Notre Dame Journal of Formal Logic
Volume XXI, Number 2, April 1980
NDJFAM

SIGNIFICANCE, NECESSITY, AND VERIFICATION

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Some philosophical positions cannot be properly assessed unless we have suitable three-valued modal logics.

Consider, for example, the positivist criterion of meaning: that a sentence is significant (meaningful) if, and only if, it is analytic, contradictory, or empirical. Writing S for 'is significant', L for 'is analytic', and E for 'is empirical', the criterion can be expressed formally[1] by: for all p, $Sp \equiv . Lp \vee L{\sim}p \vee Ep$. As such, we might expect it to be a thesis of an appropriately augmented modal logic, or perhaps we might expect to show that it cannot be a thesis of an appropriate modal logic which satisfies other positivist conditions (e.g., if we think that positivism is an inconsistent theory). But in either case the variable p will have to take both meaningful and meaningless sentences as values. If the range is restricted to meaningful sentences, the criterion is vacuous and the question is begged against positivism, for such a semantic restriction amounts to the same thing as assuming the formal thesis: for all p, Sp.

Well-known problems connected with the positivist position also point to the need for an appropriate three-valued modal logic. Thus, the criterion of meaning has to be elucidated by supplementary criteria of analyticity and empiricalness, and it is evident that if the criterion of analyticity is to be effective, it should entail certain modal principles—perhaps, for example, that if p is analytic, so is $p \vee q$; i.e., $Lp \supset L(p \vee q)$. But does this hold if q is meaningless? Even the criterion of verification, which stands as the supplementary criterion of empiricalness, leads to problems of this sort. For it is often expressed in terms of entailment principles which are intended to apply in a three-valued context, yet standard criticisms of it make use of two-valued modal principles which are assumed, without argument, to hold in the three-valued case.

A different kind of problem which seems to be independent of positivist principles but which also calls for a three-valued modal logic is posed by Kripke ([5], p. 31) in a discussion of identity: "when I say 'Hesperus is Phosphorous' is necessarily true, I of course do not deny that there may have been situations in which there was no such planet as Venus at all, and

Received May 10, 1976; Revised October 1, 1979

therefore no Hesperus and no Phosphorous. In that case, there is a question of whether the identity statement 'Hesperus is Phosphorous' would be true, false, or neither true nor false. And if we take the last option, is 'Hesperus = Phosphorous' necessary because it is never false, or should we require that a necessary truth be *true* in all possible worlds?". Surprisingly, perhaps, this problem—or at least one closely connected to it, the problem of essences—is not entirely independent of problems about positivism, but I shall not take up this question here.

In general, my concern in this paper is to develop some three-valued modal logics in order to get clear what the options are. Some of the results bear directly on the problems mentioned above, and in Section **7** there is a preliminary discussion of standard criticisms of the verification criterion, but mostly the application to philosophical issues is left for a subsequent paper.[2]

1 *Significance logics* The minimum semantic condition for a formal system to be interpreted as a significance logic is that it should contain sentential variables, say p, q, p', q', . . ., which range over the three values: t (truth), f (falsity), and n (nonsignificance). Such logics may contain other variables restricted in range to subsets of $\{t, f, n\}$, the most important being significance-restricted variables, say r, s, r', s', . . ., which range over $\{t, f\}$; and they may contain constants which take just one value from the set, for example t_0 which always takes the value t.

Characteristically, but not essentially, significance logics contain the classical significance connectives, $\sim$, $\vee$, &, $\supset$, and $\equiv$. These have the standard interpretation put on them in usual two-valued logics (i.e., $\sim$ reads 'not', & reads 'and', etc.) and they are defined over the values in such a way as to satisfy ordinary two-valued conditions over $\{t, f\}$ plus the further condition that when a variable takes the value n, any compound which contains it, and which is formed using just classical connectives, also takes the value n. These features are guaranteed if $\sim$ and $\vee$ are characterized by the following matrices and &, $\supset$, and $\equiv$ are defined in the usual way:

	$\sim$
t	f
f	t
n	n

$\vee$	t	f	n
t	t	t	n
f	t	f	n
n	n	n	n

When actual sentences are substituted for the variables, the classical connectives therefore yield a principle which has often been adopted in philosophical discussions of significance, namely that a compound sentence is meaningless if, and only if, at least one of its components is meaningless (cf., [4], p. 102).

Significance logics can contain other connectives which fail to satisfy the above principle, i.e., which are such that compound sentences containing them are meaningful even though they contain meaningless components.

In general, a connective is *unlimited* if its matrix values do not include the value n for any values of its components, otherwise *limited*. The classical significance connectives are thus a special case of limited connectives. One important and immediate result which follows by induction over the number of connectives, given the usual definition of wff, is the following:

Theorem I *A significance logic which contains no unlimited connectives and no significance-restricted variables or constants, and for which t is specified to be the only designated value, contains no theses.*

Significance logics which contain theses must therefore fail to satisfy one or more of these conditions, i.e., they must contain at least one unlimited connective, or significance-restricted variables (constants), or t must not be the only designated value. Where t is the only designated value, the resultant logic is called an **S**-logic; where both t and n are designated, we have a **C**-logic.

Several unlimited connectives have a natural interpretation. For example, the connective T defined by:

	T
t	t
f	f
n	f

if interpreted as 'is true', satisfies the principle that it is always meaningful (i.e., true-or-false) to say of any sentence, whether true, false, or meaningless, that it is true. One way of justifying this use of 'true' would be to claim that it is always meaningful to say of a meaningless sentence that it is not true; but if it is meaningful to say that it is not true, it is meaningful, though false, to say that it is true: i.e., if p takes the value n, Tp takes the value f.

Given T, other unlimited connectives can then be defined as follows:

$FA =_{df} T{\sim}A$, (A is false)

$SA =_{df} TA \vee FA$, (A is significant (meaningful))

$A \simeq B =_{df} (TA \;\&\; TB) \vee (FA \;\&\; FB) \vee ({\sim}SA \;\&\; {\sim}SB)$, ($A$ is strongly equivalent to B, or, A and B have the same significance value);

and they are characterized by the following derived matrices:

	F
t	f
f	t
n	f

	S
t	t
f	t
n	f

$\simeq$	t	f	n
t	t	f	f
f	f	t	f
n	f	f	t

Hence the significance-connective S can be defined in a system in which T is taken as primitive, but the converse is not true, i.e., given S as the only primitive unlimited connective, T cannot be defined.

These results, and a detailed investigation of various **S** and **C** significance logics, are found in [3], Part II. They are sketched in here as a background to the development of a series of modal significance logics.

2 *Modal significance logics* Just as there are many different modal logics which can be constructed on a two-valued sentential base, depending on the postulates or semantic conditions which are adopted to characterize the modal connectives, so there are various ways of modalizing significance logics. An initial decision which has to be made, however, which does not arise in the two-valued case, is whether the basic modal connective L (is logically true) should be limited or unlimited. Given a significance logic containing T, this is not a real problem, since if we take a limited L as primitive (i.e., one such that the value of LA is n for some values of A), an unlimited (two-valued) L^+ can be defined in terms of it by means of $L^+A =_{df} TLA$ (cf., [3], p. 444). A significance logic which contained only a limited L, however, together with the classical sentential connectives, and which contained no unlimited connectives and no variables restricted in range to the set $\{t,f\}$ would have no theses if t were taken as the only designated value. This is a consequence of Theorem I.

Modal systems containing only primitive limited modal connectives but other unlimited connectives are discussed in [3], section **7.2**; there is, however, no discussion in [3] of systems containing primitive unlimited modal connectives, and some of these are of interest. In particular, it is worth noting that the positivist criterion of meaning, expressed in the form $SA \equiv . LA \vee L \sim A \vee EA$, cannot be a thesis of any system in which t is the only designated value and $\vee$ and $\equiv$ are classical significance connectives, if L is a limited connective. For if LA takes the value n for some values of the variables in A, so does the whole disjunction on the right and so, too, does the equivalence; hence the equivalence would not always take the value t and so could not be a thesis. The same conclusion follows if the criterion is expressed in terms of $\simeq$ rather than $\equiv$, for although there can be theses of the form $A \simeq B$ where A and B can both take the value n, $SA \simeq . LA \vee L \sim A \vee EA$ cannot be one of them. As before, if LA takes the value n for some values of the variables in A then, for those values, the disjunction takes value n; but since SA never takes value n, the equivalence takes value f in such cases and is not a thesis. The same argument holds for E. So if we are not to beg the question against positivism, we must consider systems in which LA and EA cannot take the value n, i.e., systems in which L and E are unlimited connectives.

