Notre Dame Journal of Formal Logic
Volume XXI, Number 2, April 1980
NDJFAM

FRAMES VERSUS MINIMALLY RESTRICTED STRUCTURES

JACK C. BOUDREAUX

0 *Introduction* I have argued in [2] that the semantic theory of higher-order languages should be based on what I called the class of minimally restricted structures, *cf.* definition (4) below, rather than the more conventionally acceptable class of frames, *cf.* definition (9) below. In this paper, I will prove that these superficially similar classes of higher-order structures are in fact *semantically* distinguishable from one another and that the latter is isomorphically representable as a *proper subclass* of the former.

1 *Syntactic preliminaries* Let **ST** denote the type-theoretic language which is based on the following system of *type indices*: **i** is the primitive index, i.e., the index assigned to individuals; and if **B** is a finite, but unempty, sequence of indices, then (**B**) is a nonprimitive index. The primitive, i.e., unabbreviated, *lexicon* of **ST** admits the following symbols:

(1) variables of type **a**: u^a, v^a, x^a, y^a, z^a, . . ., with and without numeric subscripts
sentential connectives: $\sim$ (negation), $\rightarrow$ (conditional)
universal quantifier: $\forall$
punctuation: (,)

I will say that $\overline{X}$ (read: X bar) is a **B**-sequence of variables iff **B** is a finite, but unempty, sequence of indices; the length of **B** (i.e., $\mathrm{lh}(\mathbf{B})$), is equal to the length of $\overline{X}$ (i.e., $\mathrm{lh}(\overline{X})$); and for all j, $0 \leqslant j < \mathrm{lh}(\mathbf{B})$, $\overline{X}(j)$ is a variable of type $\mathbf{B}(j)$.

(2) *Well-formed formula, wff*

(i) if $\overline{X}$ is a **B**-sequence of variables, then $x^{(\mathbf{B})}(\overline{X})$ is an atomic wff
(ii) if p is a wff, then $\sim p$ is a wff
(iii) if p and q are wffs, then $(p \rightarrow q)$ is a wff
(iv) if p is a wff, then $\forall x^a p$ is a wff
(v) and nothing is an unabbreviated wff unless its being so follows from (i) through (iv).

Received November 26, 1977

At this point, all of the usual syntactic terminology can be introduced, namely: a quantifier's scope, free and bound occurrences of a variable in a wff, closed wffs (i.e., statements), open wffs, and so on.

I will say that a term $\mathbf{t}^a$ is *free for* x^a in the wff p iff x^a has zero or more free occurrences in p and no free occurrence of x^a in p lies in the scope of a quantifier which uses $\mathbf{t}^a$ as its variable of generalization. If $\mathbf{t}^a$ is free for x^a in p, then $\mathbf{S}^{x^a}_{\mathbf{t}^a}p$ denotes the wff obtained from p by replacing every free occurrence of x^a in p with an occurrence of $\mathbf{t}^a$.

T is a *higher-order theory* iff (i) T contains all instances of the axiom schemas

FI $(p \rightarrow (q \rightarrow p))$
FII $((p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))$
FIII $((\sim p \rightarrow \sim q) \rightarrow ((\sim p \rightarrow q) \rightarrow p))$
FIV $\left(\forall x^a p \rightarrow \mathbf{S}^{x^a}_{\mathbf{t}^a}p\right)$
FV $(\forall x^a(p \rightarrow q) \rightarrow (p \rightarrow \forall x^a q))$, provided x^a has no free occurrence in p

and (ii) T is closed under the *inferential rules*:

MP (Modus Ponens): From p and $(p \rightarrow q)$, infer q
Gen (Generalization): From p, infer $\forall x^a p$.

Let $\mathbf{F}$ denote the inferential closure of the axiom schemas FI through FV alone. Clearly, $\mathbf{F}$ is the 'smallest' higher-order theory; and, for obvious reasons, I will call it the *core* of every higher-order theory.

Finally, in order to avoid unnecessarily long wffs, I will adopt the following customary abbreviations,

(3) $(p \vee q)$ for $(\sim p \rightarrow q)$
$(p \wedge q)$ for $\sim(p \rightarrow \sim q)$
$(p \leftrightarrow q)$ for $\sim((p \rightarrow q) \rightarrow \sim(q \rightarrow p))$
$\exists x^a p$ for $\sim \forall x^a \sim p$.

2 *Minimally restricted structures* Every referential, or Tarskian, semantic theory for **ST** is based on a well-defined class of higher-order structures. In [2], I gave what I think is conclusive evidence that the most reasonable—if not the most natural—semantic basis for **ST** is the class of *minimally restricted* structures, i.e.,

(4) *Definition* Let μ be any nonzero finite ordinal or any initial ordinal (in the sense of von Neuman [4], pp. 269-273), i.e., μ is any ordinal in the series

$$1, 2, 3, 4, \ldots, \omega, \omega_1, \omega_2, \ldots, \omega_\omega, \ldots, \omega_\alpha, \ldots .$$

Then,

(i) S^μ is the *standard structure based on* μ iff

1. $S^\mu[\mathbf{i}] = \mu$
2. $S^\mu[(\mathbf{B})] = \left\{f\colon \mathrm{dm}(f) = \underset{(\mathbf{B})}{\mathbf{X}} S^\mu \wedge \mathrm{rg}(f) \subseteq \{\mathbf{t}, \mathbf{f}\}\right\}$,

where $\bigtimes_{(\mathbf{B})} S^{\mu}$ abbreviates the Cartesian product $\bigtimes_{0 \leq j < \mathrm{lh}(\mathbf{B})} S^{\mu}[\mathbf{B}(j)]$, $\mathbf{t}$ is the truth value "true", and $\mathbf{f}$ is the truth-value "false",

(ii) $\mathfrak{M}$ is a *minimally restricted structure based on* S^{μ}, i.e., an **MR** *structure*, iff for every index $\mathbf{a}$, $\emptyset \neq \mathfrak{M}[\mathbf{a}] \subseteq S^{\mu}[\mathbf{a}]$.

