference rules as required by Anderson and Belnap.

The Hilbert-style implication fragments with which we are concerned have been given natural deduction formulations before, both as Fitch-style systems and Gentzen systems. However, the Gentzen formulations as given in Anderson and Belnap ([1], pp. 50-68) involve restrictions on both the structural rules for $\vdash$ and the inference rules for arguments. Hence it is difficult to compare the properties of $\supset$ relative to two of the systems when $\vdash$ in each system has changed properties.

The usual interpretation of $A_1, \ldots, A_n \vdash_{\mathbf{S}} A$ is that there is a proof from $A_1, \ldots, A_n$ to A in the system $\mathbf{S}$. A deduction theorem for $\mathbf{S}$ tells us what conditions are sufficient for asserting that $A_1, \ldots, A_{n-1} \vdash A_n \supset A$. In the classical propositional calculus $A_1, \ldots, A_n \vdash A$ iff $A_1, \ldots, A_{n-1} \vdash A_n \supset A$. This tells us that we may assert that A_n implies A, given the premises $A_1, \ldots, A_{n-1}$, iff A may be asserted on the hypothesis $A_1, \ldots, A_n$. In other words, the classical calculus is deductively complete.

But it must be remembered that a very weak relationship is being asserted between $A_1, \ldots, A_n$ and A by $\vdash$, and that, correspondingly, a weak relationship is being asserted between A_n and A by $\supset$. The sense in which A is proved using $A_1, \ldots, A_n$ requires only that *something* in that set be used, and that something, of course, may even be the null set. If we keep this weak sense of $\vdash$, but strengthen our notion of $\supset$, different deduction relationships will obtain between $\vdash$ and $\supset$ than in the classical calculus.

The deduction theorems for $\mathbf{R}\rightarrow$, $\mathbf{S4}\rightarrow$, $\mathbf{E}\rightarrow$, and $\mathbf{T}\rightarrow$, tell us what conditions are sufficient for asserting that A_n entails A in these systems given that $\vdash$ only requires that some of the hypotheses be used to deduce A. Suppose we wanted a relevance logic in the sense that $A \supset B$ would be asserted in that system if both B followed from A and A was relevant to B. We might claim that if $A_1, \ldots, A_n \vdash B$, then A_i, $1 \leqslant i \leqslant n$, is relevant to B if A_i is actually used in the proof of B from the hypotheses $A_1, \ldots, A_n$. If so, we may assert that $\{A_1, \ldots, A_n\} - \{A_i\} \vdash A_i \supset B$. As the deduction theorem for $\mathbf{R}\rightarrow$ shows, such is the case in $\mathbf{R}\rightarrow$. Of course, to formalize the intuitive idea, it is necessary to replace the concept of "is used in the proof of B" with the formal concept of $\{A_1, \ldots, A_n\} - \{A_i\} \nvdash B$. Thus the deduction theorem for **R**→ tells us under what circumstances **R**→ is deductively complete. Or alternatively we could view it as redefining the notion of deductive completeness to take into account relevance.

In a like manner, the deduction theorems for **S4**→, **E**→, **T**→ give us certain conditions in which we are allowed to assert that the implication relation of the system holds (given that certain $\vdash$ relations hold). In **S4**→, we are allowed to assert that $A_1, \ldots, A_{n-1} \vdash A_n \supset A$ on the basis of $A_1, \ldots, A_n \vdash A$, if each of $A_1, \ldots, A_{n-1}$ is "necessary". This shows that **S4**→ excludes certain members of $A_1, \ldots, A_n$ on the basis of their modal characteristics, just as **R**→ excludes certain members on the basis of relevance. **E**→, of course, considers both requirements. Finally, **T**→ tells us that inference patterns used in the inference from $A_1, \ldots, A_n$ to A are not to be thought of as being hypotheses themselves. They belong to a different category of things: propositions (that are not inference patterns) do not follow from them, instead the inference patterns allow us to show that a proposition follows from other propositions.