We therefore begin with the simplest possible modal logic characterized by:

(i) variables which range over the full set $\{t,f,n\}$

(ii) no variables (or constants) restricted in range to the set $\{t,f\}$ or subsets of it

(iii) the classical significance connectives

(iv) L as the only primitive unlimited connective

(v) t as the only designated value.

Such a characterization is not strictly accurate for a modal logic since the conditions have to be world-relativized (see below) but it is intended here simply as an explanatory guide.

A justification, similar to the argument which supports an unlimited T as primitive, can be given in support of an unlimited primitive L. For if L is interpreted as 'is logically true', then it is false, therefore meaningful, to say of a meaningless sentence that it is logically true. This is not of course an argument for repudiating systems which contain a limited L (any more than the argument for an unlimited T is a repudiation of systems containing a limited T), but simply a justification for saying that there is an unlimited sense of L and, therefore, that systems containing only an unlimited L can be given a natural interpretation.

Well-formed formulas are defined as follows:

W1 Sentential variables are wff.
W2 If A is a wff, so are $\sim A$ and LA.
W3 If A and B are wff, so is $(A \vee B)$.
W4 Defined equivalents of wff are wff.
W5 These are the only wff.

We take the usual conventions with respect to parentheses as understood and add the following standard definitions:

D1 $(A \supset B) =_{df} (\sim A \vee B)$
D2 $(A \& B) =_{df} \sim(\sim A \vee \sim B)$
D3 $(A \equiv B) =_{df} (A \supset B) \& (B \supset A)$
D4 $MA =_{df} \sim L \sim A$
D5 $(A \prec B) =_{df} L(A \supset B)$
D6 $(A = B) =_{df} (A \prec B) \& (B \prec A)$.

M, $\prec$, and $=$ have their usual interpretations: respectively, 'is possible', 'strictly implies', and 'is strictly equivalent to'.

The set of wff and the "ordinary-language" interpretation of the connectives is thus exactly the same as in standard two-valued Lewis systems. The difference between the systems developed here and the usual Lewis systems will initially arise only from the introduction of the third value n to the formal semantics and the conditions which are put on it.

We now characterize semantic theses (valid wff); and to keep the interpretation in line with two-valued systems we impose the condition that the semantic rules should "contain" or "extend" the two-valued case in the sense that, where a rule or a part of a rule refers only to values from the set $\{t, f\}$, it should be identical with the corresponding rule for a standard two-valued system. This corresponds to a similar condition imposed on the one-world semantics of **S** and **C** logics, namely that the three-valued matrices (e.g., for $\sim$ and $\vee$) should "contain" the two-valued matrices in the sense that, if the value n is everywhere deleted, the result is a standard two-valued matrix for the appropriate connective. We secure this kind of feature in the modal systems by giving a world-by-world extension of the three-valued matrices for $\sim$ and $\vee$, and by adopting the standard two-valued semantic rule for L augmented by a clause which determines the value of LA at a world when A takes the value n at a related world. This clause, in effect, corresponds to the intuitive condition that it is false to say of a meaningless sentence that it is logically true.

A model is a triple $\langle W, R, v\rangle$, where W is a set of worlds, R, a relation between worlds, and v a valuation function, characterized by the following rules:

S1 Where P is any sentential variable and $w_i \in W$:

$$v(P, w_i) = t \text{ or } f \text{ or } n.$$

S2 Where A is any wff and $w_i \in W$:

$$v(\sim A, w_i) = t \text{ iff } v(A, w_i) = f$$
$$v(\sim A, w_i) = f \text{ iff } v(A, w_i) = t$$
$$v(\sim A, w_i) = n \text{ iff } v(A, w_i) = n.$$

S3 Where A and B are any wff and $w_i \in W$:

$$v(A \vee B, w_i) = t \text{ iff}$$
$$\text{either } v(A, w_i) = t \text{ and } v(B, w_i) = t \text{ or } f,$$
$$\text{or } v(B, w_i) = t \text{ and } v(A, w_i) = t \text{ or } f;$$
$$v(A \vee B, w_i) = f \text{ iff } v(A, w_i) = f \text{ and } v(B, w_i) = f;$$
$$v(A \vee B, w_i) = n \text{ iff}$$
$$\text{either } v(A, w_i) = n,$$
$$\text{or } v(B, w_i) = n.$$

S4 Where A is any wff and $w_i \in W$:

$$v(LA, w_i) = t \text{ iff } v(A, w_j) = t \text{ for every } w_j \text{ such that } w_i R w_j$$
$$v(LA, w_i) = f \text{ iff } v(A, w_j) = f \text{ or } n \text{ for some } w_j \text{ such that } w_i R w_j.$$

Since the conditions on A on the right-hand side of the clauses in S4 are exhaustive, and since the clauses are equivalences, it follows that the conditions on LA are exhaustive. Hence,

$$v(LA, w_i) = t \text{ or } f;$$

i.e., L is characterized as an unlimited connective, as required. The consequential rule for M is:

S5 Where A is any wff and $w_i \in W$:

$$v(MA, w_i) = t \text{ iff } v(A, w_j) = t \text{ or } n \text{ for some } w_j \text{ such that } w_i R w_j;$$
$$v(MA, w_i) = f \text{ iff } v(A, w_j) = f \text{ for every } w_j \text{ such that } w_i R w_j.$$

Hence, too, M is unlimited since,

$$v(MA, w_i) = t \text{ or } f.$$

This gives an extended and curious sense to 'possible', from the point of view of ordinary language, since it entitles us to say (when reflexivity is imposed on R) that meaningless sentences express possibilities. But in the purely technical sense of 'not-necessary-not' it is acceptable since the negation of a meaningless sentence, being itself meaningless, is not necessary (i.e., not logically true). Thus logics containing an M which satisfies S5 would not be acceptable to those who wish to claim that meaningless sentences should be classified as logically false.

The world-relativized extension of the condition that t should be the only designated value is secured by an appropriate condition for validity, namely

Definition: *Val:* A wff A is valid iff, for every model $\langle W, R, v\rangle$ and every $w_i \in W$, $v(A, w_i) = t$.

Strictly speaking this is a validity-schema rather than a criterion of validity, since it only yields a specific criterion when conditions are put upon R. What we have so far described, therefore, is a set of modal logics each of which satisfies the initial conditions (i)-(v). Specific logics then arise by characterizing R.

3 *The logic* $\mathbf{L}_0$ The logic $\mathbf{L}_0$ is determined by requiring only that R be reflexive. Thus:

Definition: *Val* $\mathbf{L}_0$: A wff A is $\mathbf{L}_0$-valid iff it is valid when R is reflexive.

Where this condition is imposed in the two-valued case we obtain the set of semantic theses of von Wright's system **M** or, equivalently, Feys' system **T**. Thus, $\mathbf{L}_0$ differs from **M** only in the extra conditions which determine the role of the third value n; i.e., $\mathbf{L}_0$ is a three-valued interpretation of **M**.

There are some immediate results (proofs are only indicated: in each case a full proof can be obtained by a simple inductive argument). We first define 'cover' as follows:

Definition A variable P is covered in a wff A iff P occurs in A and every occurrence of P in A is within the scope of an unlimited connective; otherwise P is uncovered in A.

We then have:

Theorem II *No formula which contains an uncovered variable is valid.*

Proof: For let A be a wff which contains an uncovered variable P. Then there is at least one occurrence of P in A which is not within the scope of an unlimited connective. Assign the value n, in some world w_i, to this occurrence. Then by S2 and S3, A takes the value n for this assignment in w_i. So there is a model such that A fails to take the value t in every world.