Henceforth, let "the **MR** class" denote the class of all $\mathbf{MR}^{\mu}$ structures, for every ordinal μ satisfying the antecedent of definition (4).

Suppose that $\mathfrak{M}$ is a member of the **MR** class, then φ is an *assignment* to $\mathfrak{M}$ iff φ is a type-preserving map from the variables of **ST** into the *typed universes* of $\mathfrak{M}$; that is, for every variable x^a, $\varphi(x^a) \in \mathfrak{M}[\mathbf{a}]$. If φ and ψ are assignments to $\mathfrak{M}$, then ψ is an $\overline{X}$*-variant* of φ iff for every variable y^b, except possibly for the variables in the sequence $\overline{X}$, $\psi(y^b) = \varphi(y^b)$.

Let φ be an assignment to $\mathfrak{M}$, then the *satisfaction relation* "$\models$" is defined by induction on the length of wffs:

(5) (i) $(\mathfrak{M}, \varphi) \models x^a(y^b \ldots z^c)$ iff $\varphi(x^a)(\varphi(y^b), \ldots, \varphi(z^c)) = \mathbf{t}$
(ii) $(\mathfrak{M}, \varphi) \models \sim p$ iff not $(\mathfrak{M}, \varphi) \models p$
(iii) $(\mathfrak{M}, \varphi) \models (p \rightarrow q)$ iff either not $(\mathfrak{M}, \varphi) \models p$ or $(\mathfrak{M}, \varphi) \models q$
(iv) $(\mathfrak{M}, \varphi) \models \forall x^a p$ iff for every x^a-variant ψ of φ, $(\mathfrak{M}, \psi) \models p$.

It is a straightforward exercise to confirm that if φ and ψ are any assignments to $\mathfrak{M}$ which *agree* on the free variables of p—i.e., for every variable x^a, if x^a is a free variable of p, $\varphi(x^a) = \psi(x^a)$—then $(\mathfrak{M}, \varphi) \models p$ iff $(\mathfrak{M}, \psi) \models p$; *cf.* Mendelson [5], p. 52. Hence, if p is a closed wff, then for all assignments φ and ψ to $\mathfrak{M}$, $(\mathfrak{M}, \varphi) \models p$ iff $(\mathfrak{M}, \psi) \models p$.

$\mathfrak{M}$ *verifies* the wff p, henceforth, $\mathfrak{M} \models p$, iff for every assignment φ to $\mathfrak{M}$, $(\mathfrak{M}, \varphi) \models p$. $\mathfrak{M}$ *falsifies* the wff p, henceforth, $\mathfrak{M} \mid\not\models p$, iff for every assignment φ to $\mathfrak{M}$, not $(\mathfrak{M}, \varphi) \models p$. Obviously, if p is a closed wff, then p is either verified or falsified by every **MR** structure.

If $\mathfrak{M}$ and $\mathfrak{N}$ are **MR** structures which verify the same wffs, i.e.,

(6) $\mathrm{Th}(\mathfrak{M}) = \{p: \mathfrak{M} \models p\} = \{p: \mathfrak{N} \models p\} = \mathrm{Th}(\mathfrak{N})$,

where "$\mathrm{Th}(\mathfrak{M})$" is to be read "the theory of $\mathfrak{M}$", then $\mathfrak{M}$ and $\mathfrak{N}$ will be said to be *elementarily equivalent.*

The wff p is *strongly valid* iff for every **MR** structure $\mathfrak{M}$, $\mathfrak{M} \models p$. The wff p is *strongly invalid* iff for every structure $\mathfrak{M}$, $\mathfrak{M} \mid\not\models p$. If p is a closed wff and is neither strongly valid nor strongly invalid, then p is *strongly factual.* The **MR** class of structures induces a three fold *partition* in the set of closed wffs, i.e.,

Strongly Valid	Strongly Factual	Strongly Invalid

I claimed above that the **MR** class is the most 'reasonable' semantic basis for **ST**. One strong piece of evidence that I can put forward in support of this claim is the following theorem, which I proved in [2],

(7) The Characterization Theorem for Strong Validity *The wff p is strongly valid iff p is derivable in* **F**.

Proof: Cf. [2], Theorem (14). QED

That is, if q is a wff which is derivable in **F**, then q is verified by every **MR** structure. But if q is not derivable in **F**, then there is at least one **MR** structure which does not verify q. Since **F** is the 'smallest' higher-order theory, point (7) implies that, as long as one is investigating *classical* higher-order theories, the **MR** class cannot be significantly 'widened'.[1]

3 *Frames* The reader has probably already noticed that many—in fact, almost all—**MR** structures have a rather unusual property, namely:

(8) if $\mathfrak{M}$ is a *nonstandard* **MR** structure, then there is at least one index $(\mathbf{B})$ such that

$$\forall f\left(f \in \mathfrak{M}[(\mathbf{B})] \rightarrow \underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{M} \subset \mathrm{dm}(f)\right);$$

that is, the product $\underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{M}$ is a *proper subset* of the domain of the functions $f \in \mathfrak{M}[(\mathbf{B})]$.