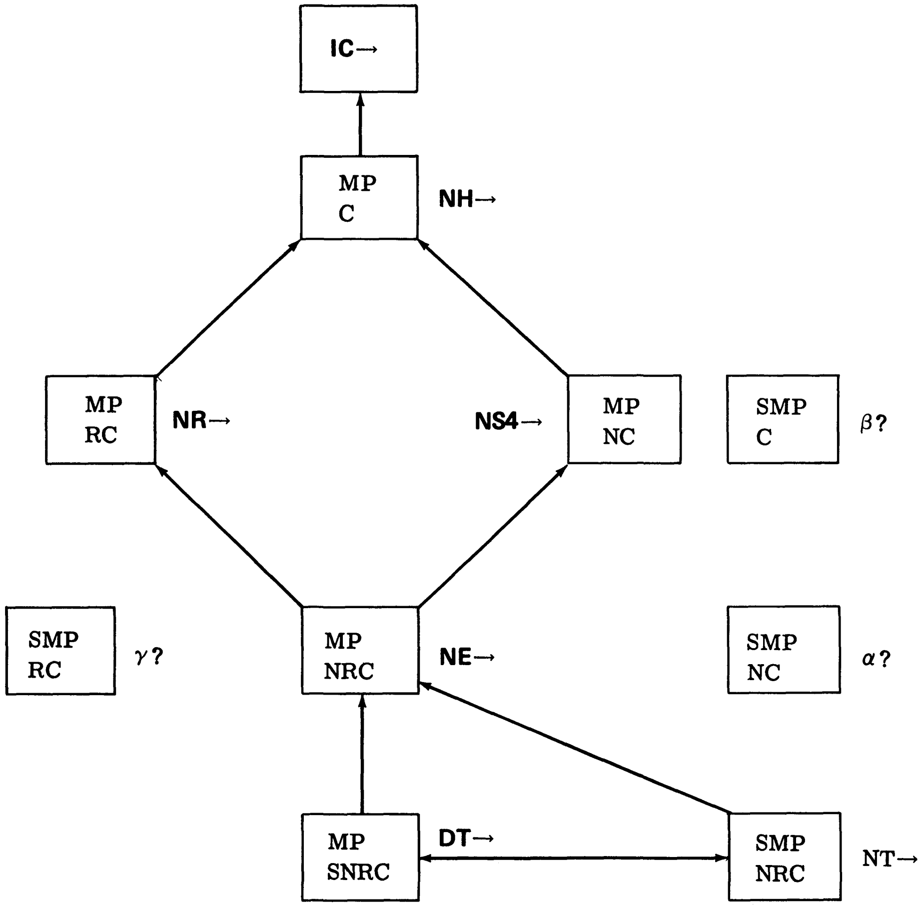
Exhibiting **R**→ as a natural deduction system using only MP and relevant conditionalization shows that the requisite concept of deductive completeness entails the entire theory of relevant implication. That is, if we define relevant deductive completeness as: if $A_1, \ldots, A_n \vdash B$ and $A_n \nvdash B$ then $A_1, \ldots, A_{n-1} \vdash A_n \supset B$, then relevant deductive completeness entails the entire theory of relevant implication. In a like manner, the natural deduction formulations given here for **S4**→, **E**→, and **T**→ show that the concept of deductive completeness, as expressed by their respective deduction theorems, entails the entire theories of strict implication, entailment, and ticket entailment, respectively.

Of course many logicians might claim that it is counterintuitive to suppose that there might be a proof from A to B while we were unwilling to assert that A entailed B. I would agree that this is the case as long as we have the proper notion of "proof of B from A". That is, if "$\vdash$" is given the classical interpretation, then we might be willing to assert $A \vdash B$, but not to assert that A entails B. On the other hand, in constructing a system of entailment, it would be desirable to have a simple correlation between valid entailments and proof, and this can be done for **S4**→, **R**→, **E**→, and **T**→. The so-called "Merge formulations" of these systems in Anderson and Belnap each has the rule

$$\frac{X, A \vdash B}{X \vdash A \supset B},$$

while modal, relevance, and ticket fallacies are avoided by various restrictions on the other rules (including the structural rules for $\vdash$).

3 *Natural deduction systems with a restriction on MP* If we make a diagram of the inclusion relationships holding between the **N**-systems, with → representing "is contained in", we see that each of the **N**-systems resulted from adding a further restriction to conditionalization, except for **NT**→ which resulted from adding a further restriction to MP in **NE**→.