Hence, no analogues of standard two-valued sentential theses are valid. For example, $p \supset (p \vee q)$, $p \vee \sim p$, etc., are not semantic theses of $\mathbf{L}_0$. Similarly, no analogues of two-valued modal theses of **M** containing uncovered variables are valid in $\mathbf{L}_0$: e.g., $Lp \supset p$ and $p \supset Mp$ are not valid in $\mathbf{L}_0$. These consequences do not constitute a serious disadvantage, however, since certain substitution instances of formulas such as $p \vee \sim p$ and $Lp \supset p$ are valid in $\mathbf{L}_0$. To show this we first define *covered variants* of a wff A as follows:

Definition Where A is a wff containing uncovered variables $P_1, \ldots, P_m$, and A^+ is a wff obtained from A by uniform substitution of wffs $C_1, \ldots, C_m$ (where these need not all be distinct) for $P_1, \ldots, P_m$, then, if all the variables in $C_1, \ldots, C_m$ are covered, A^+ is a covered variant of A.

Thus, $LLq \supset Lq$ is a covered variant of $Lp \supset p$; $(Lp \vee \sim Lp) \supset L(p \vee \sim p)$ is a covered variant of $(q \vee \sim q) \supset L(p \vee \sim p)$, etc. We then have:

Theorem III *Where* (i) *A is a* $\mathbf{L}_0$*-wff* all *of whose variables are uncovered (i.e., there is at least one occurrence of each variable in the formula which is not within the scope of an unlimited connective); and* (ii), *A is valid in* $\mathbf{M}$; *then all covered variants of A are valid in* $\mathbf{L}_0$.

Proof: For let $A^+ (C_1, \ldots, C_m)$ be a covered variant of A $(P_1, \ldots, P_m)$ where A is valid in $\mathbf{M}$ and $P_1, \ldots, P_m$ are all the distinct variables in A and each is uncovered in A. Since A is valid in $\mathbf{M}$, $v(A(P_1, \ldots, P_m), w_i) = t$ for every assignment of values from $\{t, f\}$, for every w_i. Hence, $v(A^+(C_1, \ldots, C_m), w_i) = t$, for every w_i, for each assignment of values from $\{t, f, n\}$ such that $v(C_j, w_i) = t$ or f, for each C_j. But for every assignment of values from $\{t, f, n\}$ to the variables in each C_j, $v(C_j, w_i) = t$ or f, since C_j contains no uncovered variables. Hence $v(A^+(C_1, \ldots, C_m), w_i) = t$ in every w_i and for all assignments from $\{t, f, n\}$ to the variables in A^+.

Thus, if A^+, B^+ are wff of $\mathbf{L}_0$ which contain only covered variables (effectively two-valued wff), every **RSL**-thesis[3] can be "represented" in terms of them as a valid schema of $\mathbf{L}_0$. For example, $A^+ \supset (A^+ \vee B^+)$, $A^+ \vee \sim A^+$, etc., are valid schemata of $\mathbf{L}_0$. In particular, then, we have such valid schemata as $LA \supset (LA \vee MB)$, $LA \vee \sim LA$, etc., even if A and B contain uncovered variables. Similarly, $LA^+ \supset A^+$, $A^+ \supset MA^+$, etc., are valid schemata of $\mathbf{L}_0$. Hence $LLp \supset Lp$, $Lp \supset MLp$, etc., are valid wff.

The crucial proviso on Theorem III is that the initial wff, from which a covered variant is developed, must be such that *all* of its variables are uncovered. For it is only in this case that III guarantees the $\mathbf{L}_0$-validity of the covered variant. In fact, if the initial wff A contains one or more covered variables, as well as uncovered variables, covered variants of A are not necessarily valid in $\mathbf{L}_0$, though A is valid in $\mathbf{M}$. Thus, for example, let A be $(q \vee \sim q) \supset L(p \vee \sim p)$, in which q is uncovered but p is not, and let A^+ be $(Lq \vee \sim Lq) \supset L(p \vee \sim p)$; then A^+ is not valid in $\mathbf{L}_0$ since $Lq \vee \sim Lq$ takes the value t in every world but $L(p \vee \sim p)$ takes the value f in a related world (see Theorem VI). Hence, even though a wff contains no uncovered variables and is valid in $\mathbf{M}$, this is no guarantee that it is valid in $\mathbf{L}_0$. Theorems IV-VII establish this result generally for special classes of cases:

Theorem IV *No formula of the form LA is valid, where A contains an uncovered variable.*

Proof: Let P be uncovered in A. Then, as in Theorem II, there is an occurrence of P in A such that the assignment of n to P in some world w_j

determines the value of A in w_j as n. By S4, therefore, LA takes the value f in any w_i such that $w_i R w_j$. Hence LA is not valid.

So, for example, formulas such as $L(p \vee \sim p)$ which are valid in **M** are not valid in $\mathbf{L}_0$. In general, if α is a valid **RSL**-wff, $L\alpha$ is a valid wff of **M**, but if A is the three-valued analogue of α, neither A nor LA is valid in $\mathbf{L}_0$.

In spite of these results, all theses of **RSL**, and many theses of **M**, can be restored in a weak form in $\mathbf{L}_0$ simply by prefixing certain formulas with M. Thus,

Theorem V *If A is valid in* **M**, *and if for every assignment to the variables in A which includes the assignment n to at least one variable in any world* w_i *the value of A is t or n in every* w_j *such that* $w_i R w_j$, *then MA is valid in* $\mathbf{L}_0$.

Proof: Since A is valid in **M**, it takes the value t in every world for every assignment from $\{t, f\}$ to the variables in A. So by S5, the value of MA is t for every such assignment in every world. Consider now assignments which include the value n. If, for all such assignments in a world w_i, the value of A is t or n in every w_j such that $w_i R w_j$, then the value of MA is t in every w_i such that $w_i R w_j$. But we are given that this is so for every assignment which includes the value n in any world w_i. Hence the value of MA is t for all assignments in every world provided R is reflexive. So MA is $\mathbf{L}_0$-valid.

In consequence, all theses of **RSL** and certain theses of **M** which are not theses of $\mathbf{L}_0$ come through in the form MA, where A is the **RSL**- or **M**-thesis in question: e.g., $M(p \vee \sim p)$, $M(p \supset . \ p \vee q)$, $M(Lp \supset p)$, and $M(p \supset Mp)$ are valid in $\mathbf{L}_0$. However, not all theses of **M** which are invalid in $\mathbf{L}_0$ can be restored in this way and more interesting differences between the two sets of theses arise from the following result:

Theorem VI *No formula of the form* $B \supset LA$ *is valid in* $\mathbf{L}_0$ *where,* (i) *all the variables in B are covered;* (ii) *B does not take the value f in every world;* (iii) *A contains at least one uncovered variable which does not occur in B.*

Proof: There will be a world w_i for which values can be assigned to the variables in B such that B takes the value t in w_i. For since all the variables in B are covered, B cannot take the value n in any world (by S4, S5, and the first two clauses of S2 and S3), and we are given that B does not take the value f in every world. Now let P be the variable which occurs in A but not in B. The value assignment to B in w_i does not determine an assignment to P since P does not occur in B. Moreover, at least one occurrence of P in A is not within the scope of an unlimited connective, by (iii). This occurrence may therefore be assigned the value n in w_i. Hence, as in IV, A takes the value n in w_i and LA takes the value f. Hence $B \supset LA$ takes the value f in w_i.

So, for example, $Lp \supset L(p \vee q)$ is not valid in $\mathbf{L}_0$, though it is valid in **M**.

But it cannot be restored in $\mathbf{L}_0$ in the form $M(Lp \supset L(p \vee q))$ since this is not valid either. Thus, it follows from VI that there is an assignment of values in some world w_i such that the value of $Lp \supset L(p \vee q)$ is f in w_i; but if $M(Lp \supset L(p \vee q))$ were valid, $Lp \supset L(p \vee q)$ would have to take the values t or n for every assignment in every world.