The reader should be able to convince himself that this property —though "odd"—is essentially harmless. But even so, he may believe that it would be preferable to revise definition (4.ii) along the following more conventionally acceptable lines:[2]

(9) *Definition* Let μ be an ordinal which satisfies the antecedent of definition (4). Then $\mathfrak{D}$ is a member of *the class of frames based on* μ, henceforth, "the $\mathbf{FM}^{\mu}$ class", iff

(i) $\mathfrak{D}[\mathbf{i}] = \mu$

(ii) $\emptyset \neq \mathfrak{D}[(\mathbf{B})] \subseteq \left\{f\colon \mathrm{dm}(f) = \underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{D} \wedge \mathrm{rg}(f) \subseteq \{\mathbf{t}, \mathbf{f}\}\right\}$.

Superficially, the $\mathbf{FM}^{\mu}$ class seems to be simply a minor variant of the $\mathbf{MR}^{\mu}$ class, which, unlike the latter, does not possess the 'odd' property mentioned in (8). However, a closer examination of both classes will show that they differ from one another in very important ways. In order to investigate this topic more thoroughly, I need to introduce some additional metalogical terminology.

(10) *Definition* If $\mathfrak{M}$ is a higher-order structure, i.e., either an **MR** structure or a frame, then "$\bigcup_{\mathrm{ind}} \mathfrak{M}$" will abbreviate "$\bigcup\{\mathfrak{M}[\mathbf{a}]\colon \mathbf{a}$ is an index$\}$". Let $\mathfrak{M}$ and $\mathfrak{N}$ be higher-order structures. Then:

(i) θ is an *into homomorphism* from $\mathfrak{M}$ to $\mathfrak{N}$ iff

1. $\theta\colon \bigcup_{\mathrm{ind}} \mathfrak{M} \xrightarrow[\mathrm{into}]{} \bigcup_{\mathrm{ind}} \mathfrak{N}$,
2. θ is a type-preserving map, i.e., if $f \in \mathfrak{M}[\mathbf{a}]$, then $\theta(f) \in \mathfrak{N}[\mathbf{a}]$;
3. for all $f \in \mathfrak{M}[(\mathbf{B})]$ and all $\langle g_0, \ldots, g_n\rangle \in \underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{M}$,

$$f(g_0, \ldots, g_n) = \theta(f)(\theta(g_0), \ldots, \theta(g_n)).$$

(ii) θ is an *onto homorphism* from $\mathfrak{M}$ to $\mathfrak{N}$ iff $\theta\colon \bigcup_{\mathrm{ind}} \mathfrak{M} \xrightarrow[\mathrm{onto}]{} \bigcup_{\mathrm{ind}} \mathfrak{N}$, and θ satisfies (i.2) and (i.3).

(iii) θ is an *isomorphism* from $\mathfrak{M}$ to $\mathfrak{N}$ iff θ: $\bigcup_{\text{ind}} \mathfrak{M} \xrightarrow[\text{onto}]{1-1} \bigcup_{\text{ind}} \mathfrak{N}$, and θ satisfies (i.2) and (i.3).

(iv) $\mathfrak{M}$ is into homomorphic (resp., onto homomorphism, isomorphic) to $\mathfrak{N}$ iff there is an into homorphism (resp., onto homomorphism, isomorphism) θ from $\mathfrak{M}$ to $\mathfrak{N}$.

Since the **MR** class cannot be significantly 'widened', one suspects that every frame will be identical to some **MR** structure up to isomorphism. The following theorem shows that this suspicion is correct.

(11) Theorem *Let* $\mathfrak{D}$ *be a frame in the* $\mathbf{FM}^{\mu}$ *class. Then there is an* $\mathbf{MR}^{\mu}$ *structure* $\mathfrak{M}$ *such that* $\mathfrak{D}$ *is isomorphic to* $\mathfrak{M}$.

Proof: It will be sufficient to establish the existence of a map θ such that

(i) θ: $\bigcup_{\text{ind}} \mathfrak{D} \xrightarrow[\text{into}]{1-1} \bigcup_{\text{ind}} S^{\mu}$ and θ satisfies (10.i.2) and (10.i.3).

If such a map does exist, then the $\mathbf{MR}^{\mu}$ structure required by the theorem is the structure $\mathfrak{D}^{\theta}$ defined as follows:

(ii) for every index $\mathbf{a}$, $\mathfrak{D}^{\theta}[\mathbf{a}] = \{f\colon \exists g(g \in \mathfrak{D}[\mathbf{a}] \wedge \theta(g) = f)\}$.

The map θ can be defined by induction on the *order* of the type indices, i.e., the order of the index $\mathbf{i}$ is 0, and for every nonprimitive index $(\mathbf{B})$, the order of $(\mathbf{B})$ is one greater than the highest-order index in $\{\mathbf{B}(j)\colon 0 \leq j < \mathrm{lh}(\mathbf{B})\}$.

Case 0. If the order of index $\mathbf{a}$ is 0, then $\mathbf{a}$ is the primitive index. In this case, define θ^0 as the identity function on μ. θ^0 trivially satisfies the conditions in point (i).

Inductive Hypothesis: Assume that a map θ^k satisfying point (i) has been defined for all indices of order $k \leq n$; moreover, assume that $\theta^n = \bigcup_{0 \leq k \leq n} \theta^k$.

