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CONSTRUCTIVELY NONPARTIAL RECURSIVE FUNCTIONS

BRUCE M. HOROWITZ

Rose and Ullian [3] called a total function $f(x)$ constructively nonrecursive iff for some recursive function $g(x)$, $f(g(n)) \neq \varphi_n(g(n))$ for all $n \in N$, where $\varphi_n(x)$ is the partial recursive function with index n. We define a partial function $f(x)$ to be constructively nonpartial recursive iff for some recursive $g(x)$, $f(g(n)) \not\simeq \varphi_n(g(n))$, where $\simeq$ is equality for partial functions. We say that $f(x)$ is constructively nonpartial recursive via $g(x)$. Note that for total functions, the two concepts coincide.

An example of a constructively nonpartial recursive function which is a total function is:

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_x(x) \text{ is defined} \\ 0 & \text{otherwise} \end{cases} .$$

Indeed, letting $g(x) = x$, we have

$$f(g(n)) = f(n) = \begin{cases} \varphi_n(n) + 1 \not\simeq \varphi_n(n) = \varphi_n(g(n)) & \text{if } \varphi_n(n) \text{ is defined} \\ 0 \not\simeq \varphi_n(n) = \varphi_n(g(n)) & \text{otherwise} \end{cases}$$

As an example of a constructively nonpartial recursive function which is not total, we have:

$$h(x) = \begin{cases} \text{undefined} & \text{if } \varphi_x(x) \text{ is defined} \\ x & \text{otherwise} \end{cases}$$

$h(x)$ is constructively nonpartial recursive via $g(x) = x$.

The theory of constructively nonpartial recursive functions is intimately connected with the theory of productive sets. As an analogue to the fact that any 1-1 recursive function is the productive function for some set, we have the following:

Theorem 1 *For every* 1-1 *recursive function* $g(x)$, *there is a function* $f(x)$ *which is constructively nonpartial recursive via* $g(x)$.

Proof: Suppose $g(x)$ is a 1-1 recursive function. Let $g^{-1}(x) = (\mu y)(g(y) = x)$; $g^{-1}(x)$ is partial recursive. Define

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$$f(x) = \begin{cases} \varphi_{g^{-1}(x)}(x) + 1 & \text{if } \varphi_{g^{-1}(x)}(x) \text{ is defined} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$f(g(n)) = \begin{cases} \varphi_{g^{-1}(g(n))}(g(n)) + 1 = \varphi_n(g(n)) + 1 & \text{if } \varphi_n(g(n)) \text{ is defined} \\ 0 & \text{otherwise} \end{cases}$$

Thus, $f(x)$ is constructively nonpartial recursive via $g(x)$.

Rose and Ullian showed that the characteristic function of a productive set is constructively nonrecursive. We show a more general form of this theorem for partial functions. Let Df be the domain of $f(x)$.

Theorem 2 *If Df is productive, then $f(x)$ is constructively nonpartial recursive.*

Proof: Suppose $Df = A$ is productive. Then A is completely productive via some recursive function $h(x)$. Let the recursively enumerable sets be defined so that $\omega_n = D\varphi_n$. Now by definition of $h(x)$, if $h(n) \in A$ then $h(n) \notin \omega_n$. Thus, $\varphi_n(h(n))$ is undefined. But $h(n) \in A$ implies $f(h(n))$ is defined. Alternatively, $h(n) \in \widetilde{A}$ implies $h(n) \in \omega_n$, and so $\varphi_n(h(n))$ is defined. But $h(n) \in \widetilde{A}$ implies $f(h(n))$ is undefined. Thus, $f(h(n)) \not\simeq \varphi_n(h(n))$ for all $n \in N$.

Let $\overline{C}_A(x)$ be the partial characteristic function for A, i.e.,

$$\overline{C}_A(x) = \begin{cases} 0 & \text{if } x \in A \\ \text{undefined} & \text{if } x \notin A \end{cases} .$$

Corollary 2a The partial characteristic function of a productive set is constructively nonpartial recursive.

Proof: If A is productive, then $D\overline{C}_A$ is productive.

Corollary 2b There are $2^{\aleph_0}$ constructively nonpartial recursive functions.

Proof: There are $2^{\aleph_0}$ productive sets.

At this point it would be instructive to inquire whether the usual arithmetical operations on functions preserve constructive nonpartial recursiveness. We have:

Theorem 3 *The following do not necessarily preserve constructive nonpartial recursive functions:*

a. *addition*
b. *multiplication*
c. *functional composition.*

Proof: Let A be such that A, $\widetilde{A}$ are productive. Then $C_A(x)$, $C_{\widetilde{A}}(x)$ are total constructively nonpartial recursive functions.

a. $C_A(x) + C_{\widetilde{A}}(x) = 1$, a recursive function.
b. $C_A(x) \cdot C_{\widetilde{A}}(x) = 0$, a recursive function.