A further consequence of VI is that $\mathbf{L}_0$ does not contain the paradoxes of strict implication since $L \sim p \supset L(p \supset q)$ and $Lq \supset L(p \supset q)$ are invalid—though the special cases which arise by taking q to be p are valid (such formulas fail to satisfy condition (iii) of VI, so it is not the case that, e.g., $L \sim A \supset L(A \supset B)$ is invalid for all B). On the other hand, the resultant temptation to regard $L(p \supset q)$ as a better characterization of entailment than two-valued strict implication has to be tempered by the fact that we also lose all laws of addition such as $L(p \supset q) \supset L(p \,\&\, p' \,.\!\supset\!.\, q)$, $L(p \supset q) \supset L(p \vee p' \,.\!\supset\!.\, q \vee p')$, and $L(p \supset q) \supset L(p \,\&\, p' \,.\!\supset\!.\, q \,\&\, p')$, and these are often regarded as good entailment principles. However, since some of them have been challenged in the case of 'if', rather than 'necessary if' (entailment), $L(p \supset q)$ may have some desirable features if interpreted as a straightforward conditional.

It should be noticed that formulas of the form $B \supset LA$ do not automatically fail to be valid if B contains more variables than A: for example, $L(p \,\&\, q) \supset Lp$ is valid. The condition which shows, say, $Lp \supset L(p \vee q)$ to be invalid, namely the assignment of value n to the "extra" variable q in some world w_i, does not here have the same effect. If q is assigned value n in w_i, then $L(p \vee q)$ has value f in w_i (by S3 and S4), and this is not inconsistent with Lp having the value t in w_i; but though the assignment of n to q in w_i will similarly determine the value of $L(p \,\&\, q)$ to be f in w_i, this then establishes the value of the whole formula $L(p \,\&\, q) \supset Lp$ as t in w_i, whatever the value of Lp, since Lp cannot take the value n.

Similarly, formulas of the form $B \supset MA$ are not automatically ruled out whether A contains more variables than B, or B contains more than A. Suppose P occurs in A but not in B. Then if there is an assignment which gives B the value t in some world w_i, we shall not automatically ensure the falsity of MA in w_i by taking that assignment together with the assignment of n to P. On the contrary, in such a case A takes the value n if P is uncovered in A, so MA takes the value t in w_i. As an example, $Lp \supset M(p \vee q)$ is valid. Again, if B contains more variables than A, then just as formulas of the form $B \supset LA$ are not automatically invalid, neither are those of the form $B \supset MA$, and in fact there are theses of this type, e.g., $L(p \,\&\, q) \supset Mp$. What does follow as a corollary from VI, however, is:

Theorem VII *No formula of the form $MA \supset B$ is valid in $\mathbf{L}_0$ where,* (i) *all the variables in B are covered;* (ii) *B does not take the value t in every world;* (iii) *A contains at least one uncovered variable which does not occur in B.*

Thus, $M(p \,\&\, q) \supset Mp$ is not valid in $\mathbf{L}_0$, though it is valid in $\mathbf{M}$.

In general, then, formulas of the form $C \supset D$, in which all the variables are covered, may be theses whether C contains more variables than D or D

more than C. The results VI and VII automatically eliminate certain special cases where the variables are not the same on both sides, but generally speaking the "relevance" requirement of shared variables in $\mathbf{L}_0$, though strong enough to rule out some objectionable cases, is a comparatively weak one.

Having said mainly which formulas are not valid in $\mathbf{L}_0$, it is time to consider which formulas are. We first establish the obvious result:

Theorem VIII *Every valid formula of* $\mathbf{L}_0$ *is a valid formula of* $\mathbf{M}$.

Proof: For any formula which is valid over the full range $\{t, f, n\}$ will be valid if the range is restricted to $\{t, f\}$. But the valuation rules and the definition of validity for $\mathbf{L}_0$ are identical with the valuation rules and definition of validity for $\mathbf{M}$ over the range $\{t, f\}$.

Hence, since II establishes that not all $\mathbf{M}$-valid formulas are $\mathbf{L}_0$-valid, and in fact limits the set of $\mathbf{L}_0$-valid formulas to what might be called $\mathbf{M}^+$-wff (i.e., wff of $\mathbf{M}$ which contain no uncovered variables), and since IV, VI, and VII establish that not all $\mathbf{M}^+$-wff valid in $\mathbf{M}$ are $\mathbf{L}_0$-valid, we have that the set of valid wff of $\mathbf{L}_0$ is a proper subset of the valid $\mathbf{M}^+$-wff which is in turn a proper subset of the valid wff of $\mathbf{M}$. Thus $\mathbf{L}_0$ is a very weak system.

As a test for validity we adapt the two-valued *reductio* method. There are two differences. First, in assuming for the purposes of *reductio* that there is an assignment in some world w_i such that the formula fails to take the value t in w_i, there are always two cases to consider: (a) that the formula takes value n in w_i; (b) that the formula takes value f in w_i. Then, for the *reductio* to be established, it has to be shown that *both* (a) and (b) lead to inconsistent assignments. However, since we know that all valid wff occur among $\mathbf{M}^+$-formulas, and since the assignment n to such formulas is always inconsistent in every world, by S4, step (a) is immediate. Secondly, if LA is assigned value f in some world w_i, there is a world w_j, related to w_i, in which A has value f or n. Both cases have then to be considered, and both have to be shown to lead to inconsistency. Thus, the technique is similar to that employed in the two-valued case in which, e.g., an equivalence formula is assigned value f in a world, since this gives rise to two cases each of which has to be shown to lead to inconsistency. Three examples are given below to illustrate the method.

$\mathbf{L}_0 1$. $LA \supset MA$

(a) $LA \supset MA$
$\quad\quad n$
inconsistent by S4, S5

(b) $LA \supset MA$

$$w_1 \begin{cases} t \quad f\, f & \text{S2, S3} \\ \;\; t \quad\; f & \text{S4, S5} \end{cases}$$

inconsistent.

Here, the value n plays no part in (b). It does, however, play a part in the (b)-case of the following:

$L_0$2. $L(A \supset B) \supset (MA \supset MB)$

(a) -case inconsistent as before.

(b)
$$\begin{array}{lllllll} & L(A \supset B) & \supset & (MA & \supset & MB) & \\ w_1 & t & f & t & f & f & \text{S2, S3} \\ w_2 & \left\{ \; t \right. & & \begin{array}{c} t \\ \text{or} \\ n \end{array} & & f & \text{S4, S5} \end{array}$$

inconsistent: for if $v(A, w_2) = t$ and $v(B, w_2) = f$, $v(A \supset B, w_2) = f$, by S2 and S3 (this establishes the two-valued **M**-thesis); while if $v(A, w_2) = n$ and $v(B, w_2) = f$, $v(A \supset B, w_2) = n$, by S2 and S3.

As an example of an invalid formula we choose a case which is in fact automatically excluded by VI, since it serves to illustrate the general argument of VI, namely

$$LA \supset L(A \vee B).$$

(a) -case inconsistent as before.

(b)
$$\begin{array}{llllll} & LA & \supset & L(A & \vee B) & \\ w_1 & t & f & f & & \text{S2, S3} \\ w_2 & \left\{ \; t \right. & & & \begin{array}{c} f \\ \text{or} \\ n \end{array} & \text{S4.} \end{array}$$

Now, if $v(A, w_2) = t$ and $v(B, w_2) = t$ or f, w_2 is inconsistent since $v(A \vee B, w_2) = t$ (this again establishes the two-valued **M**-thesis). But if $v(A, w_2) = t$ and $v(B, w_2) = n$, then $v(A \vee B, w_2) = n$; so this assignment is not inconsistent.

Other semantic theses are as follows:

Distributive Laws:

$L_0$3. $L(A \& B) \equiv. LA \& LB$

$L_0$4. $M(A \vee B) \equiv. MA \vee MB$

However, neither of the half-distributive laws, $LA \vee LB .\supset L(A \vee B)$ and $M(A \& B) \supset. MA \& MB$, is valid. A set of values which falsifies the first of these is: $v(LA, w_i) = t$, consequently $v(A, w_i) = t$ and $v(LA \vee LB, w_i) = t$; $v(B, w_i) = n$, consequently $v(LB, w_i) = f$, $v(A \vee B, w_i) = n$ and $v(L(A \vee B), w_i) = f$. Similarly, the values $v(MA, w_i) = f$ and $v(B, w_i) = n$, falsify the second. It should be noticed that neither formula is excluded by any of the general results which have been established.