Case $n+1$. Let $(\mathbf{B})$ be any index of order $n+1$. Then the following relation can be defined

(iii) for all $f, g \in S^{\mu}[(\mathbf{B})]$, $f \equiv g(\bmod\ \theta^n)$ iff for all $\langle h_0, \ldots, h_m\rangle \in \bigtimes_{(\mathbf{B})} \mathfrak{D}$,

$$f(\theta^n(h_0), \ldots, \theta^n(h_m)) = g(\theta^n(h_0), \ldots, \theta^n(h_m))$$

$S^{\mu}[(\mathbf{B})]$ is the set of *all* functions from $\bigtimes_{(\mathbf{B})} S^{\mu}$ into $\{\mathbf{t}, \mathbf{f}\}$; hence, if A is any subset of $\bigtimes_{(\mathbf{B})} \mathfrak{D}$, A has a θ^n-*characteristic* function in $S^{\mu}[(\mathbf{B})]$, that is, a function f^* such that

(iv) for all $\langle h_0, \ldots, h_m\rangle \in \bigtimes_{(\mathbf{B})} \mathfrak{D}$,

$$f^*(\theta^n(h_0), \ldots, \theta^n(h_m)) = \begin{cases} \mathbf{t}, \text{ if } \langle h_0, \ldots, h_m\rangle \in A \\ \mathbf{f}, \text{ if } \langle h_0, \ldots, h_m\rangle \notin A. \end{cases}$$

Moreover, if f^* is a θ^n-characteristic function for $A \subseteq \bigtimes_{(\mathbf{B})} \mathfrak{D}$ and $f^* \equiv g^*(\bmod\ \theta^n)$, then g^* is a θ^n-characteristic function for A. In the other direction, if f^* is any element of $S^{\mu}[(\mathbf{B})]$, then there is one and only one

$A \subseteq \underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{D}$ such that f^* is the θ^n-characteristic function for A. Now suppose that $d \in \mathfrak{D}[(\mathbf{B})]$, then there is one and only one $A^d \subseteq \underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{D}$ such that

(v) for all $\langle h_0, \ldots, h_m\rangle \in \underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{D}$,

$$d(h_0, \ldots, h_m) = \begin{cases} \mathbf{t}, \text{ if } \langle h_0, \ldots, h_m\rangle \in A^d \\ \mathbf{f}, \text{ if } \langle h_0, \ldots, h_m\rangle \notin A^d. \end{cases}$$

Therefore, we can define the function θ^{n+1} for the index $(\mathbf{B})$ as any function which maps every $d \in \mathfrak{D}[(\mathbf{B})]$ onto a θ^n-characteristic function for A^d. At this point, it is a straightforward exercise to prove that θ^{n+1}, so defined, satisfies all of the conditions in (i). Since $(\mathbf{B})$ was selected arbitrarily, θ^{n+1} is defined for all indices of order $n + 1$; hence, our inductive procedure is concluded. QED

If two structures are identical up to isomorphism, then they are identical in all important respects; e.g., since all isomorphic structures are elementarily equivalent, it is impossible to distinguish one from the other from a purely semantic point of view. Hence, this theorem conclusively establishes that the *addition* of frames to the **MR** class can serve no useful purpose; i.e., since every frame is isomorphic to an **MR** structure, the **FM** *class is isomorphically representable as a subclass of the* **MR** *class*.

Generally speaking, the converse of Theorem (11) is the case only if the **MR** structure in question satisfies a rather strong set-theoretical condition; specifically,

(12) Theorem *Let* $\mathfrak{D}$ *be a frame and* $\mathfrak{M}$ *be an* **MR** *structure. Then* $\mathfrak{M}$ *is isomorphic to* $\mathfrak{D}$ *iff* $\mathfrak{M}$ *satisfies the Extensionality Condition*:

$$\forall f \forall g \left(f, g \in \mathfrak{M}[(\mathbf{B})] \rightarrow \left(\forall \eta \left(\eta \in \underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{M} \rightarrow f(\eta) = g(\eta)\right) \leftrightarrow f = g\right)\right).$$

Proof: From the left to the right, assume that the **MR** structure is isomorphic to the frame $\mathfrak{D}$. Then, by definition (10), there is a map θ such that

(i) $\theta: \bigcup_{\text{ind}} \mathfrak{M} \xrightarrow[\text{onto}]{1-1} \bigcup_{\text{ind}} \mathfrak{D}$ and θ satisfies (10.i.2) and (10.i.3).

Suppose that $f, g \in \mathfrak{M}[(\mathbf{B})]$ and that

(ii) $\forall \eta \left(\eta \in \underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{M} \rightarrow f(\eta) = g(\eta)\right)$.

If we assume that $f \neq g$, then, necessarily, $\theta(f) \neq \theta(g)$. But this implies that there is at least one ordered tuple $\langle h_0, \ldots, h_m\rangle \in \underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{D}$ such that

(iii) $\theta(f)(h_0, \ldots, h_m) \neq \theta(g)(h_0, \ldots, h_m)$,

otherwise, given the definition of frames, $\theta(f) = \theta(g)$, which contradicts (i). But, by our original assumption, $\mathfrak{M}$ is isomorphic to $\mathfrak{D}$; hence,

(iv) $f(\theta^{-1}(h_0), \ldots, \theta^{-1}(h_m)) \neq g(\theta^{-1}(h_0), \ldots, \theta^{-1}(h_m))$,

which contradicts (ii). Therefore, for every $f, g \in \mathfrak{M}[(\mathbf{B})]$, if (ii) is the case, then $f = g$. Since the converse of this result is trivially true, we can conclude that if the **MR** structure $\mathfrak{M}$ is isomorphic to the frame $\mathfrak{D}$, then $\mathfrak{M}$ satisfies the Extensionality Condition. In the other direction, assume that $\mathfrak{M}$ satisfies the Extensionality Condition, and, to simplify the proof, assume that $\mathfrak{M}[\mathbf{i}] = \mu$, for some ordinal satisfying the antecedent of definition (4). Then what must now be proven is that there exists a frame $\mathfrak{D}$ such that $\mathfrak{M}$ is isomorphic to $\mathfrak{D}$. We will construct the frame $\mathfrak{D}$ by induction on the order of the type indices.