We thus see that even the added criterion of totality does not preserve constructive nonpartial recursiveness.

c. Suppose again, that A, $\widetilde{A}$ are productive, and further, that $0 \in \widetilde{A}$. Now

$$\overline{C}_A(x) = \begin{cases} 0 & \text{if } x \in A \\ \text{undefined} & \text{if } x \notin A \end{cases} \quad \text{and} \quad \overline{C}_{\widetilde{A}}(x) = \begin{cases} 0 & \text{if } x \notin A \\ \text{undefined} & \text{if } x \in A \end{cases} .$$

$\overline{C}_A(x)$, $\overline{C}_{\widetilde{A}}(x)$ are constructively nonpartial recursive. Consider $\overline{C}_A(\overline{C}_{\widetilde{A}}(x))$. If $x \in A$, then $\overline{C}_{\widetilde{A}}(x)$ is undefined, hence $\overline{C}_A(\overline{C}_{\widetilde{A}}(x))$ is undefined. If $x \notin A$, then $\overline{C}_{\widetilde{A}}(x) = 0$. Since $0 \in \widetilde{A}$, $\overline{C}_A(\overline{C}_{\widetilde{A}}(x))$ is undefined. Thus, $\overline{C}_A(\overline{C}_{\widetilde{A}}(x))$ is the completely undefined function, and is not constructively nonpartial recursive.

Note that if we also require $1 \in A$ then the total functions C_A and $C_{\widetilde{A}}$ may be used to show $C_A(C_{\widetilde{A}}(x)) = 1$, a total recursive function.

While the first example of this paper shows the converse of Theorem 2 to be false, we do have the following:

Theorem 4 (a) *If $f(x)$ is constructively nonpartial recursive via $g(x)$, such that $\varphi_x(g(x))$ is defined implies $f(g(x))$ is undefined, then Df is productive;* (b) *If, in addition, $f(x)$ is an onto function, then $\widetilde{Df}$ is creative.*

Proof: (a) Assume that $f(x)$ is constructively nonpartial recursive via $g(x)$. Also assume that $\varphi_x(g(x))$ is defined implies $f(g(x))$ is undefined. Let $\omega_n \subseteq Df$. Then $D\varphi_n(x) \subseteq Df$. If $g(n) \in Df$ then $f(g(n))$ is defined; hence $\varphi_n(g(n))$ is undefined, and so, $g(n) \notin D\varphi_n(x) = \omega_n$. Df is productive via $g(x)$.

(b) For any constructively nonpartial recursive function $f(x)$, it must be that $f(g(x))$ is undefined implies $\varphi_x(g(x))$ is defined. Thus, we have $f(g(x))$ is undefined iff $\varphi_x(g(x))$ is defined. Since $g(x)$ is onto N, $f(x)$ is undefined iff $f(g(g^{-1}(x)))$ is undefined iff $\varphi_{g^{-1}(x)}(g(g^{-1}(x)))$ is defined iff $\varphi_{g^{-1}(x)}(x)$ is defined. The ontoness of $g(x)$ guarantees $g^{-1}(x)$ is recursive. Thus $\widetilde{Df}$ is recursively enumerable, and thus creative.

In [2], we showed, directly, that if a set is completely productive via an onto recursive function, then its complement is creative. We now use results of this paper to obtain an interesting proof of this.

Theorem 5 *If A is completely productive via an onto recursive function, then $\widetilde{A}$ is creative.*

Proof: Suppose A is completely productive via $f(x)$, an onto recursive function. Consider $\overline{C}_A(x)$. We know that $D\overline{C}_A = A$. $\widetilde{A} = \{x \mid \overline{C}_A(x) \text{ is undefined}\}$. Since A is productive, by Corollary 2a, $\overline{C}_A(x)$ is constructively nonpartial recursive. In fact, examination of the proof of Theorem 2 shows that $\overline{C}_A(x)$ is constructively nonpartial recursive via $f(x)$, the complete productivity function for A. It is also seen that if $\varphi_n(f(n))$ is defined, then $f(n) \in \omega_n$; hence by the complete productivity of $f(x)$, $f(n) \notin A$. Therefore, $C_A(f(n))$ is undefined. Application of Theorem 4 allows us to conclude $\widetilde{A}$ is recursively enumerable. Hence $\widetilde{A}$ is creative.

We also showed in [2] that the complement of a creative set is completely productive via a recursive permutation. Thus, we have:

Theorem 6 *If A is creative, then $\overline{C}_{\tilde{A}}(x)$ is constructively nonpartial recursive via a recursive permutation.*

Proof: $\tilde{A}$ is completely productive via a recursive permutation $h(x)$. By the proof of Theorem 2, $\overline{C}_{\tilde{A}}(x)$ is constructively nonpartial recursive via $h(x)$.

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Rutgers University
Newark, New Jersey