Transitivity Laws:

$L_0$5. $(A \prec B) \& (B \prec C) .\supset (A \prec C)$

$L_0$6. $(A \prec B) .\supset ((B \prec C) \supset (A \prec C))$

Reductio Laws:

$L_0$7. $(\sim A \prec A) \equiv LA$

$\mathbf{L}_0 8.$ $(\sim A \dashv . B \ \& \sim B) \supset LA$

The converse of $\mathbf{L}_0 8$ fails, being a case of VI.

Modus Ponens Laws:

$\mathbf{L}_0 9.$ $LA \ \& \ (A \dashv B) . \supset LB$

$\mathbf{L}_0 10.$ $MA \ \& \ (A \dashv B) . \supset MB$

Elimination of Necessary Antecedent:

$\mathbf{L}_0 11.$ $LA \ \& \ (A \ \& \ B . \dashv C) . \supset (B \dashv C)$

Disjunctive Syllogism:

$\mathbf{L}_0 12.$ $L(A \ \& \ (\sim A \vee B)) \supset LB$

$\mathbf{L}_0 13.$ $LA \ \& \ L(\sim A \vee B) . \supset LB$

These, of course, are simply variants of $\mathbf{L}_0 9$.

Weak Law of Noncontradiction:

$\mathbf{L}_0 14.$ $\sim L(A \ \& \sim A)$

The strong law $L \sim(A \ \& \sim A)$ fails, however, since A may contain uncovered variables, in which case it is eliminated by II.

Spread Law:

$\mathbf{L}_0 15.$ $L(A \ \& \sim A) \supset LB$

This is not excluded by VI since condition (ii) fails to be satisfied; for since $\sim L(A \ \& \sim A)$ takes the value t in every world ($\mathbf{L}_0 14$), $L(A \ \& \sim A)$ takes the value f in every world. In fact $\mathbf{L}_0 15$ is a straightforward consequence of the paradoxes of material implication since we have $\sim A^+ \supset (A^+ \supset B^+)$, and taking A^+ to be $L(A \ \& \sim A)$ and B^+ to be LB, $\mathbf{L}_0 15$ will follow from $\mathbf{L}_0 14$ by detachment (we show below that detachment preserves validity). Spread laws in each of the forms $(A \ \& \sim A) \supset B$, $(A \ \& \sim A) \supset LB$, and $L(A \ \& \sim A) \supset B$ are excluded by II.

Laws of Extensionality:

$\mathbf{L}_0 16.$ $L(A \equiv B) \supset (LA \equiv LB)$

$\mathbf{L}_0 17.$ $L(A \equiv B) \supset (MA \equiv MB)$.

Derived Rules:

The following rules preserve validity (there are of course others):

$\mathbf{L}_0 \mathrm{R1}.$ Adjunction: $A, B \rightarrow A \ \& \ B$

If A is valid, it takes the value t in every world; and similarly for B. Hence by S2 and S3, $A \ \& \ B$ takes the value t in every world.

$\mathbf{L}_0 \mathrm{R2}.$ Necessitation: $A \rightarrow LA$

If A takes the value t in every world, then it takes the value t in every world w_j related to an arbitrarily chosen world w_i. Hence by S4, the value of LA is t in an arbitrarily chosen world w_i, i.e., in every world.

$\mathbf{L}_0$R3. *Modus Ponens:* $A, A \supset B \rightarrow B$

If A takes the value t in every world and $A \supset B$ takes the value t in every world, then by S2 and S3, so does B.

$\mathbf{L}_0$R4. Extensionality: $(A \equiv B) \rightarrow (LA \equiv LB)$
If $A \equiv B$ is valid, then by the rule of necessitation, so is $L(A \equiv B)$. But $L(A \equiv B) \supset (LA \equiv LB)$ is valid, $\mathbf{L}_0$16. Hence, by the *modus ponens* rule, so is $LA \equiv LB$.

4 *Extensions of* $\mathbf{L}_0$ Systems containing $\mathbf{L}_0$ may be obtained by extending the semantic conditions or the set of wff. In particular, we obtain systems $\mathbf{L}_{10}$, $\mathbf{L}_{20}$ and $\mathbf{L}_{30}$ analogous, respectively, to the Lewis Systems **S4**, **S5**, and the Brouwerian System **B** by imposing appropriate further conditions on R. Thus,

Definition *Val* $\mathbf{L}_{10}$: A wff A is $\mathbf{L}_{10}$-valid iff it is valid when R is both reflexive and transitive.

Definition *Val* $\mathbf{L}_{20}$: A wff A is $\mathbf{L}_{20}$-valid iff it is valid when R is reflexive, transitive, and symmetric.

Definition *Val* $\mathbf{L}_{30}$: A wff A is $\mathbf{L}_{30}$-valid iff it is valid when R is reflexive and symmetric.

Then, just as $\mathbf{L}_0 \subset \mathbf{M}$, so $\mathbf{L}_{10} \subset \mathbf{S4}$, $\mathbf{L}_{20} \subset \mathbf{S5}$, and $\mathbf{L}_{30} \subset \mathbf{B}$; moreover, the relations between $\mathbf{L}_0$, $\mathbf{L}_{10}$, $\mathbf{L}_{20}$, and $\mathbf{L}_{30}$ are exactly the same as the corresponding relations between **M**, **S4**, **S5**, and **B**. Similarly, by requiring R to be connected and/or discrete, other systems containing $\mathbf{L}_0$ can be obtained. In general, for any two-valued modal system a corresponding three-valued semantics can be set up such that the resultant system stands to the original two-valued system as $\mathbf{L}_0$ stands to **M**. There is, that is to say, a network of three-valued systems which exactly parallels the two-valued network and which is connected to the latter at every point by an inclusion relation.

An alternative way of extending $\mathbf{L}_0$ is to introduce further wff, and this can be done in two ways, either separately or combined. The first way consists in adding a further set or sets of variables or constants, while the second consists in adding new primitive connectives. We here look briefly at one simple extension of the first kind which arises by adding a set of restricted variables to $\mathbf{L}_0$. Extensions of the second kind open up a wide field, some aspects of which will be discussed in more detail elsewhere,[2] since any sentential significance logic can be used to provide the base for a set of modal logics.

If we add to the formation rules of $\mathbf{L}_0$,

WO. Restricted sentential variables $(r, s, r', s', \ldots .)$ are wff,

then the set of wff, excluding those which contain L, is exactly the same as that which characterizes the sentential significance logics $\mathbf{S}_0$ and $\mathbf{C}_0$ (see [3], section **5.8**). The semantic rule for restricted variables is simply:

SO. Where R is any restricted variable and $w_i \in W$: $v(R, w_i) = t$ or f.

The other rules remain the same. If we now adopt *Val* as the criterion of validity, then the set of modal logics so defined are modalized versions of $\mathbf{S}_0$ (not of $\mathbf{C}_0$ since both t and n are designated in $\mathbf{C}_0$). In particular, adopting the special criterion of validity *Val* $\mathbf{L}_0$, which requires only reflexivity, we obtain $\mathbf{L}_0\mathbf{S}_0$.

Within $\mathbf{L}_0\mathbf{S}_0$, all valid formulas of **M** come through in terms of restricted variables: e.g., $r \vee \sim r$, $Lr \supset r$, $L(r \supset r)$, etc., are all theses, though the corresponding unrestricted versions, $p \vee \sim p$, $Lp \supset p$, etc., remain invalid. Obviously, too, all valid formulas of $\mathbf{L}_0$ are valid in $\mathbf{L}_0\mathbf{S}_0$. But the set of valid $\mathbf{L}_0\mathbf{S}_0$ formulas is not simply the union of the set of **M**-valid formulas (expressed in terms of restricted variables) and $\mathbf{L}_0$-valid formulas. There are, as well, "mixed" formulas which contain both restricted and unrestricted variables and some of these are valid in $\mathbf{L}_0\mathbf{S}_0$: e.g., $Lp \supset L(p \vee r)$. In particular, all valid formulas of **M** which are invalid in $\mathbf{L}_0$ by virtue of VI or VII are restored in $\mathbf{L}_0\mathbf{S}_0$ for the special case in which the "extra" variables are restricted. For the results, VI and VII both depend on assigning the value n to a variable which occurs uncovered in A, but does not occur at all in B, in formulas of the form $B \supset LA$ and $MA \supset B$. Hence if these variables cannot be assigned the value n, the results fail.