Case 0. If $\mathbf{a}$ is an index of order 0, then $\mathbf{a}$ is the primitive index. Since we have assumed that $\mathfrak{M}[\mathbf{i}] = \mu$, in this case, we can let $\mathfrak{M}[\mathbf{i}] = \mathfrak{D}[\mathbf{i}]$. Furthermore, let θ^0 be the identity function on μ, which trivially satisfies the condition of definition (10).

Inductive Hypothesis: Assume that $\mathfrak{D}[\mathbf{a}]$ has been constructed for all indices $\mathbf{a}$ whose order is less than or equal to n; moreover, assume that θ^n is the map such that

(v) $\theta^n: \mathfrak{M}[\mathbf{a}] \xrightarrow[\text{onto}]{1-1} \mathfrak{D}[\mathbf{a}]$ and for all $f \in \mathfrak{M}[\mathbf{a}]$ and $\langle h_0, \ldots, h_m \rangle \in \underset{\mathbf{a}}{\mathsf{X}}\, \mathfrak{M}$,

$$f(h_0, \ldots, h_m) = \theta^n(f)(\theta^n(h_0), \ldots, \theta^n(h_m)).$$

Case $n + 1$. Let $\mathbf{a} = (\mathbf{b}_0, \ldots, \mathbf{b}_m)$ be any index of order $n + 1$. Then, according to the Inductive Hypothesis,

(vi) $\theta^n: \mathfrak{M}[\mathbf{b}_j] \xrightarrow[\text{onto}]{1-1} \mathfrak{D}[\mathbf{b}_j]$,

for every j, $0 \leqslant j \leqslant m$. Define the function Ψ as follows,

(vii) for every $\langle h_0, \ldots, h_m \rangle \in \underset{\mathbf{a}}{\mathsf{X}}\, \mathfrak{M}$,

$$\Psi(\langle h_0, \ldots, h_m \rangle) = \langle \theta^n(h_0), \ldots, \theta^n(h_m) \rangle.$$

Since θ^n satisfies (vi), Ψ is a *one-one onto* function from $\underset{\mathbf{a}}{\mathsf{X}}\, \mathfrak{M}$ to $\underset{\mathbf{a}}{\mathsf{X}}\, \mathfrak{D}$. Thus, for every element $f^* \in \mathfrak{M}[\mathbf{a}]$ there is one and only one element of $\{\mathbf{t}, \mathbf{f}\}^{\underset{\mathbf{a}}{\mathsf{X}} \mathfrak{D}}$, say $\xi^{\mathbf{a}}(f^*)$, such that

(viii) for all $\eta \in \underset{\mathbf{a}}{\mathsf{X}}\, \mathfrak{M}$, $f^*(\eta) = \xi^{\mathbf{a}}(f^*)(\Psi(\eta))$.

Moreover, assume that f^* and g^* are set-theoretically distinct elements of $\mathfrak{M}[\mathbf{a}]$. Since $\mathfrak{M}$ satisfies the Extensionality Condition, there is at least one $\eta \in \underset{\mathbf{a}}{\mathsf{X}}\, \mathfrak{M}$ such that $f^*(\eta) \neq g^*(\eta)$; hence, $\xi^{\mathbf{a}}(f^*)(\Psi(\eta)) \neq \xi^{\mathbf{a}}(g^*)(\Psi(\eta))$, which implies that $\xi^{\mathbf{a}}(f^*) \neq \xi^{\mathbf{a}}(g^*)$. We can now conclude that $\xi^{\mathbf{a}}$ is a *one-one* function from $\mathfrak{M}[\mathbf{a}]$ *into* $\{\mathbf{t}, \mathbf{f}\}^{\underset{\mathbf{a}}{\mathsf{X}} \mathfrak{D}}$. Consequently, the corresponding typed universe of $\mathfrak{D}$ may be defined as follows,

(ix) $\mathfrak{D}[\mathbf{a}] = \{\xi^{\mathbf{a}}(f): f \in \mathfrak{M}[\mathbf{a}]\}$, i.e., $\mathfrak{D}[\mathbf{a}]$ is the image of $\mathfrak{M}[\mathbf{a}]$ under the function $\xi^{\mathbf{a}}$. At this point, it is a straightforward exercise to prove that $\mathfrak{D}[\mathbf{a}]$ and $\xi^{\mathbf{a}}$ have the following properties

(x) 1. $\xi^{\mathbf{a}}: \mathfrak{M}[\mathbf{a}] \xrightarrow[\text{onto}]{1-1} \mathfrak{D}[\mathbf{a}]$,

and

2. for every $f \in \mathfrak{M}[\mathbf{a}]$ and for all $\langle h_0, \ldots, h_m \rangle \in \bigtimes_{\mathbf{a}} \mathfrak{M}$,

$$f(h_0, \ldots, h_m) = \xi^{\mathbf{a}}(f)(\theta^n(h_0), \ldots, \theta^n(h_m)).$$

Since $\mathbf{a}$ was an arbitrarily selected index of order $n + 1$, this function—and, hence, the corresponding typed universe of $\mathfrak{D}$—may be legitimately assumed to be defined for every index of order $n + 1$. Let θ^{n+1} be defined as the set theoretical union of θ^n with $\bigcup\{\xi^{\mathbf{b}}: \mathbf{b}$ is an index of order $n + 1\}$, which completes the inductive construction of $\mathfrak{D}$. Having thus defined $\mathfrak{D}$, it is an easy exercise to confirm that