What about the **N**-systems other than **NT→** containing SMP as a rule? We might suppose that the system consisting of SMP and NC would be included in **NS4→**, and perhaps wonder what its relationship might be to **NE→**. Let α be the system with rules SMP and NC, γ the system with rules SMP and RC, and β the system with SMP and C. We may prove:

Lemma 4 *α and* **NS4→** *are equivalent, γ and* **NR→** *are equivalent, β and* **NH→** *are equivalent.*

This lemma tells us that the addition of the restriction on MP that X be empty or the antecedent be simple has *no effect* on our stock of theorems when added to **NH→**, **NR→**, or **NS4→**. But surprisingly, if we add the restriction to **NE→** we lose some theorems. Why is this the case?

We must consider what effect SMP has on proofs in **NE→**. For example, we know that permutation can be proved in **NR→**.

1. $A \supset (B \supset C), B, A \vdash A \supset (B \supset C)$ hyp.
2. $A \supset (B \supset C), B, A \vdash B$ hyp.

3. $A \supset (B \supset C), B, A \vdash A$	hyp.
4. $A \supset (B \supset C), B, A \vdash B \supset C$	1, 3 MP
5. $A \supset (B \supset C), B, A \vdash C$	2, 4 MP
6. $A \supset (B \supset C), B \vdash A \supset C$	5 RC
7. $A \supset (B \supset C) \vdash B \supset (A \supset C)$	6 RC
8. $\vdash A \supset (B \supset C) \supset. B \supset (A \supset C)$	7 RC.

In steps 5, 6, and 7, RC is applicable since, for instance, $A \supset (B \supset C) \nvdash A \supset C$ in **IC→**. If we add the condition that our premises be strict, however, step 6 would be an error. Since B is not strict in line 5, NRC may *not* be used to arrive at line 6. Hence, permutation may not be proved in **NS4→** or **NE→**. Nonetheless, a form of permutation called restricted permutation may be proved in NS4→ and NE→.

1. $A \supset (B' \supset C), B', A \vdash A \supset (B' \supset C)$	hyp.
2. $A \supset (B' \supset C), B', A \vdash B'$	hyp.
3. $A \supset (B' \supset C), B', A \vdash A$	hyp.
4. $A \supset (B' \supset C), B', A \vdash B' \supset C$	1, 3 MP
5. $A \supset (B' \supset C), B', A \vdash C$	2, 4 MP
6. $A \supset (B' \supset C), B' \vdash A \supset C$	5 NRC
7. $A \supset (B' \supset C) \vdash B' \supset (A \supset C)$	6 NRC
8. $\vdash A \supset (B' \supset C) \supset. B' \supset (A \supset C)$	7 NRC.

In this proof, since B' is strict, we may apply the rule NRC (or the rule NC) to achieve the desired result. Now if we add our condition of simplicity to MP, the last proof will fail, because B' was used to produce C from $B' \supset C$. Thus, restricted permutation, as well as permutation, fails in **NT→**. However, the addition of the same restriction to MP in **NR→** fails to block restricted permutation. It does not fail because we can reproduce the proof given above for restricted permutation. On the contrary, that proof is not a valid proof in γ. But we can bypass this restriction by noting that restricted permutation is an instance of permutation. Thus the proof given earlier for permutation in **NR→** is also a proof of restricted permutation in **NR→**. Since that proof does not violate the restriction imposed on MP in γ, it is also a proof of both permutation *and* restricted permutation in γ. The key is, I believe, that the restriction on MP in SMP blocks the proofs of the "restricted" versions of certain theorems, such as:

restricted permutation: $A \supset (B' \supset C) \supset. B' \supset (A \supset C)$
restricted conditioned *modus ponens*: $B' \supset. (A \supset. B' \supset C) \supset. A \supset C$
restricted assertion: $A \supset B \supset. A \supset B \supset C \supset C$
specialized assertion: $A \supset A \supset B \supset B$.

The simple versions of the theorems are blocked by considerations of relevance or necessity. In **T→**, neither the simple nor the restricted versions are provable, while in **E→**, only the restricted versions are provable. However, in the absence of the combination of necessity and relevance, the simple versions are provable and the added restriction to *modus ponens* in SMP is bypassed.

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