Extended systems $\mathbf{L}_{10}\mathbf{S}_0$, $\mathbf{L}_{20}\mathbf{S}_0$, etc., can then be obtained by imposing further conditions on R, as above.

5 *Variations on* $\mathbf{L}_0$ By varying the semantic conditions on $\mathbf{L}_0$, various related systems can be obtained.

The simplest change is to turn $\mathbf{L}_0$ into a **C**-type logic by varying the definition of validity so that both t and n become designated values, i.e., by adopting, in place of *Val*

Definition *Val'*: A wff A is V'-valid iff, for every model $\langle W, R, v\rangle$ and every $w_i \in W$, $v(A, w_i) = t$ or n.

In particular, if we leave everything else unchanged and impose only reflexivity on R, we obtain a system $\mathbf{L}_{01}$ characterized by:

Val'$\mathbf{L}_{01}$. A wff A is $\mathbf{L}_{01}$-valid iff it is *Val'*-valid when R is reflexive.

Similarly, by imposing transitivity and symmetry conditions as before, we obtain systems $\mathbf{L}_{101}$, $\mathbf{L}_{201}$, $\mathbf{L}_{301}$, which are *Val'*-validity variants of, respectively, $\mathbf{L}_{10}$, $\mathbf{L}_{20}$, and $\mathbf{L}_{30}$.

In each of these systems the result II fails and standard sentential and modal laws containing uncovered variables, e.g., $p \vee \sim p$ and $Lp \supset p$, are restored. The result IV still stands, however; for since S4 remains unchanged, LA cannot be valid if A is such that it can take the value n at a world. So we have the position that although, say, $p \vee \sim p$ is valid, $L(p \vee \sim p)$ is not. Hence, the rule of necessitation fails. So, too, does the rule of extensionality, since $p \vee \sim p \;.\!\equiv.\; Mp \vee \sim Mp$, for example, is now valid but

$L(p \vee \sim p) \equiv L(Mp \vee \sim Mp)$ is not. Similarly, the *modus ponens* rule fails since $p \vee \sim p$ and $(p \vee \sim p) \supset L(p \vee \sim p)$ are valid but $L(p \vee \sim p)$ is not. The results VI and VII still stand, in consequence of S4, so the paradox laws and laws of composition remain invalid.

There is of course a sense in which these systems are interpretationally inconsistent. For if n is a designated value, it seems more reasonable to take the value of LA to be t (or n) at a world when the value of A is n at a related world. Thus one option we might take in place of S4 is:

S′4. Where A is any wff and $w_i \in W$:

$v(LA, w_i) = t$ iff $v(A, w_j) = t$ or n for every w_j such that $w_i R w_j$
$v(LA, w_i) = f$ iff $v(A, w_j) = f$ for some w_j such that $w_i R w_j$.

The consequential rule for M is then:

S′5. $v(MA, w_i) = t$ iff $v(A, w_j) = t$ for some w_j such that $w_i R w_j$
$v(MA, w_i) = f$ iff $v(A, w_j) = f$ or n for every w_j such that $w_i R w_j$.

Since both L and M are still unlimited connectives under these rules, they could be taken in conjunction with the original definition of validity *Val*, as well as with *Val′*, and perhaps more appropriately so, since to take n as a designated value is effectively to destroy its intended interpretation as "nonsignificance".

Taking *Val* with S′4, and varying the conditions on R, we obtain a series of systems $\mathbf{L}_{02}$ (where only reflexivity is imposed), $\mathbf{L}_{102}$ (where transitivity is also required), and similarly, $\mathbf{L}_{202}$ and $\mathbf{L}_{302}$. For each such system, the result II holds, since formulas containing uncovered variables still take the value n for some assignment in some world, and n is not a designated value; but the arguments for IV, VI, and VII all fail. Thus, for example, where A contains an uncovered variable, $A \vee \sim A$ takes the value t or n for every assignment in every world; hence by S′4, $L(A \vee \sim A)$ takes the value t in every world and is therefore valid in terms of *Val*. So the argument for IV collapses and, in a similar way, so do the arguments for VI and VII.

Taking *Val′* with S′4, and again varying the conditions on R, we obtain $\mathbf{L}_{03}$-$\mathbf{L}_{303}$. In such systems both t and n are designated, so II fails; but again, as the above argument shows, IV, VI, and VII also fail, as a consequence of S′4. Nevertheless, $\mathbf{L}_{03}$-$\mathbf{L}_{303}$ do not coincide, respectively, with **M**, **S4**, **S5**, and **B**. In particular, there are formulas which are valid in $\mathbf{L}_0$, hence in **M**, but which are not valid in $\mathbf{L}_{03}$: e.g., $L(A \,\&\, B) \supset. LA \,\&\, LB$. A consistent set of values which falsifies this is: $v(L(A \,\&\, B), w_i) = t$, $v(LA, w_i) = f$, $v(A, w_i) = f$, and $v(B, w_i) = n$; the converse, however, is valid. Hence, although $\mathbf{L}_0$ and $\mathbf{L}_{03}$ intersect, neither one is contained in the other: as a further example, the paradox laws are valid in $\mathbf{L}_{03}$ but the spread law is invalid; the opposite, however, is true of $\mathbf{L}_0$.

With the introduction of *Val′* it becomes possible to develop systems containing primitive limited modal connectives. Like sentential **C**-logics, however, many of them are simply inconsistent. To illustrate, we consider just one of the many possible valuation rules for a limited L:

S″4. For any wff A and any $w_i \in W$:

$v(LA, w_i) = t$ iff $v(A, w_j) = t$ for every w_j such that $w_i R w_j$
$v(LA, w_i) = f$ iff $v(A, w_j) = f$ for some w_j such that $w_i R w_j$ and $v(A, w_j) = t$ or f for every w_j such that $w_i R w_j$
$v(LA, w_i) = n$ iff $v(A, w_j) = n$ for some w_j such that $w_i R w_j$.

The consequential rule for M is then:

S″5. $v(MA, w_i) = t$ iff $v(A, w_j) = t$ for some w_j such that $w_i R w_j$ and $v(A, w_j) = t$ or f for every w_j such that $w_i R w_j$
$v(MA, w_i) = f$ iff $v(A, w_j) = f$ for every w_j such that $w_i R w_j$
$v(MA, w_i) = n$ iff $v(A, w_j) = n$ for some w_j such that $w_i R w_j$.

Here, the first two clauses in each rule are not only similar in form to the corresponding **M**-rules, but are also exactly the same in content, though embedded in a three-valued context, since they require explicitly what is achieved automatically in the two-valued case, that A be truth-valued in every world related to a world in which LA or MA is truth-valued. This requirement is not essential however, and different conditions for limited connectives could be obtained by dropping the second conjunct in the second clause for L and replacing the last clause by $v(LA, w_i) = n$ iff $v(A, w_i) = n$ (cf., [3], p. 448ff).

Given Val' and S″4, we define the systems $\mathbf{L}_{04}$-$\mathbf{L}_{304}$ in the usual way by imposing appropriate conditions on R. However, like functionally complete sentential **C**-logics (see [3], Ch.5.13, p. 354), these are simply inconsistent, though not absolutely inconsistent. Thus, since $p \vee \sim p$ takes the value t or the value n in every world, so $L(p \vee \sim p)$ never takes the value f in a world. Hence it is valid in terms of Val'. But since there is an assignment in some world w_j such that $p \vee \sim p$ takes the value n, so by S″4 $L(p \vee \sim p)$ does not take the value t in any world w_i such that $w_i R w_j$, and in fact never takes the value t since for an arbitrary choice of w_i there will be a world w_j such that $w_i R w_j$ and $v(p \vee \sim p, w_j) = n$. Hence $L(p \vee \sim p)$ takes the value n in every world. But then, by S2, so does $\sim L(p \vee \sim p)$; i.e., $\sim L(p \vee \sim p)$ is also valid. The inconsistency does not spread, however, since the *modus ponens* rule fails. Thus $L(p \vee \sim p)$ & $\sim L(p \vee \sim p)$ is valid and so is $L(p \vee \sim p)$ & $\sim L(p \vee \sim p) .\supset B$, where B is arbitrary, for by S2 and S3 it always takes the value n since the antecedent always takes the value n; because B is arbitrary, however, it may be assigned value f in some world and hence is not valid.