(xi) $\mathfrak{D}$ is a frame in the sense of definition (9)

and

(xii) $\theta = \bigcup_{0 \leqslant n < \omega} \theta^n$ is an isomorphism from $\mathfrak{M}$ to $\mathfrak{D}$. QED

Since the Extensionality Condition is rather strong, it should hardly be surprising that there are very many **MR** structures which do not satisfy it; e.g., let ${}^2\mathfrak{M}$ be the **MR** structure defined as follows

(13) ${}^2\mathfrak{M}[\mathbf{i}] = \{2\alpha: 0 \leqslant \alpha < \omega\}$
${}^2\mathfrak{M}[\mathbf{a}] = S^{\omega}[\mathbf{a}]$, for every nonprimitive index $\mathbf{a}$.

Suppose that $f \in {}^2\mathfrak{M}[(\mathbf{i})]$, then there are infinitely many—in fact, *continuum* many—elements of ${}^2\mathfrak{M}[(\mathbf{i})]$, say g, such that for all $\alpha \in {}^2\mathfrak{M}[\mathbf{i}]$, $f(\alpha) = g(\alpha)$ but $f \neq g$, i.e., the functions f and g assign the same values to the even ordinals but different values to the odd ordinals which are not elements of ${}^2\mathfrak{M}[\mathbf{i}]$. Thus, ${}^2\mathfrak{M}$ does not satisfy the Extensionality Condition, which means that *there can be no frame* $\mathfrak{D}$ *such that* ${}^2\mathfrak{M}$ *is isomorphic to* $\mathfrak{D}$.

The combined effect of both of the preceding theorems is to establish that the **FM** class—in its entirety—is isomorphically representable as a *proper subclass* of the **MR** class, that is, as the class of all **MR** studies which satisfy the Extensionality Condition; henceforth, "*the class of extensional* **MR** *structures*". I will now prove that the class of extensional **MR** structures is not essentially identical to the **MR** class. First, let us define the following schema:

The Generalized Extensionality Schema: Let p be any atomic wff such that: (i) u^a has at least one occurrence in p, (ii) every variable which has at least one occurrence in p has at most one occurrence in p, (iii) there is at least one variable of type $\mathbf{b}$ in p such that the order of $\mathbf{b}$ is greater than the order of $\mathbf{a}$, (iv) "$\forall *$" is a block of quantifiers using all the variables of p other than u^a, and (v) the variables x^a and y^a are foreign to p. Then

$$\mathbf{GEx}\quad \forall x^a \forall y^a \left(\forall \overline{X}(x^a(\overline{X}) \leftrightarrow y^a(\overline{X})) \rightarrow \forall^* \left(\mathsf{S}^{u^a}_{x^a} p \leftrightarrow \mathsf{S}^{u^a}_{y^a} p\right)\right)$$

is the Generalized Extensionality Schema.[3]

(14) Theorem *Let $\mathfrak{M}$ be an extensional* **MR** *structure. Then $\mathfrak{M}$ verifies every instance of* **GEx**.

Proof: Assume that $\mathfrak{M}$ is an extensional **MR** structure and that φ is any assignment to $\mathfrak{M}$ such that

(i) $(\mathfrak{M}, \varphi) \models \forall \overline{X}(x^a(\overline{X}) \leftrightarrow y^a(\overline{X}))$.

Then for all $\overline{X}$-variants of φ, say ψ,

(ii) $(\mathfrak{M}, \psi) \models (x^a(\overline{X}) \leftrightarrow y^a(\overline{X}))$;

hence,

(iii) for every $\eta \in \bigtimes_{\mathbf{a}} \mathfrak{M}$, $\varphi(x^a)(\eta) = \varphi(y^a)(\eta)$.

But according to the Extensionality Condition, (iii) implies

(iv) $\varphi(x^a) = \varphi(y^a)$.

Now assume that

(v) not $(\mathfrak{M}, \varphi) \models \forall^* \left(\mathsf{S}^{u^a}_{x^a} p \leftrightarrow \mathsf{S}^{u^a}_{y^a} p\right)$,

then for at least one $\forall^*$-variant of φ, say ψ^*,

(vi) not $(\mathfrak{M}, \psi^*) \models \left(\mathsf{S}^{u^a}_{x^a} p \leftrightarrow \mathsf{S}^{u^a}_{y^a} p\right)$,

which contradicts (iv). QED

(15) Theorem *No instance of* **GEx** *is strongly valid*, i.e., *if q is an instance of* **GEx**, *then there is an* **MR** *structure which falsifies q.*

Proof: Let q be an instance of **GEx** which has the following properties: the initial quantifiers of q use variables of type $\mathbf{a} = (\mathbf{b}_0, \ldots, \mathbf{b}_n)$, and the highest-order variable in the block of quantifiers $\forall^*$ is a variable of the type $(\mathbf{AaB})$, where A and B are possibly empty sequences of indices. (N.B.: The antecedent conditions of **GEx** guarantee that a variable of this kind does exist; specifically, it is the variable occupying the "relation" place in the atomic wff represented by the schematic letter "p" in **GEx**. Having thus selected the types $\mathbf{a}$ and $(\mathbf{AaB})$, the instance of **GEx** is uniquely determined up to the relettering of bound variables.) We can now define an **MR** structure $\mathfrak{N}$ in two stages.

Stage I. Let $\mathbf{b}_j$, $0 \leq j \leq n$, be the highest-order index in $\mathbf{a}$, or if there is more than one such index, the one with the smallest subscript. Then:

(i) if $\mathbf{c}$ is an index whose order is less than or equal to the order of $\mathbf{b}_j$ and $\mathbf{c}$ is not identical to $\mathbf{b}_j$, then ${}^{\omega}\mathfrak{N}[\mathbf{c}] = S^{\omega}[\mathbf{c}]$;

(ii) if $\mathbf{c}$ is identical to $\mathbf{b}_j$, then ${}^{\omega}\mathfrak{N}[\mathbf{c}] \neq S^{\omega}[\mathbf{c}]$;

(iii) ${}^{\omega}\mathfrak{N}[\mathbf{a}] = S^{\omega}[\mathbf{a}]$; and

(iv) ${}^{\omega}\mathfrak{N}[(\mathbf{AaB})] = S^{\omega}[(\mathbf{AaB})]$.

Stage II. Assume that $\mathbf{c}$ is an index whose order is greater than the order of $\mathbf{b}_j$ and that $\mathbf{c}$ is not identical to $\mathbf{a}$ or to $(\mathbf{AaB})$. Then ${}^{\omega}\mathfrak{N}[\mathbf{c}]$ is defined by induction on the order of $\mathbf{c}$. That is, suppose that $\mathbf{c} = (\mathbf{d}_0, \ldots, \mathbf{d}_m)$ and that ${}^{\omega}\mathfrak{N}[\mathbf{d}_k]$, $0 \leqslant k \leqslant m$, has already been defined. Then define the following equivalence relation on $S^{\omega}[\mathbf{c}]$,

(v) for all $f, g \in S^{\omega}[\mathbf{c}]$, $f \equiv g(\mathrm{mod}\,{}^{\omega}\mathfrak{N})$ iff for all $\eta \in \underset{\mathbf{c}}{\mathsf{X}}\, {}^{\omega}\mathfrak{N}\, f(\eta) = g(\eta)$.

Then define ${}^{\omega}\mathfrak{N}[\mathbf{c}]$ as follows,

(vi) ${}^{\omega}\mathfrak{N}[\mathbf{c}]$ is a *set of representatives* of the equivalence relation "$\equiv(\mathrm{mod}\,{}^{\omega}\mathfrak{N})$" in the field of $S^{\omega}[\mathbf{c}]$, *cf.* Kuratowski and Mostowski [4], p. 69.[4]

It must now be shown that ${}^{\omega}\mathfrak{N}$ does falsify the selected instances of **GEx**. By point (ii), ${}^{\omega}\mathfrak{N}[\mathbf{b}_j] \subset S^{\omega}[\mathbf{b}_j]$. Hence, there is at least one f^* in $S^{\omega}[\mathbf{b}_j]$ which is not an element of ${}^{\omega}\mathfrak{N}[\mathbf{b}_j]$. Since ${}^{\omega}\mathfrak{N}[\mathbf{a}] = S^{\omega}[\mathbf{a}]$, there are functions $h^{\#}$, $h^0 \in {}^{\omega}\mathfrak{N}[\mathbf{a}]$, such that

(vii) for all $\langle g_0, \ldots, g_j, \ldots, g_n\rangle \in \underset{\mathbf{a}}{\mathsf{X}}\, {}^{\omega}\mathfrak{N}$,

$$h^{\#}(g_0, \ldots, g_j, \ldots, g_n) \neq h^0(g_0, \ldots, g_j, \ldots, g_n) \text{ iff } g_j = f^*.$$

Thus, if φ is an assignment to ${}^{\omega}\mathfrak{N}$ such that $\varphi(x^a) = h^{\#}$ and $\varphi(y^a) = h^0$, then

(viii) $(\mathfrak{M}, \varphi) \models \forall\overline{X}(x^a(\overline{X}) \longleftrightarrow y^a(\overline{X}))$.

On the other hand, since ${}^{\omega}\mathfrak{N}[(\mathbf{AaB})] = S^{\omega}[(\mathbf{AaB})]$, there is a function $d^* \in {}^{\omega}\mathfrak{N}[(\mathbf{AaB})]$ such that for every $\eta \in \underset{(\mathbf{AaB})}{\mathsf{X}}\, {}^{\omega}\mathfrak{N}$, $d^*(\eta) = \mathbf{t}$ iff the $\mathrm{lh}(A)$th component of η is identical to f^*. If the assignment φ maps the variable occupying the "relation" place in the atomic wff represented by the schematic letter "p" in **GEx** onto d^*, then, trivially,

(ix) not $(\mathfrak{M}, \varphi) \models \left(\mathsf{S}^{ua}_{x^a}p \longleftrightarrow \mathsf{S}^{ua}_{x^a}p\right)$,

which in turn implies

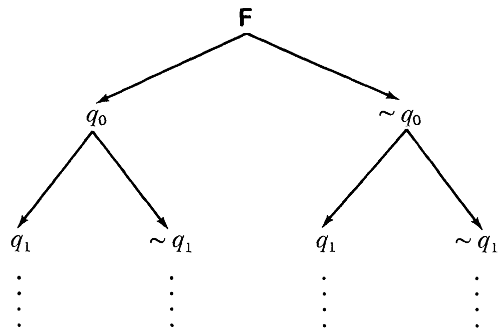
(x) not $(\mathfrak{M}, \varphi) \models \forall^*\, \mathsf{S}^{ua}_{x^a}p \longleftrightarrow \mathsf{S}^{ua}_{x^a}p\big)$.