Other systems arise by breaking the definitional connection between L and M, a course which is rational in terms of Val' since if n is a designated value it is inconsistent with the sense of 'possible' to take the value of MA to be f at a world when the value of A is n at a related world, as in S′5. Thus, we might adopt S′4 along with,

S‴5. $v(MA, w_i) = t$ iff $v(A, w_j) = t$ or n for some w_j such that $w_i R w_j$
$v(MA, w_i) = f$ iff $v(A, w_j) = f$ for every w_j such that $w_i R w_j$.

This gives rise to multiple modal systems of various kinds in each of which $LA \supset MA$ and $\sim M \sim A \supset LA$ are valid but $LA \supset \sim M \sim A$ is not. Other kinds of multiple modal systems can be developed if both limited and unlimited primitive modal connectives are introduced.

All the variants of $\mathbf{L}_0$ can be extended, as $\mathbf{L}_0$ itself can, by adding to the class of wff and/or the class of primitive connectives.

6 *Family resemblances* Semantically speaking, the most closely related of the well-known systems to $\mathbf{L}_0$ is Prior's system **Q** ([7], pp. 41-54). Prior's third value of "unstatability" could be taken as the interpretation of n since it satisfies the same conditions as n so far as the sentential connectives are concerned. However, the criterion of validity for **Q** is *Val'*, not *Val*, and the modal connectives L and M are limited, though they are not of course characterized by S″4 and S″5. In fact **Q** is a multiple modal system in L and M. **Q**-variants of the **L**-systems could be developed within the general framework set out here by dropping D4 and adopting appropriate independent semantic rules for L and M.

In making comparisons with other systems, however, it is not necessary to establish semantic connections; one might instead simply compare the sets of theses and rules of derivation. For although the semantic conditions S1-S5 are intended to provide a natural interpretation for the three values t, f, and n, there is no formal reason why they should be interpreted as truth, falsity, and nonsignificance, and indeed no formal reason why formulas containing uncovered variables have to be regarded as three-valued. We could simply regard all formulas as being restricted in application to two-valued sentences and take the semantics as providing a slightly eccentric device for distinguishing valid and invalid formulas. Such an attitude would not be very different from the regularly adopted technical trick of using many-valued matrices as a test for invalidity and consistency in two-valued systems. The only point being made here is the obvious one that any formal semantics is itself independent of an actual interpretation, or application, of the formal system it is designed to elucidate, but looked at from this point of view there are a number of similarities, as well as differences, between the **L**-systems and others. Thus, since $LA \supset A$ is not valid in $\mathbf{L}_0$, though $LA \supset MA$ is, $\mathbf{L}_0$ shares some features with two-valued deontic systems. Again, since LA is not valid, when A is **RSL**-valid, $\mathbf{L}_0$ has some characteristics in common with Lemmon's systems ([6], pp. 176-86). Whether there is a satisfactory two-valued application of $\mathbf{L}_0$, or one of its variants, however, is an open question.

7 *An application of* $\mathbf{L}_0$: *the criterion of verification* Although there may be two-valued applications of $\mathbf{L}_0$, its intended application is three-valued, and in this respect there is one area of philosophical interest on which it seems to throw some light. This concerns the various criticisms and defenses of Ayer's formulation of the criterion of verification:

> A statement is directly verifiable if it is itself either an observation-statement, or is such that in conjunction with one or more observation-statements it entails at least one observation-statement which is not deducible from these other premises alone.

> A statement is indirectly verifiable if it satisfies the following conditions:
>
> first, that in conjunction with certain other premises it entails one or more directly verifiable statements which are not deducible from these other premises alone;
>
> secondly, that these other premises do not include any statement that is not either analytic, or directly verifiable, or capable of being independently established as indirectly verifiable.
>
> A statement is verifiable if, and only if, it is either directly verifiable or indirectly verifiable. (*cf.* [1], p. 13)

This criterion is proposed in a context in which statements can be either true, false, or meaningless; and the whole point of it is to serve as a discriminating device for distinguishing meaningful and meaningless statements. Yet almost every criticism of it has been made using principles of two-valued logic; and this begs the question against the criterion *qua* discriminating device. For it presupposes that the statement to be tested itself satisfies two-valued principles and hence is meaningful. It is not surprising, then, that it can be shown, by apparently impeccable logical arguments, that the criterion entails the meaningfulness of every statement and so fails to distinguish the meaningful from the meaningless. For that conclusion is already built into the very use of two-valued logical principles in the assessment of the criterion's effectiveness. The criticisms which have been made could only be justified on the assumption that two-valued principles carry over unchanged to a three-valued context. But that assumption is false.

A second assumption which has commonly been made is that the entailment relation which plays such a crucial role in the criterion is (two-valued) strict implication. There is, however, no reason to suppose that even if the criterion fails in some respects when entailment is construed as strict implication, so it will fail for a good entailment. On the contrary, the objection that every contradiction is directly verifiable (since a contradiction in conjunction with any observation-statement will entail any other observation-statement) will not stand in the case of a good entailment even in the two-valued case. What is needed, then, for an adequate evaluation of the criterion is a three-valued entailment logic, where the values are t, f, and n, but as yet we do not have one. However, even using the three-valued version of strict implication characterized by $\mathbf{L}_0$, many of the standard criticisms fail. This is most easily shown in terms of $\mathbf{L}_0\mathbf{S}_0$ which contains both restricted and unrestricted variables since observation-statements are truth-valued, and can therefore be taken as values of the restricted variables, while the statements to be tested by the criterion must be taken as values of the unrestricted variables. Thus, using the previous notation, where $r, s, r', s', \ldots,$ take as values sentences which are known to be meaningful (true-or-false) and $p, q, p', q', \ldots,$ take as values sentences which may be either true, false, or meaningless, the condition for direct verifiability can be written in general terms, excluding the specific reference to observation sentences, as:

$$DVp \text{ iff for some } r, \text{ some } s,\ L(p \mathbin{\&} r \mathbin{.\supset} s) \mathbin{\&} {\sim}L(r \supset s).$$

This captures the structure of the condition and the essential requirement that a sentence is meaningful iff in conjunction with a meaningful sentence r it entails another meaningful sentence s, provided r itself does not entail s.

This is no doubt an oversimplification for most purposes, for it only picks out that characteristic of the verification criterion which is essential to it as a condition of significance. But the criterion is meant also to be a condition of empiricalness, and from that point of view it is equally essential that r and s should be observation-statements and not merely any truth-valued statements, including those which are analytic or contradictory. But the main concern here is with the criterion considered as a condition of significance and in this respect the simplification is of value. Moreover, if we make the same kind of simplification in the criterion for indirect verifiability, by lumping together statements which have been shown to be directly verifiable, independently indirectly verifiable, or analytic, and simply classify them all as truth-valued, then we have:

$$IDVp \text{ iff for some } r, \text{ some } s,\ L(p \mathbin{\&} r \mathbin{.\supset} s) \mathbin{\&} {\sim}L(r \supset s).$$

Thus, *qua* significance condition, the pattern of the criterion is the same for both direct and indirect verifiability—not surprisingly, since it was exactly this form which Ayer initially took as the sole criterion ([1], pp. 38-9) and which was only modified and complicated by the distinction between direct and indirect verifiability to meet some of the early criticisms. In fact, however, most of these early criticisms fail, so it is not obvious that the later complication was necessary.

The main objections to the simple version, which is represented by either of the above conditions, are:

1. Since for all p, $L(p \mathbin{\&} {\sim}p \mathbin{.\supset} s)$, and for some s, ${\sim}L({\sim}p \supset s)$, so, for all p, some r and some s, $L(p \mathbin{\&} r \mathbin{.\supset} s) \mathbin{\&} {\sim}L(r \supset s)$. Hence, for all p, p is verifiable.