Therefore, all selected instances of **GEx**, i.e., all alphabetic variants of q, are falsified by ${}^{\omega}\mathfrak{N}$. (N.B. The construction of ${}^{\omega}\mathfrak{N}$ entitles us to draw an even stronger conclusion, namely:

(xi) the instances of **GEx** are *strongly independent* of one another in the **MR** class, i.e., if q is any instance of **GEx**, then there is an **MR** structure, e.g., ${}^{\omega}\mathfrak{N}$, which falsifies q—and all alphabetic variants of q—and which verifies every other instance of **GEx**.

The proof of point (xi) is a relatively straightforward exercise, the details of which I will omit.) QED

Suppose that $Q = \{q_n\colon 0 \leqslant n < \omega\}$ is an enumeration of closed wffs such that: (i) for every n, $0 \leqslant n < \omega$, q_n is an instance of **GEx**, and (ii) for every n and m, $0 \leqslant n, m < \omega$, q_n is an alphabetic variant of q_m iff $n = m$. We can use this enumeration to construct the following tree:



Thus, according to (15.xi), if $\mathbf{P}'$ is *any branch* of this tree, then there is an **MR** structure, say ${}^{P}\mathfrak{N}$, which *models* the higher-order theory $\mathbf{F} \cup \mathbf{P}$, i.e., $\mathbf{F} \cup \mathbf{P} \subseteq \mathrm{Th}({}^{P}\mathfrak{N})$. On the other hand, only the left-most branch of the tree has a model in the **FM** class or, equivalently, the class of extensional **MR** structures. Hence,

(16) if the semantic theory of **ST** is based on the **FM** class, then infinitely many—in fact, *continuum many*—consistent higher-order theories have no models in the semantic theory.[5]

It seems to me that this is a conclusive argument in favor of basing the semantic theory of **ST** on the **MR** class.

NOTES

1. Type-theoretic calculi were originally created to provide logically secure foundations for classical mathematics. Since **F** is woefully inadequate as a foundational system, it is obvious that Strong Validity is much too strong, i.e., it rejects too many wffs. Fortunately, the remedy for this apparent defect is quite simple, specifically, in order to "weaken" Strong Validity, one need only impose some additional conditions, say Φ, on the **MR** class, thereby reducing the member of available **MR** structures. Then one can re-define Validity as follows: a wff p is *Φ-ly valid* iff p is verified by all **MR** structures which satisfy condition Φ. Thus, e.g., if we restrict the available **MR** structures to the class of standard structures, we can obtain the notion "standardly valid". Just as Strong Validity is too strong, so Standard Validity is too weak, i.e., it accepts so many wffs that it is demonstrably impossible to solve the Characterization Problem for Standard Validity with respect to any *axiomatizable* extension of **F**. Taken together, these two notions of validity provide us with a useful criterion for determining whether or not a given notion of Validity is 'reasonable', i.e., Φ Validity is 'reasonable' iff every strongly valid wff is Φ-ly valid, and every Φ-ly valid wff is standardly valid. Thus,

$$\text{Strong Validity} \rightarrow \Phi \text{ Validity} \rightarrow \text{Standard Validity.}$$

The notion "General Validity," as defined in [2], is a clear example of a 'reasonable' notion of Validity; see also note 5 below.

2. This family of higher-order structures was implicitly defined in Henkin's original paper, *cf.* [3], p. 86, from whence it passed into the literature.

3. This schema includes the usual axiom schema of Extensionality

$$\mathbf{Ex}\ \forall x^a \forall y^a (\forall \bar{X}(x^a(\bar{X}) \longleftrightarrow y^a(\bar{X})) \to \forall z^{(a)}(z^{(a)}(x^a) \longleftrightarrow z^{(a)}(y^a)))$$

as a special case. Some authors, e.g., Beth [1], p. 226, express **Ex** as a bicondition. However, as I have already observed in [2], this obscures logically important differences between **Ex** and its converse.

4. The existence of such a set of representatives depends upon the Axiom of Choice; *cf.* [4], p. 69.

5. This point can be rephrased in several ways, but perhaps one of the most interesting is the following: every frame is elementarily equivalent to an **MR** structure, but there are **MR** structures which are elementarily equivalent to no frames, e.g., all **MR** structures which falsify any instance of **GEx**. Thus, if we redefine the notion "valid" in the following way: a wff p is *extensionally valid* iff p is verified by every member of the **FM** class, or equivalently, p is verified by every member of the class of extensional **MR** structures, then Extensional Validity is 'reasonable' in the sense of note 1. It turns out that Extensional Validity is a surprisingly strong notion of validity; in fact, it is strong enough to reject every instance of the *converse* of **GEx**, i.e.,

$$\forall x^a \forall y^a (\forall^* \left(\mathbf{S}^{u^a}_{x^a} p \longleftrightarrow \mathbf{S}^{u^a}_{y^a} p\right) \to \forall \bar{X}(x^a(\bar{X}) \longleftrightarrow y^a(\bar{X})))$$

as the reader can easily confirm for himself.

REFERENCES

[1] Beth, E. W., *The Foundations of Mathematics: A Study in the Philosophy of Science*, 2nd ed., North-Holland, Amsterdam, 1965.

[2] Boudreaux, J. C., "Defining general structures," *Notre Dame Journal of Formal Logic*, vol. XX (1979), pp. 465-488.

[3] Henkin, L., "Completeness in the theory of types," *The Journal of Symbolic Logic*, vol. 15 (1950), pp. 81-91.

[4] Kuratowski, K. and A. Mostowski, *Set Theory*, North-Holland, Amsterdam, 1968.

[5] Mendelson, E., *Introduction to Mathematical Logic*, van Nostrand, Princeton, 1964.

The Catholic University of America
Washington, District of Columbia