This fails, however, if we take $\mathbf{L_0S_0}$ as the background logic since $L(p \mathbin{\&} {\sim}p \mathbin{.\supset} s)$ is not valid. Thus, the attempt to cash in on the paradoxes of strict implication does not succeed simply because the paradoxes do not hold in $\mathbf{L_0S_0}$. There is, however, a version of this criticism which does go through, namely that if p is itself logically false, then it is verifiable. The reason for this is simply that if p is logically false, it is meaningful and so satisfies the two-valued paradoxes of strict implication when taken in conjunction with other meaningful sentences. Thus,

2. $L{\sim}p \supset L(p \mathbin{\&} r \mathbin{.\supset} s)$ is valid in $\mathbf{L_0S_0}$ (it is not eliminated by VI since the "extra" variables in the consequent are not unrestricted). Hence,

$$ {\sim}L(r \supset s) \supset (L{\sim}p \supset (L(p \mathbin{\&} r \mathbin{.\supset} s) \mathbin{\&} {\sim}L(r \supset s)))$$

is valid since it follows from $L{\sim}p \supset L(p \mathbin{\&} r \mathbin{.\supset} s)$ by *modus ponens* using the valid schema $(A^+ \supset B^+) \supset (C^+ \supset (A^+ \mathbin{\supset .} B^+ \mathbin{\&} C^+))$. But since we can certainly find some r and some s such that ${\sim}L(r \supset s)$, we have,

$L \sim p \supset$. for some r, some s, $L(p \;\&\; r \;.\!\supset s) \;\&\; \sim L(r \supset s)$; consequently, $L \sim p \supset$ (p is verifiable), for all p.

This objection does not tell against the criterion as a condition of significance simply because it depends on the fact that if p is logically false, it is meaningful; but it does tell against it as a condition of empiricalness. Hence the criterion has to be modified to read:

$$Vp \text{ iff } \sim L \sim p \;\&\; \text{for some } r, \text{ some } s, \; L(p \;\&\; r \;.\!\supset s) \;\&\; \sim L(r \supset s).$$

It is just this extra clause which one would expect to be unnecessary given a good entailment in place of strict implication. It is worth noting, however, that the criterion is not automatically satisfied by taking r to be logically false, for we do not have $L \sim r \supset L(p \;\&\; r \;.\!\supset s)$: this is eliminated by VI and fails for precisely the case for which it is required to fail, namely when p is meaningless.

A different argument which is supposed to establish that every sentence is verifiable is as follows:

3. Since for all p, $L(p \;\&\; (p \supset s) \;.\!\supset s)$ and for some s, $\sim L(p \supset s \;.\!\supset s)$, so for all p, p is verifiable.

This, too, fails in $\mathbf{L_0S_0}$ since $L(p \;\&\; (p \supset s) \;.\!\supset s)$ is not valid. This is not to say, however, that we can never employ a premiss of the form $p \supset s$ to establish the verifiability of p; we can in fact do so if we can show that p entails s, i.e., if $L(p \supset s)$. For $L(p \supset s) \supset L(p \;\&\; (p \supset s) \;.\!\supset s)$ is valid. But that only leads to the conclusion: $L(p \supset s) \supset$. p is verifiable, for all p; and that is unexceptionable since if $L(p \supset s)$ is true, p must be truth-valued. In fact Ayer mentions $L(p \supset s)$ as a possible sole criterion of verifiability, where s is an observation-statement, but rejects it as inadequate since it would eliminate all hypotheticals from the class of verifiable statements ([1], p. 12).

In each of the objections (1) and (3), the condition $L(p \;\&\; r \;.\!\supset s) \;\&\; \sim L(r \supset s)$, which is expressed essentially in terms of a restricted variable r, has been misapplied by taking as an instantiation case of r a wff which contains an unrestricted variable (in the first case $\sim p$, in the second, $p \supset s$). Quite apart from the oddity of supposing that p could in any case be used nonviciously in a test of its own verifiability, the use of p in a wff which is taken to be a proper instantiation case of a restricted variable amounts to the assumption that such a wff satisfies two-valued conditions and hence that p does. Thus the argument in each case presupposes that p is verifiable. That $L(p \;\&\; \sim p \;.\!\supset s)$ and $L(p \;\&\; (p \supset s) \;.\!\supset s)$ are not valid in a context in which the significance of p is not already known is a simple reflection of the fallacy in the original arguments. Certainly, $L(r \;\&\; \sim r \;.\!\supset s)$ and $L(r \;\&\; (r \supset s) \;.\!\supset s)$ are valid, but this only establishes that every two-valued statement r is two-valued: hardly a surprising discovery.

An objection which is held to establish a slightly weaker conclusion than (1) and (3) is that if p is verifiable, so is every conjunction which contains it:

4. Since for all q, $L(p \,\&\, r \,.\!\supset s) \supset L((p \,\&\, q) \,\&\, r \,.\!\supset s)$, we have,

$$L(p \,\&\, r \,.\!\supset s) \,\&\, \sim L(r \supset s) \,.\!\supset L((p \,\&\, q) \,\&\, r \,.\!\supset s) \,\&\, \sim L(r \supset s).$$

So if for some r, some s, the antecedent condition is satisfied, i.e., p is verifiable, so is the consequent, i.e., $p \,\&\, q$ is verifiable, for all q: e.g., 'It is raining and the Absolute is green' is verifiable.

This fails, however, since $L(p \,\&\, r \,.\!\supset s) \supset L((p \,\&\, q) \,\&\, r \,.\!\supset s)$ is not valid; it is in fact a special case of the composition laws eliminated by VI. What does hold is $L(p \,\&\, r \,.\!\supset s) \supset L((p \,\&\, r') \,\&\, r \,.\!\supset s)$, where r' is another restricted variable, but that only establishes that if p is verifiable so is any conjunction containing p and other truth-valued sentences: i.e., if p is verifiable, so is $p \,\&\, r'$. But 'It is raining and the Absolute is green' is not a case of this unless it has already been established that 'The Absolute is green' is meaningful. In fact, once again, just those cases which need to be eliminated are eliminated.

Criticisms of the more complex version in terms of direct and indirect verifiability fail in a similar way. Thus, consider Church's objection ([2], p. 53):

5. Where r_1, r_2, r_3 are three observation sentences no one of which entails any of the others, we have $L((\sim r_1 \,\&\, r_2 .\!\vee\!. r_3 \,\&\, \sim p) \,\&\, r_1 .\!\supset r_3)$, so $(\sim r_1 \,\&\, r_2 .\!\vee\!. r_3 \,\&\, \sim p)$ is directly verifiable, for all p. The argument then proceeds by showing that this directly verifiable statement, call it R, in conjunction with p entails r_2. Hence p is indirectly verifiable provided R does not entail r_2; but if it does entail r_2, then so does $\sim p \,\&\, r_3$, in which case $\sim p$ is directly verifiable. So either p or $\sim p$ is verifiable for all p.

Only the first part of the argument need be considered, however, since, like the earlier criticisms, it fails because the initial entailment principle is invalid. It is valid when, and only when, p is not meaningless; but then, its use begs the question.

It seems clear that the more complex version of the verification criterion is unnecessary insofar as it was designed to meet criticisms of the simple version. In general, the structure of the criterion, when considered simply as a condition of significance, is the same in both versions, and standard criticisms of either fail for the same reasons: that they beg the question against it. The only objection which stands, in terms of an $\mathbf{L_0S_0}$ background, is (2); but what that establishes is the inadequacy of the criterion as a condition of *empirical* significance. Given a better entailment, however, or the extra clause $\sim L \sim p$, (2) is a trivial matter; and in fact it can be shown[2] that the criterion which includes the additional clause $\sim L \sim p$ is a sufficient condition for empirical significance. Whether or not it is also a necessary condition, and hence an adequate criterion of empiricalness, raises different questions.

NOTES

1. Strictly, *S, L* and *E* have to be understood as absorbing quotation functions to justify the required metalinguistic interpretation. *L* is sometimes read as 'is logically true' instead of 'is analytic'; occasionally, when the interpretation is inessential, it is read 'is necessary'.

2. *Logical Empiricism and Essentialism,* to be published.

3. **RSL** is restricted (i.e. classical two-valued) sentential Logic.

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